A Class of Refinement Schemes With Two Shape Control Parameters

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ABSTRACT A subdivision scheme defines a smooth curve or surface as the limit of a sequence of successive refinements of given polygon or mesh. These schemes take polygons or meshes as inputs and produce smooth curves or surfaces as outputs. In this paper, a class of combine refinement schemes with two shape control parameters is presented. These even and odd rules of these schemes have complexity three and four respectively. The even rule is designed to modify the vertices of the given polygon, whereas the odd rule is designed to insert a new point between every edge of the given polygon. These schemes can produce high order of continuous shapes than existing combine binary and ternary family of schemes. It has been observed that the schemes have interpolating and approximating behaviors depending on the values of parameters. These schemes have an interproximate behavior in the case of non-uniform setting of the parameters. These schemes can be considered as the generalized version of some of the interpolating and B-spline schemes. The theoretical as well as the numerical and graphical analysis of the shapes produced by these schemes are also presented.

INDEX TERMS Combined refinement schemes, continuous curves, interpolation, approximation, shape parameters, non-uniform parameters.

I. INTRODUCTION

The refinement schemes also known as subdivision schemes are widely used in the design of curves and surfaces. Initially, schemes were introduced without parameters. Later on, to improve the flexibility of designing the curves, some of the schemes were introduced with parameters. Here we present a brief survey of the schemes with parameters. The first scheme with parameter was introduced by Dyn *et al.* [21] in 1987. Then in 1990, Dyn *et al.* [1] noticed that the smoothness of the curves can be increased by using parameters in the scheme. Later on, a class of 4-point subdivision schemes with two parameters was presented in 2004 by [2]. In 2007, Siddiqi and Ahmad [3] presented a 3-point approximating C^2 scheme with single parameter. Shen and Huang [4] introduced a class of curve schemes with several parameters in 2007. Siddiqi and Rehan [5] also presented a 4-point approximating scheme

with one parameter in 2010. Mustafa *et al.* [6] presented a 3-point scheme with three parameters in 2011. In 2012, Ghaffar *et al.* [7] presented a class of 3-point *a*-ary scheme with one parameter. In 2013, Cao and Tan [8] presented a binary 5-point relaxation scheme with one parameter. Tan *et al.* [9] presented a 4-point C^3 scheme with two parameters in 2014. In 2014, Tan *et al.* [10] also presented a 5-point scheme with one parameter. Zheng *et al.* [11] introduced a scheme with multi-parameters in 2014. Mustafa *et al.* [12] introduced the families of interpolating schemes with parameters in 2014. In 2017, Feng *et al.* [13] presented a family of non-uniform schemes with variable parameters. Tan *et al.* [14] presented a new 5-point binary approximating scheme with two parameters in 2017. In 2018, Asghar and Mustafa [15] presented a family of *a*-ary univariate subdivision schemes with single parameter.

Another trend to introduce the combined schemes was evoked. These schemes have interpolating and approximating behaviors. Beccaria *et al.* [16] introduced a unified

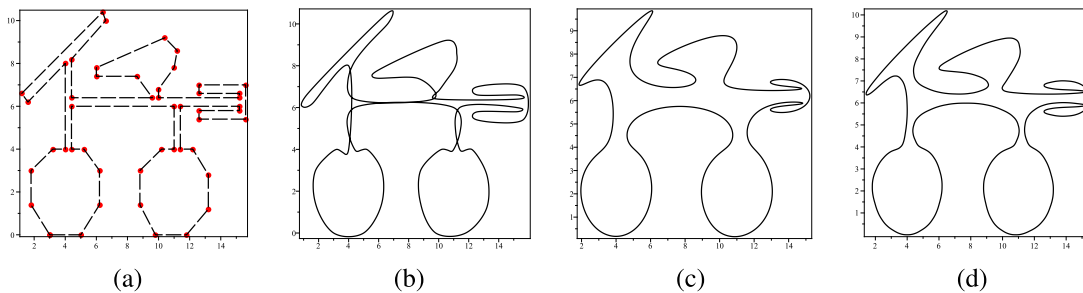The associate editor coordinating the review of this manuscript and approving it for publication was Yeliz Karaca[ID].

FIGURE 1. Comparison of the curves fitted by the binary refinement schemes after five subdivision levels. (a) is the initial sketch with initial control points.
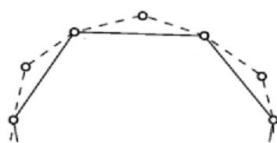

FIGURE 2. Procedure of refinement scheme. Solid lines show initial sketch while doted lines show the refined sketch.

framework for interpolating and approximating schemes with a parameter in 2010. Rehan and Siddiqi [17] presented a combined binary 6-point scheme with a parameter in 2015. Hameed and Mustafa [18] presented a class of schemes with non-uniform setting of the parameter in 2016. In 2012, Pan *et al.* [19] presented a C^2 continuous combined ternary approximating and interpolating schemes. The new family of ternary 6-point combined schemes with C^3-continuity was derived by Shi *et al.* [20] in 2018.

A. OUR CONTRIBUTION

The main purpose of this work is to increase the number of choices, for end user, of the scheme for curve modeling with less complexity and maximum smoothness in the shapes. Our scheme has less complexity and generates curve of maximum degree of smoothness than its counterpart (i.e., 4-point) schemes. It is proved in the last section of this paper. It is evidenced that there are situations or data when our scheme is more appropriate to use than the well-known schemes. This type of data is represented by Figure 1(a). It is also called initial sketch. Figure 1(b) and Figure 1(c) are generated by [22] and B-spline of degree 5 respectively, while the fitted curve by presented refinement scheme for $\alpha = -0.2e\text{-}1$ and $\beta = 0.088542$ is shown in Figure 1(d). The curve generated by [22] is not compatible with the initial sketch due to its oscillating behaviour, while the gentle behaviour of the B-spline of degree 5 pushes the limit curve away from its initial sketch. The presented refinement scheme gives good result in this case.

This paper is divided into following sections. In Section 2, we present a class of schemes with two parameters. We also discuss the theoretical analysis of the shapes produced by these schemes in this section. In Section 3, we present the numerical and graphical analysis of the shapes produced by

the schemes. Section 4, is devoted for the comparison and conclusion.

II. A CLASS OF REFINEMENT SCHEMES

If we have an initial sketch of any shape obtained by joining the 2D points f_i^0, $i \in \mathbb{Z}$ then to refine the sketch, we suggest the following refinement scheme

$$\begin{cases} f_{2i}^{k+1} = \alpha f_{i-1}^k + (1 - 2\alpha) f_i^k + \alpha f_{i+1}^k, \\ f_{2i+1}^{k+1} = \beta(1-\alpha) f_{i-1}^k + \left(\frac{1}{2} - \beta(1-\alpha) \right) f_i^k \\ \qquad + \left(\frac{1}{2} - \beta(1-\alpha) \right) f_{i+1}^k + \beta(1-\alpha) f_{i+2}^k, \end{cases} \quad (1)$$

where k represents the refinement level while α and β are the shape parameters. The refinement scheme consists of two refinement rules. One of the rules (called even rule) is used to update the vertices of the initial sketch and it uses three points of the initial sketch to insert a new one, hence its complexity is three. While the other rule (called odd rule) is used to subdivide the edges and it uses four points of the initial sketch to insert a new one, hence its complexity is four. Since there are two rules in this scheme, therefore it is called binary scheme. Graphical representation is shown in Figure 2. We can get a class of refinement schemes from (1) by assigning different values to the parameters. If we arrange the points involved in odd and even rules of (1) as

$$\left\{ \ldots, f_{i-1}^k, f_{i-1}^k, f_i^k, f_i^k, f_{i+1}^k, f_{i+1}^k, f_{i+2}^k, f_{i+2}^k, \ldots \right\}$$

then the sequence of coefficients of these points in odd and even rules is

$$\left\{ \ldots, 0, 0, 0, \beta(1-\alpha), \alpha, \left(\frac{1}{2} - \beta(1-\alpha) \right), (1 - 2\alpha), \right.$$
$$\left. \left(\frac{1}{2} - \beta(1-\alpha) \right), \alpha, \beta(1-\alpha), 0, 0, 0, 0, \ldots \right\}.$$

This sequence can be represented in terms of the following Laurent polynomial

$$a(z) = z^{-3} \left[\beta(1-\alpha)z^0 + \alpha z^1 + \left(\frac{1}{2} - \beta(1-\alpha) \right) z^2 \right.$$
$$+ (1 - 2\alpha)z^3 + \left(\frac{1}{2} - \beta(1-\alpha) \right) z^4 + \alpha z^5$$
$$\left. + \beta(1-\alpha)z^6 \right].$$

Generally, Laurent polynomial is expressed in the form $\ldots + a_{-n}z^{-n} + a_{-(n-1)}z^{-(n-1)} + \ldots + a_{-1}z^{-1} + a_0 + a_1z + \ldots + a_{n-1}z^{n-1} + a_nz^n + \ldots$, where a_i are constants and only finitely many a_i are nonzero. The detailed information about Laurent polynomial can be find in [25], [26]. Laurent polynomial or z-transform is a main tool to analyze the schemes.

Since there are two parameters in the scheme therefore we may express the one in terms of the other to explore the properties of the scheme. Classically, the schemes are analyzed by checking their degree of continuity and degrees of polynomial generation and reproduction under some conditions. These conditions can be used to express the parameters in terms of the others. The parameters α and β can be expressed as $\alpha = \frac{1}{8}\left(\frac{16\beta+1}{2\beta+1}\right)$ and $\beta = -\frac{1}{16}\left(\frac{8\alpha-1}{\alpha-1}\right)$ which can be obtained from the solution of differential equation $\frac{d^2}{dz^2}a(z) = 0$ at $z = -1$.

We get subclasses of (1) by substituting values of α and β in (1). Hence the refinement rules corresponding to $\beta = -\frac{1}{16}\left(\frac{8\alpha-1}{\alpha-1}\right)$ are

$$
\begin{cases}
f_{2i}^{k+1} = \alpha f_{i-1}^k + (1-2\alpha)f_i^k + \alpha f_{i+1}^k, \\
f_{2i+1}^{k+1} = \frac{1}{16}(8\alpha - 1)f_{i-1}^k + \frac{1}{16}(9 - 8\alpha)f_i^k \\
\qquad\quad + \frac{1}{16}(9 - 8\alpha)f_{i+1}^k + \frac{1}{16}(8\alpha - 1)f_{i+2}^k,
\end{cases} \tag{2}
$$

and its Laurent polynomial is

$$
u(z) = \frac{(1+z)^4}{16z^3}[(8\alpha - 1) + (4 - 16\alpha)z + (8\alpha - 1)z^2]. \tag{3}
$$

While the refinement rules corresponding to $\alpha = \frac{1}{8}\left(\frac{16\beta+1}{2\beta+1}\right)$ are

$$
\begin{cases}
f_{2i}^{k+1} = \left(\frac{16\beta+1}{16\beta+8}\right)f_{i-1}^k + \left(\frac{-8\beta+3}{8\beta+4}\right)f_i^k \\
\qquad + \left(\frac{16\beta+1}{16\beta+8}\right)f_{i+1}^k, \\
f_{2i+1}^{k+1} = \left(\frac{7\beta}{16\beta+8}\right)f_i^k + \left(\frac{\beta+4}{16\beta+8}\right)f_i^k \\
\qquad + \left(\frac{\beta+4}{16\beta+8}\right)f_{i+1}^k + \left(\frac{7\beta}{16\beta+8}\right)f_{i+2}^k,
\end{cases} \tag{4}
$$

and its Laurent polynomial is

$$
v(z) = \frac{(1+z)^4}{(16\beta+8)z^3}\left[7\beta - (12\beta - 1)z + (7\beta)z^2\right]. \tag{5}
$$

The following special cases show that the scheme (1) is the generalized version of B-spline of degree 1, 3 and 5. It is also the generalized version of the interpolatory schemes of [21] and [22].

- For $\alpha = 0$ and $\beta = -\omega$, we get 4-point interpolatory scheme of [21].
- For $\alpha = 0$ and $\beta = -\frac{1}{16}$, it changes into 4-point interpolatory scheme of [22].
- For $\alpha = 0$ and $\beta = 0$, it shrinks to 2-point interpolatory scheme of [22] which is also called B-spline scheme of degree 1.

- For $\alpha = \frac{1}{8}$ and $\beta = 0$, it becomes B-spline scheme of degree-3.
- For $\alpha = \frac{3}{16}$ and $\beta = \frac{1}{26}$, we get B-spline scheme of degree-5.

Through out the paper, S_u, S_v S_{c_r}, $S_{c_r^L}$ and S_{b_r} are the schemes corresponding to the polynomials $u(z)$, $v(z)$, $c_r(z)$, $c_r^L(z)$ and $b_r(z)$ for $r \geq 0$, $L \geq 2$ respectively. Some of these polynomials are defined in coming section.

A. THEORETICAL ANALYSIS OF THE SHAPES

In the similar way to the arguments in [25], we can identify the ranges of parameters to get the shapes of different degree of smoothness (i.e. order of continuity) produced by the scheme.

Lemma 1: Let $\{f_i^0\}_{i\in\mathbb{Z}}$ be the initial sketch of the shape then the refinement scheme S_u defined by (2) produces the C^0-continuous shape for the parametric interval $\frac{7}{16} - \frac{1}{16}\sqrt{137} < \alpha < \frac{5}{24} + \frac{1}{12}\sqrt{91}$.

Proof: To find out the C^0-continuity of the refinement scheme (2), we rewrite equation (3) as

$$
u(z) = \left(\frac{1+z}{2}\right)^0 b_0(z),
$$

where

$$
b_0(z) = \frac{1}{16z^3}(1+z)^4\left[(8\alpha - 1) + (4 - 16\alpha)z + (8\alpha - 1)z^2\right].
$$

Since

$$
b_0(z) = (1+z)c_0(z),
$$

where

$$
c_0(z) = \left(\frac{1}{2}\alpha - \frac{1}{16}\right)z^{-3} + \left(\frac{1}{2}\alpha + \frac{1}{16}\right)z^{-2} + \left(-\alpha + \frac{1}{2}\right)z^{-1} + \left(-\alpha + \frac{1}{2}\right)z^0 + \left(\frac{1}{2}\alpha + \frac{1}{16}\right)z^1 + \left(\frac{1}{2}\alpha - \frac{1}{16}\right)z^2. \tag{6}
$$

This implies that

$$
c_0(z) = \sum_{i=-3}^{2} r_0^i z^i,
$$

where $r_0^{-3} = r_0^2 = \frac{1}{2}\alpha - \frac{1}{16}$, $r_0^{-2} = r_0^1 = \frac{1}{2}\alpha + \frac{1}{16}$ and $r_0^{-1} = r_0^0 = -\alpha + \frac{1}{2}$. The infinity norm of c_0 is calculated as

$$
||c_0||_\infty = \max\left\{\sum_{j=-2}^{0}|r_0^{2j+1}|, \sum_{j=-1}^{1}|r_0^{2j}|\right\}.
$$

This implies

$$
||c_0||_\infty = \max\left\{\left|\frac{1}{2}\alpha - \frac{1}{16}\right| + \left|-\alpha + \frac{1}{2}\right| + \left|\frac{1}{2}\alpha + \frac{1}{16}\right|, \left|\frac{1}{2}\alpha + \frac{1}{16}\right| + \left|-\alpha + \frac{1}{2}\right| + \left|\frac{1}{2}\alpha - \frac{1}{16}\right|\right\}.
$$

It is clear that $||c_0||_\infty < 1$ for $-\frac{1}{4} < \alpha < \frac{3}{4}$. So, the scheme S_{c_0} is contractive. We may improve the range of parameter by taking $c_0^L(z) = c_0(z)c_0(z^2)\ldots c_0(z^{2^{L-1}})$, where $L > 0$. For simplicity, we may take $L = 2$

$$c_0^2(z) = c_0(z)c_0(z^2),$$

where $c_0(z)$ is defined in (6) and $c_0(z^2)$ is obtained by replacing z^2 in the place of z in (6). Hence we get

$$
\begin{aligned}
c_0^2(z) =& \left(\frac{1}{4}\alpha^2 - \frac{1}{16}\alpha + \frac{1}{256}\right)z^{-9} + \left(\frac{1}{4}\alpha^2 - \frac{1}{256}\right) \\
& \times z^{-8} + \left(-\frac{1}{4}\alpha^2 + \frac{5}{16}\alpha - \frac{9}{256}\right)z^{-7} + \left(-\frac{1}{4}\right. \\
& \left. \times \alpha^2 + \frac{3}{8}\alpha - \frac{7}{256}\right)z^{-6} + \left(-\frac{3}{4}\alpha^2 + \frac{1}{2}\alpha\right. \\
& \left. - \frac{1}{256}\right)z^{-5} + \left(-\frac{3}{4}\alpha^2 + \frac{5}{16}\alpha + \frac{17}{256}\right)z^{-4} \\
& + \left(\frac{3}{4}\alpha^2 - \frac{5}{8}\alpha + \frac{57}{256}\right)z^{-3} + \left(\frac{3}{4}\alpha^2 - \frac{13}{16}\alpha +\right. \\
& \left. \frac{71}{256}\right)z^{-2} + \left(\frac{3}{4}\alpha^2 - \frac{13}{16}\alpha + \frac{71}{256}\right)z^{-1} \\
& + \left(\frac{3}{4}\alpha^2 - \frac{5}{8}\alpha + \frac{57}{256}\right)z^0 + \left(-\frac{3}{4}\alpha^2 + \frac{5}{16}\alpha\right. \\
& \left. + \frac{17}{256}\right)z^1 + \left(-\frac{3}{4}\alpha^2 + \frac{1}{2}\alpha - \frac{1}{256}\right)z^2 \\
& + \left(-\frac{1}{4}\alpha^2 + \frac{3}{8}\alpha - \frac{7}{256}\right)z^3 + \left(-\frac{1}{4}\alpha^2 + \frac{5}{16}\alpha\right. \\
& \left. - \frac{9}{256}\right)z^4 + \left(\frac{1}{4}\alpha^2 - \frac{1}{256}\right)z^5 + \left(\frac{1}{4}\alpha^2 - \frac{1}{16}\right. \\
& \left. \times \alpha + \frac{1}{256}\right)z^6.
\end{aligned}
\tag{7}
$$

This implies that

$$c_0^2(z) = \sum_{i=-9}^{6} R_0^i z^i,$$

where R_0^i are the coefficients of z^i for $i = -9, -8, \ldots, 6$ in (7) respectively. The infinity norm of c_0^2 is

$$
||c_0^2||_\infty = \max\left\{\sum_{j=-3}^{0}|R_0^{4j+3}|, \sum_{j=-2}^{1}|R_0^{4j}|, \sum_{j=-2}^{1}|R_0^{4j+1}|,\right.
$$
$$
\left. \sum_{j=-2}^{1}|R_0^{4j+2}|\right\},
$$

This implies that

$$
\begin{aligned}
||c_0^2||_\infty = \max &\left\{\left|\frac{1}{4}\alpha^2 - \frac{1}{16}\alpha + \frac{1}{256}\right| + \left|-\frac{3}{4}\alpha^2 + \frac{1}{2}\alpha\right.\right. \\
& \left.- \frac{1}{256}\right| + \left|\frac{3}{4}\alpha^2 - \frac{13}{16}\alpha + \frac{71}{256}\right| + \left|-\frac{1}{4}\alpha^2\right. \\
& \left.+ \frac{3}{8}\alpha - \frac{7}{256}\right|, \left|\frac{1}{4}\alpha^2 - \frac{1}{256}\right| + \left|-\frac{3}{4}\alpha^2 + \frac{5}{16}\alpha\right.
\end{aligned}
$$

$$
\begin{aligned}
& \left.+ \frac{17}{256}\right| + \left|\frac{3}{4}\alpha^2 - \frac{5}{8}\alpha + \frac{57}{256}\right| + \left|-\frac{1}{4}\alpha^2+\right. \\
& \left.\frac{5}{16}\alpha - \frac{9}{256}\right|, \left|-\frac{1}{4}\alpha^2 + \frac{5}{16}\alpha - \frac{9}{256}\right| + \left|\frac{3}{4}\right. \\
& \left.\times\alpha^2 - \frac{5}{8}\alpha + \frac{57}{256}\right| + \left|-\frac{3}{4}\alpha^2 + \frac{5}{16}\alpha + \frac{17}{256}\right| \\
& + \left|\frac{1}{4}\alpha^2 - \frac{1}{256}\right|, \left|-\frac{1}{4}\alpha^2 + \frac{3}{8}\alpha - \frac{7}{256}\right| + \left|\frac{3}{4}\right. \\
& \left.\times\alpha^2 - \frac{13}{16}\alpha + \frac{71}{256}\right| + \left|-\frac{3}{4}\alpha^2 + \frac{1}{2}\alpha - \frac{1}{256}\right| \\
& + \left|\frac{1}{4}\alpha^2 - \frac{1}{16}\alpha + \frac{1}{256}\right|\bigg\}.
\end{aligned}
$$

Since $||c_0^2||_\infty < 1$ for

$$\frac{7}{16} - \frac{1}{16}\sqrt{137} < \alpha < \frac{1}{3} + \frac{1}{24}\sqrt{349}$$

and

$$\frac{5}{16} - \frac{1}{16}\sqrt{129} < \alpha < \frac{5}{24} + \frac{1}{12}\sqrt{91}.$$

Therefore, $||c_0^2||_\infty < 1$ for common range of the parameter

$$\frac{7}{16} - \frac{1}{16}\sqrt{137} < \alpha < \frac{5}{24} + \frac{1}{12}\sqrt{91}.$$

So $S_{c_0^2}$ is contractive and the scheme S_{b_0} is convergent and the scheme S_u is C^0-continuous. $\qquad\square$

Lemma 2: Let $\{f_i^0\}_{i\in\mathbb{Z}}$ be the initial sketch of the shape then the refinement scheme S_u defined by (2) produces the C^1-continuous shape for the parametric interval $\frac{5}{16} - \frac{1}{16}\sqrt{33} < \alpha < \frac{5}{16} + \frac{1}{16}\sqrt{33}$.

Proof: To find out the C^1-continuity of the refinement scheme (2), we rewrite equation (3) as

$$u(z) = \left(\frac{1+z}{2}\right)^1 b_1(z),$$

where

$$b_1(z) = \frac{1}{8z^3}(1+z)^3\left[(8\alpha - 1) + (4 - 16\alpha)z + (8\alpha - 1)z^2\right].$$

Since

$$b_1(z) = (1+z)c_1(z),$$

where

$$c_1(z) = \left(\alpha - \frac{1}{8}\right)z^{-3} + \left(\frac{1}{4}\right)z^{-2} + \left(\frac{3}{4} - 2\alpha\right)z^{-1} + \left(\frac{1}{4}\right)z^0 + \left(\alpha - \frac{1}{8}\right)z^1.$$

This implies that

$$c_1(z) = \sum_{i=-3}^{1} r_1^i z^i,$$

where $r_1^{-3} = r_1^1 = \alpha - \frac{1}{8}$, $r_1^{-2} = r_1^0 = \frac{1}{4}$ and $r_1^{-1} = \frac{3}{4} - 2\alpha$.
The infinity norm of c_1 is calculated as

$$||c_1||_\infty = \max\left\{\sum_{j=-2}^{0} |r_1^{2j+1}|, \sum_{j=-1}^{0} |r_1^{2j}|\right\}.$$

So we get

$$||c_1||_\infty = \max\left\{\left|\alpha - \frac{1}{8}\right| + \left|\frac{3}{4} - 2\alpha\right| + \left|\alpha - \frac{1}{8}\right|, \left|\frac{1}{4}\right| + \left|\frac{1}{4}\right|\right\}.$$

Hence $||c_1||_\infty < 1$ for $0 < \alpha < \frac{1}{2}$. So, the scheme S_{c_1} is contractive. Now we use $c_1^2(z) = c_1(z)c_1(z^2)$ to improve the range of parameter. This implies

$c_1^2(z)$

$$= \left(\alpha^2 - \frac{1}{4}\alpha + \frac{1}{64}\right)z^{-9} + \left(\frac{1}{4}\alpha - \frac{1}{32}\right)z^{-8}$$
$$+ \left(-2\alpha^2 + \frac{5}{4}\alpha - \frac{1}{8}\right)z^{-7} + \left(\frac{1}{4}\alpha + \frac{1}{32}\right)z^{-6} + \left(-\alpha^2\right.$$
$$+ \frac{1}{4}\alpha + \frac{7}{64}\right)z^{-5} + \left(-\frac{1}{2}\alpha + \frac{1}{4}\right)z^{-4} + \left(4\alpha^2 - \frac{5}{2}\alpha\right.$$
$$+ \frac{1}{2}\right)z^{-3} + \left(-\frac{1}{2}\alpha + \frac{1}{4}\right)z^{-2} + \left(-\alpha^2 + \frac{1}{4}\alpha + \frac{7}{64}\right)$$
$$\times z^{-1} + \left(\frac{1}{4}\alpha + \frac{1}{32}\right)z^0 + \left(-2\alpha^2 + \frac{5}{4}\alpha - \frac{1}{8}\right)z^1$$
$$+ \left(\frac{1}{4}\alpha - \frac{1}{32}\right)z^2 + \left(\alpha^2 - \frac{1}{4}\alpha + \frac{1}{64}\right)z^3. \quad (8)$$

This implies that

$$c_1^2(z) = \sum_{i=-9}^{3} R_1^i z^i,$$

where R_1^i are the coefficients of z^i for $i = -9, -8, \ldots, 3$ in (8) respectively.
The infinity norm of c_1^2 is

$$||c_1^2||_\infty = \max\left\{\sum_{j=-3}^{0} |R_1^{4j+3}|, \sum_{j=-2}^{0} |R_1^{4j}|, \sum_{j=-2}^{0} |R_1^{4j+1}|,\right.$$
$$\left.\sum_{j=-2}^{0} |R_1^{4j+2}|\right\},$$

Hence we get

$||c_1^2||_\infty$
$$= \max\left\{\left|\alpha^2 - \frac{1}{4}\alpha + \frac{1}{64}\right| + \left|-\alpha^2 + \frac{1}{4}\alpha + \frac{7}{64}\right|\right.$$
$$+ \left|-\alpha^2 + \frac{1}{4}\alpha + \frac{7}{64}\right| + \left|\alpha^2 - \frac{1}{4}\alpha + \frac{1}{64}\right|, \left|\frac{1}{4}\alpha - \frac{1}{32}\right|$$
$$+ \left|-\frac{1}{2}\alpha + \frac{1}{4}\right| + \left|\frac{1}{4}\alpha + \frac{1}{32}\right|, \left.+\left|-2\alpha^2 + \frac{5}{4}\alpha - \frac{1}{8}\right| + \left|4\alpha^2\right.\right.$$

$$-\frac{5}{2}\alpha + \frac{1}{2}\right| + \left|-2\alpha^2 + \frac{5}{4}\alpha - \frac{1}{8}\right|, \left|\frac{1}{4}\alpha - \frac{1}{32}\right| + \left|-\frac{1}{2}\alpha\right.$$
$$+ \frac{1}{4}\right| + \left|\frac{1}{4}\alpha + \frac{1}{32}\right|\right\}.$$

Hence $||c_1^2||_\infty < 1$ for

$$\frac{1}{8} - \frac{1}{4}\sqrt{5} < \alpha < \frac{1}{8} + \frac{1}{4}\sqrt{5}, \quad -\frac{3}{4} < \alpha < \frac{5}{4}$$

and

$$\frac{5}{16} - \frac{1}{16}\sqrt{33} < \alpha < \frac{5}{16} + \frac{1}{16}\sqrt{33}.$$

Hence the common range for C^1-continuity of the scheme S_u is

$$\frac{5}{16} - \frac{1}{16}\sqrt{33} < \alpha < \frac{5}{16} + \frac{1}{16}\sqrt{33}.$$

This completes the proof. □

Lemma 3: Let $\{f_i^0\}_{i\in\mathbb{Z}}$ be the initial sketch of the shape then the refinement scheme S_u produces the C^2-continuous shape for the parametric interval $0 < \alpha < \frac{1}{2}$.

Proof: To find out the C^2-continuity of the refinement scheme (2), we rewrite equation (3) as

$$u(z) = \left(\frac{1+z}{2}\right)^2 b_2(z),$$

where

$$b_2(z) = \frac{1}{4z^3}(1+z)^2\left[(8\alpha - 1) + (4 - 16\alpha)z + (8\alpha - 1)z^2\right].$$

Moreover

$$b_2(z) = (1+z)c_2(z),$$

where

$$c_2(z) = \left(2\alpha - \frac{1}{4}\right)z^{-3} + \left(-2\alpha + \frac{3}{4}\right)z^{-2} + (-2\alpha + \frac{3}{4})z^{-1} + \left(2\alpha - \frac{1}{4}\right)z^0.$$

We take infinity norm of c_2

$$||c_2||_\infty = \max\left\{\left|2\alpha - \frac{1}{4}\right| + \left|-2\alpha + \frac{3}{4}\right|, \left|-2\alpha + \frac{3}{4}\right| + \left|2\alpha - \frac{1}{4}\right|\right\}.$$

It is to be noted that $||c_2||_\infty < 1$ for $0 < \alpha < \frac{1}{2}$. Therefore, the scheme S_{c_2} is contractive and for further improvement take $c_2^2(z) = c_2(z)c_2(z^2)$. This implies

$$c_2^2(z) = \left(4\alpha^2 - \alpha + \frac{1}{16}\right)z^{-9} + \left(-4\alpha^2 + 2\alpha - \frac{3}{16}\right)$$
$$\times z^{-8} + \left(-8\alpha^2 + 4\alpha - \frac{3}{8}\right)z^{-7} + \left(8\alpha^2 - 4\alpha - \frac{5}{8}\right)$$
$$\times z^{-6} + \left(-\alpha + \frac{3}{8}\right)z^{-5} + \left(-\alpha + \frac{3}{8}\right)z^{-4} + \left(8\alpha^2\right.$$

$$\left. - 4\alpha - \frac{5}{8} \right) z^{-3} + \left(-8\alpha^2 + 4\alpha - \frac{3}{8} \right) z^{-2} + \left(-4\alpha^2 \right.$$

$$\left. + 2\alpha - \frac{3}{16} \right) z^{-1} + \left(4\alpha^2 - \alpha + \frac{1}{16} \right) z^0.$$

Now by taking infinity norm of c_2^2, we get

$$||c_2^2||_\infty$$

$$= \max \left\{ \left| 4\alpha^2 - \alpha + \frac{1}{16} \right| + \left| -\alpha + \frac{3}{8} \right| + \left| -4\alpha^2 \right. \right.$$

$$+ 2\alpha - \frac{3}{16} \left| , \left| -4\alpha^2 + 2\alpha - \frac{3}{16} \right| + \left| -\alpha + \frac{3}{8} \right| + \left| 4\alpha^2 \right. \right.$$

$$- \alpha + \frac{1}{16} \left| , \left| -8\alpha^2 + 4\alpha - \frac{3}{8} \right| + \left| 8\alpha^2 - 4\alpha + \frac{5}{8} \right| , \left| 8\alpha^2 \right. \right.$$

$$-4\alpha + \frac{5}{8} \left| + \left| -8\alpha^2 + 4\alpha - \frac{3}{8} \right| \right\}.$$

Which is less than 1 for

$$\frac{1}{4} - \frac{1}{8}\sqrt{7} < \alpha < \frac{1}{8} + \frac{1}{8}\sqrt{10} \text{ and } 0 < \alpha < \frac{1}{2}$$

The common range is

$$0 < \alpha < \frac{1}{2}.$$

In this case, further improvement in the range has not been seen. This completes the proof. □

Similarly, wet get the following result

Lemma 4: Let $\{f_i^0\}_{i\in\mathbb{Z}}$ be the initial sketch of the shape then the refinement scheme S_u produces the C^3-continuous shape for the interval $\frac{1}{8} < \alpha < \frac{3}{16} + \frac{1}{16}\sqrt{5}$.

From Lemmas 2 - 4, we get the following

Theorem 5: Let $\{f_i^0\}_{i\in\mathbb{Z}}$ be the initial sketch of the shape then the refinement scheme S_u produces the C^0-continuous shape for the parametric interval $\frac{7}{16} - \frac{1}{16}\sqrt{137} < \alpha < \frac{5}{24} + \frac{1}{12}\sqrt{91}$. Moreover, it produces C^1 and C^2-continuous shapes for the parametric interval $\frac{5}{16} - \frac{1}{16}\sqrt{33} < \alpha < \frac{5}{16} + \frac{1}{16}\sqrt{33}$ and $0 < \alpha < \frac{1}{2}$ respectively. Furthermore, it produces C^3-continuous shape for the interval $\frac{1}{8} < \alpha < \frac{3}{16} + \frac{1}{16}\sqrt{5}$.

Similarly, we have

Theorem 6: Let $\{f_i^0\}_{i\in\mathbb{Z}}$ be the initial sketch of the shape then the refinement scheme S_u produces the C^4-continuous shape for $\alpha = \frac{3}{16}$.

Theorem 7: The refinement scheme S_v defined by (4) produces C^0-continuous shape for the parametric interval $-\frac{143}{62} - \frac{7}{62}\sqrt{349} < \beta < -\frac{81}{8} + \frac{7}{8}\sqrt{129}$. Moreover, it produces C^1 and C^2-continuous shapes over the parametric intervals $\frac{3}{8} - \frac{7}{88}\sqrt{33} < \beta < \frac{3}{8} + \frac{7}{88}\sqrt{33}$ and $-\frac{1}{16} < \beta < \frac{3}{8}$ respectively. Furthermore, it produces C^3-continuous shape for the interval $0 < \beta < \frac{9}{164} + \frac{7}{164}\sqrt{5}$. In addition, it produces C^4-continuous shape for $\beta = \frac{1}{26}$.

Proof: The Laurent polynomial (5) can further be written as

$$v(z) = \left(\frac{1+z}{2} \right)^i \underline{b}_i(z), \quad i = 0, 1, 2, 3, 4, \quad (9)$$

where $\underline{b}_i(z) = (1 + z)\underline{c}_i(z)$: $i = 0, 1, 2, 3, 4$. We further calculate

$$\underline{c}_i^2(z) = \underline{c}_i(z)\underline{c}_i(z^2), \quad i = 0, 1, 2, 3, 4. \quad (10)$$

This implies that

$$\underline{c}_0^2(z)$$

$$= \left(\frac{1}{16\beta + 8} \right)^2 [(49\beta^2)z^{-9} + (63\beta^2 + 7\beta)z^{-8}$$

$$+ (7\beta^2 + 28\beta)z^{-7} + (25\beta^2 + 39\beta + 1)z^{-6} + (-65\beta^2$$

$$+ 47\beta + 3)z^{-5} + (-95\beta^2 + 38\beta + 6)z^{-4} + (89\beta^2$$

$$- 9\beta + 10)z^{-3} + (55\beta^2 - 22\beta + 12)z^{-2} + (55\beta^2 - 22$$

$$\times \beta + 12)z^{-1} + (89\beta^2 - 9\beta + 10)z^0 + (-95\beta^2 + 38$$

$$\times \beta + 6)z^1 + (-65\beta^2 + 47\beta + 3)z^2 + (25\beta^2 + 39\beta$$

$$+ 1)z^3 + (7\beta^2 + 28\beta)z^4 + (63\beta^2 + 7\beta)z^5 + (49\beta^2)z^6]$$

$$= \sum_{i=-9}^{6} C_0^i z^i,$$

$$\underline{c}_1^2(z)$$

$$= \left(\frac{1}{8\beta + 4} \right)^2 [\left(49\beta^2 \right) z^{-9} + \left(14\beta^2 + 7\beta \right) z^{-8} +$$

$$\left(-56\beta^2 + 21\beta \right) z^{-7} + \left(18\beta^2 + 11\beta + 1 \right) z^{-6} + \left(-41\beta^2 \right.$$

$$+ 8\beta + 2) z^{-5} + \left(-16\beta^2 - 2\beta + 3 \right) z^{-4} + \left(128\beta^2 - 26 \right.$$

$$\times \beta + 4) z^{-3} + \left(-16\beta^2 - 2\beta + 3 \right) z^{-2} + \left(-41\beta^2 + 8\beta \right.$$

$$+ 2) z^{-1} + \left(18\beta^2 + 11\beta + 1 \right) z^0 + \left(-56\beta^2 + 21\beta \right) z^1$$

$$+ \left(14\beta^2 + 7\beta \right) z^2 + \left(49\beta^2 \right) z^3] = \sum_{i=-9}^{3} C_1^i z^i,$$

$$\underline{c}_2^2(z)$$

$$= \left(\frac{1}{4\beta + 2} \right)^2 [\left(49\beta^2 \right) z^{-9} + \left(-35\beta^2 + 7\beta \right) z^{-8}$$

$$+ \left(-70\beta^2 + 14\beta \right) z^{-7} + \left(74\beta^2 - 10\beta + 1 \right) z^{-6} + (-10$$

$$\times \beta^2 - 3\beta + 1) z^{-5} + \left(-10\beta^2 - 3\beta + 1 \right) z^{-4} + \left(74\beta^2 \right.$$

$$- 10\beta + 1) z^{-3} + \left(-70\beta^2 + 14\beta \right) z^{-2} + \left(-35\beta^2 + 7\beta \right)$$

$$\times z^{-1} + \left(49\beta^2 \right) z^0] = \sum_{i=-9}^{0} C_2^i z^i,$$

$$\underline{c}_3^2(z)$$

$$= \left(\frac{1}{2\beta + 1} \right)^2 [\left(49\beta^2 \right) z^{-9} + \left(-84\beta^2 + 7\beta \right) z^{-8}$$

$$+ \left(-35\beta^2 + 7\beta \right) z^{-7} + \left(144\beta^2 - 24\beta + 1 \right) z^{-6} + (-35$$

$$\times \beta^2 + 7\beta) z^{-5} + \left(-84\beta^2 + 7\beta \right) z^{-4} + \left(49\beta^2 \right) z^{-3}]$$

$$= \sum_{i=-9}^{-3} C_3^i z^i,$$

and if we put $i = 4$ and $\beta = \frac{1}{26}$ in (10), we get

$$c_4(z) = \frac{1}{4}z^{-9} + \frac{1}{4}z^{-8} + \frac{1}{4}z^{-7} + \frac{1}{4}z^{-6}$$

$$= \sum_{i=-9}^{-6} C_4^i z^i.$$

Thus we have

$$||\underline{c_0^2}||_\infty = \max \left\{ \sum_{j=-3}^{0} |C_0^{4j+3}|, \sum_{j=-2}^{1} |C_0^{4j}|, \right.$$
$$\left. \sum_{j=-2}^{1} |C_0^{4j+1}|, \sum_{j=-2}^{1} |C_0^{4j+2}| \right\},$$

$$||\underline{c_1^2}||_\infty = \max \left\{ \sum_{j=-3}^{0} |C_1^{4j+3}|, \sum_{j=-2}^{0} |C_1^{4j}|, \right.$$
$$\left. \sum_{j=-2}^{0} |C_1^{4j+1}|, \sum_{j=-2}^{0} |C_1^{4j+2}| \right\},$$

$$||\underline{c_2^2}||_\infty = \max \left\{ \sum_{j=-3}^{-1} |C_2^{4j+3}|, \sum_{j=-2}^{0} |C_2^{4j}|, \right.$$
$$\left. \sum_{j=-2}^{-1} |C_2^{4j+1}|, \sum_{j=-2}^{-1} |C_2^{4j+2}| \right\},$$

$$||\underline{c_3^2}||_\infty = \max \left\{ \sum_{j=-3}^{-2} |C_3^{4j+3}|, \sum_{j=-2}^{-1} |C_3^{4j}|, \right.$$
$$\left. \sum_{j=-2}^{-1} |C_3^{4j+1}|, \sum_{j=-2}^{-2} |C_3^{4j+2}| \right\},$$

and

$$||\underline{c_4^2}||_\infty = \max \left\{ \sum_{j=-3}^{-3} |C_4^{4j+3}|, \sum_{j=-2}^{-2} |C_4^{4j}|, \right.$$
$$\left. \sum_{j=-2}^{-2} |C_4^{4j+1}|, \sum_{j=-2}^{-2} |C_4^{4j+2}| \right\}.$$

This further implies that $||\underline{c_0^2}||_\infty < 1$ for $-\frac{143}{62} - \frac{7}{62}\sqrt{349} < \beta < -\frac{81}{8} + \frac{7}{8}\sqrt{129}$, $||\underline{c_1^2}||_\infty < 1$ for $\frac{3}{8} - \frac{7}{88}\sqrt{33} < \beta < \frac{3}{8} + \frac{7}{88}\sqrt{33}$, $||\underline{c_2^2}||_\infty < 1$ for $-\frac{1}{16} < \beta < \frac{3}{8}$, $||\underline{c_3^2}||_\infty < 1$ for $0 < \beta < \frac{9}{164} + \frac{7}{164}\sqrt{5}$ and $||\underline{c_4}||_\infty < 1$. This completes the proof. □

For the special value of $\alpha = 0$ in (1), we get the following interpolating scheme

$$\begin{cases} f_{2i}^{k+1} = f_i^k, \\ f_{2i+1}^{k+1} = \beta f_{i-1}^k + (\frac{1}{2} - \beta)f_i^k + (\frac{1}{2} - \beta)f_{i+1}^k + \beta f_{i+2}^k. \end{cases} \quad (11)$$

The Laurent polynomial of (11) is

$$w(z) = \frac{(1+z)^2}{2z^3}[2\beta z^4 - 4\beta z^3 + (1+4\beta)z^2 - 4\beta z + 2\beta]. \quad (12)$$

Theorem 8: The interpolating scheme defined by (11) produces C^0 and C^1-continuous shapes for $\frac{1}{8} - \frac{1}{8}\sqrt{13} < \beta < \frac{3}{8}$ and $\frac{1}{8} - \frac{1}{8}\sqrt{5} < \beta < 0$ respectively.

Proof: Since (12) can be written as

$$w(z) = \left(\frac{1+z}{2}\right)^i m_i(z), \quad i = 0, 1.$$

where $m_i(z) = (1+z)n_i(z)$: $i = 1, 2$. We further calculate the expression

$$n_i^2(z) = n_i(z)n_i(z^2), \quad i = 0, 1.$$

The infinity norm $||n_0^2||_\infty$ and $||n_1^2||_\infty$ are less than one for $\frac{1}{8} - \frac{1}{8}\sqrt{13} < \beta < \frac{3}{8}$ and $\frac{1}{8} - \frac{1}{8}\sqrt{5} < \beta < 0$ respectively. Hence the required result is proved. □

For $\beta = 0$ in (1), we have the following scheme.

$$\begin{cases} f_{2i}^{k+1} = \alpha f_{i-1}^k + (1 - 2\alpha)f_i^k + \alpha f_{i+1}^k, \\ f_{2i+1}^{k+1} = 0f_{i-1}^k + \frac{1}{2}f_i^k + \frac{1}{2}f_{i+1}^k + 0f_{i+2}^k. \end{cases} \quad (13)$$

The Laurent polynomial corresponding to (13) is

$$y(z) = \frac{(1+z)^2}{2z^2}[2z^2\alpha + (1 - 4\alpha)z + 2\alpha]. \quad (14)$$

Theorem 9: The approximating scheme defined by (13) produces C^0 and C^1-continuous shapes for the parametric interval $-\frac{1}{4} < \alpha < \frac{3}{4}$ and $0 < \alpha < \frac{1}{8} + \frac{1}{8}\sqrt{5}$ respectively. The proof of this theorem is similar as of the Theorem 8.

B. POLYNOMIAL GENERATION AND REPRODUCTION

In this section, we discuss another feature of the scheme. If the initial data is sampled from the polynomial of degree d then we are interested to see whether or not the new data obtained from the scheme lie on the graph of same polynomial. If the new data lie on the graph of same polynomial then we say that the scheme reproduces polynomial of degree d. If the new data lie on the graph of another polynomial but with degree d then we say that the scheme generates polynomial of same degree. Mathematically, polynomial generation of degree d is equivalent to $u(z) = (1 + z)^{d+1}b(z)$, where $b(z)$ is a polynomial. Since $\frac{d}{dz}u(z) = 0$ at $z = 1$ so $\tau = \frac{u'(1)}{2} = 0$. This implies $t_i^k = -\tau + \frac{i+\tau}{2^k} = \frac{i+0}{2^k} = \frac{i}{2^k}$. This means the scheme has primal parametrization. In the similar way to the arguments in [23], we can get the degree of polynomial generation and reproduction with respect to primal parametrization of the scheme.

Lemma 10: The degree of polynomial generation of the scheme (2) is 3.

Proof: Since Laurent polynomial $u(z)$ of the scheme (2) is

$$u(z) = (1+z)^{3+1}b(z),$$

where

$$b(z) = \frac{1}{16z^3}\left[(8\alpha - 1) + (4 - 16\alpha)z + (8\alpha - 1)z^2\right].$$

This completes the proof. □

Theorem 11: A refinement scheme (2) reproduces polynomials of degree 1 with respect to the primal parameterizations with $\tau = 0$ if and only if

$$u^{(k)}(1) = 2\prod_{j=0}^{k-1}(\tau - j) \quad and \quad u^{(k)}(-1) = 0, \ \ k = 0, 1,$$

where $u^{(k)}(1)$ is the kth derivative of $u(z)$ at $z = 1$.

Proof: The Laurent polynomial (3) of the scheme (2) and its derivative with respect to z are

$$u(z) = u^{(0)}(z) = \frac{(1+z)^4}{16z^3}\Big[(8\alpha - 1) + (4 - 16\alpha)z$$
$$+ (8\alpha - 1)z^2\Big].$$

$$u^{(1)}(z) = \frac{(1+z)^3}{16z^4}\Big[(24\alpha - 3)z^3 - (40\alpha - 9)z^2$$
$$+ (40\alpha - 9)z^1 - (24\alpha - 3)z^0\Big].$$

$$u^{(2)}(z) = \frac{(1+z)^2}{8z^5}\Big[(24\alpha - 3)z^4 + (-32\alpha + 6)z^3$$
$$+ (40\alpha - 9)z^2 + (-48\alpha + 12)z^1$$
$$+ (48\alpha - 6)\Big].$$

Taking $z = -1$ in all above, we get

$$u^{(k)}(-1) = 0, \quad k = 0, 1, 2.$$

It is easy to see that

$$u^{(0)}(1) = 2, \quad u^{(1)}(1) = 0, \ \ u^{(2)}(1) = 16\alpha.$$

This further implies that

$$u^{(0)}(1) = 2, \quad u^{(1)}(1) = 2\prod_{j=0}^{1-1}(0 - j),$$

$$u^{(2)}(1) \neq 2\prod_{j=0}^{2-1}(0 - j).$$

Thus

$$u^{(k)}(1) = 2\prod_{j=0}^{k-1}(0 - j)$$

and

$$u^{(k)}(-1) = 0, \quad k = 0, 1.$$

This completes the proof. $\square$

Lemma 12: The degree of polynomial generation of the scheme (4) is 3.

Theorem 13: A refinement scheme (2) reproduces polynomials of degree 1 with respect to the primal parameterizations with $\tau = 0$ if and only if

$$u^{(k)}(1) = 2\prod_{j=0}^{k-1}(\tau - j) \quad and \quad u^{(k)}(-1) = 0, \ \ k = 0, 1,$$

where $u^{(k)}(1)$ is the kth derivative of $u(z)$ at $z = 1$.

Proof: The Laurent polynomial (3) of the scheme (2) and its derivative with respect to z are

$$u(z) = u^{(0)}(z) = \frac{(1+z)^4}{16z^3}[(8\alpha - 1) + (4 - 16\alpha)z$$
$$+ (8\alpha - 1)z^2].$$

$$u^{(1)}(z) = \frac{(1+z)^3}{16z^4}\Big[(24\alpha - 3)z^3 - (40\alpha - 9)z^2 + (40\alpha$$
$$-9)z^1 - (24\alpha - 3)z^0\Big].$$

$$u^{(2)}(z) = \frac{(1+z)^2}{8z^5}\Big[(24\alpha - 3)z^4 + (-32\alpha + 6)z^3$$
$$+ (40\alpha - 9)z^2 + (-48\alpha + 12)z^1 + (48\alpha - 6)\Big].$$

Taking $z = -1$ in all above, we get

$$u^{(k)}(-1) = 0, \quad k = 0, 1, 2.$$

It is easy to see that

$$u^{(0)}(1) = 2, \quad u^{(1)}(1) = 0, \ \ u^{(2)}(1) = 16\alpha.$$

This further implies that

$$u^{(0)}(1) = 2, \quad u^{(1)}(1) = 2\prod_{j=0}^{1-1}(0 - j),$$

$$u^{(2)}(1) \neq 2\prod_{j=0}^{2-1}(0 - j).$$

Thus

$$u^{(k)}(1) = 2\prod_{j=0}^{k-1}(0 - j) \quad and \quad u^{(k)}(-1) = 0, \ \ k = 0, 1.$$

This completes the proof. $\square$

Lemma 14: The degree of polynomial generation of the scheme (4) is 3.

Theorem 15: A refinement scheme (4) reproduces polynomials of degree 1 with respect to the primal parameterizations with $\tau = 0$ if and only if

$$v^{(k)}(1) = 2\prod_{j=0}^{k-1}(\tau - j) \quad and \quad v^{(k)}(-1) = 0, \ \ k = 0, 1,$$

where $v^{(k)}(1)$ is the kth derivative of $v(z)$ at $z = 1$.

Theorem 16: A refinement scheme (4) reproduces polynomials of degree 1 with respect to the primal parameterizations with $\tau = 0$ if and only if

$$v^{(k)}(1) = 2\prod_{j=0}^{k-1}(\tau - j) \quad and \quad v^{(k)}(-1) = 0, \ \ k = 0, 1,$$

where $v^{(k)}(1)$ is the kth derivative of $v(z)$ at $z = 1$.

Lemma 17: The degree of polynomial generation of the scheme (11) is 1.

Theorem 18: A refinement scheme (11) reproduces polynomials of degree 1 with respect to the primal parameterizations with $\tau = 0$ if and only if

$$w^{(k)}(1) = 2 \prod_{j=0}^{k-1} (\tau - j) \quad and \; w^{(k)}(-1) = 0, \; k = 0, 1,$$

where $w^{(k)}(z)$ is the kth derivative of the Laurent polynomial of the scheme (11).

Lemma 19: The degree of polynomial generation of the scheme (13) is 1.

Theorem 20: A refinement scheme (13) reproduces polynomials of degree 1 with respect to the primal parameterizations with $\tau = 0$ if and only if

$$y^{(k)}(1) = 2 \prod_{j=0}^{k-1} (\tau - j) \quad and \; y^{(k)}(-1) = 0, \; k = 0, 1,$$

where $y^{(k)}(z)$ is the kth derivative of the Laurent polynomial of the scheme (13).

C. LIMIT STENCILS OF THE SCHEMES

Since the limit curves produced by the refinement schemes do not have closed form so the traditional methods fail to compute the points on the curve. In this case, we compute the limit stencils of the schemes. This is just the sequence of scalers. If we consider the initial points of the polygon as a sequence of vectors. Then making the linear combination of these vector and scalars, we get the point on the limit curve produced by the refinement scheme.

Theorem 21: The limit stencil of the scheme (2) is

$$\left[\frac{1}{3} \frac{\alpha(-1 + 8\alpha)}{4\alpha + 3}, -\frac{16}{3} \frac{\alpha(-1 + \alpha)}{4\alpha + 3}, \frac{1}{3} \frac{16\alpha^2 - 18\alpha + 9}{4\alpha + 3}, \right.$$
$$\left. -\frac{16}{3} \frac{\alpha(-1 + \alpha)}{4\alpha + 3}, \frac{1}{3} \frac{\alpha(-1 + 8\alpha)}{4\alpha + 3} \right].$$

Proof: By taking $i = -1$ and 0 in even and odd rules and $i = 1$ in even rule of the scheme (2), we get

$$f_{-2}^{k+1} = \alpha f_{-2}^k + (1 - 2\alpha)f_{-1}^k + \alpha f_0^k,$$

$$f_{-1}^{k+1} = \frac{1}{16}(8\alpha - 1)f_{-2}^k + \frac{1}{16}(9 - 8\alpha)f_{-1}^k + \frac{1}{16}(9$$
$$- 8\alpha)f_0^k + \frac{1}{16}(8\alpha - 1)f_1^k.$$

$$f_0^{k+1} = \alpha f_{-1}^k + (1 - 2\alpha)f_0^k + \alpha f_1^k,$$

$$f_1^{k+1} = \frac{1}{16}(8\alpha - 1)f_{-1}^k + \frac{1}{16}(9 - 8\alpha)f_0^k + \frac{1}{16}(9$$
$$- 8\alpha)f_1^k + \frac{1}{16}(8\alpha - 1)f_2^k.$$

$$f_2^{k+1} = \alpha f_0^k + (1 - 2\alpha)f_1^k + \alpha f_2^k.$$

Its matrix representation is: $f^{j+1} = Sf^j$, where $f^{j+1} = [f_{-2}^{j+1}, f_{-1}^{j+1}, f_0^{j+1}, f_1^{j+1}, f_2^{j+1}]^T, f^j = [f_{-2}^j, f_{-1}^j, f_0^j, f_1^j, f_2^j]^T$

and

$$S$$
$$= \begin{pmatrix} \alpha & 1 - 2\alpha & \alpha & 0 \\ \frac{1}{16}(8\alpha - 1) & \frac{1}{16}(9 - 8\alpha) & \frac{1}{16}(9 - 8\alpha) & \frac{1}{16}(8\alpha - 1) \\ 0 & \alpha & 1 - 2\alpha & \alpha \\ 0 & \frac{1}{16}(8\alpha - 1) & \frac{1}{16}(9 - 8\alpha) & \frac{1}{16}(9 - 8\alpha) \\ 0 & 0 & \alpha & 1 - 2\alpha \end{pmatrix}$$
$$\begin{matrix} 0 \\ 0 \\ 0 \\ \frac{1}{16}(8\alpha - 1) \\ \alpha \end{matrix}.$$

Eigenvalues of this matrix are

$$\lambda_i = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{4} - \alpha.$$

The matrix of eigenvectors V corresponding to these eigenvalues is

$$V = \begin{pmatrix} 1 & -2 & \frac{4(2\alpha - 3)}{-3 + 8\alpha} & -\frac{8(-1 + 2\alpha)}{-1 + 8\alpha} & 1 \\ 1 & -1 & 1 & -1 & -\frac{1}{4}\frac{32\alpha^2 - 28\alpha + 3}{10\alpha - 3} \\ 1 & 0 & \frac{8\alpha}{-3 + 8\alpha} & 0 & -\frac{2\alpha(-1 + 8\alpha)}{10\alpha - 3} \\ 1 & 1 & 1 & 1 & -\frac{1}{4}\frac{32\alpha^2 - 28\alpha + 3}{10\alpha - 3} \\ 1 & 2 & \frac{4(2\alpha - 3)}{-3 + 8\alpha} & -\frac{8(-1 + 2\alpha)}{-1 + 8\alpha} & 1 \end{pmatrix}.$$

Its inverse V^{-1} is, as shown at the bottom of the next page. The diagonal matrix D of the eigenvalues can be written as

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{8} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} - \alpha \end{pmatrix}.$$

Since $S = VDV^{-1}$ therefore $S^j = VD^jV^{-1}$. Also

$$f^{j+1} = Sf^j = S^2 f^{j-1} = S^3 f^{j-2} = \cdots .S^j f^0.$$

So we have, $f^{j+1} = (VD^jV^{-1})f^0$. This implies $f^\infty = V(\lim_{j\to\infty}D^j)V^{-1}f^0$. So

$$\begin{pmatrix} f^\infty_{-2} \\ f^\infty_{-1} \\ f^\infty_0 \\ f^\infty_1 \\ f^\infty_2 \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{1}{3}\dfrac{\alpha(-1+8\alpha)}{4\alpha+3} & -\dfrac{16}{3}\dfrac{\alpha(-1+\alpha)}{4\alpha+3} & \dfrac{1}{3}\dfrac{16\alpha^2-18\alpha+9}{4\alpha+3} \\[2ex] \dfrac{1}{3}\dfrac{\alpha(-1+8\alpha)}{4\alpha+3} & -\dfrac{16}{3}\dfrac{\alpha(-1+\alpha)}{4\alpha+3} & \dfrac{1}{3}\dfrac{16\alpha^2-18\alpha+9}{4\alpha+3} \\[2ex] \dfrac{1}{3}\dfrac{\alpha(-1+8\alpha)}{4\alpha+3} & -\dfrac{16}{3}\dfrac{\alpha(-1+\alpha)}{4\alpha+3} & \dfrac{1}{3}\dfrac{16\alpha^2-18\alpha+9}{4\alpha+3} \\[2ex] \dfrac{1}{3}\dfrac{\alpha(-1+8\alpha)}{4\alpha+3} & -\dfrac{16}{3}\dfrac{\alpha(-1+\alpha)}{4\alpha+3} & \dfrac{1}{3}\dfrac{16\alpha^2-18\alpha+9}{4\alpha+3} \\[2ex] \dfrac{1}{3}\dfrac{\alpha(-1+8\alpha)}{4\alpha+3} & -\dfrac{16}{3}\dfrac{\alpha(-1+\alpha)}{4\alpha+3} & \dfrac{1}{3}\dfrac{16\alpha^2-18\alpha+9}{4\alpha+3} \end{pmatrix}$$

(continued at right)

$$\cdots \begin{pmatrix} -\dfrac{16}{3}\dfrac{\alpha(-1+\alpha)}{4\alpha+3} & \dfrac{1}{3}\dfrac{\alpha(-1+8\alpha)}{4\alpha+3} \\[2ex] -\dfrac{16}{3}\dfrac{\alpha(-1+\alpha)}{4\alpha+3} & \dfrac{1}{3}\dfrac{\alpha(-1+8\alpha)}{4\alpha+3} \\[2ex] -\dfrac{16}{3}\dfrac{\alpha(-1+\alpha)}{4\alpha+3} & \dfrac{1}{3}\dfrac{\alpha(-1+8\alpha)}{4\alpha+3} \\[2ex] -\dfrac{16}{3}\dfrac{\alpha(-1+\alpha)}{4\alpha+3} & \dfrac{1}{3}\dfrac{\alpha(-1+8\alpha)}{4\alpha+3} \\[2ex] -\dfrac{16}{3}\dfrac{\alpha(-1+\alpha)}{4\alpha+3} & \dfrac{1}{3}\dfrac{\alpha(-1+8\alpha)}{4\alpha+3} \end{pmatrix} \begin{pmatrix} f^0_{-2} \\ f^0_{-1} \\ f^0_0 \\ f^0_1 \\ f^0_2 \end{pmatrix}.$$

Hence the limit stencil is

$$\left[\dfrac{1}{3}\dfrac{\alpha(-1+8\alpha)}{4\alpha+3},\ -\dfrac{16}{3}\dfrac{\alpha(-1+\alpha)}{4\alpha+3},\ \dfrac{1}{3}\dfrac{16\alpha^2-18\alpha+9}{4\alpha+3}, \right.$$
$$\left. -\dfrac{16}{3}\dfrac{\alpha(-1+\alpha)}{4\alpha+3},\ \dfrac{1}{3}\dfrac{\alpha(-1+8\alpha)}{4\alpha+3} \right].$$

□

Similarly, we get the following results.

Theorem 22: The limit stencil of the scheme (4) is

$$\left[\dfrac{1}{6}\dfrac{(16\beta+1)\beta}{8\beta^2+6\beta+1},\ \dfrac{1}{6}\dfrac{16\beta+1}{8\beta^2+6\beta+1},\ \dfrac{1}{3}\dfrac{8\beta^2+\beta+2}{8\beta^2+6\beta+1},\ \dfrac{1}{6} \right.$$
$$\left. \times \dfrac{16\beta+1}{8\beta^2+6\beta+1},\ \dfrac{1}{6}\dfrac{(16\beta+1)\beta}{8\beta^2+6\beta+1} \right].$$

$$\left(\begin{array}{ccc} \dfrac{1}{3}\dfrac{\alpha(-1+8\alpha)}{4\alpha+3} & -\dfrac{16}{3}\dfrac{\alpha(-1+\alpha)}{4\alpha+3} & \dfrac{1}{3}\dfrac{-18\alpha+16\alpha^2+9}{4\alpha+3} \\[2.5ex] \dfrac{1}{12}-\dfrac{2}{3}\alpha & -\dfrac{2}{3}+\dfrac{4}{3}\alpha & 0 \\[2.5ex] -\dfrac{1}{96}\dfrac{(-3+8\alpha)(-1+8\alpha)}{\alpha} & \dfrac{1}{24}\dfrac{(-3+8\alpha)(4\alpha-1)}{\alpha} & -\dfrac{1}{48}\dfrac{(-3+8\alpha)^2}{\alpha} \\[2.5ex] -\dfrac{1}{12}+\dfrac{2}{3}\alpha & \dfrac{1}{6}-\dfrac{4}{3}\alpha & 0 \\[2.5ex] -\dfrac{1}{8}\dfrac{10\alpha-3}{\alpha(4\alpha+3)} & \dfrac{1}{2}\dfrac{10\alpha-3}{\alpha(4\alpha+3)} & -\dfrac{3}{4}\dfrac{10\alpha-3}{\alpha(4\alpha+3)} \\ & & -\dfrac{16}{3}\dfrac{\alpha(-1+\alpha)}{4\alpha+3} \\[2.5ex] & & \dfrac{2}{3}-\dfrac{4}{3}\alpha \\[2.5ex] & & \dfrac{1}{24}\dfrac{(-3+8\alpha)(4\alpha-1)}{\alpha} \\[2.5ex] & & -\dfrac{1}{6}+\dfrac{4}{3}\alpha \\[2.5ex] & & \dfrac{1}{2}\dfrac{10\alpha-3}{\alpha(4\alpha+3)} \end{array}\right.$$

(continued, right columns)

$$\left.\begin{array}{c} \dfrac{1}{3}\dfrac{\alpha(-1+8\alpha)}{4\alpha+3} \\[2.5ex] -\dfrac{1}{12}+\dfrac{2}{3}\alpha \\[2.5ex] -\dfrac{1}{96}\dfrac{(-3+8\alpha)(-1+8\alpha)}{\alpha} \\[2.5ex] \dfrac{1}{12}-\dfrac{2}{3}\alpha \\[2.5ex] -\dfrac{1}{8}\dfrac{10\alpha-3}{\alpha(4\alpha+3)} \end{array}\right).$$

Theorem 23: The limit stencil of the scheme (11) is $[0, 0, 1, 0, 0]$.

Theorem 24: The limit stencil of the scheme (13) is $[0, \frac{2\alpha}{4\alpha+1}, \frac{1}{4\alpha+1}, \frac{2\alpha}{4\alpha+1}, 0]$.

The results presented in Section 2, are graphically presented in Section 3.

III. NUMERICAL AND GRAPHICAL ANALYSIS OF THE SHAPES

In this section, we present the numerical and graphical examples to examine the features of the scheme (1). For this purpose, we explore the influence of the parameters on the final shapes produced by the refinement procedures. Here we take different initial sketches made by joining the points by straight lines. Then we apply the refinement procedure on these sketches to get smooth shapes. The values of the parameters have been randomly taken from the parametric ranges for the different order of continuities of the schemes. The refinement scheme defined in (1) only deals with the closed polygons (sketches). For open polygon, by introducing two auxiliary points $f_{-1}^0 = 2f_0^0 - f_1^0$ and $f_{n+1}^k = 2f_n^k - f_{n-1}^k$, we suggest the following rules to refine first and last edges of the polygon

$$f_0^{k+1} = f_0^0$$
$$f_1^{k+1} = \left(\beta - \alpha\beta + \frac{1}{2}\right)f_0^0 + \left(2\alpha\beta - 2\beta + \frac{1}{2}\right)f_1^0$$
$$+ (\beta - \alpha\beta)f_2^0,$$

and

$$f_{2n-2}^{k+1} = \alpha f_{n-2}^k + (1 - 2\alpha)f_{n-1}^k + \alpha f_n^k$$
$$f_{2n-1}^{k+1} = (1 - \alpha)\beta f_{n-2}^k + \left(-2\beta + 2\alpha\beta + \frac{1}{2}\right)f_{n-1}^k$$
$$+ \left(\beta - \alpha\beta + \frac{1}{2}\right)f_n^k.$$

Example 25: Figure 3 shows the interpolatory behaviors of the scheme up to three refinement steps whereas the doted polygon is the initial sketch which is made by six initial control points $(x, y) = [(5, 0), (10, 0), (20, 10), (10, 20), (5, 20), (-5, 10)]$ *denoted by red solid circles. Here we fix the values of parameters as* $\alpha = 0$ *and* $\beta = -0.05$. *We observe that the final shape lies outside the initial sketch. We also observe that by eliminating one parameter* α, *the scheme (1) produce curves which always passes through the initial control points, while the values of second parameter may vary from* $\frac{1}{8} - \frac{1}{8}\sqrt{5}$ *to 0 (see Theorem 8) to get different shapes.*

Example 26: Figure 4 shows the approximating behaviors of the scheme up to three refinement steps whereas the doted polygon is the same initial sketch used in Example 25. Here we settle the value of parameters $\alpha = \frac{1}{12}$ *and* $\beta = -\frac{1}{44}$. *These values may vary to get different types of smooth approximating shapes. To get an approximating shape we can eliminate parameter* β *but can not eliminate parameter* α.
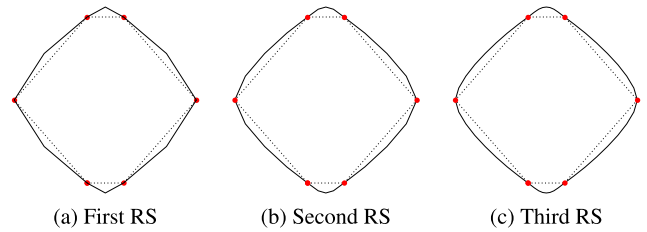

FIGURE 3. Initial sketch along with refined sketches produced by interpolatory scheme.

(a) First RS (b) Second RS (c) Third RS
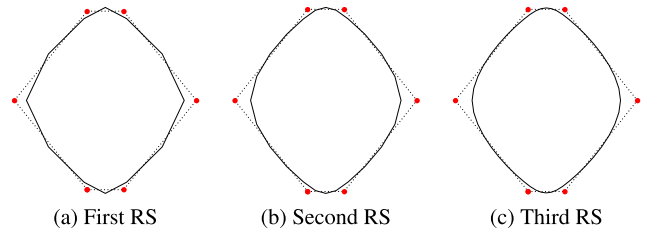

(a) First RS (b) Second RS (c) Third RS

FIGURE 4. Initial sketch along with refined sketches produced by approximating scheme.

If we get the values of these parameters from the statements of Theorems 5-7 and Theorem 9. We observe that the presented final shape lies inside the initial sketch and does not interpolate any of the initial control point.

To see another interesting performance of the presented refinement scheme, we apply subclasses of refinement scheme (1), that are given in (2), (4), (11) and (13), on the same initial sketch and get the smooth shapes of various types of our choice. This performance can be seen in Examples 3-4 and Figures 4-5.

Example 27: Figure 5 shows the interproximate (interpolate some points while approximate the other points) behaviors of the scheme in non-uniform setting of the parameters. Here we choose two initial points for interpolation and the other four initial points for approximation. The values of the parameters at each initial point are set as: $(\alpha_i, \beta_i) = \left[(\frac{1}{12}, -\frac{1}{44}), (\frac{1}{12}, -\frac{1}{44}), (\frac{1}{12}, -\frac{1}{44}), (0, -\frac{3}{50}), (0, -\frac{3}{50}), (\frac{1}{12}, -\frac{1}{44})\right]$.

Example 28: Figure 6 also shows the interproximate behaviors in non-uniform setting of the parameters of the scheme up to three refinement steps. Here we choose four initial points for interpolation and the the other two initial points for approximation. Here the setting of parametric values are: $(\alpha_i, \beta_i) = [(0, -\frac{3}{50}), (0, -\frac{3}{50}), (\frac{1}{8}, 0), (0, -\frac{3}{50}), (0, -\frac{3}{50}), (\frac{1}{8}, 0)]$.

Example 29: Figures 7(a)-(c) show the interpolatory and approximating performance of the scheme on an open polygon. We show the limit curves after three subdivision steps produced by the scheme at three different values of the shape parameters $(\alpha, \beta) = (0, -0.05)$, $(\alpha, \beta) = \left(\frac{1}{12}, \frac{1}{44}\right)$ *and* $(\alpha, \beta) = \left(\frac{3}{16}, \frac{1}{26}\right)$ *respectively.*
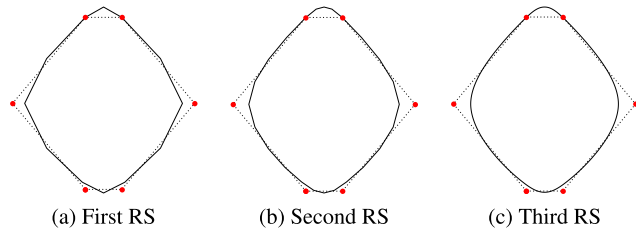
From Figures 3-6, RS is used for the Refinement Step.

(a) First RS (b) Second RS (c) Third RS

FIGURE 5. Interproximate behaviour of the scheme for non-uniform setting of the parameters.

(a) First RS (b) Second RS (c) Third RS
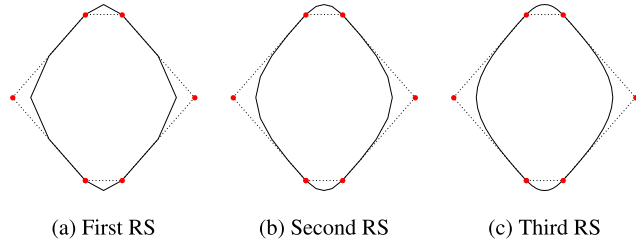
FIGURE 6. Interproximate behaviour of the scheme for non-uniform setting of the parameters.

(a) $(\alpha, \beta) = (0, -0.05)$ (b) $(\alpha, \beta) = \left(\frac{1}{12}, \frac{1}{44}\right)$ (c) $(\alpha, \beta) = \left(\frac{3}{16}, \frac{1}{26}\right)$
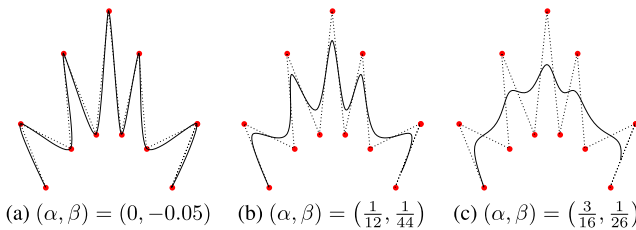
FIGURE 7. Bullets are initial points. Dotted polygons are initial polygons. Solid curves shows limit curves produced by refinement scheme after 3 refinement steps.

IV. COMPARISON AND CONCLUSION

In this paper, we have presented a unified refinement scheme with two parameters. This scheme has unified interpolating, approximating and interproximate schemes. One shape parameter controls the interpolating property of the refinement scheme while the other controls the approximating property of the scheme. Interproximate scheme can be

TABLE 1. Properties of the scheme (1) for different values of α and β. Let DPG, DPR and AO denote the degree of polynomial generation, degree of polynomial reproduction and approximation order respectively.

Schemes	α	β	DPG	DPR	AO
Scheme (2)	$\forall \alpha$	$-\frac{1}{16}\left(\frac{8\alpha-1}{\alpha-1}\right)$	3	1	2
Scheme (2)	$\frac{3}{16}$	$-\frac{1}{16}\left(\frac{8\alpha-1}{\alpha-1}\right)$	5	1	2
Scheme (4)	$\frac{1}{8}\left(\frac{16\beta+1}{2\beta+1}\right)$	$\forall \beta$	3	1	2
Scheme (4)	$\frac{1}{8}\left(\frac{16\beta+1}{2\beta+1}\right)$	$\frac{1}{26}$	5	1	2
Scheme (11)	0	$\forall \beta$	1	1	2
Scheme (13)	$\forall \alpha$	0	1	1	2

achieved by using non-uniform (the refinement rules for each edge are different) setting of the parameters. It is also observed that our schemes are the generalized version of some of the interpolating as well as B-spline schemes. It can produce different shapes with different degree of smoothness (i.e. different order of continuity). We have also presented the theoretical, numerical and graphical analysis of the shapes produced by the scheme. The scheme for $\beta = -\frac{1}{16}\left(\frac{8\alpha-1}{\alpha-1}\right)$ produces

- C^0-continuous shape for $\frac{7}{16} - \frac{1}{16}\sqrt{137} < \alpha < \frac{5}{24} + \frac{1}{12}\sqrt{91}$.
- C^1-continuous shape for $\frac{5}{16} - \frac{1}{16}\sqrt{33} < \alpha < \frac{5}{16} + \frac{1}{16}\sqrt{33}$.
- C^2-continuous shape for $0 < \alpha < \frac{1}{2}$.
- C^3-continuous shape for $\frac{1}{8} < \alpha < \frac{3}{16} + \frac{1}{16}\sqrt{5}$.
- C^4-continuous shape for $\alpha = \frac{3}{16}$.

The scheme for $\alpha = \frac{1}{8}\left(\frac{16\beta+1}{2\beta+1}\right)$ produces

- C^0-continuous shape for $-\frac{143}{62} - \frac{7}{62}\sqrt{349} < \beta < -\frac{81}{8} + \frac{7}{8}\sqrt{129}$.
- C^1-continuous shape for $\frac{3}{8} - \frac{7}{88}\sqrt{33} < \beta < \frac{3}{8} + \frac{7}{88}\sqrt{33}$.
- C^2-continuous shape for $-\frac{1}{16} < \beta < \frac{3}{8}$.
- C^3-continuous shape for $0 < \beta < \frac{9}{164} + \frac{7}{164}\sqrt{5}$.
- C^4-continuous shape for $\beta = \frac{1}{26}$.

The scheme for $\alpha = 0$ produces

- C^0-continuous shape for $\frac{1}{8} - \frac{1}{8}\sqrt{13} < \beta < \frac{3}{8}$.
- C^1-continuous shape for $\frac{1}{8} - \frac{1}{8}\sqrt{5} < \beta < 0$.

TABLE 2. Comparison: Here "inter" and "appr" means interpolating and approximating.

Schemes	Nature	Continuity	Range	Continuity	Range
4-point	inter [21]	C^1	$w \in (0, \frac{1}{8})$	.	.
4-point	inter [1]	C^1	$w \in (0, \frac{\sqrt{5}-1}{8})$	.	.
3-point	appr [3]	C^2	$w = \frac{1}{2}$	.	.
4-point*	appr [5]	C^4	$\mu \in (0, \frac{1}{32})$	C^5	$\mu = \frac{1}{64}$
3-point	appr [6]	C^3	$\mu = 4\nu + 1, \nu = \frac{1}{2}, w = \frac{3}{4}$	.	.
3-point	appr [7]	C^3	$\mu_0 = 1$	.	.
4-point	appr [9]	C^3	$\alpha = 1, \beta = \frac{1}{32}$	.	.
3-point	appr [11]	C^3	$a = d = \frac{1}{16}, b = c = \frac{3}{16}$	.	.
3-point	appr [15]	C^3	$w = \frac{1}{4}$	.	.
4-point**	appr [15]	C^4	$w \in (0, \frac{1}{4})$	C^5	$w = \frac{1}{4}$
4-point	appr (1)	C^4	$\alpha = \frac{3}{16}, \beta = \frac{1}{16}$	.	.
4-point	inter (1)	C^1	$\alpha = 0, \forall \beta$	.	.

The scheme for $\beta = 0$ produces
- C^0-continuous shape for $-\frac{1}{4} < \alpha < \frac{3}{4}$.
- C^1-continuous shape for $0 < \alpha < \frac{1}{8} + \frac{1}{8}\sqrt{5}$.

In Table 1, we summarize the properties of the schemes for different values of shape parameters α and β. Since the degree of polynomial generation of all the schemes is one therefore by [26] the approximation order of all schemes is two. Since the odd and even rules of our schemes have complexity 3 and 4 therefore we have compared the order of continuity of the shapes produced by our schemes and existing schemes having the complexity 3 and 4. The comparison is presented in Table 2. The schemes presented in this table with labels * and ** have complexity 4 in both odd and even rules but in our schemes the complexity in odd rule is less than 4. The schemes with labels * and ** can produce one order extra continuous shapes but with the high cost of complexity comparative to our approximating schemes. Moreover, the difference between the shapes with order of continuities 4 and 5 can not be observed by our naked eyes so one extra order of continuity in the shapes with high cost of complexity has no use. Furthermore, our schemes produce higher order continuous shape then ternary combined family of schemes [19], [20]. Our interpolating and exiting schemes in the combined family produce the same order of continuous shapes. Overall, we conclude that the proposed schemes are better in the sense of complexity and continuity than the combined binary and ternary family of schemes as well as other individual schemes. In this paper, we have also suggested the method to compute the points on the limit curves. In future, we will study that how theoretical results of the presented refinement schemes are potentially applicable to the design of graph filters [24].

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