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Iterative Learning Control for Linear Differential Systems With Additional Performance Requirements

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ABSTRACT This paper develops the systematic procedure for designing of iterative learning control (ILC) algorithms through the differential repetitive process setting. This means that the proposed approach can be directly applied to plants with differential dynamics and allows to satisfy the additional requirements on the resulting dynamics. In particular, the proposed design procedure enforces a required frequency attenuation over a finite frequency range and includes regional pole constraints. Additionally, an important result extension to the plants with relative degree greater than unity is presented. The sufficient conditions for the existence of the controllers are derived in terms of linear matrix inequalities, which are immediately extended to deal with time varying uncertainties. Finally, the simulations for a typical actuator of tracking servo system prove that the design is effective and brings some advantages when compared to the existing alternatives.

INDEX TERMS Convergence analysis, iterative learning control, linear differential repetitive processes, regional pole constraints.

I. INTRODUCTION

Iterative learning control (ILC) is a popular control scheme applicable to the systems that perform a given task iteratively [1], [2]. Each iteration of a given task is known as a trial, or pass, and when a trial is complete, the system resets to the same initial conditions and the next trial can begin. This allows to use the information or data collected during the previous trials, such as control input and error signals, to modify the current control input signals, aiming to track the desired trajectories of the controlled plant and hence the control performance is successively improved. Specifically, ILC scheme aims to construct the control input signal such that the output tracks the reference as accurately as possible. Hence, the basic ILC problem is to design both feedback and learning controllers which produce a such control signal to ensure that the error sequence generated over the trials converges to the prescribed value, see, e.g. [2]–[4]. Driven by this important feature, ILC has been extensively employed in high precision control systems and other important industrial applications. Reported applications of ILC mainly include robotics [5] and chemical batch processing [6].

Clearly, the design of the feedback and learning controllers must include some requirements on transient dynamics and trial-to-trial error convergence. As recognized, see, e.g. [7], for details, an efficient way to do this is to formulate ILC problem as the two-dimensional (2D) stability problem. Within this formulation, the finite time domain behavior and discrete trial number domain behavior can be easily captured and hence both time and trial number domain objectives are easily imposed with the design procedure. Therefore the trial-to-trial convergence and transient response goals can be simultaneously satisfied and hence the high performance tracking control is achieved as shown in [7] and [8]. Although ILC design over 2D/repetitive setting simplifies the integrated synthesis of both feedback and learning controllers in an

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ILC scheme for error convergence and performance, known results on applying this approach, e.g. [8], prefer discrete-time models of a plant since the control algorithm uses information from previous iterations and this information can be only recorded using suitable digital devices. However, as was demonstrated in some previous work on ILC, see [3] or [9], convergence results obtained for discrete-time models can be sometimes over-optimistic and hide problems (e.g. with robustness) when the discrete-time algorithm is applied on a plant that is originally a continuous-time system. Correspondingly, although discrete-time domain seems to be preferred by many authors, e.g. [10], [11], continuous-time model of a plant is more useful for ILC designs. This means that the hybrid (continuous-discrete) nature of the ILC problem should be considered, as it is done when differential repetitive process setting is applied and finite time domain behavior along a given trial and discrete trial domain behaviour are directly captured. Additionally, when designing ILC schemes it is preferred to impose specific performance requirements (for both transient and convergence). Often these requirements are defined over the complete frequency spectrum. This is a very strict condition since design requirements and specifications are mostly defined for different frequency ranges of relevance. For example, a closed-loop feedback control system should have small sensitivity in a low frequency range and small complementary sensitivity in the high frequency range. Moreover, since the bandwidth of the reference signal has the strongest influence on the convergence rate, learning over this frequency range should only occur when the ILC is applied.

The contribution of this paper is to provide new insights into the currently known ILC design procedures with the two-dimensional/repetitive setting. Specifically, systematic guidelines for designing of ILC laws for continuous-time systems subject to restricted frequency-domain specifications and/or structured time-varying uncertainty are proposed. Additionally, the regional pole constraints are included for transient dynamics response shaping. The generalized version of Kalman-Yakubovich-Popov (KYP) lemma [12] is extensively used to permit control law design over selected frequency ranges. These controller design procedures can also include multiple design specifications (e.g. reject disturbances at specific frequencies), whereas the vast majority of currently known designs cannot impose many relevant additional performance specifications. In particular, the developed results allows a designer to specify and/or maximize, frequency ranges where the error convergence condition has to be satisfied. Moreover, this allows design procedures over convex sets and therefore they are amenable to effective algorithmic solution in terms of linear matrix inequalities (LMIs) [13]. Additionally, by the introduction of additional decision variables in the final LMI forms the reduced conservatism may be achieved and hence improve the applicability of developed results. Finally, a numerical example illustrates the effectiveness of the obtained results. To highlight the potential interest in our approach the tracking performance is compared with some known results.

The notation used throughout this paper is as follows. The null and identity matrices with appropriate dimensions are denoted by 0 and I, respectively. Also, for a matrix W, its transpose, complex conjugate transpose and the orthogonal complement are denoted by W^T, W^* and $W_\perp$, respectively. Additionally, for hermitian (symmetric) matrices, $W \succ 0$ ($W \prec 0$) means that W is positive (negative) definite. The operator $\mathrm{sym}\{W\} = W + W^T$ is used to shorten formulas and a symmetric term in a matrix defined by blocks is denoted by ($\star$). The symbol $\otimes$ denotes the Kronecker matrix product. Also, two specific regions of the complex plane are defined: $\mathbb{C}_{hp} = \{s \in \mathbb{C} | \ \mathrm{Re}(s) < 0\}$ (open left half plane) and $\mathbb{C}_{uc} = \{z \in \mathbb{C} | \ |z| < 1\}$ (an open disc centered at the origin and radius 1).

Finally, the following lemmas are used in the proof of the main results.

Lemma 1 [14]: Suppose that the matrices $\Upsilon = \Upsilon^T$, Λ and Σ are given. Then the following statements are equivalent

- there exists an unstructured matrix W that satisfies
$$\Upsilon + \mathrm{sym}\{\Lambda^T W \Sigma\} \prec 0,$$
- for arbitrary matrices $\Lambda_\perp$ and $\Sigma_\perp$ whose columns form, respectively, a basis for the null spaces of Λ and Σ the below inequalities hold
$$\Lambda_\perp^T \Upsilon \Lambda_\perp \prec 0, \text{ and } \Sigma_\perp^T \Upsilon \Sigma_\perp \prec 0. \tag{1}$$

Lemma 2 [15]: Given matrices X, Y, $\Phi = \Phi^T$, $\delta(t)$ of compatible dimensions, then

$$\Phi + \mathrm{sym}\{X\delta(t)Y\} \prec 0,$$

for all $\delta(t)$ satisfying $\delta^T(t)\delta(t) \preceq I$ if, and only if, there exists $\varepsilon > 0$ such that

$$\Phi + \varepsilon XX^T + \varepsilon^{-1} Y^T Y \prec 0.$$

II. BACKGROUND AND PROBLEM FORMULATION

Consider an uncertain linear differential plant executing a given task repeatedly over a finite time interval $[0, T]$, $T > 0$. The plant dynamics can be described as below

$$\begin{aligned} \frac{\partial x_k(t)}{\partial t} &= [A + \Delta A(t)]x_k(t) + [B + \Delta B(t)]u_k(t), \\ y_k(t) &= Cx_k(t), \end{aligned} \tag{2}$$

where $k \geq 0$ denotes the iteration or trial number and t is the continuous time variable such that $t \in [0, T]$. Also, for all $t \in [0, T]$, y_k is the plant output, x_k denotes the state vector and u_k is the control input applied to a plant at trial k. Matrices A, B and C represent the nominal plant dynamics and they are assumed to be time and iteration invariant. Matrices $\Delta A(t)$ and $\Delta B(t)$ denote time-varying uncertainties which are assumed to satisfy

$$\Delta A(t) = E\delta(t)F_a, \ \ \Delta B(t) = E\delta(t)F_b, \tag{3}$$

where E, F_a and F_b are known real constant matrices of compatible dimensions, and $\delta(t)$ is an unknown and time-varying perturbation satisfying $\delta^T(t)\delta(t) \preceq I, \forall t \in [0, T]$. Also, $\delta(t)$ is assumed to be independent of the trial number k and therefore the allowable uncertainties can vary along each trial but are assumed constant from trial to trial. No loss of generality arises from assuming $x_k(t) = x_k(0)$ on each trial.

Let us first consider the nominal dynamics of (2), that is $\delta(t) = 0, \forall t \in [0, T]$. Then this system is said to have the relative degree $\kappa \in \mathbb{N}$, where κ is characterized as

$$\min_{\kappa}\{CA^{\kappa-1}B \neq 0\}.$$

Clearly, if the relative degree $\kappa = 1$, then $CB \neq 0$ and the following formula can be derived along the trial k

$$\frac{\partial y_k(t)}{\partial t} = C\frac{\partial x_k(t)}{\partial t} = CAx_k(t) + CBu_k(t).$$

Consequently, if the relative degree $\kappa = 2$ then $CB = 0$ and $CAB \neq 0$ and hence

$$\frac{\partial^2 y_k(t)}{\partial t^2} = CA^2x_k(t) + CABu_k(t).$$

In general, if the relative degree of the nominal dynamics in (2) is $\kappa > 1$, we get

$$\begin{aligned}\frac{\partial^i y_k(t)}{\partial t^i} &= CA^i x_k(t), 0 \leq i \leq \kappa - 1,\\ \frac{\partial^\kappa y_k(t)}{\partial t^\kappa} &= CA^\kappa x_k(t) + CA^{\kappa-1}Bu_k(t).\end{aligned}$$

Based on the above formula, κ denotes the lowest derivative order of the output $y_k(t)$ that is explicitly fed by the control input $u_k(t)$.

The control problem that we are dealing with is stated as follows. Given plant (2) of the relative degree κ, iteratively find a suitable control input $u(t)$, $t \in [0, T]$ such that the plant output $y(t)$ tracks a desired output trajectory denoted as Y_d over a finite time interval $t \in [0, T]$ where T is known and finite (i.e. $T < \infty$) trial length. Also, let

$$e_k(t) = Y_d(t) - y_k(t), \quad t \in [0, T],\ k > 0$$

be the tracking error of the kth trial generated by the plant based disturbances and transient errors form feedback controller. Therefore, the control objective is to find a control sequence such that tracking errors are minimized (to within a specified tolerance) or removed completely. Furthermore, some additional performance specification can be considered. Specifically, for the prescribed finite frequency performance specification and the pre-specified region of the complex plane, find a control sequence such that the following requirements are simultaneously satisfied:

- the resulting control scheme is convergent over limited frequency domain,
- the poles of the controlled dynamics lie in the prescribed region of the complex plane.

Remark 1: The desired output trajectory Y_d is assumed to be smooth over time interval $[0, T]$. This means that Y_d is differentiable and hence the higher order derivatives of Y_d exist.

Now, let us suppose that a standard form of ILC law (i.e. the means of updating the control vector from trial-to trial) given as

$$u_{k+1}(t) = u_k(t) + \Delta u_k(t) \tag{4}$$

is applied to a plant described by (2) where $\Delta u_k(t)$ is the correction term. This form of control law constructs the input for the next trial as the sum of the previous trial input and $\Delta u_k(t)$, where this last term is computed using the previous trial error. Introducing additional vector-valued variables $\eta_k(t)$ and $\xi_k(t)$ such that

$$\begin{aligned}\frac{\partial \eta_k(t)}{\partial t} &= x_{k+1}(t) - x_k(t),\\ \frac{\partial \xi_k(t)}{\partial t} &= \Delta u_k(t)\end{aligned}$$

one can find that (for plants of the relative degree $\kappa = 1$)

$$\begin{aligned}e_{k+1}(t) - e_k(t) &= -CA\eta_k(t) - CB\xi_k(t),\\ \frac{\partial \eta_k(t)}{\partial t} &= A\eta_k(t) + B\xi_k(t).\end{aligned}$$

In addition, suppose that the control law correction term $\Delta u_k(t)$ in (4) is defined as

$$\Delta u_k(t) = K_1\frac{\partial \eta_k(t)}{\partial t} + K_2\frac{\partial e_k(t)}{\partial t}, \tag{5}$$

where K_1 and K_2 are matrices of compatible dimensions to be designed. Application of the control law given in (5) allows the controlled dynamics to be written as

$$\begin{aligned}\frac{\partial \eta_k(t)}{\partial t} &= \mathcal{A}\eta_{k+1}(t) + \mathcal{B}_0 e_k(t),\\ e_{k+1}(t) &= \mathcal{C}\eta_{k+1}(t) + \mathcal{D}_0 e_k(t),\end{aligned} \tag{6}$$

where

$$\begin{aligned}&\mathcal{A} = A + BK_1, \quad \mathcal{B}_0 = BK_2,\\ &\mathcal{C} = -C(A + BK_1), \quad \mathcal{D}_0 = I - CBK_2.\end{aligned} \tag{7}$$

The boundary conditions are

$$\eta_k(0) = 0, \quad \forall k > 0,\ e_0(t) = Y_d(t).$$

An important point to note about the state-space model (6) is that it has the form of a differential repetitive process where the vector $\eta_{k+1}(t)$ plays the role of the state vector and the vector $e_k(t)$ plays the role of pass profile (output) vector. This means that the problem of feedback and learning gains, i.e. K_1 and K_2 in (5) can be transformed to an equivalent stability problem for repetitive processes - see [7] for detailed discussion on this topic. Anyway, this theory can be applied to the ILC dynamics (6), resulting in the required control design algorithms.

To proceed, define the trial-to-trial shift operator as z and use s as the Laplace transform variable. The Laplace transform can be applied for the along the trial dynamics since it is routine to argue that the trial length is suitably extended from T to ∞ - check [2] and the cited references on how detrimental effects due to the finite trial length are avoided. Consequently, based on transformation applied to (6) we have

$$E_{k+1}(s) = G(s)E_k(s),$$

where $G(s)$ is the so-called intertrial transfer function mapping $e_k(t)$ to $e_{k+1}(t)$ and it is expressed by

$$G(s)=\mathcal{C}(sI-\mathcal{A})^{-1}\mathcal{B}_0+\mathcal{D}_0. \tag{8}$$

Additionally, based on definition of $G(s)$ the model (6) can also be extended to include the case when $\kappa \geq 2$. It is important since the term $CA^{\kappa-1}B$ is non-zero while CB up to $CA^{\kappa-2}B$ are zero and hence a new formulation for the matrices in (7) can be derived for a general relative degree $\kappa \geq 1$. Specifically, consider the term $s^\kappa(sI-\mathcal{A})^{-1}$ and assuming $(sI-\mathcal{A})^{-1}$ is nonsingular we get that $(sI-\mathcal{A})(sI-\mathcal{A})^{-1} = I$ and hence

$$s(sI-\mathcal{A})^{-1} = I+\mathcal{A}(sI-\mathcal{A})^{-1}.$$

This immediately implies that

$$s^\kappa(sI-\mathcal{A})^{-1} = \sum_{j=0}^{\kappa-1} s^{\kappa-1-j}\mathcal{A}^j+\mathcal{A}^\kappa(sI-\mathcal{A})^{-1}.$$

Furthermore, one can find that

$$\begin{aligned}\mathcal{C}\mathcal{A}^{\kappa-1} &= \mathcal{C}A^{\kappa-1} = -CA^{\kappa-1},\\ \mathcal{C}\mathcal{A}^{\kappa} &= -CA^{\kappa-1}\mathcal{A} = -CA^{\kappa-1}(A+BK_1)\end{aligned}$$

and then the matrices in (7) transform into

$$\begin{aligned}\mathcal{A} &= A+BK_1, \quad \mathcal{B}_0=BK_2,\\ \mathcal{D}_0 &= I-CA^{\kappa-1}\mathcal{B}_0, \quad \mathcal{C}=-CA^{\kappa-1}\mathcal{A}.\end{aligned} \tag{9}$$

A. STABILITY ANALYSIS

As discussed earlier, the representation (6) can facilitate stability analysis and control synthesis. Specifically, based on the results provided in [7], [16], [17], the process (6) is said to be stable along the trial if

$$\det(sI-\mathcal{A})\det(zI-\mathcal{D}_0) \neq 0, \quad \text{and}$$
$$\det\left(\begin{bmatrix}\mathcal{A}-sI & \mathcal{B}_0\\ \mathcal{C} & \mathcal{D}_0-zI\end{bmatrix}\right) \neq 0, \quad \forall (s,z)\in \mathbb{C}_{hp}\times\mathbb{C}_{uc},$$

where the state-space quadruple $\{\mathcal{A},\mathcal{B}_0,\mathcal{C},\mathcal{D}_0\}$ are defined in (7) for $\kappa = 1$ or in (9) for $\kappa \geq 2$. It turns out that the above condition is the exact algebraic characterization for stability along the trial. Anyway, by exploring some determinant formulas the above conditions can in turn be reformulated as the following lemma.

Lemma 3 [7]: The differential linear repetitive process (6) representing ILC dynamics is robustly stable along the trial for all admissible uncertainties if, and only if

i) eig $(\mathcal{D}_0) \subset \mathbb{C}_{uc}$,
ii) eig $(\mathcal{A}) \subset \mathbb{C}_{hp}$,
iii) eig $(G(j\omega)) \subset \mathbb{C}_{uc}$, $\forall\omega \in [0,\infty)$.

Checking the third condition of this last result for a numerical example requires computations for the entire frequency range, i.e., all points on imaginary axis in the complex plane. Anyway, one can note that condition *iii*) is equivalent to

$$\rho\left(G(j\omega)\right) < 1, \quad \forall\omega\in[0,\infty), \tag{10}$$

where $\rho(\cdot)$ denotes the spectral radius of its matrix argument. Alternatively, the above result can be expressed by requiring for each $\omega \in [0,\infty)$ the existence of a $R(j\omega) \succ 0$ of the Lyapunov inequality

$$G(j\omega)^*R(j\omega)G(j\omega)-\gamma^2R(j\omega)\prec 0, \quad \forall\omega\in[0,\infty)$$

and a scalar γ satisfying $0 < \gamma \leq 1$. Unfortunately, still the stability along the trial is characterized by a convex feasibility test over the infinite-dimensional space. Furthermore, the function $R(j\omega)$ depends on ω and hence this inequality cannot be easily solved. In what follows, the repetitive process theory shows that the above condition may be not sufficient to achieve reasonable transients during the convergence process. Specifically, $e_k(t)$ may not decrease over some number of trials when (10) is satisfied only. To avoid these problems, a stronger convergence criteria is required for engineering practice. One can ensure that the Euclidean norm of the tracking error decreases monotonically for every trial if $G(j\omega)$ satisfies the sufficient stability condition

$$\overline{\sigma}(G(j\omega)) < \gamma, \quad \forall\omega\in[0,\infty), \tag{11}$$

where $\overline{\sigma}(\cdot)$ stands for the maximum singular value of its matrix argument. Although (10) is the true stability condition, (11) is sometimes more practical. Moreover, (11) is advantageous in dealing with convergence analysis since

$$\overline{\sigma}(G(j\omega)) \leq \sup_{\omega\in[0,\infty)}\overline{\sigma}(G(j\omega)) = \|G(j\omega)\|_\infty.$$

This means that for a given scalar $\gamma \in (0,1]$ we have

$$\overline{\sigma}(G(j\omega)) < \gamma \;\Leftrightarrow\; \|G(j\omega)\|_\infty < \gamma, \; \forall\omega\in[0,\infty).$$

Furthermore, let $||\cdot||_2$ denote the $\mathcal{L}_2$ norm and then

$$\|e_k(t)\|_2 \leq \|G(s)\|_\infty^k \|e_0(t)\|_2\,.$$

Therefore if (11) holds then monotonic trial-to-trial error convergence, i.e., $||e_{k+1}|| < ||e_k||, k \geq 1$, occurs in $\mathcal{L}_2$ for $k \to \infty$. Simply, this means that monotonic convergence in the sense of the $\mathcal{L}_2$-norm occurs.

Additionally, some simple multipliers (e.g. $R(j\omega) = R$ or $R(j\omega) = I$) can be used to avoid computational problems when multipliers with direct dependence on ω are considered. These simple multipliers allow us to apply the generalized KYP lemma and then we can turn our problem into finite-dimensional LMI which are relatively easy to solve and directly leads to the controller design procedures. Note that the generalized KYP lemma and its dual version (given below) provides a necessary and sufficient condition for $G(j\omega)$ to satisfy a specified frequency domain property over a finite frequency ranges (i.e. ω belongs to a subset of $[0,\infty)$)

in terms of a matrix inequality form and hence in can be directly applied to address the paper problem.

Lemma 4 [12]: Let $\mathcal{A}$, $\mathcal{C}$ and Θ be given. Then if $\det(j\omega I - \mathcal{A}) \neq 0$ for all $\omega \in [0, \infty)$ the following conditions are equivalent:

i) The frequency domain inequality

$$\begin{bmatrix} (j\omega I - \mathcal{A}^T)^{-1}\mathcal{C}^T \\ I \end{bmatrix}^* \Theta \begin{bmatrix} (j\omega I - \mathcal{A}^T)^{-1}\mathcal{C}^T \\ I \end{bmatrix} \prec 0 \quad (12)$$

holds $\forall \omega \in \Omega$ where Ω is the frequency range, i.e. ω belongs to a subset of real numbers denoted by Ω and specified as in Table 1.

ii) There exist matrices $Q \succ 0$ and a symmetric matrix P such that

$$\begin{bmatrix} \mathcal{A} & I \\ \mathcal{C} & 0 \end{bmatrix} \left(\Psi^* \otimes Q + \Phi^* \otimes P\right) \begin{bmatrix} \mathcal{A} & I \\ \mathcal{C} & 0 \end{bmatrix}^T + \Theta \prec 0, \quad (13)$$

where

$$\Phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \Psi = \begin{bmatrix} \tau & \upsilon^* \\ \upsilon & \varsigma \end{bmatrix}. \quad (14)$$

The values of τ, υ and ς for specified choices of $\omega \in \Omega$ are shown in Table 1 and LF, MF and HF stand for low, middle and high frequency ranges, respectively.

TABLE 1. Frequency ranges of interest.

	LF	MF	HF
Ω	$\lvert\omega\rvert < \omega_l$	$\omega_1 \leq \omega \leq \omega_2$	$\lvert\omega\rvert > \omega_h$
τ	-1	-1	1
υ	0	$j\frac{\omega_1+\omega_2}{2}$	0
ς	ω_l^2	$-\omega_1\omega_2$	$-\omega_h^2$

Importantly, appropriate choices for the matrix Θ in Lemma 4 allow to analyse the various system properties. In particular, we have to consider two cases concerning the application of different convergence criteria, i.e. (11) and (10), for $\gamma \in (0, 1]$ in the frequency intervals and hence the matrix Θ is fixed as

$$\Theta = \begin{bmatrix} \mathcal{B}_0 & 0 \\ \mathcal{D}_0 & I \end{bmatrix} (\Pi \otimes R) \begin{bmatrix} \mathcal{B}_0 & 0 \\ \mathcal{D}_0 & I \end{bmatrix}^T, \quad (15)$$

where $R \succ 0$ and Π is a given real symmetric matrix (defined later).

Firstly, let us consider the case when $R = I$. This choice means that we are concerning with satisfying (11) for $\gamma \in (0, 1]$ in frequency intervals. Furthermore, according to the work in [18], one can note that introducing two scalar variables $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ satisfying $\varepsilon_1 \leq \varepsilon_2$ then the requirement that $\gamma \in (0, 1]$ can be equivalently replaced by defining $\gamma = \varepsilon_1 \varepsilon_2^{-1}$. This means that the condition $\|G(j\omega)\|_\infty < \gamma \leq 1$ can be expressed as

$$\|\varepsilon_2 G(j\omega)\|_\infty < \varepsilon_1, \quad \forall \omega \in \Omega, \quad (16)$$

where

$$\varepsilon_2 G(j\omega) = \mathcal{C}(j\omega I - \mathcal{A})^{-1}\varepsilon_2\mathcal{B}_0 + \varepsilon_2\mathcal{D}_0, \quad \forall \omega \in \Omega.$$

Let us choose the matrix Π in (15) as $\Pi = \mathrm{diag}\{\varepsilon_1^{-1}, \varepsilon_1\}$ and then immediately (12) yields

$$\varepsilon_2^2 G(j\omega)G(j\omega)^* < \varepsilon_1^2, \quad \forall \omega \in \Omega,$$

which is equivalent to the condition (16). Additionally, it is worthwhile noting that the choice of Π as $\mathrm{diag}\{\varepsilon_1^{-1}, \varepsilon_1\}$ makes possible to rewrite the inequality (13) as

$$\begin{bmatrix} \mathcal{A}^T & \mathcal{C}^T \\ I & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} \Upsilon_1 & \begin{bmatrix} 0 \\ \varepsilon_1^{-1}\varepsilon_2^2\mathcal{B}_0\mathcal{D}_0^T \end{bmatrix} \\ (\star) & \varepsilon_1^{-1}\varepsilon_2^2\mathcal{D}_0\mathcal{D}_0^T - \varepsilon_1 I \end{bmatrix} \begin{bmatrix} \mathcal{A}^T & \mathcal{C}^T \\ I & 0 \\ 0 & I \end{bmatrix} \prec 0, \quad (17)$$

where

$$\Upsilon_1 = \left(\Psi^* \otimes Q + \Phi^* \otimes P\right) + \begin{bmatrix} 0 & 0 \\ 0 & \varepsilon_1^{-1}\varepsilon_2^2\mathcal{B}_0\mathcal{B}_0^T \end{bmatrix}.$$

Furthermore, an equivalent condition to (17) can be directly obtained by means of Lemma 1 (or the Finsler Lemma which is a specialized version of Lemma 1). In particular, the following result is obtained

$$\begin{bmatrix} \Psi^* \otimes Q + \Phi^* \otimes P & 0 & \begin{bmatrix} 0 \\ \varepsilon_2\mathcal{B}_0 \end{bmatrix} \\ 0 & -\varepsilon_1 I & \varepsilon_2\mathcal{D}_0 \\ \begin{bmatrix} 0 & \varepsilon_2\mathcal{B}_0^T \end{bmatrix} & \varepsilon_2\mathcal{D}_0^T & -\varepsilon_1 I \end{bmatrix} + \mathrm{sym}\left\{ \begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ 0 \end{bmatrix} \begin{bmatrix} -I & \mathcal{A}^T & \mathcal{C}^T & 0 \end{bmatrix} \right\} \prec 0, \quad (18)$$

where W_1, W_2 and W_3 are the slack matrix variables. At this point, it is worthwhile noting that ensuring $\varepsilon_2^2\mathcal{D}_0\mathcal{D}_0^T - \varepsilon_1^2 I \prec 0$ (i.e. $\mathcal{D}_0\mathcal{D}_0^T - \gamma^2 I \prec 0$ for $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ satisfying $\varepsilon_1 \leq \varepsilon_2$) imposes that W_3 can be zeroed without loss of generality. Fortunately, $\varepsilon_2^2\mathcal{D}_0\mathcal{D}_0^T - \varepsilon_1^2 I \prec 0$ is not a strong assumption since it holds even for $\mathcal{D}_0 = 0$. Moreover, the matrix variable W_1 can be eliminated from (18) if and only if the block (1,1) of $\Psi^* \otimes Q + \Phi^* \otimes P$ is negative definite for $Q \succ 0$ (always imposed) and Φ as in (14).

Secondly, let us consider the case when $R \neq I$ is fixed in (15) and hence we are concerning with satisfying (10) in frequency intervals. This case seems to be more complicated than the previous one where $R = I$. Therefore, we shall develop a new set of transformations so that the resulting problem becomes convex. Different from the results (when $R = I$) given above, to obtain LMI formulation of design conditions we propose to rewrite the inequality (13) as

$$\Upsilon_2 \Lambda \Upsilon_2^T \prec 0, \quad (19)$$

where

$$\Upsilon_2 = \begin{bmatrix} \mathcal{A} & I & 0 & \mathcal{B}_0 \\ \mathcal{C} & 0 & I & \mathcal{D}_0 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \Psi^* \otimes Q + \Phi^* \otimes P & 0 \\ 0 & \Pi \end{bmatrix} \quad (20)$$

and the matrix Π is chosen as $\Pi = \mathrm{diag}\{-\gamma^2 R, R\}$ for $\gamma \in (0, 1]$.

IMPOSING REGIONAL POLE CONSTRAINTS

In the sequel we turn our attention to extended performance specification to involve the regional pole constraints. Our motivation here is the fact that the transient behavior along a given trial k is closely related to locations all eigenvalues of $\mathcal{A}$. This means that some bounds can be put on the time-domain objective (such as the rise time, the settling time and so on) when assigning all eigenvalues of $\mathcal{A}$ to particular locations in the open left-half of the complex plane. In particular, the region of interest for pole assignment is the interior of the circle of radius $r > 0$ with center at c denoted by $C(c, r)$ and given by

$$C(c, r) := \{x + jy \in \mathbb{C} : |x + jy - c| < r\}.$$

To guarantee that the interior of this circle is located in open left-half complex plane requires $c < 0$ and $|c| >$ r. Clearly, let $\lambda = -\zeta\omega_n \pm j\omega_d$ be a pair of eigenvalues of $\mathcal{A}$, where $0 < \zeta < 1$ is the damping ratio, ω_n is the undamped natural frequency, and $\omega_d := \omega_n\sqrt{1-\zeta^2}$ is the damped natural frequency. Further, suppose that every λ is placed in $C(c, r)$. Then the following conditions are satisfied

$$\zeta > \sqrt{1-\left(\frac{r}{c}\right)^2},\ \omega_d < r,$$
$$-c - r < \omega_n < -c + r,$$
$$-c - r < \zeta\omega_n < -c + r.$$

Moreover, to include the eigenvalue assignment constraints, as an additional design specification, the following condition can be used to give an additional constraint in the design equations discussed previously

$$\begin{bmatrix}\mathcal{A}^T & I\end{bmatrix}\left(\begin{bmatrix}1 & -c\\ -c & |c|^2-r^2\end{bmatrix}\otimes Z\right)\begin{bmatrix}\mathcal{A}\\ I\end{bmatrix}\prec 0, \tag{21}$$

where $Z \succ 0$ is a matrix variable. Clearly, it can be verified that the above inequality is a particular version of Lemma 5 in [19].

III. DESIGN OF MONOTONICALLY CONVERGENT ILC SCHEMES FOR A NOMINAL PLANT MODEL

In this section, the problem of designing the corresponding matrices in the control law (5), such that the resulting ILC scheme is monotonically convergent and meets the considered design specifications, is studied. More specifically, we assume that the system (2) of the relative degree $\kappa \geq 1$ is operated repeatedly in the iteration domain with a desired output Y_d over a finite time interval $t \in [0, T]$ and let updating law (5) be applied. Then we study the condition under which the resulting repetitive process (6) is stable along the trial (and hence ensures monotonic convergence of the ILC scheme in sense of $\mathcal{L}_2$-norm) for given design specifications over finite frequency range together with the regional pole constraints. In view of this condition, feasible ILC controller gain matrices can be given.

The following theorem presents a sufficient condition for the solvability of the considered problem.

Theorem 1: Consider a nominal dynamics of (2) (that is $\delta(t) = 0$) with relative degree $\kappa \geq 1$, and let updating law (5) be applied. Also, let β, c, r, p, q be given scalars where $\beta > 0$, $c > r > 0$ and p, q satisfy $p^2 - 2cpq + q^2(|c|^2 - r^2) < 0$. Then an ILC scheme described as a repetitive process of the form (6) has the stability along the trial property over the finite frequency ranges defined in Table 1 and all eigenvalues of $\mathcal{A}$ are located in the circle of radius r with center at $(-c, 0)$. Moreover, monotonic trial-to-trial error convergence occurs over the same frequency intervals if there exist matrices $Q \succ 0$, $Z \succ 0$, W_2, Y_1, Y_2, a symmetric P, positive scalars ε_1 and ε_2 such that $\varepsilon_2 \geq \varepsilon_1$ and the following LMIs hold

$$\left(\begin{bmatrix}1 & -c\\ -c & |c|^2-r^2\end{bmatrix}\otimes Z\right) + \mathrm{sym}\left\{\begin{bmatrix}-W_2\\ (AW_2^T+BY_1)^T\end{bmatrix}\begin{bmatrix}qI & -pI\end{bmatrix}\right\}\prec 0, \tag{22}$$

$$\begin{bmatrix}\tau Q & \upsilon^*Q+P & 0 & 0\\ (\star) & \varsigma Q & 0 & BY_2\\ (\star) & (\star) & -\varepsilon_1 I & (\varepsilon_2 I - CA^{\kappa-1}BY_2)\\ (\star) & (\star) & (\star) & -\varepsilon_1 I\end{bmatrix} + \mathrm{sym}\left\{\begin{bmatrix}\beta I\\ I\\ 0\\ 0\end{bmatrix}\begin{bmatrix}-W_2 & (AW_2^T+BY_1)^T & \Upsilon_3^T & 0\end{bmatrix}\right\}\prec 0, \tag{23}$$

where a pair of Ψ and Φ is defined by (14) and

$$\Upsilon_3 = -CA^{\kappa}W_2^T - CA^{\kappa-1}BY_1.$$

In addition, if the above LMIs are feasible, the corresponding matrices in the control law (5) can be selected as

$$K_1 = Y_1 W_2^{-T},\quad K_2 = \varepsilon_2^{-1}Y_2. \tag{24}$$

Proof: First of all, it follows immediately that the feasibility of (22) ensures that W_2 is non-singular and hence invertible. Next, the LMI (22) rewritten as a version of the first inequality of Lemma 1 where

$$\Upsilon = \begin{bmatrix}Z & -cZ\\ -cZ & (|c|^2-r^2)Z\end{bmatrix},\ \Lambda = \begin{bmatrix}-I & \mathcal{A}^T\end{bmatrix},\ \Sigma = \begin{bmatrix}qI & -pI\end{bmatrix}$$

and $Y_1 = K_1 W_2^T$. Since $\Sigma_\perp^T \Upsilon \Sigma_\perp \prec 0$ holds for any p, q satisfying $p^2 - 2cpq + q^2(|c|^2 - r^2) < 0$ then the equivalence between (22) and (21) follows from the Lemma 1. Next, we have that LMI (23) is transformed into (19) where $W_1 = \beta W_2$. Clearly, we can arbitrary choose $\beta = 0$ for low and middle frequency ranges since $\tau = -1$ for these frequency ranges - see Table 1. For the high frequency range we have $\tau = 1$ and hence $\beta \neq 0$ must be chosen. Note that (23) becomes the LMI when β is fixed and given. Then the result follow directly from Lemmas 3 and 4. ∎

IV. DESIGN PROCEDURE FOR $R \neq I$

In this section, the conditions of Theorem 1 will be further developed. Since it is required to select the multiplier $R \neq I$, we propose to introduce additional slack matrix variables, with the aid of which, new conditions for designing ILC schemes are proposed. To proceed, the following notations are first defined for future use

$$\begin{bmatrix} \mathcal{A} & \mathcal{B}_0 \\ \mathcal{C} & \mathcal{D}_0 \end{bmatrix} = \begin{bmatrix} A & 0 \\ -CA^\kappa & I \end{bmatrix} + \begin{bmatrix} B \\ -CA^{\kappa-1}B \end{bmatrix} [K_1 \; K_2] = \mathbb{A} + \mathbb{B}\mathbb{K},$$

$$\Upsilon_4 = \begin{bmatrix} \tau Q & 0 \\ 0 & 0 \end{bmatrix}, \; \Upsilon_5 = \begin{bmatrix} \varsigma Q & 0 \\ 0 & -\gamma^2 R \end{bmatrix}.$$

Then the theorem established next gives new LMI-based conditions for monotonic trial-to-trial error convergence.

Theorem 2: Consider a nominal dynamics of (2) (that is $\delta(t) = 0$) with relative degree $\kappa \geq 1$, and let updating law (5) be applied. Also, let γ, β, c, r, p, q be given scalars where $0 < \gamma \leq 1$, $\beta > 0$, $c > r > 0$ and p, q satisfy $p^2 - 2cpq + q^2(|c|^2 - r^2) < 0$. Then an ILC scheme described as a repetitive process of the form (6) has the stability along the trial property over the finite frequency ranges defined in Table 1 and all eigenvalues of $\mathcal{A}$ are located in the circle of radius r with center at $(-c, 0)$. Moreover, monotonic trial-to-trial error convergence occurs over the same frequency intervals if there exist matrices $Q \succ 0$, $Z \succ 0$, W_2, W_{32}, W_{33}, Y_1, Y_2, F_1, F_2, F_3 and a symmetric matrix P such that the LMIs (22) and

$$\begin{bmatrix} \Upsilon_4 - \mathrm{sym}\{\beta \hat{W}_2\} & (\star) & (\star) \\ \Upsilon_6 + \beta(\mathbb{A}\hat{W}_2^T + \mathbb{B}N)^T - \hat{W}_2^T & \Upsilon_5 + \mathrm{sym}\left\{\mathbb{A}\hat{W}_2^T + \mathbb{B}N\right\} & (\star) \\ F_{30} - \hat{W}_3 & -F_{12}^T + [0 \; I](\mathbb{A}\hat{W}_2^T + \mathbb{B}N) & R - F_3 - F_3^T \end{bmatrix} \prec 0, \quad (25)$$

where

$$\Upsilon_6 = \begin{bmatrix} P_1 + \upsilon Q & F_1 \\ 0 & F_2 \end{bmatrix}, F_{12} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, F_{30} = \begin{bmatrix} 0 & F_3 \end{bmatrix},$$

$$N = [Y_1 \; Y_2], \; \hat{W}_3 = [W_{32} \; W_{33}], \; \hat{W}_2 = \begin{bmatrix} W_2 & 0 \\ W_{32} & W_{33} \end{bmatrix}$$

are feasible. In addition, if the above LMIs hold, the corresponding matrices in the control law (5) can be selected as

$$[K_1 \; K_2] = N\hat{W}_2^{-T}. \quad (26)$$

Proof: Assume that the LMIs defined in (22) and (25) are feasible for some given scalars γ, β, p and q. Then it is immediate that the feasibility of (22) ensures the stability of $\mathcal{A}$ and its eigenvalues are located in the circle of radius r with center at $(-c, 0)$. Also, W_2 is non-singular and hence invertible. Next, the LMIs in (25) can be rewritten as

$$\Gamma + \mathrm{sym}\left\{ \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \beta\hat{W}_2 \\ \hat{W}_2 \\ \hat{W}_3 \end{bmatrix} \begin{bmatrix} -I & (\mathbb{A}+\mathbb{B}\mathbb{K})^T & 0 \end{bmatrix} \right\} \prec 0, \quad (27)$$

where

$$\Gamma = \begin{bmatrix} \Upsilon_4 & \Upsilon_6^T & F_{30}^T \\ \Upsilon_6 & \Upsilon_5 & -F_{12} \\ F_{30} & -F_{12}^T & R - \mathrm{sym}\{F_3\} \end{bmatrix}.$$

Introduce the matrices

$$\Lambda = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, W = \begin{bmatrix} \beta\hat{W}_2 \\ \hat{W}_2 \\ \hat{W}_3 \end{bmatrix}, \; \Sigma = \begin{bmatrix} -I & (\mathbb{A}+\mathbb{B}\mathbb{K})^T & 0 \end{bmatrix}$$

and then (27) can be reformulated by application of Lemma 1 as the second inequality in (1), i.e.

$$\Sigma_\perp^T \Gamma \Sigma_\perp \prec 0, \quad (28)$$

where by construction the matrix $\Sigma_\perp$ is

$$\Sigma_\perp = \begin{bmatrix} \mathbb{A}^T & 0 \\ I & 0 \\ 0 & I \end{bmatrix}.$$

Since $\Lambda = I$ then $\Lambda_\perp = 0$ and hence the first inequality in (1) holds. Furthermore, after some routine matrix manipulations the inequality (28) can be rewritten as

$$\Gamma_1 + \mathrm{sym}\left\{ \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \begin{bmatrix} \mathcal{B}_0^T & \mathcal{D}_0^T & -I \end{bmatrix} \right\} \prec 0, \quad (29)$$

where

$$\Gamma_1 = \begin{bmatrix} \tau\mathcal{A}Q\mathcal{A}^T + \varsigma Q + \mathrm{sym}\left\{P\mathcal{A}^T + \upsilon^* Q\mathcal{A}^T\right\} & \tau\mathcal{A}Q\mathcal{C}^T + \upsilon^* Q\mathcal{C}^T + P\mathcal{C}^T & 0 \\ \tau\mathcal{C}Q\mathcal{A}^T + \upsilon\mathcal{C}Q + \mathcal{C}P & \tau\mathcal{C}Q\mathcal{C}^T - \gamma^2 R & 0 \\ 0 & 0 & R \end{bmatrix} \prec 0$$

and by Lemma 1, feasibility of (29) implies that the inequality

$$\Sigma_{1_\perp}^T \Gamma_1 \Sigma_{1_\perp} \prec 0$$

must hold where

$$\Sigma_{1_\perp} = \begin{bmatrix} I & 0 \\ 0 & I \\ \mathcal{B}_0^T & \mathcal{D}_0^T \end{bmatrix}.$$

Finally, this last inequality is equivalent to (20) and by Lemmas 3 and 4 stability along the pass is ensured and hence the monotonic trial-to-trial error convergence is guaranteed. ■

Remark 2: It is immediate that the slack matrix variable $\hat{W}_2$ must have the specific structure and this may introduce a level of conservatism into the design. On the other side, the additional matrix variables F_1, F_2 and F_3 may reduce the level of conservativeness.

Remark 3: The design conditions provided in Theorem 2 are LMIs that can be easily and effectively solved via numerical software. In addition, optimal values of the scalar parameters β, p and q can be sought to reduce the conservatism (in terms of performance provided by γ) of the solutions.

A. SOME COMMENTS ON DEVELOPED RESULTS

While using the results of Theorems 2 and 1 some numerical problems may arise. In particular, a reader must be aware calculations for frequencies $\omega \approx 0$ since $\lim_{\omega\to 0} G(j\omega) = I$ (G is defined in (8)). Simply, when $\omega = 0$ then for any K_1 and K_2 the value $G(0)$ is

$$G(0) = -\mathcal{C}\mathcal{A}^{-1}\mathcal{B}_0 + \mathcal{D}_0 = I.$$

This means that the proposed conditions cannot produce the reasonable controller gains K_1 and K_2 for frequency ranges that include $\omega = 0$ even when feasible controller gains exist. For this type of designs, we need to exclude $\omega = 0$ and $\omega \approx 0$. However, it is not clear at the moment how to choose an appropriate range of excluded frequencies for specific systems. Simply there is no systematic procedure for choosing this frequency range. An alternative would be to put a filter $F(s)$ in series with $G(s)$ such that $|F(0)G(0)| < 1$. Some simplification in filter selection procedure can be introduced by choosing F to be frequency-independent (equal to the constant f where $f \in (0, 1)$) - see [10] for more details on this type of filtering applied to mechanical systems. For this case, $G(0) = fI$ and highlighted problems are vanished.

V. DESIGN OF MONOTONICALLY CONVERGENT ILC SCHEMES FOR AN UNCERTAIN PLANT MODEL

In this section, the design of ILC schemes for an uncertain plant model is performed by making extensive use of the previously developed results. Unfortunately, this cannot be directly done for any $\kappa \geq 1$ since higher order system relative degree $\kappa > 1$ results in uncertainty matrices coupled with each other. Therefore, the conditions for robust monotonic trial-to-trial error convergence are only provided for the case when $\kappa = 1$.

The analysis that follows in this section uses the following matrices (a particular version of the left hand side terms in (22) and (23))

$$\begin{aligned}
\Upsilon_7 &= \left(\begin{bmatrix} 1 & -c \\ -c & |c|^2 - r^2 \end{bmatrix} \otimes Z\right) \\
&\quad + \mathrm{sym}\left\{\begin{bmatrix} -W_2 \\ (AW_2^T + BY_1)^T \end{bmatrix}\begin{bmatrix} qI & -pI \end{bmatrix}\right\}, \\
\Upsilon_8 &= \begin{bmatrix} \tau Q & \upsilon^* Q + P & \beta(-CAW_2^T - CBY_1)^T & 0 \\ (\star) & \varsigma Q & (-CAW_2^T - CBY_1)^T & BY_2 \\ (\star) & (\star) & -\varepsilon_1 I & (\varepsilon_2 I - CBY_2) \\ (\star) & (\star) & (\star) & -\varepsilon_1 I \end{bmatrix} \\
&\quad + \mathrm{sym}\left\{\begin{bmatrix} \beta I \\ I \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} -W_2 & (AW_2^T + BY_1)^T & 0 & 0 \end{bmatrix}\right\}.
\end{aligned} \tag{30}$$

When uncertainty is present, i.e., the matrices $\Delta A(t)$ and $\Delta B(t)$ are present in (2) and are of the form (3). Hence, the matrices in (9) for $\kappa = 1$ transform into

$$\begin{aligned}
\mathcal{A} &= (A + \Delta A(t)) + (B + \Delta B(t))K_1, \\
\mathcal{B}_0 &= (B + \Delta B(t))K_2, \ \mathcal{D}_0 = I - C(B + \Delta B(t))K_2, \\
\mathcal{C} &= -C(A + \Delta A(t)) - C(B + \Delta B(t))K_2
\end{aligned} \tag{31}$$

and inequalities in Theorem 1 will include terms formed by multiplication of two matrices with additive uncertainties. The next result reformulates the condition of Theorem 1 to an LMI-based characterization of the robust monotonic trial-to-trial error convergence.

Theorem 3: Consider an uncertain dynamics of (2) (that is $\delta(t) \neq 0$) with relative degree $\kappa = 1$, and let updating law (5) be applied. Also, let β, c, r, p, q be given scalars where $\beta > 0$, $c > r > 0$ and p, q satisfy $p^2 - 2cpq + q^2(|c|^2 - r^2) < 0$. Then an ILC scheme described as a repetitive process of the form (6) has the robust stability along the trial property over the finite frequency ranges defined in Table 1 and all eigenvalues of $\mathcal{A}$ are located in the circle of radius r with center at $(-c, 0)$. Moreover, robust monotonic trial-to-trial error convergence occurs over the specified frequency intervals if there exist matrices $Q \succ 0$, $Z \succ 0$, W_2, Y_1, Y_2, F_1, F_2, F_3, a symmetric matrix P, positive scalars ϵ_1, ϵ_2, ε_1, ε_2 such that $\varepsilon_2 \geq \varepsilon_1$ and the following LMIs hold

$$\begin{bmatrix} \Upsilon_7 & \epsilon_1 E_7 & H_7^T \\ \epsilon_1 E_7^T & -\epsilon_1 I & 0 \\ H_7 & 0 & -\epsilon_1 I \end{bmatrix} \prec 0, \tag{32}$$

$$\begin{bmatrix} \Upsilon_8 & \epsilon_2 E_8 & H_8^T \\ \epsilon_2 E_8^T & -\epsilon_2 I & 0 \\ H_8 & 0 & -\epsilon_2 I \end{bmatrix} \prec 0, \tag{33}$$

where

$$\begin{aligned}
E_7 &= \begin{bmatrix} qE^T & -pE^T \end{bmatrix}^T, \ H_7 = \begin{bmatrix} 0 & F_a W_2^T + F_b Y_1 \end{bmatrix}, \\
E_8 &= \begin{bmatrix} 0 & E^T & -E^T C^T & 0 \end{bmatrix}^T, \\
H_8 &= \begin{bmatrix} \beta F_a W_2^T + \beta F_b Y_1 & F_a W_2^T + F_b Y_1 & 0 & F_b Y_2 \end{bmatrix}.
\end{aligned}$$

Moreover, if the above LMIs are feasible, the corresponding matrices in the control law (5) can be selected as in (24).

Proof: Suppose that the LMIs (32) and (33) are feasible. Then application of Schur's complement formula to (32) yields

$$\Upsilon_7 + \epsilon_1 E_7 E_7^T + \epsilon_1^{-1} F_7^T F_7 \prec 0$$

and by Lemma 2 this last inequality is feasible if and only if

$$\Upsilon_7 + \mathrm{sym}\{E_7 \delta(t) F_7\} \prec 0.$$

The last inequality is (22) applied to the uncertainty case. The LMI of (33) can be obtained by employing the same steps used as those above and hence a version of (23) applied to the plant with uncertainty is obtained. Finally, by the result of Theorem 1, feasibility of (32) and (33) ensures that a differential linear repetitive process of the form (6) with (31) is robustly stable along the trial. This implies the robust monotonic trial-to-trial error convergence and the proof is complete. ■

As the next step, we extend Theorem 2 results to the case of plants which include norm bounded uncertainties. To proceed, let us introduce the following notation

$$\Upsilon_9 = \begin{bmatrix} \Upsilon_4 - \mathrm{sym}\{\beta \hat{W}_2\} & (\star) & (\star) \\ \Upsilon_6 + \beta(\mathbb{A}\hat{W}_2^T + \mathbb{B}N)^T - \hat{W}_2^T & \Upsilon_5 + \mathrm{sym}\left\{\mathbb{A}\hat{W}_2^T + \mathbb{B}N\right\} & (\star) \\ F_{30} - \hat{W}_3 & -F_{12}^T + [0\ I](\mathbb{A}\hat{W}_2^T + \mathbb{B}N) & R - F_3 - F_3^T \end{bmatrix},$$

$$\mathbb{E} = \begin{bmatrix} E \\ -CE \end{bmatrix}, \ \mathbb{F} = \begin{bmatrix} F_a & 0 \end{bmatrix}$$

and then the following result extends Theorem 2 to the case of uncertain plants and hence another LMI-based condition for the robust monotonic trial-to-trial error convergence is provided.

Theorem 4: Consider an uncertain dynamics of (2) (that is $\delta(t) \neq 0$,) with relative degree $\kappa = 1$, and let updating law (5) be applied. Also, let γ, β, c, r, p, q be given scalars where $0 < \gamma \leq 1$, $\beta > 0$, $c > r > 0$ and p, q satisfy $p^2 - 2cpq + q^2(|c|^2 - r^2) < 0$. Then an ILC scheme described as a repetitive process of the form (6) has the robust stability along the trial property over the finite frequency ranges defined in Table 1 and all eigenvalues of $\mathcal{A}$ are located in the circle of radius r with center at $(-c, 0)$. Moreover, robust monotonic trial-to-trial error convergence occurs over the specified frequency intervals if there exist matrices $Q \succ 0$, $Z \succ 0$, W_2, W_{32}, W_{33}, Y_1, Y_2, F_1, F_2, F_3, a symmetric matrix P and positive scalars ϵ_1, ϵ_3 such that the LMIs (32) and

$$\begin{bmatrix} \Upsilon_9 & \epsilon_3 E_9 & H_9^T \\ \epsilon_3 E_9^T & -\epsilon_3 I & 0 \\ H_9 & 0 & -\epsilon_3 I \end{bmatrix} \prec 0, \quad (34)$$

where

$$E_9 = \begin{bmatrix} \beta\mathbb{E}^T & \mathbb{E}^T & \mathbb{E}^T[0\ I]^T \end{bmatrix}^T,$$
$$H_9 = \begin{bmatrix} 0 & \mathbb{F}\hat{W}_2^T + F_b N & 0 \end{bmatrix}$$

are feasible. In addition, if the above LMIs hold, the corresponding matrices in the control law (5) can be selected as in (26).

Proof: The result follows from a straightforward application the same steps as in proof of Theorem 3 and hence omitted. ∎

VI. SIMULATION BASED CASE STUDY

To illustrate the effectiveness and feasibility of the new ILC design, this section shows the applications of developed results.

Example 1: The first example demonstrates the control law design of (5) for a permanent magnet DC-motor using the approach given in Theorem 2. This motor might be driving (via some gears) a joint in a robot arm, which has to perform a repetitive task. The equations of the motor model have been borrowed from [20] and they are as follows

$$\begin{bmatrix} \dot{\alpha}(t) \\ \dot{\varpi}(t) \\ \dot{i}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{K_e}{J} \\ 0 & -\frac{K_e}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} \alpha(t) \\ \varpi(t) \\ i(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha(t) \\ \varpi(t) \\ i(t) \end{bmatrix}, \quad (35)$$

where α denotes the angle of the motor shaft, $\varpi = \dot{\alpha}$ is the angular velocity, u denotes the input voltage and i is the armature current. Clearly, the relative degree of the nominal dynamics is 3 ($\kappa = 3$). Also, J is the total moment of inertia of the rotor with gears, K_e represents the motor torque constant (the same as the back-electromagnetic force constant), L is the electric inductance and R stands for the electric resistance. For the numerical simulations, we assume choose the following values $J = 1.5 \cdot 10^{-3} kgm^2$, $R = 0.2\Omega$ and $L = 0.02\,H$, $K_a = k = 9 \cdot 10^{-3} Vs/rad$. Based on these data, we compute the control law matrices of (5) by executing the design procedure of this paper and the existing methods in [21] and [22], then compare the achieved performance. Note that the results of [21] and [22] require some extensive modifications since they cannot be directly applied to the considered example. Simply, originally they are inapplicable to systems with $\kappa > 1$ and without modifications they give no feasible results. On the other hand, the method in [22] allows to impose the regional pole constraint and uses a type of frequency-independent filtering and hence it avoids problems at frequencies $\omega \simeq 0$.

Executing the design procedure given in Theorem 2 for $\beta = 0.7$, $p = -13920$, $q = 1000$, $\gamma = 0.99$ and $f = 0.9$ (see sub-section IV-A) we have that the controller matrices are obtained as follows for the frequency range [1, 10] and the regional pole constraints $C(-10, 4)$

$$K_1 = [-1.3332 \ \ -0.5209 \ \ -0.2298], \ K_2 = -1.3288 \cdot 10^{-4}$$

It can be verified that the controlled system represented by the differential repetitive process model is stable along the trial. Clearly, $\rho(\mathcal{D}_0) = 0.9359$, and $\rho(G(\mathrm{j}\omega)) < 1$ for all ω in the prescribed frequency range, where $G(\mathrm{j}\omega)$ is defined in (8). This can be seen in Figure 1 and it confirms that the design specifications are met. Using the same regional pole constraints ($C(-10, 4)$) and frequency-independent filter gain ($f = 0.9$), the (modified) method in [22] gives

$$K_1 = [-2.2995 \ \ -0.7877 \ \ -0.3394], \ K_2 = -6.5883 \cdot 10^{-6}$$

From Figure 1 one can clearly see that the developed design procedure over the finite frequency range can yield less conservative results than the entire frequency method in [22]. Specifically, this paper method results in less value of $\rho(G(j\omega))$ than that one in [22] and hence the tracking error convergence rate becomes faster. However, the difference between methods is not significant. Anyway, when the frequency range changed to [20,40] then effectiveness of this

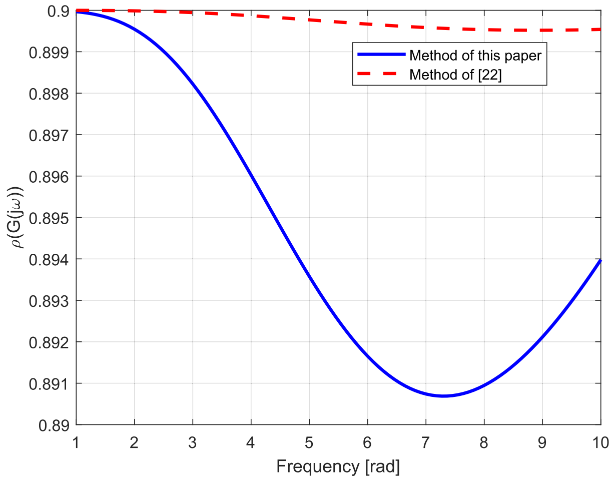

FIGURE 1. Plot of $\rho(G(j\omega))$.

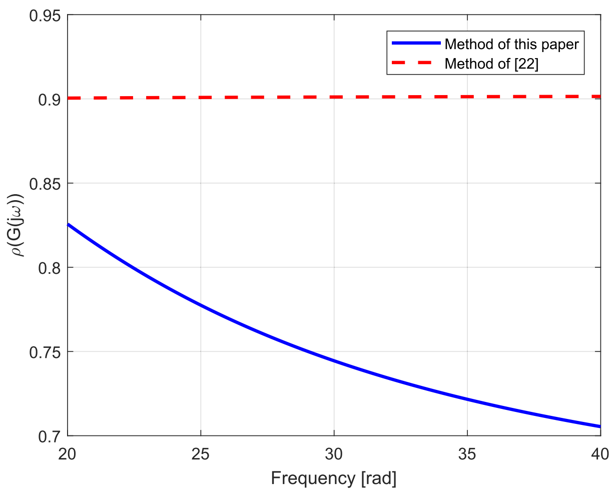

FIGURE 2. Plot of $\rho(G(j\omega))$.

paper method is strongly demonstrated - see Figure 2. Finally, the controlled dynamics were simulated over 20 trials and for each one the RMS (Root Mean Square) value of the tracking error was computed. The desired trajectory is taken as

$$Y_d(t) = \sin(60\pi t) + 0.5\sin(50\pi t), \text{ for } 0 \leq t \leq 8.$$

Figure 3 shows the RMS values of the tracking error as a function of the trial number and hence it indicates the convergence speed and accuracy for ILC control schemes. Obviously, simulations of the response of the controlled systems confirm that the tracking error of the ILC systems is monotonically convergent. More importantly, one can observe that this paper method converges the error faster than the method in [22]. Moreover, when using the LMI test provided in [16] or [21] (after obvious transformation from repetitive process results into ILC design procedures), infeasibility occurs even when the LMI computations are performed for middle frequency range only (the frequency range [1, 10] or [20, 40]). It means that the many approaches for computing the control law matrices of (5) fail while the presented approach

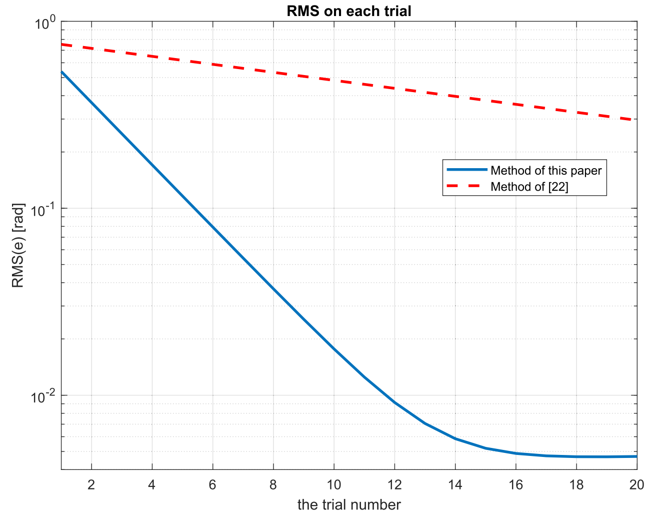

FIGURE 3. RMS values of the error over 20 trials.

succeeds. Additionally, it must be understood that unlike the most applicable design procedures provided in, e.g. [8], [11] or [18], there is no need to use discrete-time model of (35). Simply, our algorithm is applied to a plant that is originally a continuous-time system and takes into account the hybrid nature of the problem, i.e., the learning is a discrete-time process (control signal update is performed for previous iteration to the current one) and the plant in is a continuous-time system.

Example 2: In this example the uncertain system with relative degree $\kappa = 1$ is considered where the state-space model is defined by the following matrices:

$$A = \begin{bmatrix} -6 & -2 \\ 4 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0.75 \end{bmatrix}.$$

Obviously this is a pure academic example since there is almost none continuous-time models of physical systems with relative degree $\kappa = 1$. For uncertainties modeled by (3), assume that

$$E = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}, \quad F_a = \begin{bmatrix} 1 \ 0.75 \end{bmatrix}, \quad F_b = 0.5$$

Application of the design procedure given in Theorem 4 for $\beta = 1$, $p = -180$, $q = 100$, $\gamma = 0.95$ and $f = 0.9$ gives that the controller matrices are as follows for the frequency range from 1 [rad/s] to 10 [rad/s]

$$K_1 = [0.6236 \ \ 0.3226], \quad K_2 = 0.1741$$

Again, it can be verified that the controlled system represented by differential repetitive process model is stable along the trial and therefore the robust tracking performance of the resulting ILC scheme is improved. The effectiveness between the above controllers and the controllers designed by using repetitive setting approach given by [21] is compared. Note that the method of [22] can only be applied for nominal systems and hence cannot be used for robust control law

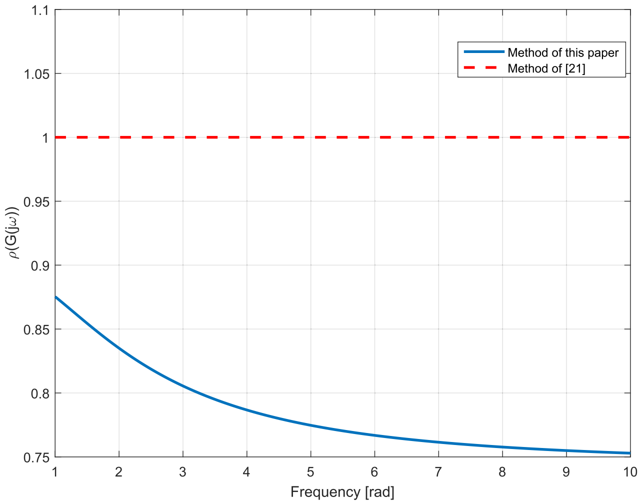

FIGURE 4. Comparison of two methods.

design here. By applying the modified version of Theorem 3 in [21] we can find the control law matrices as

$$K_1 = [-32.9240 \quad -21.2431], \quad K_2 = 5.3524 \cdot 10^{-15}$$

As it is seen, K_2 gain generated by the method of [21] is very low ($K_2 \approx 0$), and hence it results in very low convergence speed since $\rho(G(j\omega)) \approx 1$ over the chosen frequency range. On the other side, this paper method yields $\rho(G(j\omega)) < 0.87$. From comparison of two situations in Figure 4, the convergence rate of the developed method is significantly increased when comparing with the method of [21]. Obviously, the effectiveness of the presented ILC design is apparent.

VII. CONCLUSIONS

This paper has developed new results on the ILC problem for a class of linear continuous-time systems. Both nominal and uncertain dynamics of a given plant have been considered and design of ILC schemes based on the differential repetitive process theory has been developed. Sufficient conditions for the existence of a monotonically convergent ILC law have been obtained in LMI form and thereby allowing practically relevant control requirements over restricted frequency ranges to be imposed. A simulation based case study demonstrates the effectiveness of the new design method. It is visible that this paper design procedures outperform known alternatives so this work is an important progress. On the other hand, it should be noted that this paper methods involve more matrix variables than known alternatives and hence usually leads to more computational cost. However, the control law matrices are designed off-line and the tracking (convergence) performance is the main concern here. Since this paper methods can guarantee a lower spectral radius (and hence the higher speed of tracking error convergence), their more computational cost is justified.

Future research should include a detailed investigation into the $\mathcal{H}_\infty$ robust performance in the presence of external bounded non-repetitive disturbances in both the state and output vectors. Another area is design of an ILC law where the more complex controller structures are used, e.g. a dynamic controller. Also the extension to multiple state and input delays should also be investigated, together with the effects of uncertainty in the state initial vector on each trial. Finally, when appropriate, experimental validation should be undertaken.

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