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Global Quadratic Stabilization in Probability for Switched Linear Stochastic Systems

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ABSTRACT Global quadratic stabilization in probability is considered for both switched linear certain stochastic systems and switched linear uncertain stochastic systems where there are norm bounded uncertainties. Under the assumption that every single subsystem is NOT globally quadratically stable in probability (GQS-P), we propose both static and dynamic output based switching laws such that the switched system on hand is GQS-P. In the case of static output based switching, the condition is expressed by a set of matrix inequalities, while the design of dynamic output based switching is proposed with a convex combination of subsystems and a robust Luenberger observer for each subsystem. Numerical examples are presented to show validity of the design conditions and switching algorithms.

INDEX TERMS Switched linear certain stochastic systems (SLCSS), switched linear uncertain stochastic systems (SLUSS), norm bounded uncertainties, convex combination, globally quadratically stable in probability (GQS-P), output based switching, LMIs.

I. INTRODUCTION

Switched systems consist of a set of subsystems, which are continuous-time or discrete-time, and a switching law (or, switching rule/strategy/signal) specifying one subsystem which will be activated for any time instant. Since switched systems can represent a wide class of dynamical systems in real world and are applicable to intelligent system design with the framework of switching control, there has been extensive research interest in the last two decades [1]–[10]. Moreover, the direct and indirect extensions of switched systems include stochastic and LPV systems [11]–[13], network controlled systems [14]–[16], Boolean networks control [17], [18], and multi-agent systems [19].

It has been widely recognized [4], [7] that there are three basic problems in the analysis and design of switched systems. One of them, which will be handled in the present paper, is to design a stabilizing switching law (strategy) in the case that there is no single asymptotically stable subsystem. More specifically, we deal with switched systems which are composed of continuous-time linear subsystems with stochastic

noises, and suppose every single subsystem is NOT globally quadratically stable in probability (GQS-P). The control problem is to study the existence condition and design procedure of a switching law under which the switched system is GQS-P. When the subsystems are certain (without uncertainties) deterministic (without stochastic noises), it has been shown in [20] that a state based switching rule quadratically stabilizes the switched system if a convex combination of subsystem matrices is Hurwitz. Furthermore, an output based switching algorithm is proposed in [21], which is constructed by a switching Luenberger observer and a robust detectability condition for all subsystems.

The state based switching algorithms in [20], [21] are discussed for switched linear discrete-time systems in [22], and to switched linear systems with polytopic uncertainties (both continuous-time and discrete-time) in [23]. Moreover, Ref. [25] extended the discussion to quadratic stabilization for switched linear systems with norm bounded uncertainties. In that context, it is claimed that if there exists a convex combination of subsystems which is quadratically stable, then it is possible to establish a state based (dependent) switching law such that the switched system is GQS (globally quadratically stable). Finally, the convex combination based

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switching algorithms are extended to switched affine systems in [24]. It is noted that all the above mentioned references only dealt with the deterministic system, while the GQS problem for switched uncertain stochastic systems (SUSS) remains open.

Motivated by the above mentioned observation, we here aim to design output based switching laws for quadratic stabilization of switched linear stochastic systems, under the assumption that no single subsystem is GQS-P. The motivation of considering GQS-P is that quadratic stabilization has strong capability of dealing with parametric uncertainty in the system, which will be re-examined later. In the literature, Ref. [23] has shown that quadratic stabilization is powerful when there are polytopic uncertainties in switched systems. Ref. [27] extended the discussion to uncertain switched linear systems with norm-bounded uncertainties, and [28] used KKT condition in nonlinear programming to deal with quadratic stabilization. Recently, [26] adapted the convex combination approach in [20], [21], [25] for quadratic stabilization of the SLUSS with norm bounded uncertainties. And, [29] extended the discussion to switched affine systems. However, the discussion in the above references is based on state feedback or state-dependent switching. Therefore, when the system state can not be obtained, the design condition and the algorithm of output based switching laws is desired but not solved. In addition, due to the existence of stochastic process in the system, the stabilizing switching law can not be separated from the Luenberger observer design.

In this paper, we first consider switched linear certain stochastic systems (SLCSS), and propose a sufficient design condition and an output based switching law such that the resultant switched system is GQS-P. The obtained sufficient condition requires that a convex combination of subsystems should be GQS-P, and the switching law is implemented by using the state of a robust Luenberger observer for each subsystem. The discussion is also extended to switched linear uncertain stochastic systems (SLUSS) with norm bounded uncertainties, with the sufficient condition updated by dealing with the uncertainty terms.

This remaining part of this paper is organized as follows. We provide several preliminary lemmas concerning stochastic systems and robust stability in Section II, and formulate our control problem in Section III. Then, Section IV and V establish the sufficient conditions of designing output based switching laws for the SLCSS and the SLUSS, respectively. Both static and dynamic output based switching laws are discussed, while the focus is on the latter. Two numerical examples are provided to show validity of the proposed design conditions and switching algorithms. Finally, Section VI concludes the paper.

Notations: Throughout this paper, we will use the superscripts “ \top ” and “ -1 ” to denote the transpose and the inverse of a matrix with proper size, respectively. $W > 0$ ($W < 0$) means W is symmetric and positive (negative) definite, and $W_1 > W_2$ if and only if $W_1 - W_2 > 0$. $\lambda_M(\cdot)$ and $\lambda_m(\cdot)$ denote the largest and smallest eigenvalue of a symmetric matrix.

A matrix is Hurwitz if all its eigenvalues have negative real parts. As used in most textbooks, $P\{\cdot\}$ denotes the probability of a set and $E[\cdot]$ is the expectation value of a random variable.

II. PRELIMINARIES

First, several lemmas are stated, which will be used later.

Lemma 1: [30] *The matrix inequality*

$$G(s) + U(s)XV(s)^\top + V(s)X^\top U(s)^\top > 0, \quad (1)$$

with the vector s and the matrix X being variables, is feasible if and only if

$$\begin{aligned} G(s) - \sigma U(s)U(s)^\top &> 0, \\ G(s) - \sigma V(s)V(s)^\top &> 0 \end{aligned} \quad (2)$$

are satisfied with some scalar $\sigma \in \mathcal{R}$.

Lemma 2 [31] *Given any constant matrices $V \in \mathcal{R}^{n \times m}$ and $W \in \mathcal{R}^{p \times n}$, the matrix inequality*

$$VFW + W^\top F^\top V^\top \leq VV^\top + W^\top W \quad (3)$$

holds for any uncertain $F \in \mathcal{R}^{m \times p}$ satisfying $\|F\| \leq 1$.

Lemma 3 (Markov’s Inequality) [32]) *Consider a nonnegative random variable $Y : \Omega \rightarrow \mathcal{R}$. Then, for any positive number a , the following is true.*

$$P\{Y \geq a\} \leq \frac{E[Y]}{a} \quad (4)$$

Next, we consider the stochastic system [32]

$$dx = h(x) dt + k(x) dw \quad (5)$$

where $x \in \mathcal{R}^n$ is the system state, w is an r -dimensional independent standard Wiener process on a probability space, dx is the stochastic differential of x , and $h : \mathcal{R}^n \rightarrow \mathcal{R}^n$ and $k : \mathcal{R}^n \rightarrow \mathcal{R}^{n \times r}$ are locally Lipschitz functions satisfying $h(0) = 0, k(0) = 0$.

Definition 1 The equilibrium $x = 0$ of the stochastic system (5) is globally quadratically stable in probability (GQS-P) if there is a positive definite quadratic function $V(x) = x^\top Px$ with $P > 0$ such that

$$E[V(x(t))] \leq V(x(0))e^{-\eta t} \quad (6)$$

holds for $\forall t \geq 0$ and scalar $\eta > 0$. ■

Remark 1 If (6) holds, by using Lemma 3, we obtain that for any scalar $\epsilon \in (0, 1)$,

$$P \left\{ |x(t)| < \frac{1}{\epsilon} \sqrt{\frac{\lambda_M(P)}{\lambda_m(P)}} |x(0)| e^{-\frac{\eta}{2}t} \right\} \geq 1 - \epsilon, \quad (7)$$

which is defined as globally asymptotical stability in probability (GAS-P) in the literature [32]. Thus, the stability concept “GQS-P” in Definition 1 implies “GAS-P”. Such a relation is well known in stability analysis of deterministic systems, where we state “GQS” and “GAS” (without “in probability”). ■

Lemma 4 For the stochastic system (5), suppose there exist a C^2 function $V(x)$, two class \mathcal{K}_∞ functions α_1 and α_2 , and a class \mathcal{K} function α_3 , satisfying

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{8}$$

$$\mathcal{L}V(x) = \frac{\partial V}{\partial x}h(x) + \frac{1}{2}\text{Tr}\left(k^\top \frac{\partial^2 V}{\partial x^2}k\right) \leq -\alpha_3(|x|). \tag{9}$$

Then, the equilibrium $x = 0$ of (5) is GAS-P.

Next, let us focus on the linear stochastic system

$$dx = Ax dt + Bx dw \tag{10}$$

where $A, B \in \mathcal{R}^{n \times n}$ are constant matrices. Then, by setting $h(x) = Ax, k(x) = Bx$ and $V(x) = x^\top Px$ with $P \succ 0$ for the above linear system in Lemma 4, we reach the following lemma immediately.

Lemma 5 If the linear matrix inequality (LMI)

$$A^\top P + PA + B^\top PB < 0, \tag{11}$$

is satisfied with a matrix $P \succ 0$, then the equilibrium $x = 0$ of (10) is GQS-P.

When there is uncertainty in the system matrix of the stochastic system (10), we assume

$$dx = (A + DF(t)E)x dt + Bx dw \tag{12}$$

where $D \in \mathcal{R}^{n \times m}, E \in \mathcal{R}^{p \times n}$ are known time-invariant matrices and $F(t) \in \mathcal{R}^{m \times p}$ denotes the system uncertainty satisfying the norm bound $\|F(t)\| \leq 1$. Using Lemma 5, we obtain that the equilibrium $x = 0$ of (12) is GQS-P if there exists $P \succ 0$ such that the time-varying matrix inequality

$$[A + DF(t)E]^\top P + P[A + DF(t)E] + B^\top PB < 0 \tag{13}$$

holds for any $\|F(t)\| \leq 1$. Then, using the Schur complement and Lemma 2 for (13), we obtain the following lemma.

Lemma 6 If $P \succ 0$ exists satisfying the linear matrix inequality (LMI)

$$\begin{bmatrix} A^\top P + PA + B^\top PB + E^\top E & PD \\ D^\top P & -I_m \end{bmatrix} < 0, \tag{14}$$

then the equilibrium $x = 0$ of (12) is GQS-P.

III. PROBLEM FORMULATION

In this paper, we deal with two class of switched systems, i.e., the switched linear certain stochastic system (SLCSS)

$$\begin{cases} dx = A_\sigma x dt + Bx dw \\ y = C_\sigma x \end{cases} \tag{15}$$

and the switched linear uncertain stochastic system (SLUSS)

$$\begin{cases} dx = (A_\sigma + D_\sigma F(t)E_\sigma)x dt + Bx dw \\ y = C_\sigma x. \end{cases} \tag{16}$$

In the above two systems, $x \in \mathcal{R}^n$ is the stochastic state, $y \in \mathcal{R}^q$ is the measurement output, w and dx are the same as in (5). The switching law (signal) $\sigma(t) : [0, \infty) \rightarrow \mathcal{I}$ defines the index of subsystem which is activated at time t , and $\mathcal{I} =$

$\{1, 2, \dots, \mathcal{N}\}$ is the index set of subsystems. Usually, $\sigma(t)$ is assumed to be continuous from the right. Thus, there are \mathcal{N} subsystems, and the i -th subsystem's dynamics is described by the triple (A_i, B, C_i) or the quadruple (A_i, B, C_i, D_i, E_i) , $i = 1, 2, \dots, \mathcal{N}$, where the matrices $A_i, B \in \mathcal{R}^{n \times n}, C_i \in \mathcal{R}^{q \times n}, D_i \in \mathcal{R}^{n \times m}, E_i \in \mathcal{R}^{p \times n}$ are constant, and $F(t) \in \mathcal{R}^{m \times p}$ satisfying $\|F(t)\| \leq 1$ denotes the norm bounded uncertainty term.

If there is one GQS-P subsystem in the switched system, one stabilizing switching law (strategy) is to activate that subsystem for all time (without activating other subsystems), and the switched system is certainly GQS-P. Since this is a trivial case, we assume from now on that there is no single subsystem in (15) and (16) which is GQS-P.

With the above preparation, we formulate our control problem as follows:

For the SLCSS (15) and the SLUSS (16), design the output based switching law $\sigma(y)$, such that the resultant switched systems are GQS-P.

IV. QUADRATIC STABILIZATION OF SLCSS

We first deal with quadratic stabilization of the SLCSS (15). Since we have assumed that no single subsystem is GQS-P, according to Lemma 5, there does NOT exist a matrix $P \succ 0$ satisfying the LMI [30]

$$A_i^\top P + PA_i + B^\top PB < 0. \tag{17}$$

A. STATIC OUTPUT BASED SWITCHING LAW

We state the first switching law

$$SW1 : \sigma(y) = \arg \min_{i \in \mathcal{I}} \{y^\top W_i y\}, \tag{18}$$

where W_i 's are symmetric positive definite matrices to be determined. To justify this switching law, let us consider the situation when the following condition holds.

Condition 1: There exists $P \succ 0$ such that for $\forall i \in \mathcal{I}$, the inequality

$$x^\top (A_i^\top P + PA_i + B^\top PB)x < 0 \tag{19}$$

holds whenever $x \neq 0$ and

$$y^\top W_i y \leq \min_{j \in \mathcal{I}, j \neq i} y^\top W_j y, \tag{20}$$

or equivalently,

$$x^\top C_i^\top W_i C_i x \leq x^\top C_j^\top W_j C_j x, \quad \forall j \neq i. \tag{21}$$

If the above condition is satisfied, then the inequality

$$x^\top (A_\sigma^\top P + PA_\sigma + B^\top PB)x < 0 \tag{22}$$

holds for $x \neq 0$ under the switching law defined by (18). Moreover, for nonzero x , there exists a positive scalar η such that

$$x^\top (A_\sigma^\top P + PA_\sigma + B^\top PB)x < -\eta x^\top P x. \tag{23}$$

Then, the derivative of the Lyapunov function candidate $V(x) = x^T P x$ along the solutions of (15) is computed as

$$\begin{aligned} dV(x) &= \mathcal{L}V(x) dt + x^T (B^T P + PB) x dw, \\ \mathcal{L}V(x) &= x^T (A_\sigma^T P + PA_\sigma + B^T PB) x. \end{aligned} \quad (24)$$

Due to (23), we have $\mathcal{L}V(x) \leq -\eta V(x)$,

$$dV(x) \leq -\eta V(x) dt + x^T (B^T P + PB) x dw \quad (25)$$

and thus

$$\begin{aligned} d[e^{\eta t} V(x)] &= \eta e^{\eta t} V(x) dt + e^{\eta t} dV(x) \\ &\leq x^T e^{\eta t} (B^T P + PB) x dw. \end{aligned} \quad (26)$$

Integrating and taking expectation of both sides of the above inequality on the interval $[0, t]$, together with the property of Wiener process, $E[dw] = 0$, results in

$$E[V(x)] \leq V(x_0)e^{-\eta t}. \quad (27)$$

This implies that the switched system (15) under the switching law (18) is GQS-P with the Lyapunov candidate $V(x) = x^T P x$.

According to the S-Procedure [30], Condition 1 holds if there exist nonnegative scalars $\beta_{ij}, i, j \in \mathcal{I}, j \neq i$, such that

$$A_i^T P + PA_i + B^T PB + \sum_{j \in \mathcal{I}, j \neq i} \beta_{ij} (C_j^T W_j C_j - C_i^T W_i C_i) < 0. \quad (28)$$

The above discussion is summarized in the following theorem.

Theorem 1: If there are $P > 0, W_i > 0, \beta_{ij} \geq 0$ satisfying the matrix inequality (28) for all $i \in \mathcal{I}$, then the SLCSS (15) is GQS-P under the switching law (18).

Since the condition (28) is not linear with respect to the variables, it is not easy to obtain the solution globally efficiently. One may try to set an interval for the nonnegative parameters β_{ij} and then fix them temporarily by using a line search method to solve (28) with respect to P and W_i 's.

Remark 2: Using the above approach, we can deal with some robust stabilization problem for the switched system (15) with parameter perturbations. For example, let us suppose the typical polytopic perturbations in the system matrices A_i and the control input matrix B as

$$\begin{aligned} [A_i \quad B] &= \sum_{k=1}^{\ell} \mu_k [A_{ik} \quad B_k] \\ \mu_k &\geq 0, \quad \sum_{k=1}^{\ell} \mu_k = 1, \end{aligned} \quad (29)$$

where $A_{ik}, B_k, k = 1, 2, \dots, \ell$, denote the extreme points of the polytopes. Since the condition (28) is equivalent to

$$\left[\begin{array}{c|c} A_i^T P + PA_i + \sum_{j \in \mathcal{I}, j \neq i} \beta_{ij} (C_j^T W_j C_j - C_i^T W_i C_i) & B^T P \\ \hline PB & -P \end{array} \right] < 0 \quad (30)$$

which is linear with respect to A_i and B , we obtain that the condition (28) is reduced to

$$\left[\begin{array}{c|c} \sum_{j \in \mathcal{I}, j \neq i} \beta_{ij} (C_j^T W_j C_j - C_i^T W_i C_i) & B_k^T P \\ \hline PB_k & -P \end{array} \right] < 0 \quad (31)$$

for all $k = 1, \dots, \ell$. In other words, if there are $P > 0, W_i > 0, \beta_{ij} \geq 0$ satisfying the matrix inequality (31) for all $i \in \mathcal{I}$ and $k = 1, \dots, \ell$, then the SLCSS (15) with the perturbation (29) is GQS-P under the switching law (18). ■

B. DYNAMIC OUTPUT BASED SWITCHING LAW

Since the static output $y(t)$ used in the switching law (18) does not memorize the output history, it can only be adapted to a limited class of switched systems. Here, we aim to propose a dynamic output based switching law for the switched system (15). For this purpose, we define a convex combination of subsystem matrices as

$$A_\lambda = \sum_{i=1}^{\mathcal{N}} \lambda_i A_i, \quad \lambda_i \geq 0, \quad \sum_{i=1}^{\mathcal{N}} \lambda_i = 1. \quad (32)$$

Next, to propose the dynamic output based switching law, we construct the following full order Luenberger observer for the system state $x(t)$

$$\begin{aligned} \dot{\hat{x}}(t) &= A_\sigma \hat{x}(t) + L_\sigma (y(t) - C_\sigma \hat{x}(t)) \\ \text{or } d\hat{x}(t) &= [A_\sigma \hat{x}(t) + L_\sigma (y(t) - C_\sigma \hat{x}(t))] dt, \end{aligned} \quad (33)$$

where $\hat{x}(t) \in \mathcal{R}^n$ is the observer's state and L_i 's are observer gain matrices which will be determined from now. Clearly, \hat{x} is dependent on the output measurement y . If we can design a valid switching law based on the observer's state \hat{x} , then it is an output based switching law we desire.

To proceed further, we propose the following assumption, which is an integrated condition combining system stabilization, Luenberger observer design and implementation of switching law together.

Assumption 1: There are positive definite matrices Q, P and matrices L_i 's satisfying

$$\left[\begin{array}{ccc} A_\lambda^T P + PA_\lambda & PL_i C_i & B^T Q \\ C_i^T L_i^T P & \Gamma_i & B^T Q \\ QB & QB & -Q \end{array} \right] < 0 \quad (34)$$

for all $i = 1, \dots, \mathcal{N}$, where

$$\Gamma_i = (A_i - L_i C_i)^T Q + Q(A_i - L_i C_i). \quad (35)$$

According to the Schur complement lemma, (34) is equivalent to

$$\left[\begin{array}{cc} A_\lambda^T P + PA_\lambda + B^T QB & PL_i C_i + B^T QB \\ C_i^T L_i^T P + B^T QB & \Gamma_i + B^T QB \end{array} \right] < 0. \quad (36)$$

When (34) or (36) is feasible, it is always possible to find a constant $\eta > 0$ such that

$$\begin{bmatrix} A_\lambda^\top P + PA_\lambda + B^\top QB & PL_i C_i + B^\top QB \\ C_i^\top L_i^\top P + B^\top QB & \Gamma_i + B^\top QB \end{bmatrix} \prec -\eta \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}. \quad (37)$$

Remark 3 If we set $P = Q$ in the condition (34) and focus on the 1st and the 3rd rows and columns, we obtain $A_\lambda^\top P + PA_\lambda + B^\top PB \prec 0$ with $P > 0$. This implies that we require that a combination of the subsystems in (15) is GQS-P. ■

Remark 4 The condition (34) requires $\Gamma_i < 0$, and thus covers the robust detectability condition [21], [33] as special case. Actually, focusing on the last two rows and columns of the left side of (34), we have $\Gamma_i + B^\top QB = (A_i - L_i C_i)^\top Q + Q(A_i - L_i C_i) + B^\top QB \prec 0$, which can be regarded as the robust detectability condition for linear stochastic systems. ■

Using the matrix $P > 0$ satisfying Assumption 1 and the estimated state $\hat{x}(t)$ in the observer (33), we define the output based switching law as

$$SW2 : \sigma(y) = \arg \min_{i \in \mathcal{I}} f_i(\hat{x}) \quad (38)$$

$$f_i(\hat{x}) = \hat{x}^\top (A_i^\top P + PA_i) \hat{x}.$$

Here, since the observers' state \hat{x} is generated by the output y , the switching law in SW2 essentially only depends on the output information, and thus we simply write it as $\sigma(y)$.

Remark 5 In order to implement the switching law SW2 clearly, we need to deal with the case that several indexes of subsystems satisfy (38) simultaneously. If the present subsystem is included in the index, that is,

$$\sigma(y(t^-)) \in \arg \min_{i \in \mathcal{I}} f_i(\hat{x}(t)), \quad (39)$$

we choose not to switch to other subsystems (staying at the same subsystem), which definitely decreases the switching frequency. Otherwise (the present subsystem is not included in the index set satisfying (38)), we choose any subsystem index generated by (38). This rule is also applied to another switching law SW3 later. ■

With the switching law SW2, we have for any $j \in \mathcal{I}$ that

$$\hat{x}^\top (A_\sigma^\top P + PA_\sigma) \hat{x} \leq \hat{x}^\top (A_j^\top P + PA_j) \hat{x}. \quad (40)$$

Multiplying both sides of the above inequality by nonnegative scalar λ_j and then summing up for $j = 1, \dots, \mathcal{N}$, we obtain

$$\hat{x}^\top (A_\sigma^\top P + PA_\sigma) \hat{x} \leq \hat{x}^\top (A_\lambda^\top P + PA_\lambda) \hat{x}. \quad (41)$$

Theorem 2 Under Assumption 1 and the switching law SW2, the SLCSS (15) is GQS-P.

Proof: Define $e = x - \hat{x}$, $\tilde{x} = [\hat{x}^\top \ e^\top]^\top$, and write the closed-loop system composed of (15) and (33) as

$$d\tilde{x} = \tilde{A}_\sigma \tilde{x} dt + \tilde{B} \tilde{x} dw, \quad (42)$$

where

$$\tilde{A}_\sigma = \begin{bmatrix} A_\sigma & L_\sigma C_\sigma \\ 0 & A_\sigma - L_\sigma C_\sigma \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 & 0 \\ B & B \end{bmatrix}. \quad (43)$$

To analyse quadratic stability of (42), we use the Lyapunov function candidate $V(\tilde{x}) = \tilde{x}^\top \tilde{P} \tilde{x}$, where

$$\tilde{P} = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}, \quad (44)$$

P, Q are the positive definite matrices satisfying (34). Then, the derivative of $V(\tilde{x})$ along the solutions of (42) is computed as

$$dV(\tilde{x}) = \mathcal{L}V(\tilde{x}) dt + \tilde{x}^\top (\tilde{B}^\top \tilde{P} + \tilde{P} \tilde{B}) \tilde{x} dw,$$

$$\mathcal{L}V(\tilde{x}) = \tilde{x}^\top (\tilde{A}_\sigma^\top \tilde{P} + \tilde{P} \tilde{A}_\sigma + \tilde{B}^\top \tilde{P} \tilde{B}) \tilde{x}, \quad (45)$$

and

$$\tilde{A}_\sigma^\top \tilde{P} + \tilde{P} \tilde{A}_\sigma + \tilde{B}^\top \tilde{P} \tilde{B} = \begin{bmatrix} A_\sigma^\top P + PA_\sigma + B^\top QB & PL_\sigma C_\sigma + B^\top QB \\ * & \Gamma_\sigma + B^\top QB \end{bmatrix}. \quad (46)$$

Therefore,

$$\mathcal{L}V(\tilde{x}) = \hat{x}^\top (A_\sigma^\top P + PA_\sigma + B^\top QB) \hat{x} + 2\hat{x}^\top (PL_\sigma C_\sigma + B^\top QB) e + e^\top (\Gamma_\sigma + B^\top QB) e. \quad (47)$$

Under the switching law SW2, (41) holds, and thus

$$\hat{x}^\top (A_\sigma^\top P + PA_\sigma + B^\top QB) \hat{x} \leq \hat{x}^\top (A_\lambda^\top P + PA_\lambda + B^\top QB) \hat{x}. \quad (48)$$

Then, combining (47), (48) and (37), we obtain

$$\mathcal{L}V(\tilde{x}) \leq -\eta V(\tilde{x}) \quad (49)$$

and furthermore

$$dV(\tilde{x}) \leq -\eta V(\tilde{x}) dt + \tilde{x}^\top (\tilde{B}^\top \tilde{P} + \tilde{P} \tilde{B}) \tilde{x} dw. \quad (50)$$

Using the above inequality, we reach

$$d[e^{\eta t} V(\tilde{x})] = \eta e^{\eta t} V(\tilde{x}) dt + e^{\eta t} dV(\tilde{x}) \leq \tilde{x}^\top e^{\eta t} (\tilde{B}^\top \tilde{P} + \tilde{P} \tilde{B}) \tilde{x} dw. \quad (51)$$

Integrating and taking expectation of both sides of the above inequality on the interval $[0, t]$, together with the property of Wiener process, $E[dw] = 0$, results in

$$E[V(\tilde{x})] \leq V(\tilde{x}_0) e^{-\eta t}. \quad (52)$$

This completes the proof. ■

The condition (34) in Assumption 1 is a bilinear matrix inequality (BMI) with respect to λ_i 's, $Q > 0$, $P > 0$ and matrices L_i 's, and it is not easy to solve general BMIs efficiently. Here, we suggest an algorithm for solving (34) with some trial and error.

[Algorithm 1]

- 1) Find a set of λ_i 's such that A_λ is Hurwitz.
- 2) Solve

$$(A_i - L_i C_i)^\top Q + Q(A_i - L_i C_i) + B^\top QB$$

In addition to A_λ in (32), we define the convex combination involving the uncertainty terms as

$$D_\lambda D_\lambda^\top = \sum_{i=1}^{\mathcal{N}} \lambda_i D_i D_i^\top, \quad E_\lambda^\top E_\lambda = \sum_{i=1}^{\mathcal{N}} \lambda_i E_i^\top E_i, \quad (57)$$

where the coefficients λ_i 's are the same as in (32).

Remark 6 The matrices D_λ and E_λ satisfying (57) for given D_i 's, E_i 's and λ_i 's, can be computed by using the Cholesky decomposition method, which is numerically and efficiently tractable in MATLAB. ■

Due to the uncertainty term $D_i F(t) E_i$ in the subsystem matrix, we update Assumption 1 to the following assumption.

Assumption 2: There exist positive definite matrices Q , P and matrices L_i 's satisfying

$$\begin{bmatrix} A_\lambda^\top P + PA_\lambda + E_\lambda^\top E_\lambda & PD_\lambda & PL_i C_i & B^\top Q \\ D_\lambda^\top P & -I_m & 0 & 0 \\ C_i^\top L_i^\top P & 0 & \Gamma_i & B^\top Q \\ QB & 0 & QB & -Q \end{bmatrix} < 0 \quad (58)$$

for all $i = 1, \dots, \mathcal{N}$, where Γ_i is the same as in Assumption 1.

Remark 7 If we set $P = Q$ in the condition (58) as a special case and use Schur complement lemma for the first two and the 4th rows and columns, we obtain

$$\sum_{i=1}^{\mathcal{N}} \lambda_i \left(A_i^\top P + PA_i + B^\top P B + PD_i D_i^\top P + E_i^\top E_i \right) < 0, \quad (59)$$

which is exactly a convex combination of (54). Therefore, although every subsystem in the SLUSS is NOT GQS-P, the condition in Assumption 2 requires that a convex combination of the subsystems should be GQS-P. ■

We use the matrix P satisfying Assumption 2 to define the dynamic output based switching law

$$\begin{aligned} SW3 : \sigma(y) &= \arg \min_{i \in \mathcal{I}} g_i(\hat{x}) \\ g_i(\hat{x}) &= \hat{x}^\top (A_i^\top P + PA_i + PD_i D_i^\top P + E_i^\top E_i) \hat{x}, \quad (60) \end{aligned}$$

under which

$$\begin{aligned} \hat{x}^\top (A_\sigma^\top P + PA_\sigma + PD_\sigma D_\sigma^\top P + E_\sigma^\top E_\sigma) \hat{x} \\ \leq \hat{x}^\top (A_i^\top P + PA_i + PD_i D_i^\top P + E_i^\top E_i) \hat{x}, \quad (61) \end{aligned}$$

holds for any \hat{x} . Multiplying both sides of the above inequality by nonnegative scalars λ_i and then adding them for all i 's, we obtain

$$\begin{aligned} \hat{x}^\top (A_\sigma^\top P + PA_\sigma + PD_\sigma D_\sigma^\top P + E_\sigma^\top E_\sigma) \hat{x} \\ \leq \sum_{i=1}^{\mathcal{N}} \lambda_i \hat{x}^\top (A_i^\top P + PA_i + PD_i D_i^\top P + E_i^\top E_i) \hat{x} \\ = \hat{x}^\top (A_\lambda^\top P + PA_\lambda + PD_\lambda D_\lambda^\top P + E_\lambda^\top E_\lambda) \hat{x}. \quad (62) \end{aligned}$$

On the other hand, the condition (58) in Assumption 2 is equivalent to

$$\begin{bmatrix} \left(\begin{array}{c} A_\lambda^\top P + PA_\lambda + \\ PD_\lambda D_\lambda^\top P + E_\lambda^\top E_\lambda \end{array} \right) & PL_i C_i & B^\top Q \\ C_i^\top L_i^\top P & \Gamma_i & B^\top Q \\ QB & QB & -Q \end{bmatrix} < 0 \quad (63)$$

which has the same form as the condition (34) but incorporates the uncertainty matrices D_λ , E_λ .

The remaining discussion of proving quadratic stability of the closed-loop switched system with the Lyapunov function candidate \tilde{P} in (44) is similar to that in Theorem 2, and is thus omitted here.

Theorem 3 Under Assumption 2 and the switching law SW3, the SLUSS (16) is GQS-P.

Similar to (34), the condition (58) in Assumption 2 is a bilinear matrix inequality (BMI) with respect to λ_i 's, $Q > 0$, $P > 0$ and matrices L_i 's. However, we observe several necessary conditions for (58), which help us understand the meaning of the condition and then solve it step by step. First, according to (63), (58) requires

$$A_\lambda^\top P + PA_\lambda + PD_\lambda D_\lambda^\top P + E_\lambda^\top E_\lambda < 0, \quad (64)$$

which implies that the combination system

$$\dot{x} = (A_\lambda + D_\lambda F(t) E_\lambda) x \quad (65)$$

is quadratically stable for any $\|F(t)\| \leq 1$. And, it is well known [35] that (64) holds if and only if A_λ is Hurwitz and the \mathcal{H}_∞ norm of the transfer function matrix $E_\lambda (sI - A_\lambda)^{-1} D_\lambda$ is smaller than 1. Secondly, the 3rd and the 4th rows and columns of (58) implies the matrix inequality (53), which is the robust detectability condition for designing the Luenberger observers.

Based on the above observation, we suggest the following algorithm for solving (58) with some trial and error.

[Algorithm 2]

- 1) Find a set of λ_i 's such that A_λ is Hurwitz, and $\|E_\lambda (sI - A_\lambda)^{-1} D_\lambda\|_\infty < 1$.
- 2) Solve (53) with respect to $Q > 0$ and M_i . If feasible, set $L_i = Q^{-1} M_i$.
- 3) With the obtained λ_i 's, $Q > 0$ and L_i 's, solve the condition (58) with respect to $P > 0$.

Remark 8 It is noted that a common quadratic Lyapunov function $V(x) = x^\top P x$ or $V(\tilde{x}) = \tilde{x}^\top \tilde{P} \tilde{x}$ has been used for the static and dynamic output based switching laws in our switched systems design up to now. The main reason is to take advantage of the convex combination of subsystems, which is expressed by the subsystem matrices and the coefficients. However, if we generalize the convex combination by including the Lyapunov-like matrices for each subsystems, less conservative conditions could be obtained while the design condition may be more complicated. ■

Remark 9 It is noted that the switching laws (38) and (60) are called the minimum (energy) rule [36], and theoretically the so-called ‘‘chattering’’ or ‘‘Zeno’’ phenomena (switchings

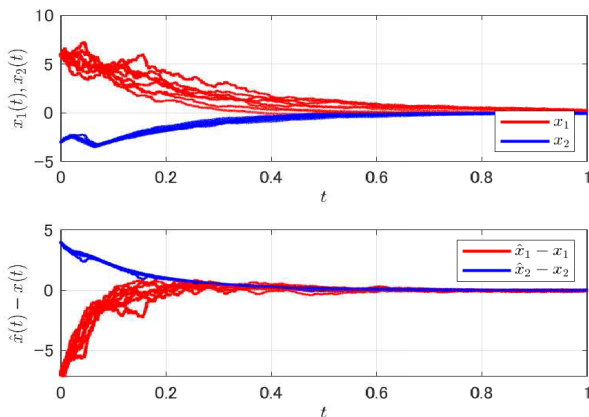


FIGURE 2. Plot of the SLUSS under SWS in Example 2.

occur infinite number of times on a finite time interval) may occur, which are not desired in any real application. To get rid of the possibility of this trouble, we may adopt a kind of hybrid switching rule for our switched system. Assuming the present activated subsystem index is i_0 ($\sigma(t) = i_0$), we do not switch to other subsystems until

$$f_{i_0}(\hat{x}) \leq \hat{x}^\top (A_\lambda^\top P + PA_\lambda) \hat{x}, \quad \text{or}$$

$$g_{i_0}(\hat{x}) \leq \hat{x}^\top (A_\lambda^\top P + PA_\lambda + PD_\lambda D_\lambda^\top P + E_\lambda^\top E_\lambda) \hat{x} \quad (66)$$

is violated. This is based on the observation that the above inequality holds on a nonzero time interval, since the function $f_i(\hat{x})$, $g_i(\hat{x})$ are continuous with respect to the state \hat{x} (and thus time t).

Furthermore, for the purpose of real applications, we may set a small enough lower bound ρ_{off} for $\|\hat{x}\|$ such that switching is stopped when $\|\hat{x}\| \leq \rho_{off}$, if necessary. ■

C. NUMERICAL EXAMPLE

Example 2 Consider the SLUSS (16) where

$$D_1 = \begin{bmatrix} 0.4 & -0.3 \\ -0.2 & 0.5 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.3 & -1.0 \\ 0.2 & -0.5 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 0.2 & 0 \\ 0.2 & 0.2 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0.3 & -0.3 \\ 0.3 & 0.3 \end{bmatrix}$$

$$F(t) = \begin{bmatrix} 0.5 \sin t - 0.1 \cos t & -0.2 \sin t - 0.3 \cos t \\ -0.3 \sin t - 0.2 \cos t & 0.1 \sin t - 0.4 \cos t \end{bmatrix},$$

and all the other matrices are the same as in Example 1. We can confirm $\|F(t)\| \leq 1$, and with the same λ_1, λ_2 as in Example 1,

$$D_\lambda = \begin{bmatrix} 0.8683 & 0 \\ 0.2810 & 0.4594 \end{bmatrix}, \quad E_\lambda = \begin{bmatrix} 0.3742 & 0.0428 \\ 0 & 0.3495 \end{bmatrix}$$

and furthermore, $\|E_\lambda(sI - A_\lambda)^{-1}D_\lambda\|_\infty < 1$.

With the matrices $Q > 0, M_1, M_2$ and L_1, L_2 obtained in Example 1, we proceed to solve (58) with respect to $P > 0$, and obtain

$$P = \begin{bmatrix} 0.1030 & 0.1652 \\ 0.1652 & 0.8849 \end{bmatrix}.$$

Finally, we activate the switching law (60) for the SLUSS with the same initial states as in Example 1. Figure 2 shows the state trajectories of the switched system and the observers, together with the tracking errors, for several patterns of random white noise. It is observed that good convergence property has been achieved.

VI. CONCLUSION

We have analyzed the QQS-P property for switched linear certain and uncertain stochastic systems, under the assumption that no single subsystem is GQS-P, and our main focus is to design the dynamic output based switching laws such that the entire switched system is GQS-P. It turns out that the approach of adopting a convex combination of subsystems (for both certain and uncertain switched systems) together with a robust Luenberger observer for each subsystem is efficient. Since the design conditions are expressed by matrix inequalities which are not linear in the variables, we have presented two algorithms for solving them efficiently.

As also mentioned in Remark 8, it is possible to adopt a kind of piecewise Lyapunov-like functions together with the convex combination of subsystems for GQS-P of the switched stochastic systems, which is in the line of our future work.

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