IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received April 24, 2020, accepted May 11, 2020, date of publication May 19, 2020, date of current version June 8, 2020.

Digital Object Identifier 10.1109/ACCESS.2020.2995679

Dynamical Complexity in a Class of Novel Discrete-Time Predator-Prey Interaction With Cannibalism

MUHAMMAD SAJJAD SHABBIR[1], QAMAR DIN[2], RANA ALABDAN[3], ASIFA TASSADDIQ[4], (Member, IEEE), AND KHALIL AHMAD[1]

[1]Department of Mathematics, Air University Islamabad, Islamabad 44000, Pakistan
[2]Department of Mathematics, University of Poonch Rawalakot, Azad Kashmir 12350, Pakistan
[3]Department of Information Systems, College of Computer and Information Sciences, Majmaah University, Al-Majmaah 11952, Saudi Arabia
[4]Department of Basic Sciences and Humanities, College of Computer and Information Sciences, Majmaah University, Al-Majmaah 11952, Saudi Arabia

Corresponding author: Muhammad Sajjad Shabbir (sajjadmust@gmail.com)

This work of Rana Alabdan was supported by the Deanship of Scientific Research at Majmaah University under Project R-1441-127.

ABSTRACT Cannibalism is ubiquitous in natural communities and has the tendency to change the functional connection among prey-predator interactions. Keeping in view the inclusion of prey cannibalism, a novel discrete nonlinear predator-prey model is proposed. Asymptotic stability is carried out around biologically feasible equilibria of proposed model. Center manifold theorem and bifurcation theory of normal form ensure the existence of bifurcation in the system. Our study reveals that periodic outbreaks may result due to incorporation of cannibalism in prey population and this periodic outbreak is limited to prey population only without leaving an effect on predation. In order to control these periodic oscillations in prey population density and other bifurcating and fluctuating behavior of the system, various chaos control strategies are implemented. Ultimately, some extensive numerical simulations are elaborated to demonstrate the effectiveness of our acquired analytical and theoretical results.

INDEX TERMS Discrete dynamical system, stability, bifurcation, chaos control, cannibalism.

I. INTRODUCTION

Cannibalism is precisely defined as the process of both eating and killing individuals of the same species. Specially, in socio-anthropological and ecological point of view, it is an abominable and stimulating phenomenon occurring worldwide. This behavioral attribute of cannibalism is merged in an extensive range of animals composed of flour beetles, locusts, insects, spider and fish, [1]–[5]. Usually, the cannibals as well as the sufferers belong to the various stages of maturation such as, teenage and adult, adolescent and developed. Such type of explanation throws back a predator-prey communication in the identical classes and the equivalent scientific representations are dissimilar [6], [7]. The great work of Polis regarding cannibalism has quoted about 1300 species, amongst which it occurs [7]. In the dynamics of predator-prey models, a cannibalism factor is frequently a process of survival, and is in fact a common place [8]. Further related studies can be found in [9], [10]. In the literature of ecology and mathematical biology, massive investigation has been carried out on cannibalism and it can influence in population dynamics due to several counterintuitive effects. Cannibalism seems to appear in numerous species have a very large population size as compared with resources [11]. Consequently, it is sometimes competition arbitrated. It can be observed in the literature that initially cannibalism is only modeled in the predator species [12], [13] regardless of ecological endorsement in both experimental and field work that it frequently becomes visible in the prey species [14]–[16]. These experimental findings greatly inspire to develop new ideas in current research. In [16] author investigated the impact of prey cannibalism and is noticed that prey cannibalism establish a density-conciliated indirect consequence that results in various predator-prey phenomenon. Several kinds of predator-prey models involving cannibalism are the focus point of mathematical literature. Such models comprise of two and three species involving stage structure, two species PDE models, discrete models and ratio dependent

The associate editor coordinating the review of this manuscript and approving it for publication was Jianquan Lu.

functional response. Furthermore, current models related to cannibalism that involve diseased predators [17]–[21] are also mentionable. In [22] authors proposed a Lotka-Volterra model with predator cannibalism and considered the control of the system. Moreover, authors investigated that this phenomena has constructive and destructive consequences on the control which depends on the dynamic performance of the original structure. Zhang *et al.* [23] reinvestigated a predator-prey of stage-structured involving anthropophagy in the predator inhabitants. This is separated into two stages, a childish and an advanced stage. They proposed a new non-dimensionalization approach. Moreover, global stability of equilibria, supercritical and subcritical Hopf bifurcation involving biological parameters are also established. Keeping in view non-overlapping generation of predator-prey interaction, Danca *et al.* [9] studied the subsequent scheme:

$$\begin{cases} x_{n+1} := rx_n\left(1-x_n\right) - bx_ny_n \\ y_{n+1} := dx_ny_n, \end{cases} \tag{1}$$

where x_n and y_n represents the number of prey and predator inhabitants in the nth age group respectively. The growth of prey is logistic having a basic growth rate "r" and carrying capacity "1" with no predation. Positive parameters r, b, d represent prey intrinsic growth rate, per capita searching efficiency of the predator and adaptation frequency. Taking into account the natural death rate for predator, system (1) can be modified as follows:

$$\begin{cases} x_{n+1} := rx_n\left(1-x_n\right) - bx_ny_n \\ y_{n+1} := dx_ny_n - cy_n, \end{cases} \tag{2}$$

where c is the death rate of predator.

Currently, there are numerous continued works to explore the complexity and chaos control, stability and bifurcation theory for some modified system of (2), for example, [24], [25]. The feedback control strategies to control the bifurcation and chaos for the Leslie-Gower prey-predator model are studied in [26]. Discrete-time dynamic model of population ecology by choosing a discrete-time version of the classical Lotka-Volterra prey-predator model is considered in [27]. In [28], the author analyzed a discrete predator-prey model with nonmonotonic functional response exhibit and proved the existence of fold bifurcation, transcritical bifurcation, flip bifurcation and Neimark-Sacker bifurcations (NSB). Similarly, some discrete-time predator-prey models and hydra effect and paradox of enrichment are studied in [29]. Furthermore, for several attracting findings associated to the qualitative analysis of difference equations, we refer to the work done by [30], [31]. Also authors in [32] proposed and investigate the dynamics of cannibalism in discrete-time predator-prey system and considering two-stage population model where cannibalism factor involving only in prey population. Additionally, a sufficient condition relies on the boundary equilibrium is explored for which the existence of both populations can ensure.

In this manuscript, we proposed the following predator-prey model with cannibalism in prey and investigate the dynamical behaviors of this system:

$$\begin{cases} x_{n+1} := rx_n\left(1-x_n\right) + ax_n - bx_ny_n - \dfrac{\alpha x_n^2}{x_n+\beta} \\ y_{n+1} := dx_ny_n - cy_{n.} \end{cases} \tag{3}$$

It must be noted that the prey species $x(t)$ is depredating on its own species. Moreover, in prey equation, the generic cannibalism factor $C\left(x\right) = \alpha \times x \times \frac{x}{x+\beta}$, is added. Clearly, the cannibalistic prey has the functional response of Holling type II where α represents cannibalism rate. Moreover, adding a term ax in the prey equation because of reproduction in prey population. We also restrict $a < \alpha$, as it takes attack of a total of prey.

The motivational aspects and novelty of this paper are further described as follows:

- Considering the non-overlapping generations for predator-prey interaction, a novel discrete predator-prey system (3) involving cannibalism in prey population is proposed and studied. Moreover, model (3) is a natural extension of systems (1) and (2) under the influence of cannibalism in prey species.
- Existence of steady-states, local stability analysis, transcritical bifurcation, period-doubling bifurcation and Neimark-Sacker bifurcation are studied for proposed model (3).
- Some chaos control methods are introduced for controlling fluctuating and chaotic behavior of system (3).

The remaining investigation of this manuscript can be summarized as follows. Section 2 is related to presence of biologically practicable equilibrium as well as stability conditions for these equilibria are also discussed. In Section 3, bifurcation analysis for system (3) is investigated. We show that trivial equilibrium of system (3) undergoes transcritical bifurcation whenever growth parameter r of prey population is taken as bifurcation parameter. Furthermore, it is shown that system (3) go through the period-doubling bifurcation on its boundary equilibrium point whenever cannibalism rate α is taken as bifurcation parameter, and it experiences both period-doubling and Neimark-Sacker bifurcations (NSB) around its interior equilibrium point. Some chaos control methods (that is, Ott-Grebogi-Yorke (OGY), hybrid control method and an exponential type chaos control method) are introduced in Section 4. Lastly, numerical imitations are provided in Section 5 to demonstrate our theoretical discussion.

II. STABILITY ANALYSIS

To obtain the equilibria of system (3), we can solve the following algebraic equations:

$$\begin{cases} x = rx\left(1-x\right) + ax - bxy - \frac{\alpha x^2}{x+\beta} \\ y = dxy - cy \end{cases}$$

It can be easily observed that system (3) has three biologically feasible equilibria; trival equilibrium point $E_T = (0, 0)$, boundary equilibrium $E_B=(k, 0)$ and unique positive equilibrium $E_U := (x^\star, y^\star)$. Moreover, k :, as shown at the bottom of this page. Furthermore, the Jacobian matrix $J(x, y)$ for system (3) calculated at any point (x, y) is given by:

$$J(x,y) := \begin{pmatrix} a+r-2rx-by-\frac{x\alpha(x+2\beta)}{(x+\beta)^2} & -bx \\ dy & dx-c \end{pmatrix}.$$

Now, we describe a general result for local stability of fixed points of system (3). If $Q^\star$ be any arbitrary fixed point of system (3) and suppose that:

$$\mathrm{J}(Q^\star) := \begin{pmatrix} \mathbb{a}_{11} & \mathbb{a}_{12} \\ \mathbb{a}_{21} & \mathbb{a}_{22} \end{pmatrix}$$

be the variational matrix evaluated at $Q^\star$. Then the quadratic characteristic polynomial is expressed by:

$$\mathbb{F}(\lambda) := \lambda^2 - \mathcal{T}\lambda + \mathcal{D},$$

where $\mathcal{T} = \mathbb{a}_{11} + \mathbb{a}_{22}$ and $\mathcal{D} = \mathbb{a}_{11}\mathbb{a}_{22} - \mathbb{a}_{12}\mathbb{a}_{21}$.

For the sake of investigating stability, the following lemma demonstrates the best interpretation regarding local stability analysis of feasible equilibria, see also [33].

Lemma 1: Let $\mathbb{F}(\xi) = \xi^2 - \mathcal{T}\xi + \mathcal{D}$, and $\mathbb{F}(1) > 0$. Moreover, ξ_1, ξ_2 are root of the equation $\xi^2 - \mathcal{T}\xi + \mathcal{D} = 0$ then, the following assumptions hold:

(i) $|\xi_1| < 1; |\xi_2| < 1 \iff \mathbb{F}(-1) > 0; \mathcal{D} < 1$
(ii) $|\xi_1| > 1; |\xi_2| > 1 \iff \mathbb{F}(-1) > 0; \mathcal{D} > 1$
(iii) $|\xi_1| < 1; |\xi_2| > 1$ or $(|\xi_1| > 1; |\xi_2| < 1) \iff \mathbb{F}(-1) < 0;$
(iv) $\xi_1 = -1; |\xi_2| \neq 1 \iff \mathbb{F}(-1) = 0; \mathcal{T} \neq 0, 2;$
(v) ξ_1, ξ_2 are complex conjugates with $|\xi_1| = 1 = |\xi_2| \iff \mathcal{T}^2 - 4\mathcal{D} < 0 \& \mathcal{D} = 1$.

Since, the eigenvalues of (3) are ξ_1, ξ_2, then to discuss the the stability of $Q^\star$, the following topological type results has been developed .

(i) Fixed point $Q^\star$ is known as sink if $|\xi_1| < 1$ and $|\xi_2| < 1$, as it is locally asymptotic stable.

(ii) Fixed point $Q^\star$ is known as source if $|\xi_1| > 1$ and $|\xi_2| > 1$, as source is repeller hence it remains unstable.

(iii) Fixed point $Q^\star$ is a saddle point if $|\xi_1| < 1$ *and* $|\xi_2| > 1$or$|\xi_1| > 1\&|\xi_2| < 1$.

(iv) Fixed point $Q^\star$ is said to be non-hyperbolic if conditions (iv) and (v) of the Lemma 1 are satisfied.

Now, using Lemma 1, we discuss the topological classification of system (3) at trivial, boundary and interior equilibrium points.

At trivial equilibrium $E_T = (0, 0)$, the variational matrix $J(x, y)$ of system (3) is expressed as:

$$J(E_T) := \begin{pmatrix} a+r & 0 \\ 0 & -c \end{pmatrix}.$$

Clearly, the following topological results hold:

- E_T is a sink $\iff 0 < a + r < 1 \; \& \; 0 < c < 1$.
- E_T is a source $\iff a + r > 1 \; \& \; c > 1$.
- E_T is saddle point $\iff a + r > 1 \; \& \; c < 1$ or $(a + r < 1 \& c > 1)$.
- E_T is non-hyperbolic at $a + r = 1$ or $c = 1$.

Moreover, in ac-plane topological classification of E_T is depicted in *Figure* 1(**a**).

Furthermore, $J(x, y)$ at E_B is given by:

$$J(E_B) := \begin{pmatrix} a+r+\alpha k\Phi(k\Phi-2)-2kr & -bk \\ 0 & dk-c \end{pmatrix},$$

where $\Phi = \frac{1}{k+\beta}$.

The following result gives the topological classification of (3) at boundary equilibrium E_B:

Theorem 1: Assume that $a > 1$, $r > 0$, $\beta > 0$, $\alpha > 0$, then the following results hold:

1) E_B is a sink if and only if $2k(r + \alpha\Phi) < 1 + a + r + \alpha k^2\Phi^2 < 2(1 + k(r + \alpha\Phi))$ and $c - 1 < dk < 1 + c$.
2) E_B is a source if and only if $a + r + \alpha k^2\Phi^2 > 1 + 2k(r + \alpha\Phi)$ and $dk > 1 + c$.
3) E_B is known as saddle point $\iff dk > 1 + c$ and $2k(r + \alpha\Phi) < 1 + a + r + \alpha k^2\Phi^2 < 2(1 + k(r + \alpha\Phi))$.
4) E_B is non-hyperbolic $\iff a+r+\alpha k^2\Phi^2 = 1+2k(r+\alpha\Phi)$ or $dk = 1 + c$.

On the other hand, topological classification of boundary equilibrium E_B in ac-plane with $r = 1.32$, $d = 0.23$, $\alpha = 0.79$ and $\beta = 1.2$ is depicted in *Figure* 1(**b**).

Furthermore, let $\mathrm{J}(E_U)$ be variational matrix of the system (3) about unique positive equilibrium E_U, then $\mathrm{J}(E_U)$ is shown at the bottom of the next page. The characteristic polynomial is given by:

$$\begin{aligned}\mathbb{F}(\lambda) &:= \lambda^2 - \left(2 - \frac{(1+c)r}{d} + \frac{(1+c)^2\alpha}{(1+c+d\beta)^2} - \frac{(1+c)\alpha}{1+c+d\beta}\right)\lambda \\ &\quad + \frac{(1+c)(d-c-2)r}{d} + \frac{(1+c)^2\alpha}{(1+c+d\beta)^2} \\ &\quad - \frac{(1+c)(2+c)\alpha}{1+c+d\beta} + a(1+c) - c. \qquad (4)\end{aligned}$$

Furthermore, by performing simple algebraic calculations and letting $\frac{(1+c)\alpha}{1+c+d\beta} + \frac{d+r(1+c)}{d} < a + r$, we get:

$$\begin{aligned}\mathbb{F}(-1) &:= \frac{2(1+c)^2\alpha}{(1+c+d\beta)^2} - \frac{(1+c)(3+c)\alpha}{1+c+d\beta} \\ &\quad + \frac{(1+c)(d-c-3)r}{d} + 3 + a - c + ac.\end{aligned}$$

$$\mathbb{F}(1) := (1+c)\left(a - 1 + \frac{(-1-c+d)r}{d} - \frac{(1+c)\alpha}{1+c+d\beta}\right) \qquad (5)$$

$$\begin{cases} k := \dfrac{a+r-\alpha-r\beta-1+\sqrt{4r(-1+a+r)\beta+(1-a-r+\alpha+r\beta)^2}}{2r}; & \text{where } a > 1,\ r > 0,\ \beta > 0, \alpha > 0. \\ x^\star := \dfrac{1+c}{d} \text{ and } y^\star := \dfrac{1}{b}\left(\dfrac{d(a-1)+(d-c-1)r}{d} - \dfrac{(1+c)\alpha}{1+c+d\beta}\right); & \text{where } \dfrac{(1+c)\alpha}{1+c+d\beta} + \dfrac{d+r(1+c)}{d} < a + r. \end{cases}$$

From (5), we see that $\mathbb{F}(1) > 0$. Therefore, the following results can be deduced by applying Lemma 1.

Theorem 2: Assume that $\frac{(1+c)\alpha}{1+c+d\beta} + \frac{d+r(1+c)}{d} < a+r$, such that the unique positive equilibrium $E_U := (x^\star, y^\star)$ of (3) exist, then the following results remain true:

(i) E_U is locally asymptotically stable if and only if

$$\left(a + r + \frac{2(1+c)\alpha}{(1+c+d\beta)^2}\right)(1+c) + 3 > \left(\frac{r}{d} + \frac{\alpha}{1+c+d\beta}\right)(1+c)(3+c) + c,$$

and

$$a+r+\frac{(1+c)\alpha}{(1+c+d\beta)^2} < 1+(2+c)\left(\frac{r}{d}+\frac{\alpha}{1+c+d\beta}\right).$$

(ii) E_U is unstable equilibrium point if and only if

$$\left(\frac{2(1+c)\alpha}{(1+c+d\beta)^2} + a + r\right)(1+c) + 3 > \left(\frac{r}{d} + \frac{\alpha}{1+c+d\beta}\right)(1+c)(3+c) + c,$$

and

$$1 + (2+c)\left(\frac{r}{d} + \frac{\alpha}{1+c+d\beta}\right) < a + r + \frac{(1+c)\alpha}{(1+c+d\beta)^2}.$$

(iii) E_U is saddle point if and only if

$$\left(\frac{2(1+c)\alpha}{(1+c+d\beta)^2} + a + r\right)(1+c) + 3 < \left(\frac{r}{d} + \frac{\alpha}{1+c+d\beta}\right)(1+c)(3+c) + c.$$

At the end of this section, topological classification for positive equilibrium in rc-plane with $a = 2.88$, $d = 1.45$, $\alpha = 2.4$ and $\beta = 2.5$ is shown in *Figure* 1(**c**).

III. BIFURCATION ANALYSIS

In this section, we investigate transcritical bifurcation of system (3) at trivial equilibrium point E_T, period-doubling bifurcation at boundary equilibrium E_B and both period-doubling and NSB at interior equilibrium E_U.

A. TRANSCRITICAL BIFURCATION AT TRIVIAL EQUILIBRIUM E_T

Initially, we investigate that trivial equilibrium (0, 0) undergoes transcritial bifurcation [31]. For this, we assume that:

$$r \equiv r_0 := 1 - a.$$

Consider the set

$$\psi_{\mathcal{T}\mathfrak{B}\mathcal{T}} := \left\{(r_0, a, b, c, d, \alpha, \beta) \in \mathcal{R}_+^7 : r_0 := 1 - a\right\}.$$

Assume that $(r_0, a, b, c, d, \alpha, \beta) \in \psi_{\mathcal{T}\mathfrak{B}}$, then map (1.3) is equivalently expressed by the map:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} (r_0 + \check{r})\, x(1-x) + ax - bxy - \dfrac{\alpha x^2}{x+\beta} \\ dxy - cy \end{pmatrix}$$

where $\check{r}$ be a small bifurcation parameter in r_0.

The elaboration of Taylor series expansion about $(x, y, \check{r}) = (0, 0, 0)$ yields:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} a+\check{r} & 0 \\ 0 & -c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f_0(x,y,\check{r}) \\ g_0(x,y) \end{pmatrix}, \quad (6)$$

where

$$\mathfrak{f}_0(x, y, \check{r}) := -\left(\check{r} + \frac{\alpha}{\beta}\right)x^2 - bxy + \frac{\alpha}{\beta^2}x^3 + x\check{r} + O\left(\left(|x| + |y| + |\check{r}|\right)^4\right),$$
$$\mathfrak{g}_0(x, y, \check{r}) := dxy.$$

Since, at $r_0 := 1-a$, the map (6) having linear part is already in canonical form and map is also at origion (0, 0), then center manifold $W^c(0,0,0)$ for (6) can be resembled to:

$$W^c(0,0,0) := \left\{\begin{array}{l} u, v, \check{r} \in \mathbb{R}^3 : \\ v = h_1 u^2 + h_2 u\check{r} + h_3\check{r}^2 + O\left(\left(|u| + |\check{r}|\right)^3\right) \end{array}\right\}.$$

Then simple calculation yields that $h_1 = h_2 = h_3 = 0$.

Further, we explicate the map and restrain it to $W^c(0,0,0)$ as :

$$F : u \to u + \check{r} + k_1u^2 + k_2u\check{r} + k_3\check{r}^2 + O\left(\left(|u| + |\check{r}|\right)^4\right),$$

where $k_1 = a - 1 - \frac{\alpha}{\beta}$ $k_2 = 1$ and $k_3 = 0$.

Now, here we establish two real numbers L_1 and L_2:

$$\mathcal{L}_1 := \left(\frac{\partial^2 F}{\partial u^2}\right)_{(0,0)} = 2\left(a - 1 - \frac{\alpha}{\beta}\right) \neq 0,$$

and

$$\mathcal{L}_2 := \left(\frac{\partial^2 F}{\partial u \partial \check{r}}\right)_{(0,0)} = 1 \neq 0.$$

Thus, aforemention analytical approach, the following result regarding transcritical bifurcation has been acquired.

Theorem 3: The system (3) undergoes transcritical bifurcation at E_T, if $r = 1 - a$ and $a - 1 - \frac{\alpha}{\beta} \neq 0$.

$$\mathrm{J}(E_U) := \begin{pmatrix} 1 - \dfrac{(1+c)r}{d} + \dfrac{(1+c)^2\alpha}{(1+c+d\beta)^2} - \dfrac{(1+c)\alpha}{1+c+d\beta} & -\dfrac{b(1+c)}{d} \\ \dfrac{d(a+r-1) - r(1+c)}{b} - \dfrac{(1+c)\alpha d}{b(1+c+d\beta)} & 1 \end{pmatrix}.$$

B. PERIOD-DOUBLING BIFURCATION AT EQUILIBRIUM E_B

Here, we discuss bifurcation of system (3) at boundary equilibrium $(k, 0)$. The Jacobian of linearized system about E_B is given by:

$$J\left(E_B\right) := \begin{pmatrix} r + a - 2kr - \dfrac{k\alpha(k+2\beta)}{(k+\beta)^2} & -bk \\ 0 & dk - c \end{pmatrix}$$

Moreover, the condition that eigen value $\lambda_1 = -1$ implies that

$$\alpha := \frac{(1+a+r-2kr)\,(k+\beta)^2}{k\,(k+2\beta)}.$$

Assume that

$$\psi_{\mathcal{P}\mathfrak{D}\mathfrak{B}} := \left\{ \begin{array}{l} (r, a, b, c, d, \alpha, \beta) \in \mathcal{R}_+^7 : \\ \alpha = \dfrac{(1+a+r-2kr)(k+\beta)^2}{k(k+2\beta)} \end{array} \right\}.$$

The boundary equilibrium E_B of map (3) undergoes period-doubling bifurcation when the parameters changes values in the small neighboring points of $\psi_{\mathcal{P}\mathfrak{D}\mathfrak{B}}$. Let $\alpha_1 = \frac{(1+c+r-2kr)(k+\beta)^2}{k(k+2\beta)}$ and taking the arbitrary parameters $(\alpha_1, r, a, b, c, d, \beta) \in \psi_{\mathcal{P}\mathfrak{D}\mathfrak{B}}$, then in terms of parameters $(\alpha_1, r, a, b, c, d, \beta)$, map (3) can be demonstrated in the following map:

$$\begin{pmatrix} \mathbb{x} \\ \mathbb{y} \end{pmatrix} \to \begin{pmatrix} r\mathbb{x}\,(1-\mathbb{x}) + a\mathbb{x} - b\mathbb{x}\mathbb{y} - \dfrac{\alpha_1 \mathbb{x}^2}{\mathbb{x}+\beta} \\ d\mathbb{x}\mathbb{y} - c\mathbb{y} \end{pmatrix} \tag{7}$$

Assume that a new small perturbation parameter $\tilde{\alpha}$, then (7) can be examined and consequently, we have the following map:

$$\begin{pmatrix} \mathbb{x} \\ \mathbb{y} \end{pmatrix} \to \begin{pmatrix} r\mathbb{x}\,(1-\mathbb{x}) + a\mathbb{x} - b\mathbb{x}\mathbb{y} - \dfrac{(\alpha_1+\check{\alpha})\mathbb{x}^2}{\mathbb{x}+\beta} \\ d\mathbb{x}\mathbb{y} - c\mathbb{y} \end{pmatrix} \tag{8}$$

where $\left|\check{\alpha}\right| \ll 1$, a small perturbation parameter.

Assume that $x = \mathbb{x} - k$ and $y = \mathbb{y}$, then map (8) is transformed to:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \mathfrak{h}_1(x, y, \check{\alpha}) \\ \mathfrak{h}_2(x, y) \end{pmatrix}, \tag{9}$$

where

$$\mathfrak{h}_1\left(x, y, \check{\alpha}\right) := c_{13}x^2 + c_{14}xy + c_{15}x^3 + c_{16}x\check{\alpha} + c_{17}x^2\check{\alpha} + O\left(\left(|x| + |y| + |\check{\alpha}|\right)^4\right),$$

$$\mathfrak{h}_2\left(x, y, \check{\alpha}\right) := c_{23}xy.$$

$$\begin{cases} c_{11} := a + r - 2kr - \dfrac{k\check{\alpha}\,(k+2\beta)}{(k+\beta)^2};\ c_{12} := -bk; \\ c_{17} := -\dfrac{\beta^2 - 4k\beta + 4k^2}{\beta^3};\ c_{21} := 0;\ c_{22} := dk - c; \\ c_{13} := \dfrac{4k\check{\alpha}}{\beta^2} - \dfrac{4k^2\check{\alpha}}{\beta^3} - \dfrac{\check{\alpha}}{\beta} - r;\ c_{14} := -b;\ c_{23} := d; \\ c_{15} := \dfrac{2\check{\alpha}\left(\beta^2 - 4k\beta + 4k^2\right)}{\beta^4};\ c_{16} := -\dfrac{2k\,(\beta-k)}{\beta^2}; \end{cases}$$

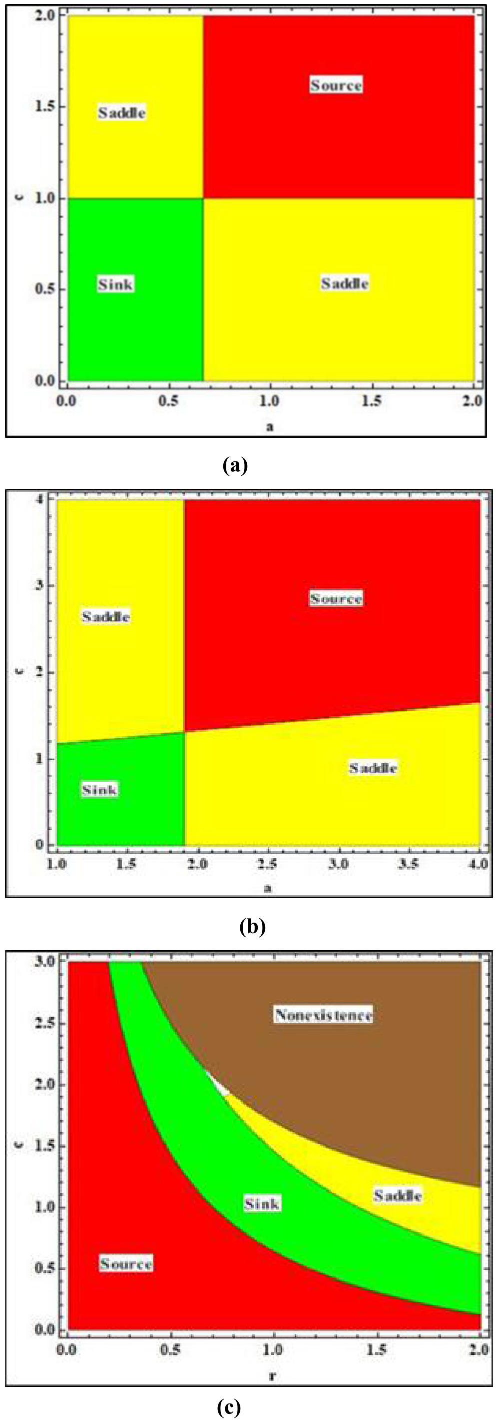


FIGURE 1. (a) Topological classification of (0,0) at $r = 0.335$. (b) Topological classification of boundary equilibrium. (c) Topological classification of unique positive equilibrium.

Now, instantly we introduce the following translation:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \mathfrak{I} \begin{pmatrix} \mathbb{u} \\ \mathbb{v} \end{pmatrix} \tag{10}$$

where a nonsingular matrix $\mathfrak{I} := \begin{pmatrix} c_{12} & c_{12} \\ -(1+c_{11}) & \lambda_2 - c_{11} \end{pmatrix}$ exists. Further, the translation (10) can be formulated under the translation (9) as follows:

$$\begin{pmatrix} \mathbb{u} \\ \mathbb{v} \end{pmatrix} \to \begin{pmatrix} -1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} h_3(\mathbb{u}, \mathbb{v}, \breve{\alpha}) \\ h_4(\mathbb{u}, \mathbb{v}, \breve{\alpha}) \end{pmatrix} \quad (11)$$

where

$$\begin{aligned} \mathfrak{h}_3(\mathbb{u}, \mathbb{v}, \breve{\alpha}) &:= \left(\frac{(\lambda_2 - c_{11})(\breve{\alpha} c_{17} + c_{13})}{c_{12}(\lambda_2 + 1)}\right) x^2 \\ &+ \left(\frac{(\lambda_2 - c_{11}) c_{14}}{(\lambda_2+1) c_{12}} - \frac{c_{23}}{\lambda_2 + 1}\right) xy \\ &+ \left(\frac{(\lambda_2 - c_{11}) c_{15}}{(\lambda_2+1) c_{12}}\right) x^3 \\ &+ \left(\frac{(\lambda_2 - c_{11}) \breve{\alpha} c_{16}}{(\lambda_2+1) c_{12}}\right) x \\ &+ O\left((|\mathbb{u}| + |\mathbb{v}| + |\breve{\alpha}|)^4\right); \end{aligned}$$

$$\begin{aligned} \mathfrak{h}_4(\mathbb{u}, \mathbb{v}, \breve{\alpha}) &:= \left(\frac{(1 + c_{11})(\breve{\alpha} c_{17} + c_{13})}{(\lambda_2 + 1) c_{12}}\right) x^2 \\ &+ \left(\frac{(1 + c_{11}) c_{14}}{(\lambda_2+1) c_{12}} + \frac{c_{23}}{\lambda_2 + 1}\right) xy \\ &+ \left(\frac{(1 + c_{11}) c_{15}}{(\lambda_2+1) c_{12}}\right) x^3 + \left(\frac{(1 + c_{11}) \breve{\alpha} c_{16}}{(\lambda_2+1) c_{12}}\right) x \\ &+ O\left((|\mathbb{u}| + |\mathbb{v}| + |\breve{\alpha}|)^4\right). \end{aligned}$$

Also, $x := c_{12}(\mathbb{u} + \mathbb{v})$; $y := (\lambda_2 - c_{11}) \mathbb{v} - (1 + c_{11}) \mathbb{u}$.

Moreover, consider center manifold $\mho^c(0, 0, 0)$ for map (11) evaluated at origion (0, 0) and $\breve{\alpha} = 0$, then $\mho^c(0, 0, 0)$ takes the form:

$$\mho^c(0,0,0) := \left\{ \begin{array}{l} \mathbb{u}, \mathbb{v}, \breve{\alpha} \in \mathbb{R}^3 : \\ \mathbb{v} = \mathbb{h}_1 \mathbb{u}^2 + \mathbb{h}_2 \mathbb{u} \breve{\alpha} + \mathbb{h}_3 \breve{\alpha}^2 + O\left((|\mathbb{u}| + |\breve{\alpha}|)^3\right) \end{array} \right\}.$$

where

$$\mathbb{h}_1 := \frac{(1 + c_{11})(c_{11} c_{14} - c_{12} c_{13} + c_{12} c_{23} + c_{14})}{\lambda_2^2 - 1},$$

$$\mathbb{h}_2 := -\frac{(1 + c_{11}) c_{16}}{\lambda_2^2 - 1}, \quad \mathbb{h}_3 := 0.$$

Consequently, the restricted map to $\mho^c(0, 0, 0)$ is prescribed by:

$$F : \mathbb{u} \to -\mathbb{u} + \mathbb{s}_1 \mathbb{u}^2 + \mathbb{s}_2 \mathbb{u} \breve{\alpha} + \mathbb{s}_3 \mathbb{u}^2 \breve{\alpha} + \mathbb{s}_4 \mathbb{u} \breve{\alpha}^2 + \mathbb{s}_5 \mathbb{u}^3 + O\left((|\mathbb{u}| + |\breve{\alpha}|)^4\right),$$

Here,

$$\begin{aligned} \mathbb{s}_1 &:= \left(\frac{c_{23} c_{12}}{1 + \lambda_2} - \frac{(\lambda_2 - c_{11}) c_{14}}{1 + \lambda_2}\right)(1 + c_{11}) \\ &\quad + c_{12} \left(\frac{\lambda_2 - c_{11}}{1 + \lambda_2}\right) c_{13} \\ \mathbb{s}_2 &= \frac{\lambda_2 - c_{11}}{\lambda_2 + 1} c_{16}, \end{aligned}$$

$$\begin{aligned} \mathbb{s}_3 &:= \left(\frac{(\lambda_2 - c_{11}) c_{14}}{1 + \lambda_2} - \frac{c_{12} c_{23}}{1 + \lambda_2}\right)(\lambda_2 - c_{11}) \mathbb{h}_2 \\ &+ \left(\frac{c_{23} c_{12}}{1 + \lambda_2} - \frac{(\lambda_2 - c_{11}) c_{14}}{1 + \lambda_2}\right)(1 + c_{11}) \mathbb{h}_2 \\ &+ \left(\frac{\lambda_2 - c_{11}}{\lambda_2 + 1}\right) c_{16} \mathbb{h}_1; \ \frac{\lambda_2 - c_{11}}{1 + \lambda_2} 2 c_{12} c_{13} \mathbb{h}_2 \\ &+ \left(\frac{\lambda_2 - c_{11}}{1 + \lambda_2}\right) c_{12} c_{17}, \\ \mathbb{s}_4 &:= \left(\frac{(\lambda_2 - c_{11}) c_{14}}{(\lambda_2 + 1)} - \frac{c_{12} c_{23}}{\lambda_2 + 1}\right)(\lambda_2 - c_{11}) \mathbb{h}_3 \\ &+ \left(\frac{c_{12} c_{23}}{\lambda_2 + 1} - \frac{(\lambda_2 - c_{11}) c_{14}}{(\lambda_2 + 1)}\right)(1 + c_{11}) \mathbb{h}_3 \\ &+ \frac{\lambda_2 - c_{11}}{\lambda_2 + 1} 2 c_{12} c_{13} \mathbb{h}_3 \\ &+ \frac{\lambda_2 - c_{11}}{\lambda_2 + 1} c_{16} \mathbb{h}_2, \\ \mathbb{s}_5 &:= \left(\frac{(\lambda_2 - c_{11}) c_{14}}{1 + \lambda_2} - \frac{c_{12} c_{23}}{1 + \lambda_2}\right)(\lambda_2 - c_{11}) \mathbb{h}_1 \\ &+ \frac{\lambda_2 - c_{11}}{\lambda_2 + 1} c_{12}^2 c_{15} + \frac{\lambda_2 - c_{11}}{\lambda_2 + 1} 2 c_{12} c_{13} \mathbb{h}_1 \\ &+ \left(\frac{c_{12} c_{23}}{\lambda_2 + 1} - \frac{(\lambda_2 - c_{11}) c_{14}}{(\lambda_2 + 1)}\right)(1 + c_{11}) \mathbb{h}_1. \end{aligned}$$

Now, here we establish nonzero real numbers $L_1 \& L_2$:

$$L_1 := \left(\frac{\partial^2 f}{\partial \mathbb{u} \partial \breve{\alpha}} + \frac{1}{2} \frac{\partial F}{\partial \breve{\alpha}} \frac{\partial^2 F}{\partial \mathbb{u}^2}\right)_{(0,0)} = \frac{(\lambda_2 - c_{11}) c_{16}}{\lambda_2 + 1} \neq 0.$$

$$L_2 := \left(\frac{1}{6} \frac{\partial^3 F}{\partial \mathbb{u}^3} + \left(\frac{1}{2} \frac{\partial^2 F}{\partial \mathbb{u}^2}\right)^2\right)_{(0,0)} = \mathbb{s}_5 + \mathbb{s}_1^2 \neq 0.$$

Consequently, we deduce the following result:

Theorem 4:For non-zero L_1, L_2, the system (3) go through period-doubling bifurcation by boundary equilibrium E_B whereas parameter α differs in small neighborhood of the point α_1. Likewise, for positive values of L_2, the period-two orbits that bifurcate from E_B are stable, and for negative values of L_2, these orbits are non-stable.

C. PERIOD-DOUBLING BIFURCATION AT EQUILIBRIUM E_U

Now, we study period-doubling and Neimark-Sacker bifucation of system (3) at interior equilibrium E_U. At first, we study the period-doubling bifurcation at E_U. Then, equation (4) with interior equilibrium $E_U := (x^\star, y^\star) = \left(\frac{1+c}{d}, \frac{1}{b}\left(\frac{d(a-1)+(d-c-1)r}{d} - \frac{(1+c)\alpha}{1+c+d\beta}\right)\right)$ is given by

$$\mathbb{F}(\lambda) := \lambda^2 - p(x^\star, y^\star)\lambda + q(x^\star, y^\star), \quad (12)$$

where

$$p(x^\star, y^\star) := 2 - \frac{(1+c) r}{d} + \frac{(1+c)^2 \alpha}{(1+c+d\beta)^2} - \frac{(1+c)\alpha}{1+c+d\beta},$$

$$q(x^\star, y^\star) := (1+c)\left(a + \frac{(d-c-2) r}{d} + \frac{(1+c)\alpha}{(1+c+d\beta)^2} - \frac{(2+c)\alpha}{1+c+d\beta}\right).$$

Assume that

$$p^2(x^\star, y^\star) > 4q\left(x^\star, y^\star\right), \tag{13}$$

and $\mathbb{F}(-1) = 0$, implies that

$$r := \frac{d\left(c-3-a(1+c)-\frac{2(1+c)^2\alpha}{(1+c+d\beta)^2}+\frac{(1+c)(3+c)\alpha}{1+c+d\beta}\right)}{(1+c)(d-c-3)}. \tag{14}$$

The roots of aforementioned equation (12), along with $\mathbb{F}(\lambda) = 0$ gives $\lambda_1 = -1$; and

$$\begin{aligned}\lambda_2 := \frac{1}{(3+c-d)(1+c+d\beta)^2}&(6-a+16c-3ac+14c^2\\ &-3ac^2+4c^3-ac^3-3d-6cd-3c^2d+\alpha+3c\alpha+3c^2\alpha\\ &+c^3\alpha+12d\beta-2ad\beta+20cd\beta-4acd\beta+8c^2d\beta\\ &-2ac^2d\beta-6d^2\beta-6cd^2\beta+d^2\alpha\beta\\ &+cd^2\alpha\beta+6d^2\beta^2-ad^2\beta^2+4cd^2\beta^2\\ &-acd^2\beta^2-3d^3\beta^2).\end{aligned}$$

Moreover, the condition $|\lambda_2| \neq 1$ implies that

$$(1+c)\left(a+\frac{(d-c-2)r}{d}+\frac{(1+c)\alpha}{(1+c+d\beta)^2}-\frac{(2+c)\alpha}{1+c+d\beta}\right)-c \neq \pm 1. \tag{15}$$

Assume that

$$\psi_{\mathcal{PB}} := \left\{(r,a,b,c,d,\alpha,\beta) \in \mathcal{R}_+^7 : (13)-(15)\text{ holds}\right\}.$$

The interior equilibrium point E_U of map (3) undergoes period-doubling bifurcation whenever the parameters varies in the small neighborhood of map $\psi_{\mathcal{PB}}$. Let $r_1 := \frac{d\left(c-3-a(1+c)-\frac{2(1+c)^2\alpha}{(1+c+d\beta)^2}+\frac{(1+c)(3+c)\alpha}{1+c+d\beta}\right)}{(1+c)(d-c-3)}$ and taking the arbitrary parameters $(r_1, a, b, c, d, \alpha, \beta) \in \psi_{PB}$, then in terms of parameters $(r_1, a, b, c, d, \alpha, \beta)$ map (3) can be discribed by:

$$\begin{pmatrix}\mathcal{P}\\ \mathcal{Q}\end{pmatrix} \to \begin{pmatrix} r_1\mathcal{P}(1-\mathcal{P})+a\mathcal{P}-b\mathcal{P}\mathcal{Q}-\frac{\alpha\mathcal{P}^2}{\mathcal{Q}+\beta}\\ d\mathcal{P}\mathcal{Q}-c\mathcal{Q}\end{pmatrix}, \tag{16}$$

Assuming a minimal bifurcation parameter $\tilde{r}$ along with a perturbation. Then (16) can be demonstrated by:

$$\begin{pmatrix}\mathcal{P}\\ \mathcal{Q}\end{pmatrix} \to \begin{pmatrix} (r_1+\tilde{r})\mathcal{P}(1-\mathcal{P})+a\mathcal{P}-b\mathcal{P}\mathcal{Q}-\frac{\alpha\mathcal{P}^2}{\mathcal{Q}+\beta}\\ d\mathcal{P}\mathcal{Q}-c\mathcal{Q}\end{pmatrix}, \tag{17}$$

where $|\tilde{r}| \ll 1$ is a small perturbation parameter.

Considering $x = \mathcal{P} - x^\star$ and $y = \mathcal{Q} - y^\star$, then system (17) can be transformed into:

$$\begin{pmatrix}x\\ y\end{pmatrix} \to \begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\cdot\begin{pmatrix}x\\ y\end{pmatrix} + \begin{pmatrix}\mathfrak{f}_1(x,y,\tilde{r})\\ \mathfrak{g}_1(x,y,\tilde{r})\end{pmatrix}, \tag{18}$$

$$\begin{aligned}\mathfrak{f}_1(x,y,\tilde{r}) &:= a_{13}x^2+a_{14}xy+a_{15}x^3+a_{16}x\tilde{r}+a_{17}x^2\tilde{r}\\ &\quad +O\left((|x|+|y|+|\tilde{r}|)^4\right),\\ \mathfrak{g}_1(x,y,\tilde{r}) &:= a_{23}xy+a_{24}x\tilde{r}+O\left((|x|+|y|+|\tilde{r}|)^4\right),\\ a_{11} &:= 1-\frac{(1+c)r}{d}+\frac{(1+c)^2\alpha}{(1+c+d\beta)^2}-\frac{(1+c)\alpha}{1+c+d\beta},\\ a_{21} &:= \frac{d(a+r-1)-r(1+c)}{b}-\frac{(1+c)\alpha d}{b(1+c+d\beta)},\\ a_{15} &:= \frac{\alpha\beta^2d^4}{(1+c+\beta d)^4}, \quad a_{12} := -\frac{b(1+c)}{d},\\ a_{16} &:= -\frac{1+c}{d},\\ a_{24} &:= \frac{d-c-1}{b}, \quad a_{17} := -1, a_{14} := -b,\\ a_{23} &:= d;\ a_{22} := 1,\\ a_{13} &:= -\frac{3\beta^2d^2\tilde{r}+3c^2\beta d\tilde{r}+6\beta cd\tilde{r}+c^3\tilde{r}+3\beta d\tilde{r}}{(1+c+\beta d)^3}\\ &\quad -\frac{3c^2\tilde{r}+3c\tilde{r}+\tilde{r}}{(1+c+\beta d)^3}\\ &\quad -\frac{\beta^3d^3\tilde{r}+\alpha\beta^2d^3+3\beta^2d^2c\tilde{r}}{(1+c+\beta d)^3},\end{aligned}$$

Moreover, we introduced the following translation:

$$\begin{pmatrix}x\\ y\end{pmatrix} \to \mathfrak{T}\begin{pmatrix}u\\ v\end{pmatrix}, \tag{19}$$

here $\mathfrak{T} := \begin{pmatrix}a_{12} & a_{12}\\ -1-a_{11} & \lambda_2-a_{11}\end{pmatrix}$ is an invertible matrix, the map (19) under the translation (18), is formulated as

$$\begin{pmatrix}u\\ v\end{pmatrix} \to \begin{pmatrix}-1 & 0\\ 0 & \lambda_2\end{pmatrix}\cdot\begin{pmatrix}u\\ v\end{pmatrix} + \begin{pmatrix}\mathfrak{f}_2(u,v,\tilde{r})\\ \mathfrak{g}_2(u,v,\tilde{r})\end{pmatrix}. \tag{20}$$

where

$$\begin{aligned}\mathfrak{f}_2(u,v,\tilde{r}) &:= \left(\frac{(\lambda_2-a_{11})a_{14}}{a_{12}(\lambda_2+1)}-\frac{a_{23}}{\lambda_2+1}\right)xy+\left(\frac{(\lambda_2-a_{11})a_{15}}{(\lambda_2+1)a_{12}}\right)x^3\\ &\quad+\left(\frac{(\lambda_2-a_{11})(\tilde{r}a_{17}+a_{13})}{a_{12}(\lambda_2+1)}\right)x^2\\ &\quad+\left(\frac{(\lambda_2-a_{11})\tilde{r}a_{16}}{a_{12}(\lambda_2+1)}-\frac{\tilde{r}a_{24}}{\lambda_2+1}\right)x+O\left((|u|+|v|+|\tilde{r}|)^4\right),\\ \mathfrak{g}_2(u,v,\tilde{r}) &:= \left(\frac{(1+a_{11})a_{14}}{(\lambda_2+1)a_{12}}+\frac{a_{23}}{\lambda_2+1}\right)xy\\ &\quad+\left(\frac{(1+a_{11})a_{15}}{a_{12}(\lambda_2+1)}\right)x^3\\ &\quad+\left(\frac{(1+a_{11})(\tilde{r}a_{17}+a_{13})}{a_{12}(\lambda_2+1)}\right)x^2\\ &\quad+\left(\frac{(1+a_{11})\tilde{r}a_{16}}{(\lambda_2+1)a_{12}}+\frac{\tilde{r}a_{24}}{\lambda_2+1}\right)x+O\left((|u|+|v|+|\tilde{r}|)^4\right),\end{aligned}$$

and $x := a_{12}(u+v)$; $y := (\lambda_2-a_{11})v-(1+a_{11})u$.

Now, considering the center manifold $\mathfrak{W}^c(0,0,0)$ of (20) in a small neighborhood of $\tilde{r}=0$, then $\mathfrak{W}^c(0,0,0)$ can be embellished by:

$$\mathfrak{W}^c(0,0,0) := \Big\{u, v, \tilde{r} \in \mathbb{R}^3 : v = h_1u^2 + h_2u\tilde{r} + h_3\tilde{r}^2 + O\left((|u|+|\tilde{r}|)^3\right)\Big\}.$$

Here,

$$\begin{cases} h_1 := \dfrac{(1+a_{11})(a_{11}a_{14} - a_{12}a_{13} + a_{12}a_{23} + a_{14})}{\lambda_2^2 - 1}, \\ h_2 := -\dfrac{a_{11}a_{16} + a_{12}a_{24} + a_{16}}{1-\lambda_2^2}, \\ h_3 := 0. \end{cases}$$

Hence, the map restricted to $W^c(0,0,0)$ is demonstrated by:

$$F : u \to -u + s_1u^2 + s_2u\tilde{r} + s_3u^2\tilde{r} + s_4u\tilde{r}^2 + s_5u^3 + O\left((|u|+|\tilde{r}|)^4\right),$$

where

$$\begin{aligned}
s_1 &:= \left(\frac{a_{12}a_{23}}{\lambda_2+1} - \frac{(\lambda_2 - a_{11})a_{14}}{\lambda_2+1}\right)(1+a_{11}) + \frac{\lambda_2 - a_{11}}{\lambda_2+1}a_{12}a_{13},\\
s_2 &:= \frac{\lambda_2 a_{16} - a_{11}a_{16} - a_{12}a_{24}}{\lambda_2+1},\\
s_3 &:= \left(\frac{(\lambda_2 - a_{11})a_{14}}{\lambda_2+1} - \frac{a_{12}a_{23}}{\lambda_2+1}\right)(\lambda_2 - a_{11})h_2 + \frac{\lambda_2 - a_{11}}{\lambda_2+1}2a_{12}a_{13}h_2 + \left(\frac{a_{12}a_{23}}{\lambda_2+1} - \frac{(\lambda_2 - a_{11})a_{14}}{(\lambda_2+1)}\right)\\
&\quad \times (1+a_{11})h_2 + \left(\frac{(\lambda_2 - a_{11})a_{16}}{\lambda_2+1} - \frac{a_{12}a_{24}}{\lambda_2+1}\right)h_1,\\
s_4 &:= \left(\frac{(\lambda_2 - a_{11})a_{14}}{\lambda_2+1} - \frac{a_{12}a_{23}}{\lambda_2+1}\right)(\lambda_2 - a_{11})h_3 + \frac{\lambda_2 - a_{11}}{\lambda_2+1}2a_{12}a_{13}h_3 + \left(\frac{a_{12}a_{23}}{\lambda_2+1} - \frac{(\lambda_2 - a_{11})a_{14}}{\lambda_2+1}\right)\\
&\quad \times (1+a_{11})h_3 + \left(\frac{(\lambda_2 - a_{11})a_{16}}{\lambda_2+1} - \frac{a_{12}a_{24}}{\lambda_2+1}\right)h_2,\\
s_5 &:= \left(\frac{a_{12}a_{23}}{\lambda_2+1} - \frac{(\lambda_2 - a_{11})a_{14}}{\lambda_2+1}\right)(1+a_{11})h_1 + \frac{\lambda_2 - a_{11}}{\lambda_2+1}2a_{12}^2a_{15} + \frac{(\lambda_2 - a_{11})2a_{12}a_{13}h_1}{\lambda_2+1}\\
&\quad + \left(\frac{(\lambda_2 - a_{11})a_{14}}{\lambda_2+1} - \frac{a_{12}a_{23}}{\lambda_2+1}\right)(\lambda_2 - a_{11})h_1.
\end{aligned}$$

Consequently, we establish nonzero real numbers:

$$\begin{cases} l_1 := \left(\dfrac{\partial^2 f}{\partial u \partial \tilde{r}} + \dfrac{1}{2}\dfrac{\partial F}{\partial \tilde{r}}\dfrac{\partial^2 F}{\partial u^2}\right)_{(0,0)} := \dfrac{\lambda_2 a_{16} - a_{12}a_{24} - a_{11}a_{16}}{\lambda_2+1} \neq 0. \\ l_2 := \left(\dfrac{1}{6}\dfrac{\partial^3 F}{\partial u^3} + \left(\dfrac{1}{2}\dfrac{\partial^2 F}{\partial u^2}\right)^2\right)_{(0,0)} := s_1^2 + s_5 \neq 0. \end{cases}$$

As a consequence of aforementioned analysis, the following result associated to period-doubling bifurcation has been obtained.

Theorem 5: For non zero values of l_1 and l_2, the system (3) go through the period-doubling bifurcation at E_U whereas the parameter r differs in a small neighborhood of the point r_1. Furthermore, for positive values of l_2, the period-two orbits that bifurcate from E_U are stable, and for negative values of l_2, these orbits are non-stable.

D. NEIMARK-SACKER BIFURCATION AT EQUILIBRIUM E_U

In the present section, we investigate that by assuming the bifurcation parameter r, the system (3) undergoes NSB at E_U. Taking into account the similar investigation analogous to the bifurcation theory of dynamical systems, we refer to [33]–[36]. On the other hand, due to the aforementioned study in the literature, it is obviously clear that the NSB is a fascinating transition for iterated maps which produced dynamically invariant closed curves whenever the varied parameter passes through the bifurcation parameter, the attracting equilibrium loses its stability. In retort, we can arrive at a few insulated orbits having periodic nature along with trajectories that densely cover these invariant closed curves [37]. We have explored the conditions for system (3) for the case of non-hyperbolic equilibria, and a pair of eigenvalues with complex conjugate. From (12), assume that $\mathbb{F}(\lambda) = 0$ has two roots with complex conjugate nature whenever the following conditions hold:

$$r := \frac{d\left(1 - a + \frac{\alpha\left((1+c)^2 + (2+c)d\beta\right)}{(1+c+d\beta)^2}\right)}{d-c-2},$$

and

$$\left|2 - \frac{(1+c)r}{d} + \frac{(1+c)^2\alpha}{(1+c+d\beta)^2} - \frac{(1+c)\alpha}{1+c+d\beta}\right| < 2. \quad (21)$$

Assume that

$$\psi_{\mathcal{NB}} := \left\{ \begin{array}{l} (r,a,b,c,d,\alpha,\beta) : (21) \text{ holds with} \\ r := \dfrac{d\left(1 - a + \dfrac{\alpha((1+c)^2 + (2+c)d\beta)}{(1+c+d\beta)^2}\right)}{d-c-2} \end{array} \right\}$$

Then interior equilibrium E_U of (3) undergoes NSB whenever a variation of parameters in a small neighborhood of the map $\psi_{\mathcal{NB}}$ developed. Let $r_2 := \frac{d\left(1-a+\frac{\alpha((1+c)^2+(2+c)d\beta)}{(1+c+d\beta)^2}\right)}{d-c-2}$ and choosing the arbitrary parameters $(r_2, a, b, c, d, \alpha, \beta)$ from $\psi_{\mathcal{NB}}$, then we have the following modified map:

$$\begin{pmatrix} \mathcal{P} \\ \mathcal{Q} \end{pmatrix} \to \begin{pmatrix} r_2\mathcal{P}(1-\mathcal{P}) + a\mathcal{P} - b\mathcal{P}\mathcal{Q} - \dfrac{\alpha\mathcal{P}^2}{\mathcal{Q}+\beta} \\ d\mathcal{P}\mathcal{Q} - c\mathcal{Q} \end{pmatrix}, \quad (22)$$

Taking a small perturbation $\bar{r}$ as a bifurcation parameter of map (22), then:

$$\begin{pmatrix} \mathcal{P} \\ \mathcal{Q} \end{pmatrix} \to \begin{pmatrix} (r_2+\bar{r})\mathcal{P}(1-\mathcal{P}) + a\mathcal{P} - b\mathcal{P}\mathcal{Q} - \dfrac{\alpha \mathcal{P}^2}{\mathcal{Q}+\beta} \\ d\mathcal{P}\mathcal{Q} - c\mathcal{Q} \end{pmatrix}, \tag{23}$$

where $|\bar{r}| \ll 1$.

Now, here we elaborate the following transformations $x = \mathcal{P} - x^\star$ and $y = \mathcal{Q} - y^\star$, where $(x^\star, y^\star)$ be the interior equilibrium of system (3), then modified form of map (22) is prescribed by:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \mathfrak{f}_3(x,y) \\ \mathfrak{g}_3(x,y) \end{pmatrix}, \tag{24}$$

where

$$\begin{aligned} \mathfrak{f}_3(x,y) &:= a_{13}x^2 + a_{14}xy + a_{15}x^3 \\ &\quad + O\left((|x|+|y|+|\bar{r}|)^4\right), \\ \mathfrak{g}_3(x,y) &:= a_{23}xy + O\left((|x|+|y|+|\bar{r}|)^4\right), \end{aligned}$$

all other parameters $a_{11}, a_{12}, a_{21}, a_{22}, a_{13}, a_{14}, a_{15}$ and a_{23} are given in (18) by replacing r_1 by $r_2+\bar{r}$. Moreover, the characteristic equation for linearized system (24) evaluated at the trivial equilibrium can be formulated by:

$$\lambda^2 - p(\bar{r})\lambda + q(\bar{r}) = 0, \tag{25}$$

where

$$\begin{aligned} p(\bar{r}) &:= \frac{(1+c)^2\alpha}{(1+c+d\beta)^2} - \frac{(1+c)(r_2+\bar{r})}{d} \\ &\quad - \frac{(1+c)\alpha}{1+c+d\beta} + 2, \\ q(\bar{r}) &:= \left(\frac{(d-c-2)(r_2+\bar{r})}{d} + \frac{(1+c)\alpha}{(1+c+d\beta)^2} \right. \\ &\quad \left. - \frac{(2+c)\alpha}{1+c+d\beta} + a\right)(1+c) - c. \end{aligned}$$

Since $(r_2, a, b, c, d, \alpha, \beta) \in \psi_{\mathcal{NB}}$, then solution of (25) are conjugate complex numbers $\lambda_1 \& \lambda_2$. Consequently:

$$\lambda_1, \lambda_2 := \frac{p(\bar{r})}{2} \pm \frac{i}{2}\sqrt{4q(\bar{r}) - p^2(\bar{r})}.$$

Thus, we have $|\lambda_1| = |\lambda_2| = \sqrt{q(\bar{r})}$, and $\left(\frac{d|\lambda_1|}{d\bar{r}}\right)_{(\bar{r}=0)}$, as shown at the bottom of this page. Furthermore, we assume that $p(0) := \left(2 - \frac{(1+c)r_2}{d} + \frac{(1+c)^2\alpha}{(1+c+d\beta)^2} - \frac{(1+c)\alpha}{1+c+d\beta}\right) \neq 0, 1$. Moreover, $(r_2, a, b, c, d, \alpha, \beta) \in \Omega_{NB}$ implies $-2 < p(0) < 2$. Therefore, $p(0) \neq \pm 2, 0, -1$ implies that $\lambda_1^m, \lambda_2^m \neq 1$, for all $m = 1, 2, 3, 4$ at $\bar{r} = 0$. Consequently, roots of (25) do not occurs in the intersection of the unit circle with the coordinate axes when $\bar{r} = 0$ and the following conditions hold:

$$\left(2 - \frac{(1+c)r_2}{d} + \frac{(1+c)^2\alpha}{(1+c+d\beta)^2} - \frac{(1+c)\alpha}{1+c+d\beta}\right) \neq 0, -1$$

or

$$\left.\begin{aligned} 3 + \frac{(1+c)^2\alpha}{(1+c+d\beta)^2} &\neq \frac{(1+c)\alpha}{1+c+d\beta} + \frac{(1+c)r_2}{d} \\ 2 + \frac{(1+c)^2\alpha}{(1+c+d\beta)^2} &\neq \frac{(1+c)\alpha}{1+c+d\beta} + \frac{(1+c)r_2}{d} \end{aligned}\right\} \tag{26}$$

In order to acquire normal form of (24) at $\bar{r} = 0$, we choose $\gamma := \frac{p(0)}{2}$, $\delta := \frac{1}{2}\sqrt{4q(0) - p^2(0)}$, and establish the following elaborations:

$$\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} a_{12} & 0 \\ -a_{11}+\gamma & -\delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{27}$$

Therefore, the normal form of (24) under transformation (27), can be written as:

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} \gamma & -\delta \\ \delta & \gamma \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \tilde{\mathfrak{f}}(u,v) \\ \tilde{\mathfrak{g}}(u,v) \end{pmatrix}. \tag{28}$$

where

$$\begin{aligned} \tilde{\mathfrak{f}}(u,v) &:= \frac{a_{15}}{a_{12}}x^3 + \frac{a_{13}}{a_{12}}x^2 + \frac{a_{14}}{a_{12}}xy + O\left((|u|+|v|)^4\right), \\ \tilde{\mathfrak{g}}(u,v) &:= \left[(\gamma - a_{11})a_{14} - a_{23}a_{12}\right]\frac{xy}{\delta a_{12}} + \frac{(\gamma - a_{11})a_{13}}{\delta a_{12}}x^2 \\ &\quad + \frac{(\gamma - a_{11})a_{15}}{\delta a_{12}}x^3 + O\left((|u|+|v|)^4\right). \end{aligned}$$

Also, $x = a_{12}u$ and $y = (\gamma - a_{11})u - \delta v$. Now, we define $\mathcal{L} \neq 0$ belongs to the set of real numbers as follows:

$$\mathcal{L} := \left(\left[-Re\left(\frac{(1-2\lambda_1)\lambda_2^2}{1-\lambda_1}\eta_{20}\eta_{11}\right) - \frac{1}{2}|\eta_{11}|^2 + |\eta_{02}|^2 + Re(\lambda_2\eta_{21})\right]\right)_{\bar{r}=0},$$

where

$$\begin{aligned} \eta_{21} &:= \frac{1}{16}\left[\tilde{\mathfrak{f}}_{uuu} + \tilde{\mathfrak{f}}_{uvv} + \tilde{\mathfrak{g}}_{uuv} + \tilde{\mathfrak{g}}_{vvv} \right. \\ &\quad \left. + i\left(\tilde{\mathfrak{g}}_{uuu} + \tilde{\mathfrak{g}}_{uvv} - \tilde{\mathfrak{f}}_{uuv} - \tilde{\mathfrak{f}}_{vvv}\right)\right], \\ \eta_{20} &:= \frac{1}{8}\left[\tilde{\mathfrak{f}}_{uu} - \tilde{\mathfrak{f}}_{vv} + 2\tilde{\mathfrak{g}}_{uv} + i\left(\tilde{\mathfrak{g}}_{uu} - \tilde{\mathfrak{g}}_{vv} - 2\tilde{\mathfrak{f}}_{uv}\right)\right], \end{aligned}$$

$$\begin{aligned} \left(\frac{d|\lambda_1|}{d\bar{r}}\right)_{(\bar{r}=0)} &:= \left(\frac{d|\lambda_2|}{d\bar{r}}\right)_{(\bar{r}=0)} \\ &= \frac{(1+c)(d-c-2)}{2d\sqrt{(1+c)\left(a + \frac{(d-c-2)r_2}{d} + \frac{(1+c)\alpha}{(1+c+d\beta)^2} - \frac{(2+c)\alpha}{1+c+d\beta}\right) - c}}. \end{aligned}$$

$$\eta_{02} := \frac{1}{8}\left[\tilde{\mathfrak{f}}_{uu} - \tilde{\mathfrak{f}}_{vv} - 2\tilde{\mathfrak{g}}_{uv} + i\left(\tilde{\mathfrak{g}}_{uu} - \tilde{\mathfrak{g}}_{vv} + 2\tilde{\mathfrak{f}}_{uv}\right)\right],$$
$$\eta_{11} := \frac{1}{4}\left[\tilde{\mathfrak{f}}_{uu} + \tilde{\mathfrak{f}}_{vv} + i\left(\tilde{\mathfrak{g}}_{uu} + \tilde{\mathfrak{g}}_{vv}\right)\right].$$

Thus, in light of aforementioned analytical approach, we can state the following result [38]–[40].

Theorem 6: Assume that (26) holds and $L \neq 0$, then system (3) undergoes NSB at E_U when the parameter r varies in a small neighborhood of $r_2 := \frac{d}{d-c-2}\left(1 - a + \frac{\alpha\left((1+c)^2+(2+c)d\beta\right)}{(1+c+d\beta)^2}\right)$. Furthermore, if $\mathcal{L}<0$, then an attracting invariant closed curve bifurcates from the equilibrium point for $r_2 < r$, and if $\mathcal{L} > 0$, then a repelling invariant closed curve bifurcates from the equilibrium point for $r_2 > r$.

IV. CHAOS CONTROL

The study of chaos control and bifurcation theory is considered as an important and vital area of the present research. It has substantial applications especially in engineering and biological sciences. Particularly, its characteristics have developed in population models, the models related to ecology and biological breeding of species. As compared to continuous-time population model, the behaviour of discrete-time models are most chaotic and complex. Therefore, it is essential to implement the suitable techniques for chaos control to avoid the unpredictable situations. In the present section, we implement three different feedback control strategies, that is, OGY method [41], Hybrid control method and an exponential type control method [37], in order to control the chaos which produced under the influence of various types of bifurcations and move the unstable trajectory towards stable one. Now, we execute a hybrid control methodology [42], and this technique was initially formulated to control the chaos under the appearance of period-doubling bifurcation, but in 2015, the same technique was implemented for controlling the chaos under the development of NSB [45]. In order to apply the hybrid control method, assuming that, system (3) undergoes bifurcation at interior equilibrium point $E_U := (x^\star, y^\star)$, then the modified controlled system can be expressed as:

$$\begin{aligned} x_{n+1} &:= \xi\left[r x_n(1-x_n) + a x_n - b x_n y_n - \frac{\alpha x_n^2}{x_n+\beta}\right] + (1-\xi)x_n \\ y_{n+1} &:= \xi\left[d x_n y_n - c y_n\right] + (1-\xi) y_n, \end{aligned} \tag{29}$$

where controlled parameter $\xi \in (0, 1)$. Note that, the controlled strategy in (29) consists of both perturbation parameter and feedback control. Furthermore, by appropriate choice of ξ, the bifurcation for E_U of system (29) can be advanced or delayed or even entirely obliterated, for detail see also [10], [40]–[44]. The Jacobian $H(\xi)$ of system (29) estimated at E_U is given by:

$$\mathcal{H}(\xi) = \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix} \tag{30}$$

The characteristic equation of (30) is given by:

$$\lambda^2 - (\eta_{11} + \eta_{22})\lambda + \mathfrak{D} = 0,$$

where

$$\begin{aligned} \mathfrak{D} &:= \left(a - \frac{r}{d} - \frac{d\alpha\beta}{(1+d\beta)^2} + c(\xi-2) - \xi - ac\xi\right)\xi \\ &+ \frac{\alpha\xi^2}{1+d\beta}\left(\frac{cd\beta}{1+d\beta} - 1\right) + \left(c^2 + r - \frac{r}{d}(1-c)\right)\xi^2 + 1, \end{aligned}$$
$$\eta_{11} := 1 + \left(a - c - \frac{r}{d} - \frac{d\alpha\beta}{(1+d\beta)^2}\right)\xi, \quad \eta_{12} := -\frac{b\xi}{d},$$
$$\eta_{21} := \left(c + r - 1 - \frac{r}{d} - \frac{\alpha}{1+d\beta}\right)\frac{d\xi}{b}, \quad \eta_{22} := 1 - c\xi.$$

The aforementioned calculations give condition for local asymptotic stability of positive equilibrium E_U of the control system (29).

Lemma 2: The interior equilibrium E_U of the controlled system (29) is locally asymptotically stable, whenever the following condition holds:

$$|\eta_{11} + \eta_{22}| < 1 + \mathfrak{D} < 2.$$

Furthermore, an application of OGY method gives the following control system [43]:

$$\left.\begin{aligned} x_{n+1} &= x_n(1-x_n)\left[r_0 - k_1(x_n - x^*) - k_2(y_n - y^*)\right] \\ &\quad + a x_n - b x_n x_n - \frac{\alpha x_n^2}{x_n+\beta} \\ y_{n+1} &= d x_n y_n - c y_n. \end{aligned}\right\} \tag{31}$$

where k_1 and k_2 are control parameter for OGY method and $(x^\star, y^\star)$ is unique positive equilibrium for system (3). The characteristic equation for the controlled system (31) can be written as; $\lambda^2 - (\sigma_{11} + \sigma_{22})\lambda + D = 0$, where

$$\begin{aligned} \sigma_{11} &= \frac{(1+c)^2 - (1+c)d(\alpha-2)\beta + d^2\beta^2}{(1+c+d\beta)^2} \\ &\quad + \left(\frac{(1+c)k_1}{d} - (r_0 + k_1)\right)\frac{(1+c)}{d}, \\ \sigma_{21} &= \frac{d\left(a - 1 + \frac{(d-1-c)r_0}{d} - \frac{(1+c)\alpha}{1+c+d\beta}\right)}{b}, \sigma_{22} = 1, \\ \sigma_{12} &= \frac{(1+c)\left((1+c)k_2 - d(b+k_2)\right)}{d^2}, \\ D &= \sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21}. \end{aligned}$$

Lemma 3: The interior equilibrium $(x^\star, y^\star)$ of the controlled system (31) is locally asymptotically stable, whenever the following condition holds:

$$|\sigma_{11} + \sigma_{22}| < 1 + D < 2.$$

Moreover, an Exponential type control method takes the form [37]:

$$\begin{aligned} x_{n+1} &= e^{-s_1(x_n - x^\star)}\left[r x_n(1-x_n) + a x_n - b x_n y_n - \frac{\alpha x_n^2}{x_n+\beta}\right] \\ y_{n+1} &= e^{-s_2(y_n - y^\star)}\left[d x_n y_n - c y_n\right]. \end{aligned} \tag{32}$$

Here, s_1 and s_2 are control parameters for (32). The characteristic equation for (32) takes the form

$$\lambda^2 - (\theta_{11} + \theta_{22})\lambda + \theta_{11}\theta_{22} - \theta_{12}\theta_{21} = 0,$$

where

$$\begin{cases} \theta_{11} := \dfrac{(1+c)^2 d(-1+2r\beta) + (1+c)(1+c+d\beta)^2 s_1}{d(1+c+d\beta)^2} \\ -\dfrac{(1+c)^3 r - d^3\beta^2 + (1+c)d^2\beta(-2+\alpha+r\beta)}{d(1+c+d\beta)^2}, \\ \theta_{22} := \dfrac{s_2}{bd(1+c+d\beta)} \\ \times \begin{bmatrix} (1+c)^2 r - d^2(-1+a+r)\beta \\ -(1+c)d(-1+a+r-\alpha-r\beta) \end{bmatrix} \\ +1; \quad \theta_{12} := -\dfrac{b(1+c)}{d}; \\ \theta_{21} := \dfrac{d\left(-1+a+\dfrac{(-1-c+d)r}{d} - \dfrac{(1+c)\alpha}{1+c+d\beta}\right)}{b}. \end{cases}$$

Lemma 4: The equilibrium $(x^\star, y^\star)$ of the controlled system (32) is locally asymptotically stable, if

$$|\theta_{11} + \theta_{22}| < 1 + \theta_{11}\theta_{22} - \theta_{12}\theta_{21} < 2.$$

V. NUMERICAL SIMULATION

In this section, some numerical simulations are provided to confirm our analytical and mathematical investigations. These simulations are further indicating the interesting complex behavior of system (3). Here, *Figure* 2**(a)** and **(b)** is the verification of period-doubling bifurcation in prey and predator population respectively, whereas *Figure* 2**(e)** identifies maximum Lyapunov exponents (MLE) which confirm bifurcating behavior. Moreover, *Figure* 2**(c,d,f,g)** represents controllable region by applying three different approaches that is; hybrid control, OGY control method and exponential type control method. From *Figure* 2**(c,d,f,g)**, it is clear that all control strategies successfully controls the bifurcation. In *Figure* 3**(a-g),** all above cases have discussed for NSB. Moreover, some phase portraits in *Figure* 4**(a-f)** specify the complex nature of system (3).

Example 1: First we choosing parameters $\beta = 2.2$, $a = 2.2$, $b = 1.5$, $c = 0.01$, $d = 0.5$, $\alpha = 0.1 r \in [0.2, 1.8]$ and with initial conditions $(x_0, y_0) = (2.02, 0.08)$, then system (3) undergoes period-doubling bifurcation as $r \approx 1.00858841796$. Moreover, the system (3) has interior equilibrium (2.02, 0.0822483434014). The characteristic equation for system (3) evaluated at equilibrium (2.02, 0.0822483434014) is expressed by:

$$\lambda^2 + 0.0623031201265\lambda - 0.937696879873 = 0, \quad (33)$$

Furthermore, the roots of (33) are $\lambda_1 = -1 \& \lambda_2 = 0.937696879873422 \implies |\lambda_2| \neq 1$. Thus the parameters $(r, a, b, c, d, \alpha, \beta) = (1.008588417960457, 2.2, 1.5, 0.01, 0.5, 0.1, 2.2) \in \psi_{PB}$.

The bifurcation diagrams and Maximum Lyapunov exponents (MLE) of the corresponding system are shown in

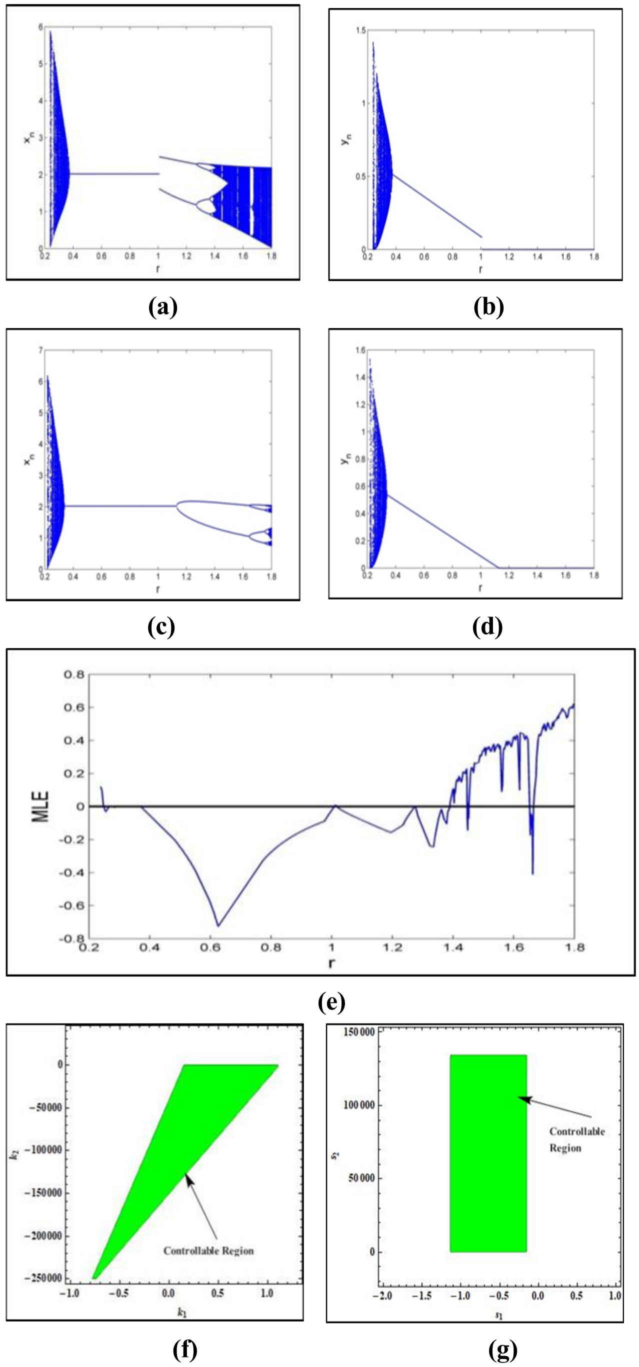


FIGURE 2. Bifurcation diagrams and MLE for system (3) with $\beta = 2.2$, $a = 2.2$, $b = 1.5$, $c = 0.01$, $d = 0.5$, $\alpha = 0.1$, $r \in [0.2, 1.8]$ and with initial conditions $(x_0, y_0) = (2.02, 0.08)$; (a) Bifurcation diagram for x_n (b) Bifurcation diagram for y_n (c) Bifurcation diagram of x_n for controlled system (34) (d) Bifurcation diagram of y_n for controlled system (34) (e) MLE (f) Controllable region for system (31) (g) Controllable region for system (32).

Figure 2**(a,b & e)**. If we again choose above parameters along with initial conditions $(x_0, y_0) = (2.02, 0.08)$, then for these numerical values, the system (3) undergoes period-doubling bifurcation. To control bifurcation, we apply hybrid control

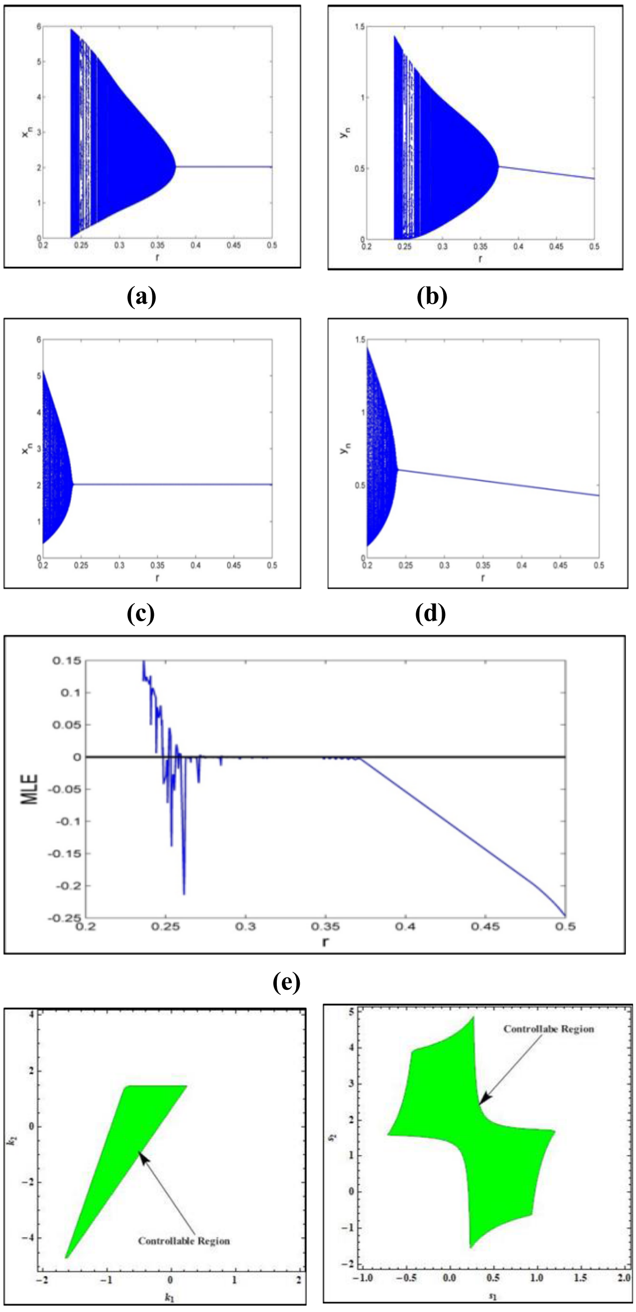


FIGURE 3. Bifurcation diagrams and MLE for system (3) with $\beta = 2.2, a = 2.2, b = 1.5, c = 0.01, d = 0.5, \alpha = 0.1, r \in [0.2, 0.5]$ and with initial conditions $(x_0, y_0) = (2.02, 0.08)$ (a) Bifurcation diagram for x_n (b) Bifurcation diagram for y_n (c) Bifurcation of prey for system (36) when $\xi = 0.55$ (d) Bifurcation of predator for (36) when $\xi = 0.55$ (e) MLE (f) Controllable region for system (31) (g) Controllable region for system (32).

method. The parallel controlled system is stated as:

$$\begin{cases} x_{n+1} := \xi \left[1.0085 x_n (1 - x_n) + 2.2 x_n - 1.5 x_n y_n - \dfrac{0.1 x_n^2}{x_n + 2.2} \right] \\ \qquad + (1 - \xi) x_n \\ y_{n+1} := \xi [0.5 x_n y_n - 0.01 y_n] + (1 - \xi) y_n. \end{cases} \tag{34}$$

The stability interval of the controlled system (34) for various parametric values of bifurcation parameter from chaotic region can be observed in the following table:

TABLE 1. Controllable interval for system (34) with various values of r in chaotic region.

Values of bifurcation parameter r from chaotic region	*Stability interval* ξ
1.10	$0 < \xi < 0.8955248064575835$
1.11	$0 < \xi < 0.8856456174528254$
1.12	$0 < \xi < 0.8760215505050334$
1.121	$0 < \xi < 0.8750727461207667$
1.124	$0 < \xi < 0.8722408713847918$
1.129	$0 < \xi < 0.8675689167854668$
1.1295	$0 < \xi < 0.8671049701275002$
1.12952	$0 < \xi < 0.8670864244507052$

Furthermore, *Figure* 2**(c, d)** represents the bifurcation diagrams for controlled system (34). Now, if we choose $\beta = 2.2$, $a = 2.2$, $b = 1.5$, $c = 0.01$, $d = 0.5$, $\alpha = 0.1$ and $r = 1.12952$. For these parameters, controllable region of map (31) with OGY method is shown in *Figure* 2**(f)**. On the other hand, the controllable region for system (32), when above parameters remain same is depicted in *Figure* 2**(g)**.

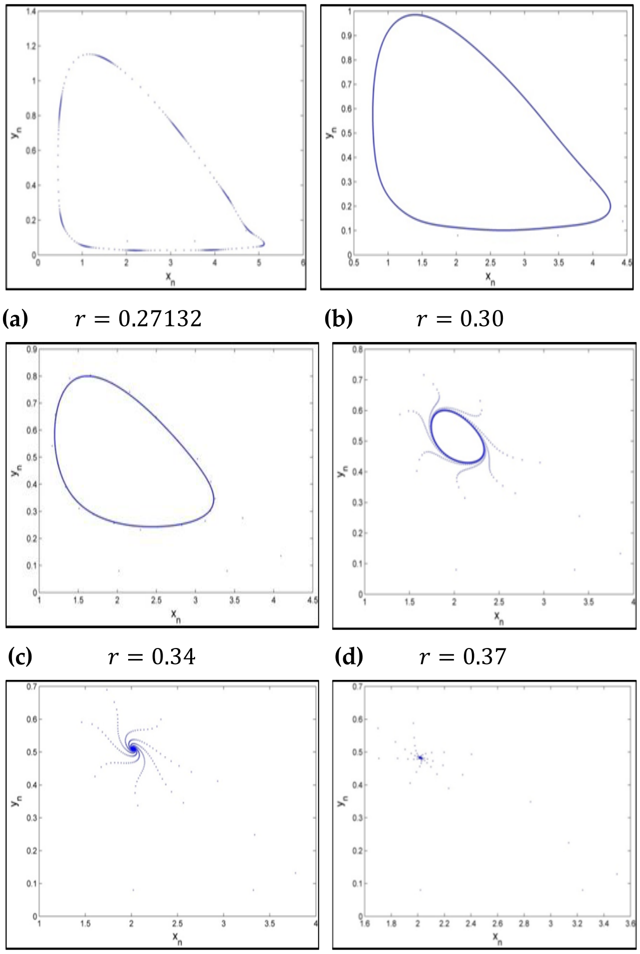
Example 2: Suppose $\beta = 2.2$, $a = 2.2$, $b = 1.5$, $c = 0.01$, $d = 0.5$, $\alpha = 0.1 r \in [0.2, 0.5]$ and with initial conditions $(x_0, y_0) = (2.02, 0.08)$, then system (3) undergoes NSB where $r \approx 0.3733196225131351$ is the bifurcation parameter. Moreover, the characteristic equation along with interior equilibrium $E_U := (2.02, 0.514231)$ of the Jacobian matrix of system (3) evaluated at E_U is expressed by:

$$\lambda^2 - 1.2209398466770125\lambda + 1 = 0. \tag{35}$$

Moreover, $\lambda_{1,2} = 0.6104699233385062 \pm 0.7920394388533177i$, are the roots of (35) along with condition $|\lambda_{1,2}| = 1$. Thus $(r, a, b, c, d, \alpha, \beta) = (0.3733196, 2.2, 1.5, 0.01, 0.5, 0.1, 2.2) \in \psi_{\mathcal{NB}}$.

On the other hand, bifurcation diagrams and MLE are shown in *Figure* 3(**a**, **b**&**e**). Furthermore, we again take $\beta = 2.2$, $a = 2.2$, $b = 1.5$, $c = 0.01$, $d = 0.5$, $\alpha = 0.1 r \in [0.2, 0.5]$ and with initial conditions $(x_0, y_0) = (2.02, 0.08)$. For these values of parameters, there exists NSB in system (3). In order to controlling the chaos which is due to appearance of bifurcation, we use hybrid control strategy. Therefore, the modified controlled system takes the form:

$$\begin{cases} x_{n+1} := \xi \left[0.3733 x_n (1 - x_n) + 2.2 x_n \right. \\ \qquad \left. - 1.5 x_n y_n - \dfrac{0.1 x_n^2}{x_n + 2.2} \right] + (1 - \xi) x_n \\ y_{n+1} := \xi [0.5 x_n y_n - 0.01 y_n] + (1 - \xi) y_n. \end{cases} \tag{36}$$



(a) $r = 0.27132$ (b) $r = 0.30$ (c) $r = 0.34$ (d) $r = 0.37$ (e) $r = 0.38$ (f) $r = 0.42$

FIGURE 4. Phase portraits for system (3) when r ∈ [0.2, 0.5].

The stability interval of the controlled system (36) for various parametric values of bifurcation parameter from chaotic region can be observed in the following table:

TABLE 2. Controllable interval for system (36) with various values of *r* in chaotic region.

Values of bifurcation parameter r from chaotic region	*Stability interval* ξ
0.24	$0 < \xi < 0.5562539966224632$
0.26	$0 < \xi < 0.6141474325604837$
0.28	$0 < \xi < 0.6747667346803521$
0.32	$0 < \xi < 0.8049910585926848$
0.36	$0 < \xi < 0.9487532371669501$
0.37	$0 < \xi < 0.9870597175370684$

Moreover, *Figure* 3(**c, d**) represents the bifurcation diagrams for controlled systems (36). Next, we choose $\beta = 2.2$, $a = 2.2$, $b = 1.5$, $c = 0.01$, $d = 0.5$, $\alpha = 0.1$ and $r = 0.24$. For these parametric values, the controlled region of system (31) is plotted in *Figure* 3(**f**). Additionally, the controllable region obtained from exponential type controlled map (32) is depicted in *Figure* 3(**g**), whenever the above parameter values remain fixed.

Furthermore, the following *Figure* 4(**a-g**), elaborates phase portraits for system (3). By taking different values of bifurcation parameter r from chaotic region while other parameters remain same in each case, that is, $(a, b, c, d, \alpha, \beta) = (2.2, 1.5, 0.01, 0.5, 0.1, 2.2)$ and initial conditions $(x_0, y_0) = (2.02, 0.08)$.

VI. CONCLUSION

The innovation of our work is to analyze the dynamical behaviour of a novel predator-prey model incorporating cannibalistic prey of Holling type II functional response with cannibalism rate α. Cannibalism is an important and significant natural phenomenon that affects the population dynamics. There exists numerous biological species in which cannibalism has been observed. Cannibalism has enormous and complex effects on population dynamics. On the other hand, cannibalism is a remedy for equilibration and regulation of population size as well. Consequently, population oscillations are observed due to emergence of cannibalism.

In this work, existence criteria of biologically meaningful equilibrium points have been investigated and their stability analysis has also been carried out. By implementation of center manifold theorem and bifurcation theory of normal forms, it is investigated that system undergoes transcritical, period-doubling and Neimark-Sacker bifurcations whenever intrinsic growth rate r of prey population is taken as bifurcation parameter. Moreover, it seems that cannibalism is an effective mechanism for emergence of periodic oscillations in populations. Numerical simulations show that periodic outbreaks may result due to incorporation of cannibalism in prey population and this periodic outbreak is limited to prey population only without leaving an effect on predation. In order to control these periodic oscillations in prey population density, and other bifurcating and fluctuating behavior of the system, three different chaos control methodologies are implemented.

At the end, it is worthwhile to mention that system (3) is a novel discrete-time model and it is not a discrete counterpart of any continuous system.

ACKNOWLEDGMENT

Rana Alabdan would like to thank Deanship of Scientific Research at Majmaah University for supporting this work under the Project No. (R-1441-127).

REFERENCES

[1] D. Claessen and A. M. de Roos, "Bistability in a size-structured population model of cannibalistic fish—A continuation study," *Theor. Population Biol.*, vol. 64, no. 1, pp. 49–65, Aug. 2003.
[2] V. Guttal, P. Romanczuk, S. J. Simpson, G. A. Sword, and I. D. Couzin, "Cannibalism can drive the evolution of behavioural phase polyphenism in locusts," *Ecol. Lett.*, vol. 15, no. 10, pp. 1158–1166, Oct. 2012.
[3] M. Lloyd, "Self regulation of adult numbers by cannibalism in two laboratory strains of flour beetles (Tribolium Castaneum)," *Ecology*, vol. 49, no. 2, pp. 245–259, Mar. 1968.

[4] M. L. Richardson, R. F. Mitchell, P. F. Reagel, and L. M. Hanks, "Causes and consequences of cannibalism in noncarnivorous insects," *Annu. Rev. Entomol.*, vol. 55, no. 1, pp. 39–53, Jan. 2010.
[5] D. H. Wise, "Cannibalism, food limitation, intraspecific competition, and the regulation of spider populations," *Annu. Rev. Entomol.*, vol. 51, no. 1, pp. 441–465, Jan. 2006.
[6] L. R. Fox, "Cannibalism in natural populations," *Ann. Rev. Ecol. Syst.*, vol. 6, no. 1, pp. 87–106, Nov. 1975.
[7] G. A. Polis, "The evolution and dynamics of intraspecific predation," *Annu. Rev. Ecol. Syst.*, vol. 12, no. 1, pp. 225–251, Nov. 1981.
[8] D. Claessen, A. M. de Roos, and L. Persson, "Population dynamic theory of size-dependent cannibalism," *Proc. Roy. Soc. London B, Biol. Sci.*, vol. 271, no. 1537, pp. 333–340, Feb. 2004.
[9] M. Danca, S. Codreanu, and B. Bakó, "Detailed analysis of a nonlinear prey–predator model," *J. Biol. Phys.*, vol. 23, no. 1, pp. 11–20, Mar. 1997.
[10] Q. Din, M. A. Khan, and U. Saeed, "Qualitative behaviour of generalised Beddington model," *Zeitschrift Naturforschung A*, vol. 71, no. 2, pp. 145–155, Feb. 2016.
[11] L. Pizzatto and R. Shine, "The behavioral ecology of cannibalism in cane toads (Bufo marinus)," *Behav. Ecol. Sociobiol.*, vol. 63, no. 1, pp. 123–133, Nov. 2008.
[12] S. Fasani and S. Rinaldi, "Remarks on cannibalism and pattern formation in spatially extended prey–predator systems," *Nonlinear Dyn.*, vol. 67, no. 4, pp. 2543–2548, Mar. 2012.
[13] G.-Q. Sun, G. Zhang, Z. Jin, and L. Li, "Predator cannibalism can give rise to regular spatial pattern in a predator–prey system," *Nonlinear Dyn.*, vol. 58, nos. 1–2, pp. 75–84, Oct. 2009.
[14] V. H. W. Rudolf, "Consequences of stage-structured predators: Cannibalism, behavioral effects, and trophic cascades," *Ecology*, vol. 88, no. 12, pp. 2991–3003, Dec. 2007.
[15] V. H. W. Rudolf, "The interaction of cannibalism and omnivory: Consequences for community dynamics," *Ecology*, vol. 88, no. 11, pp. 2697–2705, Nov. 2007.
[16] V. H. W. Rudolf, "The impact of cannibalism in the prey on predator–prey systems," *Ecology*, vol. 89, no. 11, pp. 3116–3127, Nov. 2008.
[17] S. Biswas, S. Chatterjee, and J. Chattopadhyay, "Cannibalism may control disease in predator population: Result drawn from a model based study," *Math. Methods Appl. Sci.*, vol. 38, no. 11, pp. 2272–2290, Jul. 2015.
[18] B. Buonomo, D. Lacitignola, and S. Rionero, "Effect of prey growth and predator cannibalism rate on the stability of a structured population model," *Nonlinear Anal., Real World Appl.*, vol. 11, no. 2, pp. 1170–1181, Apr. 2010.
[19] B. Buonomo and D. Lacitignola, "On the stabilizing effect of cannibalism in stage-structured population models," *Math. Biosci. Eng.*, vol. 3, no. 4, pp. 717–731, Oct. 2006.
[20] A. Basheer, E. Quansah, S. Bhowmick, and R. D. Parshad, "Prey cannibalism alters the dynamics of Holling–Tanner-type predator–prey models," *Nonlinear Dyn.*, vol. 85, no. 4, pp. 2549–2567, Sep. 2016.
[21] A. Al Basheer, R. D. Parshad, E. Quansah, S. Yu, and R. K. Upadhyay, "Exploring the dynamics of a Holling–Tanner model with cannibalism in both predator and prey population," *Int. J. Biomath.*, vol. 11, no. 1, Jan. 2018, Art. no. 1850010.
[22] H. Deng, F. Chen, Z. Zhu, and Z. Li, "Dynamic behaviors of Lotka–Volterra predator–prey model incorporating predator cannibalism," *Adv. Difference Equ.*, vol. 2019, no. 1, pp. 1–17, Dec. 2019.
[23] F. Zhang, Y. Chen, and J. Li, "Dynamical analysis of a stage-structured predator–prey model with cannibalism," *Math. Biosci.*, vol. 307, pp. 33–41, Jan. 2019.
[24] X. Liu, "A note on the existence of periodic solutions in discrete predator–prey models," *Appl. Math. Model.*, vol. 34, no. 9, pp. 2477–2483, Sep. 2010.
[25] Y. Li, T. Zhang, and Y. Ye, "On the existence and stability of a unique almost periodic sequence solution in discrete predator–prey models with time delays," *Appl. Math. Model.*, vol. 35, no. 11, pp. 5448–5459, Nov. 2011.
[26] Q. Din, "Complexity and chaos control in a discrete-time prey–predator model," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 49, pp. 113–134, Aug. 2017.
[27] M. Gámez, I. López, C. Rodríguez, Z. Varga, and J. Garay, "Ecological monitoring in a discrete-time prey–predator model," *J. Theor. Biol.*, vol. 429, pp. 52–60, Sep. 2017.
[28] J. Huang, S. Liu, S. Ruan, and D. Xiao, "Bifurcations in a discrete predator–prey model with nonmonotonic functional response," *J. Math. Anal. Appl.*, vol. 464, no. 1, pp. 201–230, Aug. 2018.
[29] V. Weide, M. C. Varriale, and F. M. Hilker, "Hydra effect and paradox of enrichment in discrete-time predator–prey models," *Math. Biosci.*, vol. 310, pp. 120–127, Apr. 2019.
[30] M. S. Shabbir, Q. Din, M. Safeer, M. A. Khan, and K. Ahmad, "A dynamically consistent nonstandard finite difference scheme for a predator–prey model," *Adv. Difference Equ.*, vol. 2019, no. 1, pp. 1–17, Dec. 2019.
[31] Q. Din, M. S. Shabbir, M. A. Khan, and K. Ahmad, "Bifurcation analysis and chaos control for a plant–herbivore model with weak predator functional response," *J. Biol. Dyn.*, vol. 13, no. 1, pp. 481–501, Jan. 2019.
[32] Y. Chow and S. R.-J. Jang, "Cannibalism in discrete-time predator–prey systems," *J. Biol. Dyn.*, vol. 6, no. 1, pp. 38–62, Jan. 2012.
[33] X. Liu and D. Xiao, "Complex dynamic behaviors of a discrete-time predator–prey system," *Chaos, Solitons Fractals*, vol. 32, no. 1, pp. 80–94, Apr. 2007.
[34] Q. Din and M. Hussain, "Controlling chaos and Neimark–Sacker bifurcation in a Host–Parasitoid model," *Asian J. Control*, vol. 21, no. 3, pp. 1202–1215, May 2019.
[35] Z. He and X. Lai, "Bifurcation and chaotic behavior of a discrete-time predator–prey system," *Nonlinear Anal., Real World Appl.*, vol. 12, no. 1, pp. 403–417, Feb. 2011.
[36] Z. Jing and J. Yang, "Bifurcation and chaos in discrete-time predator–prey system," *Chaos, Solitons Fractals*, vol. 27, no. 1, pp. 259–277, Jan. 2006.
[37] Q. Din, "A novel chaos control strategy for discrete-time Brusselator models," *J. Math. Chem.*, vol. 56, no. 10, pp. 3045–3075, Nov. 2018.
[38] C. K. Tse, Y. M. Lai, and H. H. C. Iu, "Hopf bifurcation and chaos in a free-running current-controlled cuk switching regulator," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 47, no. 4, pp. 448–457, Apr. 2000.
[39] Y. Li, M. Rafaqat, T. J. Zia, I. Ahmed, and C. Y. Jung, "Flip and neimark-sacker bifurcations of a discrete time predator-pre model," *IEEE Access*, vol. 7, pp. 123430–123435, 2019.
[40] Y.-H. Wan, "Computation of the stability condition for the hopf bifurcation of diffeomorphism on R^2," *SIAM J. Appl. Math.*, vol. 34, no. 1, pp. 167–175, Jan. 1978.
[41] E. Ott, C. Grebogi, and J. A. Yorke, "Controlling chaos," *Phys. Rev. Lett.*, vol. 64, pp. 1196–1199, Mar. 1990.
[42] X. S. Luo, G. Chen, B. H. Wang, and J. Q. Fang, "Hybrid control of period-doubling bifurcation and chaos in discrete nonlinear dynamical systems," *Chaos, Solitons Fractals*, vol. 18, no. 4, pp. 775–783, Nov. 2003.
[43] S. Song, J. Liu, and H. Wang, "Adaptive neural network control for uncertain switched nonlinear systems with time delays," *IEEE Access*, vol. 6, pp. 22899–22907, 2018.
[44] K. Rajagopal, A. Akgul, I. M. Moroz, Z. Wei, S. Jafari, and I. Hussain, "A simple chaotic system with topologically different attractors," *IEEE Access*, vol. 7, pp. 89936–89947, 2019.
[45] L.-G. Yuan and Q.-G. Yang, "Bifurcation, invariant curve and hybrid control in a discrete-time predator–prey system," *Appl. Math. Model.*, vol. 39, no. 8, pp. 2345–2362, Apr. 2015.

MUHAMMAD SAJJAD SHABBIR received the M.Phil. degree from the Mirpur University of Science and Technology (MUST), Mirpur AJK, Pakistan, in 2015. He is currently pursuing the Ph.D. degree with Air University Islamabad, Pakistan. His current area of research including stability analysis, bifurcation analysis and chaos control in a discrete-time dynamical systems.

QAMAR DIN received the Ph.D. degree in discrete approximations and optimization of evolution inclusions and equations from the Abdus Salam School of Mathematical Science, GC University, Lahore, Pakistan, in 2012. He is currently an Assistant Professor with the Department of Mathematics, University of Poonch Rawalakot, Azad Jammu and Kashmir, Pakistan. His current research interests include stability analysis, bifurcation analysis, and chaos control in discrete-time systems. He has published more than 90 research articles in reputed international journals.

RANA ALABDAN received the B.S. degree in information systems from Imam Mohammed Ibn Saud Islamic University, Riyadh, Saudi Arabia, the Master of Science degree in information systems management from Robert Morris University, Pittsburgh, PA, and the D.Sc. degree in information systems and communication from Robert Morris University, Pittsburgh, PA. She is the Vice Dean for Student Affairs with the College of Computer and Information Sciences, Majmaah University, and an Assistant Professor with the Information Systems Department. Her research interests include e-banking, mobile payment technology, machine learning, and artificial intelligence. She is also working is bifurcation analysis and chaos control in discrete-time systems.

ASIFA TASSADDIQ (Member, IEEE) is affiliated with the College of Computer and Information Sciences, Majmaah University Saudi Arabia. She is currently providing services as an Associate Professor of Mathematics. She is interested in multidisciplinary applications of mathematics. She has published several articles in reputed journals. She also has presented her research work in different conferences/workshops at national and international levels.

KHALIL AHMAD received the Ph.D. degree in geometric function theory of complex analysis from COMSATS University Islamabad, Pakistan, in 2014. He is currently an Assistant Professor with the Department of Mathematics, Air University Islamabad, Pakistan. His current research interests include stability analysis, bifurcation analysis and chaos control in discrete-time systems, approximation theory, and Complex analysis.