IEEE Access
Multidisciplinary : Rapid Review : Open Access Journal

Received April 9, 2020, accepted May 4, 2020, date of publication May 18, 2020, date of current version June 3, 2020.

Digital Object Identifier 10.1109/ACCESS.2020.2995312

Road Capacity and Throughput for Safe Driving Autonomous Vehicles

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This work was supported by the Ministry of Science and Technology (MOST) of Taiwan under Grant 108-2221-E-002-033-MY3.

ABSTRACT Safe driving is a relatively new concept that focuses on solving the responsibility attribution problem for autonomous vehicles (AVs), claiming that once the AVs follow a series of pre-defined safe driving policies, it is free from liability even when accidents happen. In this work, we propose safe driving policies for traffic configurations, including straight road, intersection, and Manhattan-like city. Base on the defined safe driving policies, we offer the concepts of safe driving capacity (SDC) and safe driving throughput (SDT) to measure the safe efficiency of the traffic configurations. The former measures the maximum AVs a traffic configuration could accommodate, and the latter measures the maximum throughputs of safe AVs of a traffic configuration. The values obtained are the fundamental limits of the traffic efficiencies for liability-free AVs under the defined conditions. The theoretical performance bounds give people insights on the potential limitations of the safe traffic efficiencies. Finally, this work provides analytical results of SDC and SDT on all the traffic configurations mentioned with explanations, implications, and trade-offs on the issues that may have effects on them.

INDEX TERMS Self-driving car, autonomous vehicles, safe driving throughput, safe driving capacity, VANET, V2V, collision avoidance, C-ITS, cooperative intersection management.

I. INTRODUCTION

The concept of safe driving is first proposed by the Intel Mobileye group [1] under the name of responsibility-sensitive safety (RSS). It uses the formal method to define the safeness of an autonomous vehicle (AV). Unlike the convention definition of safety, RSS defines the safeness of AVs upon whether it's liable when an accident happens and the liability attribution is based on a set of rules. In this work, we name these rules safe driving policies. If these rules are properly, reasonably, and completely defined, the AV-related constitutional issues studied by Merchant and Lindor in [2] and the concerns on the relations between AVs and pedestrians [3] may to some extent be relieved. The original work focused on the logic, the conditions, and the liveness of the safety rules. It covered a variety of scenarios and traffic configurations an AV might encounter but mentioned very little on the efficiencies of AVs based on such rules.

Theoretic performance bounds provide fundamental understating of systems. Shannon's work [4] lay the foundation of communications systems and gives theoretic bound on their efficiencies through the proposition of information entropy. Gupta and Kumar [5] investigated the theoretic capacity of two types of N-node wireless networks, Random Networks and Arbitrary Networks. In this work, motivated by these works, we aim to investigate the theoretic performance bound for safe-driving vehicles on the roads. We first formalize the concept of efficiency under the safe driving context and proposed two metrics, SDC and SDT, as the measure of safe driving efficiency. The former measures how many AVs that do not violate the safe driving rules an area (a traffic configuration) could accommodate, and the latter measures the number of safe AVs to pass through the traffic configuration within a fixed time. As there are hundreds of traffic configurations, in this work, we focus only on the following three types for their simplicity in theoretical studies and their universality. They are straight road, intersection, and Manhattan-like city, as shown in Fig. 1. Many related studies of vehicle behaviors in these traffic configurations, depends on the focus, have been done. For the straight road, Zhang *et al.* focused on the over-riding mechanism and proposed the idea

The associate editor coordinating the review of this manuscript and approving it for publication was Baozhen Yao.

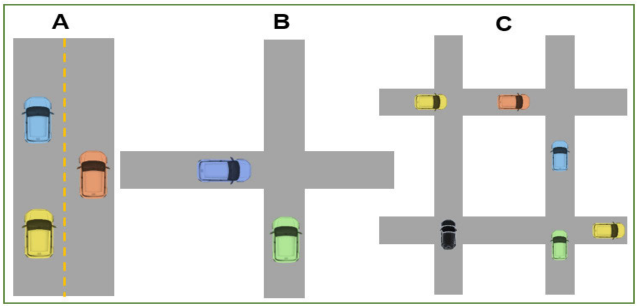

FIGURE 1. Traffic configurations: A is the straight road, B is the intersection, and C is the the Manhattan-like city.

of time-to-last-second-braking [6]. Wu *et al.* focused on the linear stability required for platoon safety in emergent situations on a straight road. Ayumu *et al.* studied the relationship between rear-end collision and automatic brake control [7]. Nunen *et al.* and Solyom *et al.* discussed the safety and limitations of truck platooning [8], [9]. Aoki and Rajkumar design algorithms for vehicle merging when two straight roads are merged [10]. Wang and Wei showed the efficiency limitation of straight road base on safe driving mechanisms proved that the enabling of V2V communication is positive to the overall efficiency [11].

For the intersection, depends on whether there is centralized signaling, the study of intersection could be classified into the signalized one and the non-signalized one. And ways of managing them varies. For the non-signalized intersection, which is the setting of this work, Chen and Englund [12] did a complete survey and categorize the studies according to the algorithm design philosophy into the following three classes: resource reservation, trajectory planning, and collision avoidance (CA). The first class treats the intersection as a resource to contend and reserve for and provide analysis and heuristics, aiming to improve passing efficiency [13], [14]. The second class designs trajectory planning algorithms to find the optimum route with minimum risk for AVs [15]. The third class focuses on avoiding collisions with brakings and steerings and, at the same time, manage to enhance the passing efficiency [16]–[20].

In general, intersection management without further restrictions is a multi-agent, and thus NP-hard problem [21]. In order to obtain an analytic result, problem relaxation is required. The approaches we used to relax the problems are motivated by the following works: The worst-case study from the work of Nilsson *et al.* [22]; The constant spacing strategy from the work of Darbha and Hedrick *et al.* [23]; The rear-end collision avoidance with brake control and emergency braking from Ayumu *et al.* and Segata *et al.* [7], [24]; The idea to quantify the DoF of AV by Rodrigues de Campos *et al.* and the safety assessment from the Ph.D. thesis of Althof [25].

The major distinction between our work and all the ones mentioned above is clear. We are the first group to propose the idea of SDC and SDT that measures the safe AVs on roads. And we are also the first group to study the fundamental efficiency limits in the context of safe driving with precise analytic results. The trade-off of the precision comes from our relatively idealistic presumptions made comparing to the rest that tries to make their work as practical as possible but could only give heuristics.

To conclude, the major contributions of this paper are:

1) We are the first group to analyze the traffic efficiencies under the context of safe driving.
2) We formalize the concept of safe driving efficiencies in both space and time domain and obtain their achievable analytic upper-bounds under the provided conditions; furthermore, some direct corollaries are given.

We organized the rest of this paper as follows. In section II, we give an overview of the problem and elaborate settings and requirements needed for later analysis. In section III, we first define the safe driving policies with math and some modal languages. Then we set the metrics SDC and SDT as the indications of the efficiencies under safe driving policies. In section IV, we analyze the traffic efficiencies of several traffic configurations and their behaviors. In section V, we discuss the direct results from the theorems proven in Section IV. We conclude the work in Section V. Most proofs of this work are left in the Appendix section.

II. PROBLEM OVERVIEW AND SETUP

A. VEHICLE REQUIREMENTS AND CAPABILITIES

In the following contexts, All AVs are presumed to have identical physical characteristics and capabilities (Same vehicle length, width, response time, acceleration, and deceleration). The maximum accelerations and decelerations are presumed to be constant. The AVs are equipped with high-precision mapping, navigating, positioning, and perceiving systems along with the wireless V2X interfaces like 4G/5G and DSRC. The perceiving system consists of all the available visual sensors such as LIDAR, camera, and radar. The inter-vehicle communication is presumed to be free from attack and is always reliable as long as received. All the traffic configurations are signal-free, and roads are orthogonal to one another if intersected, as shown in Fig. 1. All the traffic configurations have the same speed limit $[V_{min}, V_{Max}]$. No traffic facilities nor pedestrians unless otherwise specified, and no AVs would form platoons. Each of the AV makes decisions following identical decision process characterized by a four tuple $< \mathcal{S}, \mathcal{A}, \mathcal{T}, \mathcal{R} >$. $\mathcal{S}$ denotes the state space, which is the collection of all possible situations the vehicle might encounter while driving. $\mathcal{A}$ denotes the action space representing all the possible actions the agent is capable of apply under the current state. In general context, it could be referred to actions such as brake, change the heading, lane, speed, etc. However, in this work, it is confined to brake with maximum braking power for tractable analysis. $\mathcal{T}$ is the transition function with domain and co-domain $\mathcal{S} \times \mathcal{A} \times \mathcal{S}' \rightarrow [0, 1]$. $\mathcal{R}$: $\mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}$ is the reward function to evaluate the goodness of a chosen action.

B. PRESUMPTIONS AND DEFINITIONS OF SAFE DRIVING POLICIES

We denote the trace of states for the i^{th} AV on the road as $\sigma_i = <s_{i0}, s_{i1}, \ldots, s_{i\check{T}}>$, where the subscript 0 and subscript $\check{T}$ denote the current instant and a finite system-dependent time horizon, respectively. Also, let W and L be the width and length of the AVs. To address the policy-based safety, We define a set that is a sub-set of $\mathcal{A}$ and call the set best effort reaction (BER). It contains only those $a_t \in \mathcal{A}$ that minimize the probability of leading to an accident.

$$\boldsymbol{BER}(s_t) = \mathbf{arg\,min}_{a_t \in \mathcal{A}}\{\mathbf{max}_{s_{t+1}\in\{accident\}}\mathcal{T}(s_t, a_t, s_{t+1})\}.$$

In the following context, taking $a_t \in BER(s_t)$ should lead the AVs to accident with probability 0 almost surely, that is, no collision is allowed. Also, we only consider the scenarios that the BER within the time horizon never becomes empty for the logical liveness issue. Furthermore, we presume that the larger the deceleration, the better the reward. Briefly speaking, $\forall a_t \in \boldsymbol{BER}(s_t)$, the following three equations always hold

- $\mathbf{max}_{s_{t+1}\in\{accident\}}\mathcal{T}(s_t, a_t, s_{t+1}) = 0$ ***a.s.***.
- $|a_1| > |a_2| \leftrightarrow \mathcal{R}(s, |a_1|) > \mathcal{R}(s, |a_2|), \quad \forall s \in \mathcal{S}$.
- $\boldsymbol{BER}(\boldsymbol{s_t}) \neq \{\}, \forall t \in [t_0, t_0 + \check{T}]$.

The t_0 in the last equation denotes the current instance. Now we could define the safe state of an AV. We say an AV, ω_r, is safe, or is in safe state, if and only if there would be no collision between the rear AV ω_r and the front AV ω_f, when ω_f applies any possible action abruptly, and ω_1 responds to the action with the action in its BER. For straight road analysis, we define the longitudinal distance (inter-vehicle distance) of AVs on the same lane as the distance of their body center measured along the direction of the road parallel to the ground. With the safe definition given, we say a longitudinal distance is safe if and only if it's sufficient to make the rear AV stay within safe states no matter what the front AV does.

For the intersection, we define the concept of intersection distance and exposure time. The intersection distance of an AV is the euclidean distance measured along the direction parallel to the ground from the AV center to the intersection center. The red arrows in Fig. 2 illustrates the intersection distances of ω_1, ω_2 and ω_3. If the intersection distance of an AV is sufficient for it to remain in a safe state when applying its BER right after a fixed response time τ from the first time an emergency happens, it's safe. The exposure time is the first time point when vehicles from two intersected lanes of an intersection notice the coming of each other through either their perception system or the V2V communication when they are within the communication range D_{comm}. When first exposed, the AV closer to the intersection would be the prioritized vehicle, and the other one would be the yielding vehicle. The priority is relative, i.e., ω_2 might be the prioritized vehicle to ω_3 but is the yielding vehicle to ω_1, as shown in Figure.2. If there is no counter-lane vehicle detected at the intersection distance $\frac{1}{\sqrt{2}}D_{comm}$, the vehicle accessing the intersection is defaulted to be the prioritized vehicle. For

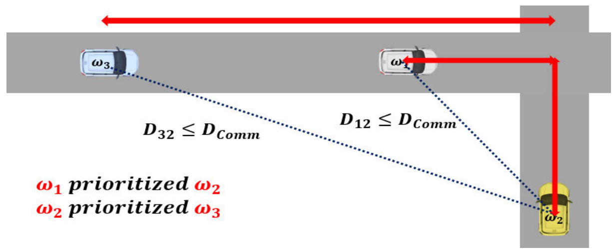

FIGURE 2. The intersection priority and exposed time illustration.

the time between an AV is exposed and the time it passes through the intersection, it's not allowed to decelerate if it's the prioritized vehicle, and it's not allowed to accelerate if it's the yielding vehicle. If an AV is both prioritized to one AV and yielding vehicle to another, it should keep constant velocity between the time of its second exposed (that make it prioritized) to the time it pass through the intersection.

Now we close this section by the definition of safe driving capacity (SDC) and safe driving throughput (SDT). The SDC of a traffic configuration is the maximum number of safe state AVs that could drive within. The SDT of a traffic configuration is the number of safe AVs that could pass through a cut of the traffic configuration within a given amount of time $\mathbf{T}$. If not otherwise specified, we set the $\mathbf{T}$ to be a second.

III. ANALYSIS ON SDC AND SDT

In this section, we will derive the two safe driving efficiency metrics: Safe Driving Capacity (SDC) and Safe Driving Throughput (SDT) in the following three scenarios.

1) Straight Road
2) Intersection
3) Manhattan-Like City

The straight road and the intersection are the building blocks for the analysis of the city. With the given definitions and policies, we first analyze the safe efficiencies of the two in terms of SDC and SDT. Base on the result, We then derive the SDC and SDT of a city.

A. STRAIGHT ROAD

The overall derivation of SDC and SDT for a straight road R is based on the concept that the longitudinal distance kept between two AVs should be no less than the minimum safe longitudinal distance $\mathbf{inf}\ d^R_{LS}$. Such distance should be sufficient so that even the front vehicle ω_f applies maximum braking power without early warnings and the rear vehicle, ω_r, without knowing the emergency, applies maximum acceleration u throughout the response time τ, ω_r would not collide with ω_f if it performs the action in its BER (Maximum braking power here). Let v_r and v_f be the velocity of ω_r and ω_f respectively and a_{max} be the maximum deceleration of the vehicle, then for ω_r not to collide with ω_f, their longitudinal distance should always be greater than L. Let $d^R_{LS}(v_r, v_f)$ be the longitudinal distance between ω_r and ω_f at the moment ω_f starts to brake with a_{max}, then until ω_r fully stops, it is

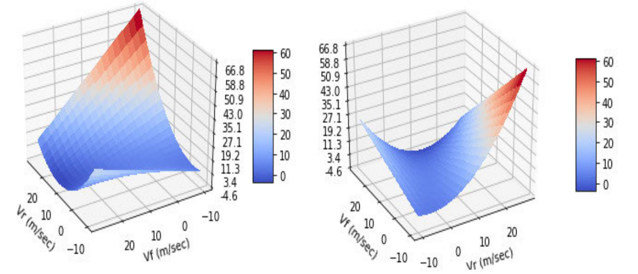

FIGURE 3. Safe longitudinal distance surface in different angles. One can clearly see a saddle point and thus non-convex.

required that

$$d^R_{LS}(v_r, v_f) + \frac{v_f^2}{2a_{max}} \geq L + \tau(v_r + \frac{u\tau}{2}) + \frac{(v_r + u\tau)^2}{2a_{max}}. \quad (1)$$

The LHS of inequality is the sum of initial inter-vehicle distance and the distance ω_f would forward throughout the deceleration. The RHS is the distance ω_r would forward before it stops. Re-organize the terms, we have

$$d^R_{LS}(v_f, v_r) \geq \tau(v_r + \frac{u\tau}{2}) + L + \frac{(v_r + u\tau)^2 - v_f^2}{2a_{max}}. \quad (2)$$

Adding the condition that the inter-vehicle distance should never be less than L that corresponds to the trivial case that ω_f stops after ω_r stops, we have the general form

$$d^R_{LS}(v_f, v_r) \geq \max\{\tau(v_r + \frac{u\tau}{2}) + L + \frac{(v_r + u\tau)^2 - v_f^2}{2a_{max}}, L\}. \quad (3)$$

Since the worst case only happens at either the moment ω_f starts to brake with a_{max} or the time that ω_r stops, we can be sure the correctness of (3).

1) SDC_R AND SDT_R

To find the SDC_R and SDT_R of a $\dot{N}$-lanes $\dot{M}$-meter road measure within the time $\boldsymbol{T}$, we first index the AVs along the road from rear to front in increasing order. Then by definitions, we have

$$SDC_R(\dot{M}, \dot{N}) = \sup_{n} \inf_{\lambda \mathcal{I} \lambda \geq 0} \dot{N}\Big(n + \lambda\{1 - \frac{\sum_i^{n-1} \inf d^R_{LS}[\omega_i]}{\dot{M}}\}\Big). \quad (4)$$

$$SDT_R(\boldsymbol{T}, \dot{N}) = \sup_{n} \inf_{\lambda \mathcal{I} \lambda \geq 0} \dot{N}\Big(n + \lambda\{1 - \frac{\sum_i^{n} \inf d^R_{LS}[\omega_i]}{\int_0^{\boldsymbol{T}} v_n(t)dt}\}\Big). \quad (5)$$

The λs on the RHS of (4) and (5) are the KKT multipliers correspond to the constraints that the results are measured within the specified space range $\dot{M}$ and time range $\boldsymbol{T}$. $d^R_{LS}[\omega_i]$ is the longitudinal distance between the i^{th} AV and the $(i+1)^{th}$ AV in its front. This primal form of the problem is intractable due to its multi-agent nature and its non-convexity in the feasible set shown in Fig. 3. We define the concept of steady-state to relax the problem. We say a road is in steady-state if all the AVs within are driving with constant velocity v for which $V_{min} \leq v \leq V_{Max}$ and each of the inter-vehicle distance is greater than $\inf d^R_{LS}(v, \cdot)$. Here $d^R_{LS}(v, \cdot) \equiv d^R_{LS}(v, v)$. In the steady-state, we have $\forall\, i, j \in \{0, 1, \ldots, n-1\}$

$$\inf d^R_{LS}[\omega_i] = \inf d^R_{LS}[\omega_j]. \quad (6)$$

(6) states the minimum safe inter-vehicle distance between different AVs, when R is steady, are independent. For R that is initially unsteady, one could have it steady by letting every AV in R to keep a constant safe distance with the AV in its front except the AV at the very front of R that has no front AV to follow and keep distance with. This procedure would terminate within finite time if the conditions in the following lemma hold. The proofs of the lemmas and theorem follows are left in the Appendix.

Lemma 1: If all the AVs in R with $|R| < \infty$ are safe initially with $u > 0$ and $a_{max} > 0$, then the procedure described above terminates within finite time.

Here $|R|$ stands for the number of AVs in R. Lemma 1 states that there exists a finite time way to make R steady. The following lemma holds true when R is steady.

Lemma 2: When a R is steady and $V_{min} \leq v \leq V_{Max}$, the following relations hold

$$\frac{V_{Max}}{\inf d^R_{LS}(V_{Max}, \cdot)} \geq \frac{v}{\inf d^R_{LS}(v, \cdot)}. \quad (7)$$

$$\sum_{\omega_i \in R} \inf d^R_{LS}[\omega_i] \geq \inf\Big(\sum_{\omega_i \in R} d^R_{LS}\Big) = |R| \inf d^R_{LS}(V_{min}, \cdot). \quad (8)$$

Base on lemma 2, (4), and (5), we have the following theorem

Theorem 1: The SDC_R of R with $\dot{M}$-meter length and $\dot{N}$-lanes in the steady-state is

$$SDC_R(\dot{M}, \dot{N}) = \dot{N}\lfloor \frac{(\dot{M})}{\inf d^R_{LS}(V_{min}, \cdot)} \rfloor = \dot{N}\lfloor \frac{(\dot{M})}{\tau(V_{min} + \frac{u\tau}{2}) + L + \frac{(V_{min}+u\tau)^2 - V_{min}^2}{2a_{max}}} \rfloor. \quad (9)$$

And the SDT_R of R with $\dot{N}$ lanes in the steady-state is

$$SDT_R(\boldsymbol{T}, \dot{N}) = \dot{N}\lfloor \frac{V_{Max}\boldsymbol{T}}{\inf d^R_{LS}(V_{Max}, \cdot)} \rfloor = \dot{N}\lfloor \frac{V_{Max}}{V_{Max}\tau + L} \rfloor. \quad (10)$$

The results are shown visually in Fig. 4, Fig 5, Fig. 6, and Fig. 7. The figures show how safe driving efficiencies are affected by changing some of the parameters while fixing others. The parameters include the maximum acceleration, the response time, and the upper and lower-speed limits set by the road.

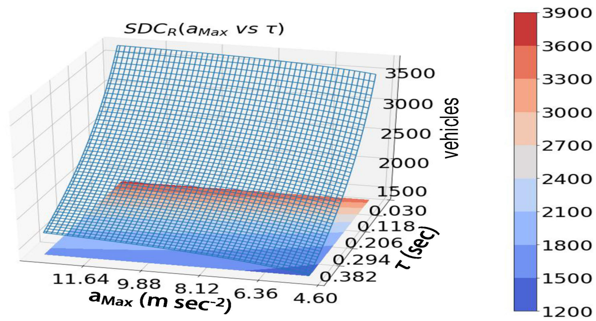

FIGURE 4. The SDC_R under different τ and a_{Max}.

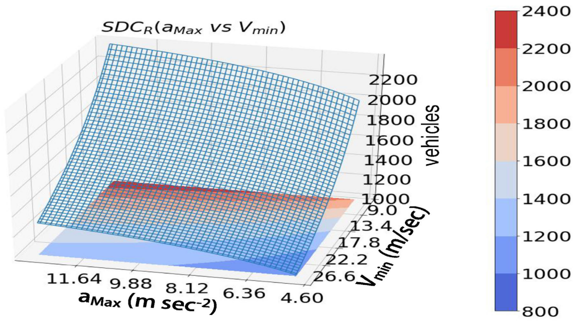

FIGURE 5. The SDC_R under different a_{Max} and V_{min}.

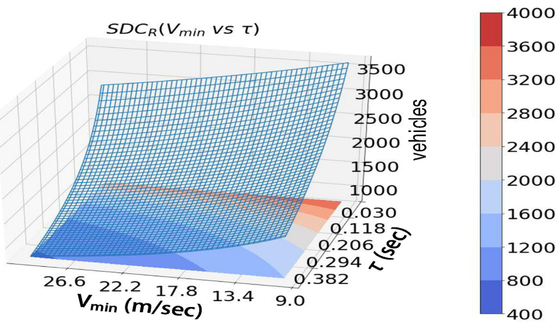

FIGURE 6. The SDC_R under different τ and V_{min}.

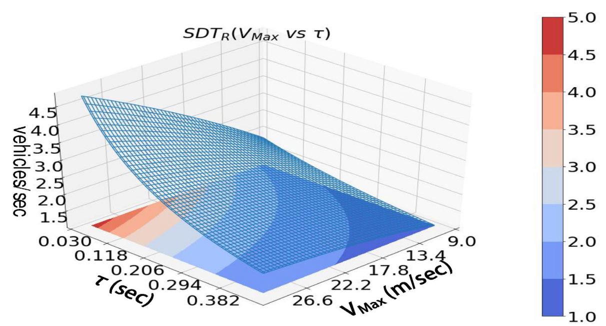

FIGURE 7. The SDT_R under different τ and V_{Max}.

B. INTERSECTION

Fig. 8 shows an illustration of the type of intersection we considered. We say an AV is safe in the intersection if its intersection distance is sufficient for it to avoid the collision with AVs on the contrary lane with higher priorities by applying maximum braking power when they make emergent moves. To achieve the optimum safe efficiencies in the intersection, we presume that there are always AVs accessing the intersection. Since the two intersected roads $l1$ and $l2$ are of equal importance in our setting, If the ith AV from $l1$ to access the intersection be labeled as ω_{2i-1} and the ith AV to access the intersection in $l2$ be ω_{2i}. For fairness, ω_i would prioritized ω_j on the counter lane if $j < i$. Under this setting, finding the SDC_I of two $\dot{M}$-meter roads $l1$ and $l2$ could be written as

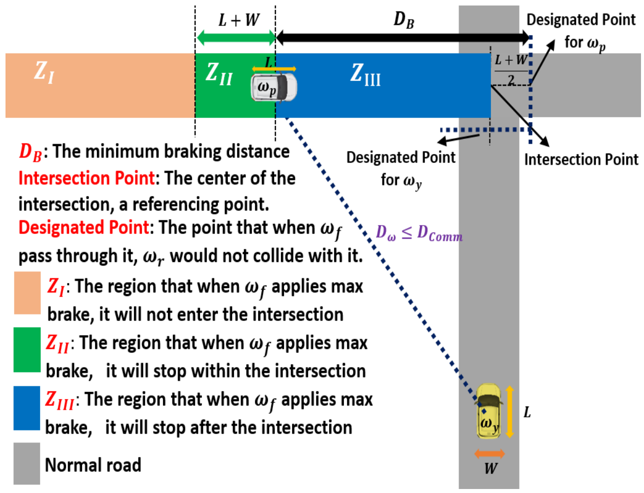

FIGURE 8. Safe driving analysis at an intersection.

$$SDC_I(\dot{M}) = \sup_{n} \inf_{\lambda\mathcal{I}\lambda\geq 0} 2n + \lambda\Big(1 - \frac{\sum_i^{n-1} d_{LS}^I[\omega_i]}{\dot{M}}\Big). \quad (11)$$

The SDT_I measured within time $\boldsymbol{T}$ could be expressed as

$$SDT_I(\boldsymbol{T}) = \sup_{n} \inf_{\lambda\mathcal{I}\lambda\geq 0} n + \lambda\Big(1 - \frac{\sum_i^{n} \inf d_{LS}^I[\omega_i]}{\int_0^{\boldsymbol{T}} v_n(t)dt}\Big). \quad (12)$$

Here $d_{LS}^I[\omega_i]$ stands for the inter-vehicle distance between the ith AV and $(i-2)$th AV on the same lane. As shown in (11) and (12), even though we have greatly simplified the problem by considering only single lane on each intersected road and also have we reduced the action space of AVs by only allowing them to accelerate or decelerate, (11) and (12) still fall in the category of multi-agent problem, which is intractable in general. To solve this issue, similar to how we relax the straight road efficiency problem, we define the steadiness of an intersection as follows: An intersection is steady if and only if both $l1$ and $l2$ are steady. By lemma 1, we know that there exists ways to make $l1$ and $l2$ steady if they are treated independently. However, they are not always independent now since they are intersected. Suppose AVs in $l1$ keep identical safe inter-vehicle distance d_H with steady-state velocities v_H and AVs in $l2$ keep d_V in between with velocities v_V, the following lemma states that only if d_H, d_V, v_H and v_V satisfy certain conditions could the intersection be steady.

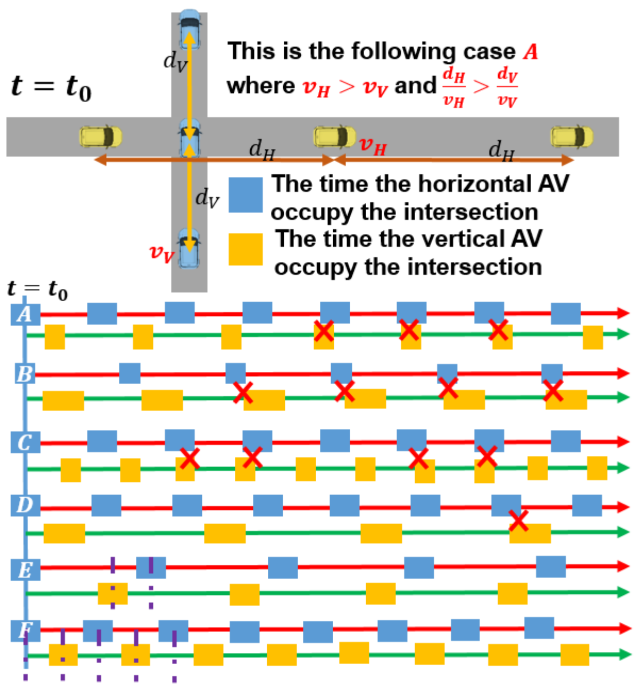

FIGURE 9. The upper half figure illustrates the passing scenario during the intersection, and the lower half, ranging from (A) to (F), shows the scenarios with different velocities and inter-vehicle distances. (A) $v_H > v_V$ and $\frac{d_H}{v_H} > \frac{d_V}{v_V}$ (B) $v_H < v_V$ and $\frac{d_H}{v_H} > \frac{d_V}{v_V}$ (C) $v_H > v_V$ and $\frac{d_H}{v_H} < \frac{d_V}{v_V}$ (D) $v_H < v_V$ and $\frac{d_H}{v_H} < \frac{d_V}{v_V}$ (E) $v_H = v_V$, $\frac{d_H}{v_H} = \frac{d_V}{v_V}$ but not symmetric (F) $v_H = v_V$, $\frac{d_H}{v_H} > \frac{d_V}{v_V}$ and symmetric. (F) is further illustrated in Fig. 10.

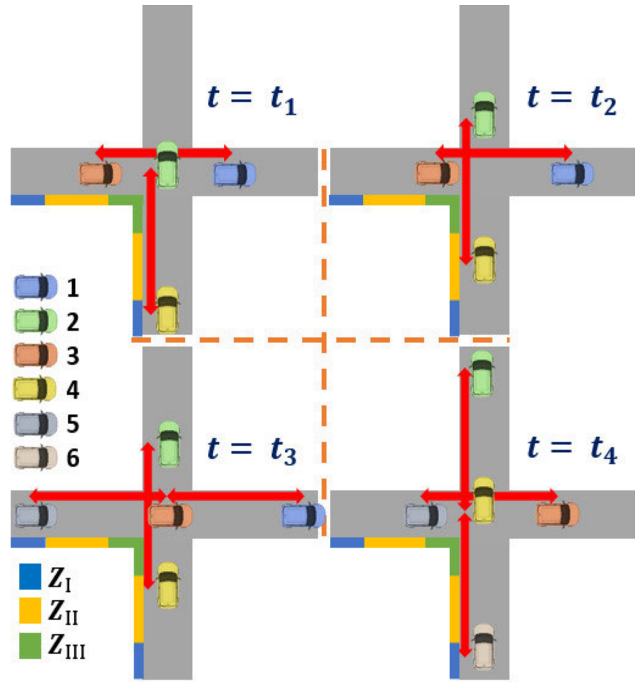

FIGURE 10. The detail illustration of case F in Fig. 9.

Lemma 3: If $d_H, d_V < \infty$ are currently the least safe inter-vehicle distance of horizontal and vertical intersected roads of an intersection respectively, then the intersection could be steady if and only if there exist $\tilde{m}, \tilde{n} \in \mathbb{N}$ such that

$$\frac{d_V}{v_V} = \tilde{m}\frac{d_H}{v_H} \tag{13}$$

or

$$\tilde{n}\frac{d_V}{v_V} = \frac{d_H}{v_H}. \tag{14}$$

Case A to case D in Fig. 9 illustrate all possible relations between v_H, v_V, d_H and d_V. The figure also shows the intuition on its proof, which is left in the Appendix. For fairness reason and to simplify the original problem, $v_H = v_V$ is required since none of the roads should work at higher velocity. The following lemma states that under such a setting, the efficiency is optimized when the passing pattern is symmetric, i.e., every AV splits equally the inter-vehicle distance of another road when it's at the intersection point, as shown in Fig. 10 and case E and case F in Fig. 9.

Lemma 4: Let $d_H < \infty$ and $d_V < \infty$ be the current least safe inter-vehicle distance of the horizontal road and the vertical road of an intersection, respectively. Then if $v_V = v_H$ and the intersection is steady, it achieves optimum efficiency at velocity v if and only if $d_V = d_H = \inf d_{LS}^I(v)$. Furthermore, any AV, when at the intersection point, splits the inter-vehicle distance of another road equally.

Lemma 3 guarantees that when the conditions in lemma 4 are met, the intersection could be steady since $\tilde{m}$ and $\tilde{n}$ in this case are both 1. Thus under the condition that $v_H = v_V$ and all the inter-vehicle distances on the same road are identical, according to lemma 4, to obtain the optimum safe efficiency of an intersection, it suffices to decide how close the safe distances between AVs on the same road could be. Such distance should be sufficient for the yielding AV to claim its safeness by applying BER whenever the pure prioritized AV has any emergency moves. In the following, we analyze the least safe inter-vehicle distance $\inf d_{LS}^I(v, \cdot)$ when AVs in the intersection are driving at velocity $V_{min} \leq v \leq V_{Max}$.

1) SAFE LONGITUDINAL DISTANCE (d_{LS}^I)

By the symmetric setting, let the pure prioritized vehicle ω_p have the intersection distance d_c, and the yielding vehicle with the least intersection distance has the intersection distance $d_c + \frac{d_p}{2}$. For d_c belongs to each of the zones (Z_I, Z_{II}, and Z_{III}) shown in Fig. 8, we find the least $\frac{d_p}{2}$ so that d_p could serve as d_{LS}^I for the yielding vehicle. In Z_I, if the prioritized vehicle ω_p applies maximum braking power a_{max}, it stops before the intersection, and so we do not need to set any constraint on the action space of the yielding vehicle ω_y. Thus the d_c corresponds to Z_I has the range

$$d_c \geq \frac{v^2}{2a_{max}} + \frac{1}{2}(W + L). \tag{15}$$

For the Z_{II} case where the prioritized vehicle ω_p stops within the intersection so that the yielding vehicle ω_y should stop before the intersection. Thus d_c should satisfy the following inequality

$$d_c + \frac{1}{2}(W+L) \geq \frac{v^2}{2a_{max}} \geq d_c - \frac{1}{2}(W+L). \quad (16)$$

Therefore, the range of d_c is

$$\frac{v^2}{2a_{max}} - \frac{1}{2}(W+L) \leq d_c \leq \frac{v^2}{2a_{max}} + \frac{1}{2}(W+L). \quad (17)$$

The inter-vehicle distance for ω_y should be sufficient so that after the response time, it could stop before the intersection. For simplicity, we replace $W+L$ with e.

$$\frac{v^2}{2a_{max}} + v\tau \leq \frac{1}{2}d_p + d_c - \frac{e}{2}. \quad (18)$$

$$d_p \geq \frac{v^2}{a_{max}} + e - 2d_c + v\tau. \quad (19)$$

Plugging-in the d_c range for Z_{II}, we have

$$d_p \geq 2(v\tau + e). \quad (20)$$

For ω_p in Z_{III}, which corresponds to that even it brakes with a_{max}, it will still slide through the intersection and so that as long as ω_y arrives at the intersection later than ω_p, there will be no collision. There are two sub-phases here: The first one is that d_c is still sufficient so that after τ, ω_y is still before the intersection. Let the time ω_y arrives at the point $(W+L)/2$ before intersection be t_y and the time ω_p arrives at the point $(W+L)/2$ after the intersection be t_p, we have

$$t_y = \frac{1}{a_{max}}\{v - [v^2 - 2a_{max}(d_m - v\tau - \frac{e}{2})]^{\frac{1}{2}}\} + \tau.$$

$$t_p = \frac{1}{a_{max}}(v - [v^2 - 2a_{max}(d_c + \frac{e}{2})]^{\frac{1}{2}}).$$

Solving the valid range of d_p, it suffices to have $t_p \leq t_y$,

Let k be $d_m - v\tau - \frac{e}{2}$ and $\beta = d_c + \frac{e}{2}$, we have

$$\sqrt{v^2 - 2a_{max}k} \leq \sqrt{v^2 - 2a_{max}\beta} + a_{max}\tau. \quad (21)$$

Squaring both sides and re-arranging the terms, we have

$$(\beta - k) - \frac{a_{max}}{2}\tau^2 \leq \tau\sqrt{v^2 - 2a_{max}\beta}. \quad (22)$$

Squaring both sides again, we get

$$(\beta - k)^2 - a_{max}\tau^2(\beta - k) + \frac{1}{4}a_{max}^2\tau^4 \leq \tau^2(v^2 - 2a_{max}\beta). \quad (23)$$

By $\beta - k = -\frac{1}{2}d_p + e + v\tau$ and $d_c - d_m = -\frac{1}{2}d_p$, we have

$$(-\frac{1}{2}d_p + e + v\tau)^2 - a_{max}\tau^2(-\frac{1}{2}d_p + e + v\tau) + \frac{1}{4}a_{max}^2\tau^4 \leq \tau^2(v^2 - 2a_{max}(d_c + \frac{e}{2})). \quad (24)$$

Expanding and re-arranging the terms, we have

$$\frac{1}{4}d_p^2 + (\frac{1}{2}a_{max}\tau^2 - e - v\tau)d_p + (2a_{max}\tau^2 d_c + e^2 + 2ev\tau + \frac{1}{4}a_{max}^2\tau^4 - a_{max}\tau^3 v) \leq 0. \quad (25)$$

Solving the quadratic inequality of d_p, we get

$$d_p \geq 2e + 2v\tau - a_{max}\tau^2 - 2\tau\sqrt{(v^2 - a_{max}(e + 2d_c))}. \quad (26)$$

Since $d_c \leq \frac{v^2}{2a_{max}} - \frac{e}{2}$ and $\tau > 0$, we have

$$d_p \geq 2e + 2v\tau - a_{max}\tau^2. \quad (27)$$

This is the second sub-phase for Z_{III}. In this sub-phase, ω_y will access the intersection during its response time. Similar to the previous analysis, for ω_y, we have

$$t_y = \frac{1}{v}(d_c + \frac{1}{2}d_p - \frac{e}{2})$$

and

$$t_p = \frac{1}{a_{max}}(v - [v^2 - 2a_{max}(d_c + \frac{e}{2})]^{\frac{1}{2}}).$$

By the requirements of this sub-phase, we have

$$\tau \geq t_y \geq t_p. \quad (28)$$

For the first inequality, we have

$$d_p \leq 2v\tau - 2d_c + e. \quad (29)$$

Comparing the result with (20), since

$$2v\tau - 2\,d_c + e < 2(v\tau + e),$$

we know that the d_p result in this sub-phase would be dominated by the result in Z_{II} as the case in $Z_{III} - 1$.

2) WORST CASE ANALYSIS AND GENERALIZED FAIRNESS

Safe driving requires the responding vehicle to take actions according to the worst possible actions the AV in its front or the AV on the contrary-lane of the intersection with prioritized status may take. In both Z_{II} and Z_{III}, it makes sense to say that ω_p with braking power a_{max} is the worse than with any other $a < a_{Max}$. Suppose it brakes with an $a < a_{max}$, we have $t_p(a) < t_p(a_{max})$, and thus $t_y > t_p(a_{max}) > t_p(a)$. It means that the d_p derived in (20) and (27) are sufficient to guarantee the safeness of ω_y. In Z_I, the worst action to be presumed is not $a = a_{max}$ but $0 < a < a_{max}$ because braking with less a would force ω_y to respond to the action instead of doing nothing as discussed in *Case* Z_I. However, the derived d_p in Z_{II} is still sufficient for the safeness of ω_y since $d_c(Z_I) \geq d_c(Z_{II})$. We plug-in this inequality into (18), we can see that the inequality for d_p still holds.

In our intersection settings, we let an AV from one lane to access the intersection after the AV from the contrary-lane has accessed it, and so on so forth. In this 1-1 manner, we say the intersection is fair since the priority level is only determined by the intersection distance when two AVs are exposed. When both lanes are full of vehicles, AVs from both lanes will take turns accessing the intersection. Now we generalize the fairness concept as follows. Let AVs crossing the intersection in a similar inter-laced manner, but now, M AVs after M AVs. Here $M \geq 1$ is a positive integer. In the long run, the intersection is still fair in terms of the AVs passing through it from both lanes. Now we analyze the efficiency of the generalization. From (35), we are able to treat the M AVs on the same lane as a larger vehicle with identical width W but longer length $L' = (M-1)(v\tau + L) + L$. Let's call the M-consecutive AVs a M-group. By the intersection results obtained, in order to achieve SDT, the inter-group distance should at least be $2(v\tau + L' + W)$. The average time required for a M-group to pass through the intersection is

$$\frac{(\frac{3}{2}M - \frac{1}{2})V_{Max}\tau + (\frac{3}{2}M - \frac{1}{2})L + W}{MV_{Max}} \geq \frac{V_{Max}\tau + L + W}{V_{Max}}. \tag{30}$$

We can analyze the SDC similarly. Since the average inter-vehicle (inter-group) distance increases for $M > 1$. That is, for integers M_1 and M_2 with the relation $1 \leq M_1 < M_2$, the safe efficiencies of M_1-group always outperforms the M_2-group provided the measuring time $\boldsymbol{T}$ of SDT and the measuring distance $\hat{M}$ of SDC be sufficient.

For insufficient measuring time or distance, there may be cases where the efficiency of M_2-group is better than the M_1-group's. In this case the intersection is degenerated into a road and not the scenario that we would liked to focus on.

Finally, by combining (20), (27), and $\mathbf{inf}\ d^R_{LS}$. One could discover that $2(v\tau + e) > 2v\tau + 2e - a_{max}\tau^2$. The minimum inter-vehicle distance to keep in a steady intersection is

$$\mathbf{inf}\ d^I_{LS}(v,\cdot) = \mathbf{max}\{\mathbf{inf}\ d^R_{LS}(v,\cdot), 2(v\tau + e)\} \tag{31}$$

(31) means that for any ω_i with current intersection distance d_c, the AV on the contrary-lane, ω_j, with intersection distance greater than $d_c + \frac{1}{2}\min d^I_{LS}$, could be in safe state before ω_i crosses the intersection. And similar argument could be applied to ω_k, the AV in ω_j's front which has intersection distance $d_c - \frac{1}{2}\inf d^I_{LS}(v,\cdot) \geq 0$. To ω_i, since ω_k is its prioritized vehicle. The difference in their intersection distance, $\frac{1}{2}\inf d^I_{LS}(v,\cdot)$, is sufficient, and thus ω_i is safe. Now by (7), (8) and (31), we have the following lemma.

Lemma 5: When an intersection is steady, we have

$$\frac{V_{Max}}{\mathbf{inf}\ d^I_{LS}(V_{Max},\cdot)} \geq \Big(\frac{v}{\mathbf{inf}\ d^I_{LS}(v,\cdot)}\Big) \tag{32}$$

and

$$\sum_{\omega_i \in I} \mathbf{inf}\ d^I_{LS}[\omega_i] \geq \mathbf{inf}\Big(\sum_{\omega_i \in I} d^I_{LS}\Big) = |I|\ \mathbf{inf}\ d^I_{LS}(V_{min},\cdot). \tag{33}$$

By (11), (12), and lemma 5, we have the following theorem.

TABLE 1. **Straight road and intersection notations.**

Variable (unit)	Description
$L\ (m)$	Length of an AV.
$W\ (m)$	Width of an AV.
$d^R_{LS}\ (m)$	Straight road safe inter-vehicle distance
$d^I_{LS}\ (m)$	Intersection safe inter-vehicle distance
$\tau\ (sec)$	Response time of AV
$v_r\ (m/s)$	Speed of rear vehicle
$v_f\ (m/s)$	Speed of front vehicle
$V_{Max}, V_{min}\ (m/s)$	The maximum and minimum speed limit of a traffic configuration
$a_{max}\ (m/s^2)$	Deceleration of full braking power
$u\ (m/s^2)$	Max acceleration of a vehicle
$d_H\ (m)$	The inter-vehicle distance at the horizontal lane of an intersection
$d_V\ (m)$	The inter-vehicle distance at the vertical lane of an intersection
$v_H\ (m/s)$	The AV velocity at the horizontal lane of an intersection
$v_V\ (m/s)$	The AV velocity at the vertical lane of an intersection
$d_c\ (m)$	The dummy variable representing the current intersection distance of the prioritized AV
$d_p\ (m)$	The presumed inter-vehicle distance at an intersection
$t_p\ (sec)$	The time the prioritized AV required to access the intersection
$t_y\ (sec)$	The time the yielding AV required to access the intersection
$e\ (m)$	The joint vehicle size, equivalent to $L + W$
$SDC_R(m,n)$	Safe driving capacity of m-meter-n-lane straight road
$SDC_I(m)$	Safe driving capacity of intersection consists of two M-meter straight roads
$SDT_R(t,n)$	Safe driving throughput of n-lane straight road within time t
$SDT_I(t)$	Safe driving throughput of intersection within time t

Theorem 2: The SDC_I of an steady intersection intersected by 2 $\dot{M}$-meter single-lane roads is

$$SDC_I(\dot{M}) = 2\Big\lfloor\frac{(\dot{M})}{\mathbf{max}\{\mathbf{inf}\ d^R_{LS}(v,\cdot), 2(v\tau + e)\}}\Big\rfloor \tag{34}$$

and similarly, the SDT_I of the intersection is

$$SDT_I(\boldsymbol{T}) = 2\Big\lfloor\frac{(V_{Max}\boldsymbol{T})}{2(V_{Max}\tau + e)}\Big\rfloor. \tag{35}$$

The SDC_I and SDT_I under different V_{min}, V_{Max}, and τ, is given in Fig. 11 and Fig. 12. As the figures indicate, the lower response time τ results in higher safe efficiencies. Also, at $v = V_{min}$, the SDC_I is achieved for all τ. At $v = V_{Max}$, the SDT_I is achieved for all τ.

C. CITY $(m_c(L_{cV}), n_c(L_{cH}))$

We extend the measure of safe efficiencies to Manhattan-like city, as illustrated in Fig. 13. Since a city is a combination

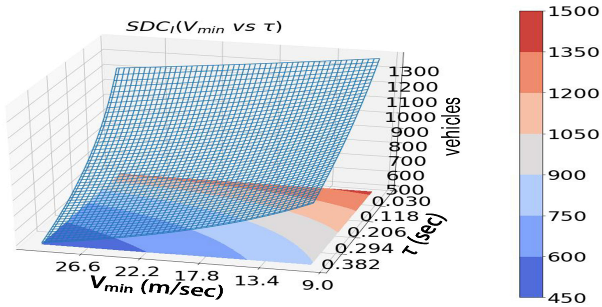

FIGURE 11. The SDC_I under different V_{min} and τ.

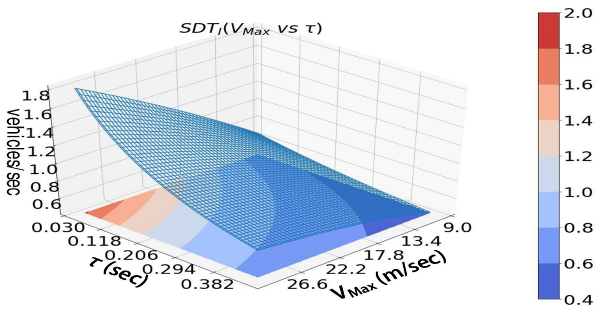

FIGURE 12. The SDT_I under different τ and V_{Max}.

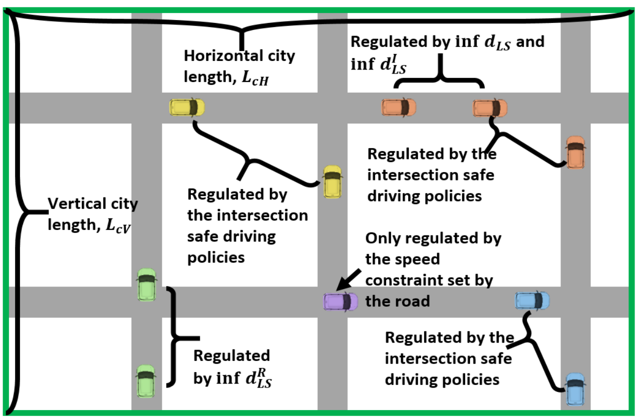

FIGURE 13. A Manhattan-like city with 2 horizontal roads ($n_c = 2$) intersected by 3 vertical roads ($m_c = 3$).

of intersections and straight roads, as in the previous cases, we first define the steadiness of a city. We say a city is steady if all the roads and intersections within the city are steady. By maximum flow min cut theory, if all of them are steady with maximum safe driving efficiencies, the city achieves its optimum efficiency. However, the inter-intersection distance may bring dependencies to the intersections. We use the following lemma to state the condition required to make all the intersections steady as if there are no dependencies.

Lemma 6 (Achievability): If all the AVs in a city are driving with velocity v and their inter-vehicle distances are identically $d_C \geq \textbf{inf}\ d_{LS}^I(v, \cdot)$, then it could always be steady.

If the conditions specified in lemma 6 is met, we could have the following theorem for SDC_C and SDT_C.

TABLE 2. Manhattan-city notations.

Variable	Description
$SDC_C(m_c(L_{cV}), n_c(L_{cH})$	SDC of a Manhattan-like city with m_c L_{cV}-meter roads intersected by n_c L_{cH}-meter roads
$SDT_C(m_c, n_c, t)$	SDT of a Manhattan city with m_c lanes intersected by n_c roads within time t

Theorem 3: If all the inter-intersection distance of a city is greater than the constant spacing inter-vehicle distance $d_c \geq d_{LS}^I$, then its SDC_C is

$$SDC_C(m_c(L_{cV}), n_c(L_{cH})) = m_c\lfloor\frac{(L_{cV})}{\textbf{inf}\ d_{LS}^I(V_{min}, \cdot)}\rfloor + n_c\lfloor\frac{(L_{cH})}{\textbf{inf}\ d_{LS}^I(V_{min}, \cdot)}\rfloor, \quad (36)$$

and its SDT_C

$$SDT_C(m_c, n_c, \boldsymbol{T}) = (m_c + n_c)\lfloor\frac{(V_{Max}\boldsymbol{T})}{2(V_{Max}\tau + W + L)}\rfloor. \quad (37)$$

Lemma 6 and Theorem 3 state that if all the AVs keep the constant spacing between one another and this spacing distance is greater than the minimum safe inter-vehicle distance required for the intersection to work in steady-state, also, the inter-intersection distance is sufficient. We could always make the city steady, and the *SDC* and *SDT* of the city could be acquired from the modification of the result of a single intersection. The resulted SDC_C represents the maximum number of AVs driving within the valid velocity range that could be safe in terms of responsibility. In other words, when any of the AV brakes with maximum braking power, there would be no collisions. The resulted SDT_C gives the number of AVs that could pass through the city with the same safety guarantee.

IV. DISCUSSIONS

We organize the *SDC* and *SDT* results in Table 3. These results imply that the upper and lower-speed limits set by the roads (V_{min} and V_{Max}) have a direct impact on the safe driving efficiency bounds. Furthermore, the results obtained show a clear relationship on how the parameters of an AV affect the overall safe efficiencies. These equations not only answer the safe efficiencies of the traffic configurations but also reveal the AV requirements when a city is asked to reach certain safe traffic efficiencies. The following corollaries are the immediate consequence of Theorem 3, and thus their proofs are omitted.

Corollary 1: Given a city of size $(m_c(L_{cV}), n_c(L_{cH}))$ and all the AVs driving within to have maximum braking $a_{max} > 0$ and maximum response time τ. If the city is required to have throughput no less than $\tilde{\rho_{\mathbb{C}}}$, the upper-speed limit V_{Max} set by the city should be no less than $\frac{2\tilde{\rho_{\mathbb{C}}}(L+W)}{(m_c+n_c)\boldsymbol{T}-2\tilde{\rho_{\mathbb{C}}}\tau}$; On the other hand, if the city is required to have a capacity of no less than $\tilde{\mathbb{C}}$, the lower-speed limit set

TABLE 3. Summary of all *SDC* and *SDT* results.

Metrics	Analytic Form
$SDC_R(M,N)$	$N\lfloor \frac{M}{\tau(V_{min}+\frac{u\tau}{2})+L+\frac{(V_{min}+u\tau)^2-V_{min}^2}{2a_{max}}} \rfloor$
$SDT_R(\boldsymbol{T},N)$	$N\lfloor \frac{V_{Max}}{V_{Max}\tau+L} \rfloor$
$SDC_I(M)$	$2\lfloor \frac{M}{\mathbf{max}\{\mathbf{inf}\ d_{LS}^R(V_{min},\cdot),2(V_{min}\tau+W+L)\}} \rfloor$
$SDT_I(\boldsymbol{T})$	$2\lfloor \frac{V_{Max}\boldsymbol{T}}{2(V_{Max}\tau+W+L)} \rfloor$
$SDC_C(m_c(L_{cV}),n_c(L_{cH}))$	$m_c\lfloor \frac{L_{cV}}{\mathbf{inf}\ d_{LS}^I(V_{min},\cdot)} \rfloor + n_c\lfloor \frac{L_{cH}}{\mathbf{inf}\ d_{LS}^I(V_{min},\cdot)} \rfloor$
$SDT_C(m_c,n_c,\boldsymbol{T})$	$(m_c+n_c)\lfloor \frac{V_{Max}\boldsymbol{T}}{2(V_{Max}\tau+W+L)} \rfloor$

by the city V_{min} should be no greater than $\frac{1}{\tau}(\frac{m_cL_{cV}+n_cL_{cH}}{\tilde{\mathbb{C}}} - L - W)$.

Corollary 2: Given a city of size $(m_c(L_{cV}), n_c(L_{cH}))$ with speed limits $[V_{min}, V_{Max}]$ and all the AVs within to have maximum braking $a_{max} > 0$. If the city is required to have throughput no less than $\tilde{\rho_{\mathbb{C}}}$, the response time τ of the AVs should not exceed $\frac{(m_c+n_c)\boldsymbol{T}}{2\tilde{\rho_{\mathbb{C}}}} - \frac{(W+L)}{V_{Max}}$; If the city is required to have capacity no less than $\tilde{\mathbb{C}}$, then the response time τ should be no greater than $\frac{1}{v}(\frac{m_cL_{cV}+n_cL_{cH}}{2\tilde{\mathbb{C}}} - L - W)$.

Both Corollary 1 and Corollary 2 are based on the implicit assumption that

$$\mathbf{inf}\ d_{LS}^R(V_{min}, \cdot) < 2(V_{min}\tau + W + L).$$

And thus we have

$$\mathbf{inf}\ d_{LS}^I(V_{min}, \cdot) = 2(V_{min}\tau + L + W).$$

Corollary 1 shows that if a certain safe traffic efficiency is to be achieved, then the speed limits set by the traffic configurations would have some requirements. Corollary 2 states similarly but in a reverse way: When the speed limits and vehicle capabilities except the maximum response time τ are fixed, if the city is required to achieve a certain level of efficiency in terms of either capacity or throughput, then there exists a minimum requirement on the response time of AVs so that only when all the AVs could respond to emergencies in a sufficiently short time that is less than a threshold could the required efficiencies be achieved.

V. CONCLUSIONS

In this work, the concepts of SDC and SDT are proposed. They measure the capacity and throughput of AVs base on the safe driving policies that guarantee all the abiding AVs to be free of liability even when an accident happens. We circumvent the multi-agent difficulty by defining steady-state of the traffic configurations so that closed-form results of SDC and SDT under several traffic configurations could be obtained. They are organized in Table 3. We also show the implication of the derived results in the discussion section that these formulae not only reveal the optimum safe efficiencies but also give the least requirements on either velocity constraints of the traffic configurations or the least response time required for the AVs. To conclude, this work shows that, on the one hand, to maximize safe transportation efficiency, the AVs should drive as fast as possible and drive as close to other AVs as possible. On the other hand, the AVs need to keep the minimum distance to other AVs for driving safety base on the responsibility settings.

APPENDIX

A. PROOF OF LEMMA 1

Given a road R with $|R|$ AVs $\omega_1, \omega_2, \ldots, \omega_{|R|}$ within with velocities $V_{min} \leq v_1, v_2, \ldots, v_{|R|} \leq V_{Max}$. Suppose the very first AV that has no front vehicle is $\omega_{|R|}$ with velocity $v_{|R|} \neq V_{Max}$, and the road R has length L_R. We sequentially make ω_i for i from 1 to $|R|-1$ to do the following: If $v_i \leq v_{i+1}$, ω_i accelerates until $v_i = v_{i+1}$. On the other hand, if $v_i > v_{i+1}$, we make ω_i decelerate with a where $0 < a \leq a_{max}$ so that when $d(\omega_i, \omega_{i+1})$ becomes $\mathbf{inf}\ d_{LS}^R(v_f, \cdot)$, it has the velocity v_{i+1}. When this procedure terminates, all the AVs would have velocities $v_{|R|}$. The time required to terminate, T_{steady}, could be written as

$$T_{steady} \leq \sum_{i}^{|R|-1} t_{steady}[i] \tag{38}$$

$t_{steady}[i]$ is the time required for the ith AV to finish the procedure and the inequality comes from the fact that the time for a procedure might overlap with another. Now it suffices to bound each $t_{steady}[i]$ in the summation term. Since for all ω_i and ω_{i+1}, their speed difference would be no more than $V_{Max} - V_{min}$, each $t_{steady}[i]$ would be bounded by $\frac{2(V_{Max}-V_{min})}{\mathbf{max}\{a,u\}}$. So the LHS of (38) could be bounded as

$$T_{steady} \leq \sum_{i}^{|R|-1} t_{steady}[i] \leq \frac{2(V_{Max} - V_{min})|R|}{\mathbf{max}\{a, u\}} < \infty \tag{39}$$

Thus we complete the proof. □

B. PROOF OF LEMMA 2 AND THEOREM 1

For (7), to check whether at $v = V_{Max}$ would be the supremum, we differentiate $\frac{\inf d_{LS}^R(v,\cdot)}{v}$ with respect to v

$$\frac{\partial}{\partial v}\frac{d_{LS}^R(v,\cdot)}{v} = \frac{-1}{v^2}\left\{\frac{\tau^2u^2}{a_{max}} + \frac{\tau^2 u}{2} + L\right\} \leq 0 \tag{40}$$

and thus for $V_{min} \leq v_i \leq v_j \leq V_{Max} - u\tau$, we have

$$\frac{\inf d_{LS}^R(v_i,\cdot)}{v_i} \geq \frac{\inf d_{LS}^R(v_j,\cdot)}{v_j} \tag{41}$$

So at $v = V_{Max} - u\tau$, $\frac{\inf d_{LS}^R(v,\cdot)}{v}$ is minimized and its reciprocal, $\frac{v}{d_{LS}^R(v,\cdot)}$, is maximized. For velocity v in $[V_{Max} - u, V_{Max}]$, since we have

$$\frac{\inf d_{LS}^R(v,\cdot)}{v} \geq \frac{\inf d_{LS}^R(v, V_{Max})}{v} \geq \frac{\inf d_{LS}^R(V_{Max},\cdot)}{V_{Max}} \tag{42}$$

It's clear that $\frac{V_{Max}}{\inf d_{LS}^R(V_{Max},\cdot)}$ is the supremum for all v within $[V_{min}, V_{Max}]$. Since SDT_R is proportional to this reciprocal, we know that SDT_R is maximized when $v = V_{Max}$ and thus we complete the proof of (7) and (10). For (8), the first inequality is simply by the fact that local optimum would not be better than the global optimum. For the equality part, since all the AVs have identical velocity for R is steady, they have identical least safe inter-vehicle distance. This distance is a function of their steady-state velocity v. To check whether at $v = V_{min}$ would be the infimum, we differentiate $d_{LS}^R(v,\cdot)$ with respect to v

$$\frac{\partial}{\partial v}\{\inf d_{LS}^R(v,\cdot)\} = \frac{u\tau}{a_{max}} + \tau > 0. \tag{43}$$

And thus it's true that at $v = V_{min}$, the infimum is achieved for (8). Base on (43), for velocity $V_{min} \leq v_i \leq v_j \leq V_{Max}$ and any positive constant c, we have

$$\frac{c}{\inf d_{LS}^R(v_j,\cdot)} \leq \frac{c}{\inf d_{LS}^R(v_i,\cdot)} \tag{44}$$

Thus (9) could be proved by substituting c with $\hat{M}$, the length of measure of SDC_R. Now we complete all the proofs. □

C. PROOF OF LEMMA 3

Suppose $\frac{d_V}{v_V} < \frac{d_H}{v_H}$ but $\frac{v_V d_H}{d_V v_H}$ is not an integer, also without loss of generality, we assume at $t = t_0 = 0$, there is an AV belongs to the horizontal road at the intersection point. Since d_V and d_H are the least safe inter-vehicle distance on each lane, the inter-vehicle distance between the first AV waiting to pass through the intersection on the vertical lane and the AV from the vertical lane that had just passed through the intersection (at $t < 0$) cannot be smaller. Similarly, the second AV in the horizontal lane that has current intersection distance d_H (since the first AV in this lane is currently at the intersection point) cannot be further closer to the intersection. Let the current intersection distance of the first AV to pass the intersection in the vertical lane be γd_H where $0 < \gamma < 1$. At time $t = \frac{d_H}{v_H}$ when the second AV from the horizontal lane is at the intersection point, the second AV from the vertical lane would have the intersection distance $d_H^{[2]}(t = \frac{d_H}{v_H})$ where

$$d_H^{[2]}(t = \frac{d_H}{v_H}) = (1+\gamma)d_V - \frac{d_H}{v_H}d_H < \gamma d_V \tag{45}$$

If $d_H^{[2]}(t = \frac{d_H}{v_H}) \geq 0$, then it indicates such distance is not sufficient to claim the safeness. Now shall we abuse the notation by allowing the intersection distance of an AV to be negative if it has passed through the intersection. Then if $(\gamma - 1)d_V < d_H^{[2]}(t = \frac{d_H}{v_H}) < 0$, it must mean that the second AV on the vertical lane who is at the intersection point at $t = \frac{d_H}{v_H}$ might have some time that is unsafe for $t < \frac{d_H}{v_H}$. For those $d_H^{[2]}(t = \frac{d_H}{v_H}) \neq (\gamma - \kappa)d_V, \kappa \in \mathbb{N}^+$, we could draw similar conclusions for either the horizontal AV or the vertical AV would be unsafe. For the case $\frac{d_V}{v_V} < \frac{d_H}{v_H}$ which corresponds to (14), the proof above could also be applied by simply swapping all the subscripts V with H. Since only the integer cases could result in steady-state, the only if direction is proved and the lemma is proved. □

D. PROOF OF LEMMA 4

When $v_V = v_H$, if $\tilde{m} > 1$ is an integer, then with $d'_V = \frac{d_V}{\tilde{m}}$, one could enhance the efficiency of the vertical road while still keep all the AVs safe in responsibility. If $\tilde{n} > 1$ is an integer, then with $d'_H = \frac{d_H}{\tilde{n}}$, one could similarly enhance the efficiency of the horizontal lane. If none of $\tilde{m}, \tilde{n}$ is an integer, then by the previous lemma, the intersection won't even stay steady. Base on the result, now we have $d_H = d_V$ as the necessary condition for optimality. For symmetry, let δ_l and δ_r be the distance between itself and the AV at its left and right respectively when it's at the intersection point. Suppose an AV doesn't split equally the inter- vehicle distance of another road, then we have $\delta_l + \delta_r = d_H = d_V$ but $\delta_l \neq \delta_r$. If in the steady-state $\delta_l > \delta_r$ could make AVs be safe, then it implies that δ_r is sufficient to be the difference in intersection distance of two AVs on intersected roads of the intersection. Thus the symmetric setting with inter-vehicle distance $2\delta_r$ is sufficient to be safe while keep steady. Similarly for the case $\delta_l < \delta_r$. Thus if $\delta_l \neq \delta_r$, the one with smaller value indicates a way to enhance the traffic efficiency and only when $\delta_l = \delta_r = \frac{1}{2}\inf d_{LS}^I(v,\cdot)$ could the optimum efficiency be reached. □

E. PROOF OF LEMMA 5 AND THEOREM 2

For (33), to check whether at $v = V_{Max}$ would be the supremum, we differentiate $\frac{\inf d_{LS}^R(v,\cdot)}{v}$ with respect to v

$$\frac{\partial}{\partial v}\frac{\inf d_{LS}^I(v,\cdot)}{v} = \frac{-(W+L)}{v^2} \leq 0. \tag{46}$$

Thus for $V_{min} \leq v_i \leq v_j \leq V_{Max}$, we have

$$\frac{\inf d_{LS}^I(v_i,\cdot)}{v_i} \geq \frac{\inf d_{LS}^I(v_j,\cdot)}{v_j}. \tag{47}$$

So at $v = V_{Max}$, $\frac{\inf d_{LS}^I(v,\cdot)}{v}$ is minimized and its reciprocal, $\frac{v}{\inf d_{LS}^I(v,\cdot)}$, is maximized. Since SDT_R is proportional to this

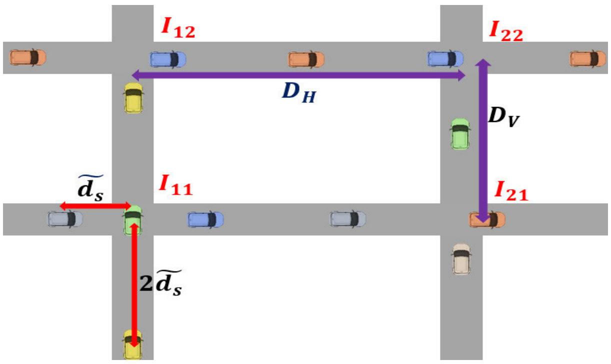

FIGURE 14. The illustration for the proof of Lemma 6.

reciprocal, by (12) and (47), we complete the proof for SDT_I in (35). For (32), the first inequality is simply by the fact that local optimum would not be better than the global optimum. For the equality part, since all the AVs have identical velocity for the intersection is steady, they have identical least safe inter-vehicle distance. This distance is a function of their steady-state velocity v. To check whether at $v = V_{min}$ would be the infimum, we differentiate $d^I_{LS}(v, \cdot)$ with respect to v

$$\frac{\partial}{\partial v}\{\inf d^I_{LS}(v, \cdot)\} = 2\tau > 0. \tag{48}$$

and thus it's true that at $v = V_{min}$, the infimum is achieved for (32). Base on (11), for velocity $V_{min} \leq v_i \leq v_j \leq V_{Max}$ and any positive constant c, we have

$$\frac{c}{\inf d^I_{LS}(v_j, \cdot)} \leq \frac{c}{\inf d^I_{LS}(v_i, \cdot)}. \tag{49}$$

Thus (34) is proved by simply substitute c with $\hat{M}$, the length of measure of SDC_I. Now we complete all the proofs. □

F. PROOF OF LEMMA 6

Since we know the AVs entering I_{ij} comes from $I_{i-1\,j}$ and I_{ij-1}. If both $I_{i-1\,j}$ and I_{ij-1} are in steady-state, then I_{ij} must satisfy some constraint so that all the AVs from $I_{i-1\,j}$ and I_{ij-1} could still pass through I_{ij} without changing their behaviours. To prove the lemma, we introduce the concept of phase. Noticing that for every $m_c \geq 2$ by $n_c \geq 2$ roads city, we could always split the city into $((m_c - 1) \times (n_c - 1))$ 2 by 2 blocks as shown in Fig. 15. There would be 4 intersections in any of the block: $I_{11}, I_{12}, I_{21}, I_{22}$ as shown in Fig. 14. We define the phase of I_{11} as

$$\Phi(I_{11}) = \frac{(\tilde{d}_s - d^{(11)}_{iH})}{\tilde{d}_s}\pi. \tag{50}$$

Here $\tilde{d}_s$ represents half of the constant spacing between AVs that should be greater or equal to $\inf \frac{1}{2} d^I_{LS}(v, \cdot)$, and $d^{(11)}_{iH}$ is the intersection distance of the first AV in the horizontal lane that's going to pass through I_{11}. Without loss of generality, let $d^{(11)}_{iH} = \tilde{d}_s$ and so $\Phi(I_{11}) = \frac{(\tilde{d}_s - \tilde{d}_s)}{\tilde{d}_s}\pi = 0$. Now we extend the definition of intersection distance so that the AV, which has passed the intersection, has a negative intersection distance. By lemma 4, we know that in order to achieve optimum efficiency, the first AV in the vertical road to cross I_{11} should have its intersection distance $d_{iV} = 0$. Similarly, we have the equivalent phase representation measuring from the vertical intersection distance d_{iV}

$$\Phi(I_{11}) = \frac{(2\tilde{d}_s - d^{(11)}_{iV})}{\tilde{d}_s}\pi. \tag{51}$$

Let $\zeta^*_{(1121)} = \mathbf{arg\,max}_{\zeta \in \mathbb{N}}\big(\mathbb{1}[D_H - (2\zeta + 1)\tilde{d}_s \geq 0] - 0.5\big)^{-1}\zeta$ where D_H is the distance between I_{11} and I_{12}. D_H is presumed to be greater than $2\tilde{d}_s$. By (50), we have

$$\Phi(I_{21}) = \frac{(\tilde{d}_s - d^{(11)}_{iH})}{\tilde{d}_s}\pi = \frac{\tilde{d}_s - D_H + (2\zeta^*_{(1121)} + 1)\tilde{d}_s}{\tilde{d}_s}\pi. \tag{52}$$

Suppose

$$d^{(21)}_{iH} = D_H - (2\zeta^*_{(1121)} + 1)\tilde{d}_s < \tilde{d}_s, \tag{53}$$

we have

$$d^{(21)}_{iV} = D_H - (2\zeta^*_{(1121)})\tilde{d}_s. \tag{54}$$

Let $\zeta^*_{(2122)} = \mathbf{arg\,max}_{\zeta \in \mathbb{N}}\big(\mathbb{1}[D_V - (2\zeta\tilde{d}_s - d^{(21)}_{iV}) \geq 0] - 0.5\big)^{-1}\zeta$, where D_V is the distance between I_{11} and I_{12} so that

$$d^{(22)}_{iV} = D_V + D_H - (2\zeta^*_{(1121)})\tilde{d}_s - (2\zeta^*_{(2122)})\tilde{d}_s. \tag{55}$$

By (51), we then have the phase of I_{22}

$$\Phi(I_{22}) = \frac{2\tilde{d}_s - (D_V + D_H) + (2\zeta^*_{(1121)} + 2\zeta^*_{(2122)})\tilde{d}_s}{\tilde{d}_s}\pi. \tag{56}$$

With similar procedure, we can have the phase of I_{12}

$$\Phi(I_{12}) = \frac{D_V - 2\zeta^*_{(1112)}\tilde{d}_s}{\tilde{d}_s}\pi. \tag{57}$$

Here $\zeta^*_{(1112)} = \mathbf{arg\,max}_{\zeta \in \mathbb{N}}\big(\mathbb{1}[D_V - (2\zeta\tilde{d}_s) \geq 0] - 0.5\big)^{-1}\zeta$
Suppose

$$d^{(12)}_{iV} = D_V - (2\zeta^*_{(1112)})\tilde{d}_s < \tilde{d}_s, \tag{58}$$

we have $d^{(12)}_{iH} = D_V - (2\zeta^*_{(1112)} - 1)\tilde{d}_s$. Now let

$$\zeta^*_{(1222)} = \mathbf{arg\,max}_{\zeta \in \mathbb{N}}\big(\mathbb{1}[D_H - (2\zeta\tilde{d}_s - d^{(12)}_{iH}) \geq 0] - 0.5\big)^{-1}\zeta$$

We have

$$d^{(22)}_{iH} = D_H + d^{(12)}_{iH} - 2\zeta^*_{(1222)}\tilde{d}_s. \tag{59}$$

$$\begin{aligned}\Phi(I_{22}) &= \frac{\tilde{d}_s - d^{(22)}_{iH}}{\tilde{d}_s}\pi \\ &= \frac{\tilde{d}_s - D_H - d^{(12)}_{iH} + 2\zeta^*_{(1222)}\tilde{d}_s}{\tilde{d}_s}\pi \\ &= \frac{-(D_H + D_V) + 2(\zeta^*_{(1112)} + \zeta^*_{(1222)})\tilde{d}_s}{\tilde{d}_s}\pi. \end{aligned} \tag{60}$$

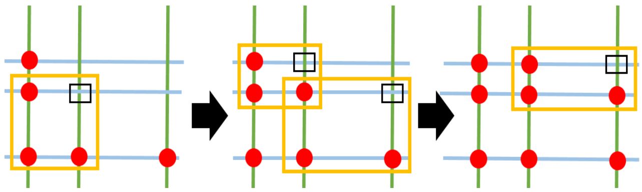

FIGURE 15. Decompose the city into 2 by 2 blocks (yellow box) and then generalize the result into the case of m_c by n_c roads. (The illustration here is the case of $m_c = 3$ and $n_c = 3$).

Since the phase derived in (56) should meet with the one in (60) to have the city steady, equating (56) and (60), we have

$$1 = \zeta^*_{(1112)} - \zeta^*_{(2122)} + \zeta^*_{(1222)} - \zeta^*_{(1121)}. \tag{61}$$

Similarly, if we consider different conditions in (53) and (58), that is, the other 3 possible conditions: $d^{(21)}_{iH} \geq \tilde{d}_s$ and $d^{(12)}_{iV} \geq \tilde{d}_s$; $d^{(21)}_{iH} \geq \tilde{d}_s$ and $d^{(12)}_{iV} < \tilde{d}_s$; $d^{(21)}_{iH} < \tilde{d}_s$ and $d^{(12)}_{iV} \geq \tilde{d}_s$, following the similar procedure by equating their phase at I_{22}, we have the following two more results:

$$0 = \zeta^*_{(1112)} - \zeta^*_{(2122)} + \zeta^*_{(1222)} - \zeta^*_{(1121)} \tag{62}$$

and

$$-1 = \zeta^*_{(1112)} - \zeta^*_{(2122)} + \zeta^*_{(1222)} - \zeta^*_{(1121)}. \tag{63}$$

Since (61), (62), and (63) cover all the possible results of $\zeta^*_{(1112)} - \zeta^*_{(2122)} + \zeta^*_{(1222)} - \zeta^*_{(1121)}$ and they're all irrelevant to D_H and D_V, we can conclude that for any 2 by 2 case as shown in Fig. 14. If I_{11}, I_{12} and I_{21} are and could be properly controlled so that they are all steady, then as long as D_H and D_V are greater than the constant spacing kept by the AVs, I_{22} would always be steady. Now in order to generalize the result into any m_c by n_c city where $m_c \geq 2$ and $n_c \geq 2$, we first label all the intersections in the similar manner as in Fig. 14. Then for any intersection $I_{ij}, \forall\, i, j\ s.t.\ 1 < i \leq m_c$ and $1 < j \leq m_c$, if $I_{i-1\,j}$, $I_{i\,j-1}$, and $I_{i-1\,j-1}$ are steady, I_{ij} is steady according to the result above. For those boundary intersections I_{ij} where at least one of i or j is 1, they could always be made steady since we can control the time the AVs enter the city. Thus, as shown in Fig. 15, we can finish the generalization simply by induction and we complete the proof. □

G. PROOF OF THEOREM 3

By lemma 6, when all the intersections are steady with optimum efficiency, they have identical $\inf d^I_{LS}(v, \cdot)$ and thus the SDC_C and SDT_C could simply be written in the form that treat all the roads within the city as if they are independent with inter-vehicle distances $\inf d^I_{LS}(v, \cdot)$. □

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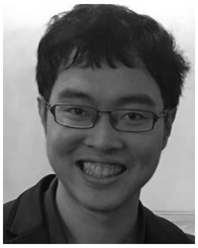

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