Received April 18, 2020, accepted May 6, 2020, date of publication May 14, 2020, date of current version May 28, 2020.

Digital Object Identifier 10.1109/ACCESS.2020.2994554

Exponential Stability Analysis of Mixed Delayed Quaternion-Valued Neural Networks via Decomposed Approach

HUAMIN WANG[1], JIE TAN[2], AND SHIPING WEN[3]

[1]Department of Mathematics, Luoyang Normal University, Luoyang 471934, China

[2]College of Mathematics, Physics and Data Science, Chongqing University of Science and Technology, Chongqing 401331, China

[3]Centre for Artificial Intelligence, Faculty of Engineering and Information Technology, University of Technology Sydney, Sydney, NSW 2007, Australia

Corresponding authors: Jie Tan (tanjie1018@163.com) and Shiping Wen (shiping.wen@uts.edu.au)

This work was supported in part by the National Natural Science Foundation of China under Grant U1804158 and Grant 61673187, in part by the Program for Science and Technology Innovation Talents in Universities of Henan Province under Grant 20HASTIT023, in part by the Science and Technology Development Program of Henan Province under Grant 172102210407, in part by the IRTSTHN under Grant 18IRTSTHN014, in part by the Key Project in Universities of Henan Province under Grant 19A110025, in part by the Core Teacher of Luoyang Normal University under Grant 2018XJGGJS-09, and in part by the Cultivation Fund of Luoyang Normal University under Grant 2017-PYJJ-007.

ABSTRACT With the application of quaternion in technology, quaternion-valued neural networks (QVNNs) have attracted many scholars' attention in recent years. For the existing results, dynamical behavior is an important studying side. In this paper, we mainly research the existence, uniqueness and exponential stability criteria of solutions for the QVNNs with discrete time-varying delays and distributed delays by means of generalized 2-norm. In order to avoid the noncommutativity of quaternion multiplication, the QVDNN system is firstly decomposed into four real-number systems by Hamilton rules. Then, we obtain the sufficient criteria for the existence, uniqueness and exponential stability of solutions by special Lyapunov-type functional, Cauchy convergence principle and monotone function. Furthermore, several corollaries are derived from the main results. Finally, we give one numerical example and its simulated figures to illustrate the effectiveness of the obtained conclusion.

INDEX TERMS Quaternion-valued neural networks, discrete and distributed delays, exponential stability, generalized 2-norm.

I. INTRODUCTION

After the models of various neural networks (NNs) (Hopfield NNs, Cohen-Grossberg NNs, memristive NNs, *etc.*) were built [1]–[3], they have been widely used to research pattern recognition, optimization problems, intelligent control, and so on. When NNs are utilized in the reality applications, dynamical characteristics of these systems are very important. Therefore, the study of dynamical behaviors has been one evergreen hot topic because of its significant influence on NNs [4]–[6], [12]. For instance, in [4], [9], the authors discussed the stability criteria of different NNs by means of diverse ways. Synchronization problems of NNs were investigated in [5], [7], [8], [11]. Passivity and Dissipativity of NNs were studied in [5], [6]. From these results, it can be seen that most of this references have studied the delayed NNs systems.

As we all know, time-delays are often the main cause of oscillation and instability, which are need to be considered when NNs models and neural circuits are constructed. Therefore, the dynamical characteristics of NNs with various time delays are necessary for the NNs researches [6], [10], [11], [13]–[21]. In particular, stability of delayed NNs (DNNs) is one of the most desirable dynamical properties when NNs models are used, which have drawn much attention of many scholars. There have been a large amount of related results in recent years [13]–[21]. For example, in [13], [15], [16], some important stability conclusions were obtained for those NNs with discrete and distributed delays. And discrete delays were considered in NNs to obtain the stability criteria in [14], [17]–[21]. Although there have been some significant stability results for the discrete delayed QVNNs (QVDNNs) systems [20]–[26], distributed delays have been rarely discussed.

The associate editor coordinating the review of this manuscript and approving it for publication was Engang Tian.

Although the signal propagation sometimes can be modeled by QVNNs with discrete delays, it can also be distributed in some certain periods. Hence, the distributed delays should also be considered simultaneously with discrete delays in a QVNNs system.

In the past several decades, NNs have been mainly studied in the real number field [6], [10], [15] and in the complex number field [9], [17]–[19], and then, in the quaternion number field [20], [22], [26]. Quaternion was firstly given by W. R. Hamilton in 1843. Recently, it has been used in many areas, such as computer graphics, 3 or 4-D data modeling, array processing and so on. As one of the most important research contents of quaternion, QVNNs have drawn many researchers' attention [20]–[33], which is a natural continuation of complex-valued NNs (QVNNs) and real-valued NNs (RVNNs). Due to the noncommutativity of quaternion multiplication, decomposition and direct approaches are usually used to research the QVDNNs. In [21], [22], [29], [32], the authors used direct approach to investigate QVDNNs, while the authors in [20], [24], [26], [31], [33] studied QVDNNs by decomposition method.

Stability is one of the most fundamental important dynamical properties for QVDNNs [20]–[26], [31]–[33]. For example, in [21], [32], homeomorphic mapping was utilized for QVDNNs to study the existence, uniqueness and stability criteria of solutions by constructing a complex Lyapunov-Krasovskii functional. LMI-form sufficient conditions were derived in these papers. The existence and stability criteria of multiple equilibrium points were obtained for QVNNs by means of different ways in [22], [31]. And in [20], [25], $\{\xi, \infty\}$-norm, a generalized ∞-norm, was firstly used to study the existence and uniqueness criteria of solutions, and the μ-stable criteria for QVDNNs. It is worthy noting that generalized norm is an useful definition for QVDNNs to get their existence and stability criteria of solutions. Apart from generalized ∞-norm, there have been generalized 1-norm and 2-norm, which are named as $\{\xi, 1\}$-norm and $\{\xi, 2\}$-norm [4], [18], respectively. Although the method of ∞-norm can not be used to study some dynamical behaviors by 1-norm or 2-norm, the similar results can be derived by generalized 1-norm if some results of QVDNNs can be obtained by generalized 2-norm. Therefore, it is worthy studying the stability of QVNNs with discrete time-varying delays and distributed delays by $\{\xi, 2\}$-norm, which remains an open problem.

Based on the above analyses, this paper focuses on the existence and exponential stability of solutions for the QVNNs with discrete time-varying delays and distributed delays by generalized 2-norm ($\{\xi, 2\}$-norm). Because of the noncommutativity of quaternion multiplication, the QVDNN system is firstly decomposed into four real-number systems by Hamilton rules. Then, the novel stability definition of QVDNNs is introduced according to the definition of $\{\xi, 2\}$-norm. Meanwhile, some assumptions of discrete time-varying delays and distributed delays are given. In addition, by constructing $\{\xi, 2\}$-norm-type Lyapunov functional, the existence, uniqueness and exponential stability sufficient criteria of the discrete-distributed-delayed QVNNs are obtained by Cauchy convergence principle and monotone function. Several corollaries are derived from the main results. Finally, one numerical example about QVNNs with discrete time-varying delays and distributed delays is given to illustrate the effectiveness of the obtained conclusions.

The rest of this paper is organized as follows. In Section II, models and preliminaries are given. Then, the existence and stability sufficient criteria for discrete and distributed delayed QVNNs is obtained in Section III. In Section IV, a numerical simulation example is shown to illustrate the validity of obtained results. Finally, conclusions are given in Section V.

II. PRELIMINARIES

Notation: $\mathbb{R}$ and $\mathbb{Q}$ show the sets of real numbers and quaternion numbers, respectively. $\mathbb{R}^n$ and $\mathbb{Q}^n$ denote the n-dimensional Euclidean and quaternion spaces, respectively. $\mathbb{R}^{n\times m}$ and $\mathbb{Q}^{n\times m}$ are the sets of $n \times m$ real matrixes and $n \times m$ quaternion matrixes, respectively. $\|\cdot\|$ denotes Euclidean vector norm and $O(\cdot)$ denotes infinitesimal of the same order. If $z = (z_1, z_2, \cdots, z_n)^T \in \mathbb{Q}^n$, then $|z| = (|z_1|, |z_2|, \cdots, |z_n|)^T$.

In this paper, we will consider the following QVNNs with discrete and distributed delays:

$$\begin{cases} \dot{x}_p(t) = -d_p x_p(t) + \sum_{q=1}^{n} a_{pq} f_q(x_q(t)) \\ \qquad + \sum_{q=1}^{n} b_{pq} g_q(x_q(t-\tau_{pq}(t))) \\ \qquad + \sum_{q=1}^{n} c_{pq} \int_0^{\infty} k_{pq}(s) g_q(x_q(t-s))ds \\ \qquad + u_p, \qquad t \geq 0, \\ x_p(s) = \varphi_p(s), \qquad s \in (-\infty, 0], \end{cases} \tag{1}$$

where $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T \in \mathbb{Q}^n$ with $x_p(t) = x_p^R(t) + \imath x_p^I(t) + \jmath x_p^J(t) + \kappa x_p^K(t) (p = 1, 2, \cdots, n)$ is the state vector, $D = \text{diag}\{d_1, d_2, \cdots, d_n\} \in \mathbb{R}^{n\times n}$ with $d_p > 0$ is the self-inhibition matrix, $f(x(t)) = (f_1(x_1(t)), f_2(x_2(t)), \cdots, f_n(x_n(t)))^T \in \mathbb{Q}^n$ and $g(x(t)) = (g_1(x_1(t)), g_2(x_2(t)), \cdots, g_n(x_n(t)))^T \in \mathbb{Q}^n$ represent the quaternion-valued neuron vector-valued activation functions, which satisfy $f_q(0) = 0$ and $g_q(0) = 0$. $A = [a_{pq}]_{n\times n}$, $B = [b_{pq}]_{n\times n} \in \mathbb{Q}^{n\times n}$ and $C = [c_{pq}]_{n\times n} \in \mathbb{Q}^{n\times n}$ are the non-delayed, discrete-delayed and distributed-delayed connective weights matrixes, respectively. $\tau_{pq}(t) > 0$ is discrete time-varying delay. $K(s) = [k_{pq}(s)]_{n\times n} \in \mathbb{R}^{n\times n}$ is the delayed kernel function matrix. $u = (u_1, u_2, \cdots, u_n)^T \in \mathbb{Q}^n$ is an external input or bias vector. The initial condition is $\varphi(s) = (\varphi_1(s), \varphi_2(s), \ldots, \varphi_n(s))^T \in C((-\infty, 0], \mathbb{Q}^n)$.

Next, some basic definitions and properties of quaternion are introduced. A quaternion $h \in \mathbb{Q}$ is defined as $h = h^R + \imath h^I + \jmath h^J + \kappa h^K \in \mathbb{Q}$ with $h^R, h^I, h^J, h^K \in \mathbb{R}$, which shows that the real quaternion field $\mathbb{Q}$ can be viewed as a 4-D vector space over $\mathbb{R}$. According to Hamilton rules, its imaginary units $\imath$, $\jmath$, and κ obey the following rules: $\imath\jmath = -\jmath\imath = \kappa$, $\jmath\kappa = -\kappa\jmath = \imath$, $\kappa\imath = -\imath\kappa = \jmath$, $\imath^2 = \jmath^2 = \kappa^2 = \imath\jmath\kappa = -1$, which means they are noncommutative. Its conjugate h^* or $\bar{h}$ is defined by $h^* = \bar{h} = h^R - \imath h^I - \jmath h^J - \kappa h^K$, and its modulus $|h|$ is defined by $|h| = \sqrt{h^* h} = \sqrt{(h^R)^2 + (h^I)^2 + (h^J)^2 + (h^K)^2}$. Let $s = s^R + \imath s^I + \jmath s^J + \kappa s^K \in \mathbb{Q}$, the addition $h + s$ and product hs of h and s can be defined as $h + s = (h^R + s^R) + \imath(h^I + s^I) + \jmath(h^J + s^J) + \kappa(h^K + s^K)$ and $hs = (h^R s^R - h^I s^I - h^J s^J - h^K s^K) + \imath(h^R s^I + h^I s^R + h^J s^K - h^K s^J) + \jmath(h^R s^J + h^J s^R - h^I s^K + h^K s^I) + \kappa(h^R s^K + h^K s^R + h^I s^J - h^J s^I)$, respectively.

Denote $M = \{R, I, J, K\}$, then the QVNN model with discrete and distributed delays (1) can be decomposed into real-valued systems with $L \in M$ as follows:

$$\begin{aligned}\dot{x}_p^L(t) = -d_p x_p^L(t) &+ \sum_{q=1}^{n} (a_{pq} f_q(x_q(t)))^L \\ &+ \sum_{q=1}^{n} (b_{pq} g_q(x_q(t - \tau_{pq}(t))))^L \\ &+ \sum_{q=1}^{n} (c_{pq} \int_0^\infty k_{pq}(s) g_q(x_q(t-s)) ds)^L + u_p^L. \quad (2)\end{aligned}$$

Definition 1 [4]:

(1) Class $L_1(\lambda_q^R, \lambda_q^I, \lambda_q^J, \lambda_q^K)$. If there exists $\lambda_q^l > 0$, such that $0 < \frac{f_q^l(x_q^l) - f_q^l(y_q^l)}{x_q^l - y_q^l} \le \lambda_q^l$ holds for any $q = 1, 2, \cdots, n$, $l \in M$ and $x_q^l, y_q^l \in R$, then $f_q^l(x_q^l)$ is said to belong to $L_1(\lambda_q^R, \lambda_q^I, \lambda_q^J, \lambda_q^K)$.

(2) Class $L_2(\gamma_q^R, \gamma_q^I, \gamma_q^J, \gamma_q^K)$. If there exists $\gamma_q^l > 0$, such that $\frac{|g_q^l(x_q^l) - g_q^l(y_q^l)|}{|x_q^l - y_q^l|} \le \gamma_q^l$ holds for any $q = 1, 2, \cdots, n$, $l \in M$ and $x_q^l, y_q^l \in R$, then $g_q^l(x_q^l)$ is said to belong to $L_2(\gamma_q^R, \gamma_q^I, \gamma_q^J, \gamma_q^K)$.

In order to study the existence and stability of the above delayed QVNN model (2), the following assumptions should be introduced:

(H1) The activation functions $f_q(x_q(t))$ and $g_q(x_q(t))$ can be separated into one real and three imaginary parts as follows:

$$\begin{aligned} f_q(x_q(t)) &= f_q^R(x_q^R(t)) + \imath f_q^I(x_q^I(t)) \\ &\quad + \jmath f_q^J(x_q^J(t)) + \kappa f_q^K(x_q^K(t)), \\ g_q(x_q(t)) &= g_q^R(x_q^R(t)) + \imath g_q^I(x_q^I(t)) \\ &\quad + \jmath g_q^J(x_q^J(t)) + \kappa g_q^K(x_q^K(t)), \end{aligned}$$

where $f_q^l(x_q^l(t)) \triangleq f_q^l(t) : \mathbb{R} \to \mathbb{R}$ belongs to class L_2 and $g_q^l(x_q^l(t)) \triangleq g_q^l(t) : \mathbb{R} \to \mathbb{R}$ belongs to class L_2 for every $l \in M$.

(H2) The discrete time-varying delays $\tau_{pq}(t) : \mathbb{R} \to \mathbb{R}^+$ are continuously differential functions and satisfy $\tau_{pq}(t) \le \tau_{pq} \le \tau$ and $|\tau'_{pq}(t)| \le \eta_{pq} < 1$ for any $p, q = 1, 2, \cdots, n$ and $t > 0$, where τ_{pq}, τ and η_{pq} are real positive constants.

(H3) For any $p, q = 1, 2, \cdots, n$, the kernel $k_{pq} : [0, +\infty) \to [0, +\infty)$ are real-valued nonnegative continuous functions and satisfy the following conditions: $\int_0^\infty k_{pq}(s) ds = 1$ and $\int_0^\infty e^{\sigma s} k_{pq}(s) ds < \infty$, where σ is positive numbers.

Remark 1: Different from assumption (H1), activation function $f_q(x_q(t))$ can be decomposed into in some references as $f_q(x_q(t)) = f_q^R(x_q^R(t), x_q^I(t), x_q^J(t), x_q^K(t)) + \imath f_q^I(x_q^R(t), x_q^I(t), x_q^J(t), x_q^K(t)) + \jmath f_q^J(x_q^R(t), x_q^I(t), x_q^J(t), x_q^K(t)) + \kappa f_q^K(x_q^R(t), x_q^I(t), x_q^J(t), x_q^K(t))$ [20], [24], [25]. Actually, the decomposed form of assumption (H1) can simplifies research process and results of QVDNNs, which is used in this paper. Furthermore, when the exponential stable criteria of QVDNNs are studied, some special restrictions should be given, as in assumption (H2), to deal with discrete time-varying delays.

Based on (H1), the system (2) can be rewritten as

$$\begin{aligned}\dot{x}_p^L(t) = -d_p x_p^L(t) &+ \sum_{q=1}^{n} \sum_{(l,w)\in M^L} \psi_{lw} a_{pq}^l f_q^w(x_q^w(t)) \\ &+ \sum_{q=1}^{n} \sum_{(l,w)\in M^L} \psi_{lw} b_{pq}^l g_q^w(x_q^w(t - \tau_{pq}(t))) \\ &+ \sum_{q=1}^{n} \sum_{(l,w)\in M^L} \psi_{lw} c_{pq}^l \int_0^\infty k_{pq}(s) g_q^w(x_q(t \\ &- s)) ds)^L + u_p^L, \quad (3)\end{aligned}$$

where $M^L \in \{M^R, M^I, M^J, M^K\}$, $M^R = \{(R,R), (I,I), (J,J), (K,K)\}$, $M^I = \{(R,I), (I,R), (J,K), (K,J)\}$, $M^J = \{(R,J), (I,K), (J,R), (K,I)\}$, $M^K = \{(R,K), (I,J), (J,I), (K,R)\}$ and $\psi_{lw} = 1$ or -1 is the sign of $a_{pq}^l f_q^w(\cdot)$, $b_{pq}^l g_q^w(\cdot)$ and $c_{pq}^l g_q^w(\cdot)$. Then the concrete forms of $\dot{x}_p^R(t)$ $\dot{x}_p^I(t)$ $\dot{x}_p^J(t)$ and $\dot{x}_p^K(t)$ can be written as follows:

$$\begin{aligned}\dot{x}_p^R(t) = -d_p x_p^R(t) &+ \sum_{q=1}^{n} \big(a_{pq}^R f_q^R(t) - a_{pq}^I f_q^I(t) \\ &- a_{pq}^J f_q^J(t) - a_{pq}^K f_q^K(t)\big) \\ &+ \sum_{q=1}^{n} \big(b_{pq}^R g_q^R(t - \tau_{pq}(t)) - b_{pq}^I g_q^I(t - \tau_{pq}(t)) \\ &- b_{pq}^J g_q^J(t - \tau_{pq}(t)) - b_{pq}^K g_q^K(t - \tau_{pq}(t))\big) \\ &+ \sum_{q=1}^{n} \big(c_{pq}^R \int_0^\infty k_{pq}(s) g_q^R(x_q(t-s)) ds \\ &- c_{pq}^I \int_0^\infty k_{pq}(s) g_q^I(x_q(t-s)) ds \\ &- c_{pq}^J \int_0^\infty k_{pq}(s) g_q^J(x_q(t-s)) ds \\ &- c_{pq}^K \int_0^\infty k_{pq}(s) g_q^K(x_q(t-s)) ds\big) + u_p^R,\end{aligned}$$

$$\begin{aligned}\dot{x}_p^I(t) = -d_p x_p^I(t) &+ \sum_{q=1}^{n} \big(a_{pq}^R f_q^I(t) + a_{pq}^I f_q^R(t) \\ &+ a_{pq}^J f_q^K(t) - a_{pq}^K f_q^J(t)\big) \\ &+ \sum_{q=1}^{n} \big(b_{pq}^R g_q^I(t - \tau_{pq}(t)) + b_{pq}^I g_q^R(t - \tau_{pq}(t))\end{aligned}$$

$$
\begin{aligned}
&+b_{pq}^{J}g_q^{K}(t-\tau_{pq}(t))-b_{pq}^{K}g_q^{J}(t-\tau_{pq}(t))\Big)\\
&+\sum_{q=1}^{n}\Big(c_{pq}^{R}\int_0^{\infty}k_{pq}(s)g_q^{I}(x_q(t-s))ds\\
&+c_{pq}^{I}\int_0^{\infty}k_{pq}(s)g_q^{R}(x_q(t-s))ds\\
&+c_{pq}^{J}\int_0^{\infty}k_{pq}(s)g_q^{K}(x_q(t-s))ds\\
&-c_{pq}^{K}\int_0^{\infty}k_{pq}(s)g_q^{J}(x_q(t-s))ds\Big),\\
\dot{x}_p^{J}(t)=&-d_px_p^{J}(t)+\sum_{q=1}^{n}\Big(a_{pq}^{R}f_q^{J}(t)-a_{pq}^{I}f_q^{K}(t)\\
&+a_{pq}^{J}f_q^{R}(t)+a_{pq}^{K}f_q^{I}(t)\Big)\\
&+\sum_{q=1}^{n}\Big(b_{pq}^{R}g_q^{J}(t-\tau_{pq}(t))-b_{pq}^{I}g_q^{K}(t-\tau_{pq}(t))\\
&+b_{pq}^{J}g_q^{R}(t-\tau_{pq}(t))+b_{pq}^{K}g_q^{I}(t-\tau_{pq}(t))\Big)\\
&+\sum_{q=1}^{n}\Big(c_{pq}^{R}\int_0^{\infty}k_{pq}(s)g_q^{J}(x_q(t-s))ds\\
&-c_{pq}^{I}\int_0^{\infty}k_{pq}(s)g_q^{K}(x_q(t-s))ds\\
&+c_{pq}^{J}\int_0^{\infty}k_{pq}(s)g_q^{R}(x_q(t-s))ds\\
&+c_{pq}^{K}\int_0^{\infty}k_{pq}(s)g_q^{I}(x_q(t-s))ds\Big),\\
\dot{x}_p^{K}(t)=&-d_px_p^{K}(t)+\sum_{q=1}^{n}\Big(a_{pq}^{R}f_q^{K}(t)+a_{pq}^{I}f_q^{J}(t)\\
&-a_{pq}^{J}f_q^{I}(t)+a_{pq}^{K}f_q^{R}(t)\Big)\\
&+\sum_{q=1}^{n}\Big(b_{pq}^{R}g_q^{K}(t-\tau_{pq}(t))+b_{pq}^{I}g_q^{J}(t-\tau_{pq}(t))\\
&-b_{pq}^{J}g_q^{I}(t-\tau_{pq}(t))+b_{pq}^{K}g_q^{R}(t-\tau_{pq}(t))\Big)\\
&+\sum_{q=1}^{n}\Big(c_{pq}^{R}\int_0^{\infty}k_{pq}(s)g_q^{K}(x_q(t-s))ds\\
&+c_{pq}^{I}\int_0^{\infty}k_{pq}(s)g_q^{J}(x_q(t-s))ds\\
&-c_{pq}^{J}\int_0^{\infty}k_{pq}(s)g_q^{I}(x_q(t-s))ds\\
&+c_{pq}^{K}\int_0^{\infty}k_{pq}(s)g_q^{R}(x_q(t-s))ds\Big). \qquad (4)
\end{aligned}
$$

Definition 2 [24]: A constant vector $x^*=(x_1^*,x_2^*,\cdots,x_n^*)^T\in\mathbb{Q}$ is called an equilibrium point of delayed QVNNs (1), if

$$
-d_px_p^*+\sum_{q=1}^{n}a_{pq}f_q(x_q^*)+\sum_{q=1}^{n}b_{pq}g_q(x_q^*)+\sum_{q=1}^{n}c_{pq}g_q(x_q^*)+u_p=0
$$

holds for any $p,q=1,2,\cdots,n$.

Definition 3 [18]: For any vector $u(t)\in\mathbb{R}^{n\times 1}$, if $\|u(t)\|_{\{\xi,2\}}=(\sum_p|\xi_pu_p(t)|^2)^{1/2}$, where $\xi_p>0$ and $p=1,2,\cdots,n$, it can be called generalized 2-norm or $\{\xi,2\}$-norm.

Remark 2: If $\xi_p=1$ holds for every $p=1,2,\cdots,n$, $\{\xi,2\}$-norm becomes the normal 2-norm $\|u(t)\|_2=(\sum_p|u_p(t)|^2)^{1/2}$. Based on the definition of generalized 2-norm, a novel exponential stable definition can be given as follows.

Definition 4: Let x^* be an equilibrium point of QVDNN (1), if there exists a real constant $\alpha>0$, such that $\|x(t)-x^*\|_{\{\xi,2\}}=O(e^{-\alpha t})$ holds for any solution $x(t)$, then QVDNN (1) is said to be globally exponentially stable.

III. MAIN RESULTS

In this section, the existence and exponential stability criteria of the QVNNs (1) with time-varying discrete delays and distributed delays are studied by utilizing its decomposed form and the definition of $\{\xi,2\}$-norm.

Theorem 1: Under assumptions (H1), (H2) and (H3), if there exist real constants $\varsigma>0$, and $\xi_p^l>0(p=1,2,\cdots,n,l\in M)$, such that

$$
\begin{aligned}
&2\xi_q^L(-d_q+\varsigma)+\sum_{p=1}^{n}\sum_{(l,w)\in M^L}\xi_q^L\Big(|a_{qp}^l|\lambda_p^w+\\
&\quad(1+\eta_{qp})e^{\varsigma\tau_{qp}}|b_{qp}^l|\gamma_p^w+|c_{qp}^l|\gamma_p^w\int_0^{\infty}k_{qp}(s)e^{\varsigma s}ds\Big)\\
&\quad+\sum_{p=1}^{n}\sum_{(l,w)\in M^L}\xi_p^L\Big(|a_{pq}^l|\lambda_q^w+\frac{(1+\eta_{pq})}{1-\eta_{pq}}e^{\varsigma\tau_{pq}}|b_{pq}^l|\gamma_q^w\\
&\quad+|c_{pq}^l|\gamma_q^w\int_0^{\infty}k_{pq}(s)e^{\varsigma s}ds\Big)\le 0 \qquad (5)
\end{aligned}
$$

holds for every $L\in M$ and $q=1,2,\cdots,n$. Then, the delayed QVNNs system (1) has an unique equilibrium point x^*, which is globally exponentially stable.

Proof: Since the system (1) has time-varying delays $\tau_{pq}(t)$, we can obtain that

$$
\begin{aligned}
\frac{d\dot{x}_p(t)}{dt}=&-d_p\dot{x}_p(t)+\sum_{q=1}^{n}a_{pq}f_q'(x_q(t))\dot{x}_q(t)\\
&+\sum_{q=1}^{n}b_{pq}g_q'(x_q(t-\tau_{pq}(t)))\dot{x}_q(t-\tau_{pq}(t))(1-\tau_{pq}'(t))\\
&+\sum_{q=1}^{n}c_{pq}\int_0^{\infty}k_{pq}(s)g_q'(x_q(t-s))\dot{x}_q(t-s)ds.
\end{aligned}
$$

Define $u(t)=e^{\varsigma t}\dot{x}(t)$, then we have

$$
\begin{aligned}
\frac{du_p(t)}{dt}=&(-d_p+\varsigma)u_p(t)+\sum_{q=1}^{n}a_{pq}f_q'(x_q(t))u_q(t)\\
&+\sum_{q=1}^{n}b_{pq}f_q'(x_q(t-\tau_{pq}(t)))e^{\varsigma\tau_{pq}(t)}u_q(t
\end{aligned}
$$

$$-\tau_{pq}(t))(1-\tau'_{pq}(t))+\sum_{q=1}^{n}c_{pq}\int_{0}^{\infty}$$
$$k_{pq}(s)e^{\varsigma s}g'_q(x_q(t-s))u_q(t-s)ds.$$

By (2), we have

$$\begin{aligned}
\frac{d(u_p^L(t))^2}{dt}
&= 2u_p^L(t)\Big((-d_p+\varsigma)u_p^L(t)+\sum_{q=1}^{n}(a_{pq}f'_q(x_q(t))u_q(t))^L\\
&\quad+\sum_{q=1}^{n}(b_{pq}f'_q(x_q(t-\tau_{pq}(t)))e^{\varsigma\tau_{pq}(t)}u_q(t-\tau_{pq}(t))\\
&\quad(1-\tau'_{pq}(t)))^L+\sum_{q=1}^{n}(c_{pq}\int_{0}^{\infty}k_{pq}(s)e^{\varsigma s}g'_q\\
&\quad(x_q(t-s))u_q(t-s)ds)^L\Big)\\
&\le 2(-d_p+\varsigma)(u_p^L(t))^2+2\sum_{q=1}^{n}\sum_{(l,w)\in M^L}|a_{pq}^l|\lambda_q^w\\
&\quad|u_p^L(t)||u_q^w(t)|+2\sum_{q=1}^{n}\sum_{(l,w)\in M^L}(1+\eta_{pq})e^{\varsigma\tau_{pq}}|b_{pq}^l|\gamma_q^w\\
&\quad|u_p^L(t)||u_q^w(t-\tau_{pq}(t))|+2\sum_{q=1}^{n}\sum_{(l,w)\in M^L}|c_{pq}^l|\gamma_q^w\int_{0}^{\infty}\\
&\quad k_{pq}(s)e^{\varsigma s}|u_p^L(t)||u_q^w(t-s)|ds\\
&\le\Big(2(-d_p+\varsigma)+\sum_{q=1}^{n}\sum_{(l,w)\in M^L}|a_{pq}^l|\lambda_q^w+\sum_{q=1}^{n}\sum_{(l,w)\in M^L}\\
&\quad(1+\eta_{pq})e^{\varsigma\tau_{pq}}|b_{pq}^l|\gamma_q^w+\sum_{q=1}^{n}\sum_{(l,w)\in M^L}|c_{pq}^l|\gamma_q^w\int_{0}^{\infty}k_{pq}\\
&\quad(s)e^{\varsigma s}ds\Big)(u_p^L(t))^2+\sum_{q=1}^{n}\sum_{(l,w)\in M^L}|a_{pq}^l|\lambda_q^w(u_q^w(t))^2\\
&\quad+\sum_{q=1}^{n}\sum_{(l,w)\in M^L}(1+\eta_{pq})e^{\varsigma\tau_{pq}}|b_{pq}^l|\gamma_q^w(u_q^w(t-\tau_{pq}(t)))^2\\
&\quad+\sum_{q=1}^{n}\sum_{(l,w)\in M^L}|c_{pq}^l|\gamma_q^w\int_{0}^{\infty}k_{pq}(s)e^{\varsigma s}(u_q^w(t-s))^2ds,
\end{aligned}$$

which means

$$\begin{aligned}
\frac{du_p^2(t)}{dt}&\le\Big(2(-d_p+\varsigma)+\sum_{q=1}^{n}\sum_{l\in M}\sum_{w\in M}|a_{pq}^l|\lambda_q^w\\
&\quad+\sum_{q=1}^{n}\sum_{l\in M}\sum_{w\in M}(1+\eta_{pq})e^{\varsigma\tau_{pq}}|b_{pq}^l|\gamma_q^w\\
&\quad+\sum_{q=1}^{n}\sum_{l\in M}\sum_{w\in M}|c_{pq}^l|\gamma_q^w\int_{0}^{\infty}k_{pq}(s)e^{\varsigma s}ds\Big)u_p^2(t)\\
&\quad+\sum_{q=1}^{n}\sum_{l\in M}\sum_{w\in M}|a_{pq}^l|\lambda_q^w(u_q^w(t))^2\\
&\quad+\sum_{q=1}^{n}\sum_{l\in M}\sum_{w\in M}(1+\eta_{pq})e^{\varsigma\tau_{pq}}|b_{pq}^l|\gamma_q^w\\
&\quad(u_q^w(t-\tau_{pq}(t)))^2+\sum_{q=1}^{n}\sum_{l\in M}\sum_{w\in M}\\
&\quad|c_{pq}^l|\gamma_q^w\int_{0}^{\infty}k_{pq}(s)e^{\varsigma s}(u_q^w(t-s))^2ds,
\end{aligned}$$

where $u_p^2(t)=\sum_{L\in M}(u_p^L(t))^2$.

Let

$$\begin{aligned}
V^L(t)&=\sum_{p=1}^{n}\Big(\xi_p^L(u_p^L(t))^2+\xi_p^L\sum_{q=1}^{n}\sum_{(l,w)\in M^L}\\
&\quad(1+\eta_{pq})e^{\varsigma\tau_{pq}}|b_{pq}^l|\gamma_q^w\int_{t-\tau_{pq}(t)}^{t}\frac{(u_q^w(s))^2}{1-\eta_{pq}}ds\\
&\quad+\xi_p^L\sum_{q=1}^{n}\sum_{(l,w)\in M^L}|c_{pq}^l|\gamma_q^w\\
&\quad\int_{0}^{\infty}k_{pq}(s)e^{\varsigma s}(\int_{t-s}^{t}(u_q^w(r))^2dr)ds\Big),
\end{aligned}$$

then, differentiating it, we have

$$\begin{aligned}
\frac{dV^L(t)}{dt}
&=\sum_{p=1}^{n}\Big(\xi_p^L\frac{d(u_p^L(t))^2}{dt}+\xi_p^L\sum_{q=1}^{n}\sum_{(l,w)\in M^L}\\
&\quad(1+\eta_{pq})e^{\varsigma\tau_{pq}}|b_{pq}^l|\gamma_q^w(\frac{(u_q^w(t))^2}{1-\eta_{pq}}\\
&\quad-\frac{(1-\tau'_{pq}(t))(u_q^w(t-\tau_{pq}(t)))^2}{1-\eta_{pq}})\\
&\quad+\xi_p^L\sum_{q=1}^{n}\sum_{(l,w)\in M^L}|c_{pq}^l|\gamma_q^w\int_{0}^{\infty}k_{pq}(s)e^{\varsigma s}\\
&\quad((u_q^w(t))^2-(u_q^w(t-s))^2)ds\Big)\\
&\le\sum_{p=1}^{n}\Big(\xi_p^L\Big(2(-d_p+\varsigma)+\sum_{q=1}^{n}\sum_{(l,w)\in M^L}|a_{pq}^l|\lambda_q^w\\
&\quad+\sum_{q=1}^{n}\sum_{(l,w)\in M^L}(1+\eta_{pq})e^{\varsigma\tau_{pq}}|b_{pq}^l|\gamma_q^w\\
&\quad+\sum_{q=1}^{n}\sum_{(l,w)\in M^L}|c_{pq}^l|\gamma_q^w\int_{0}^{\infty}k_{pq}(s)e^{\varsigma s}ds\Big)(u_p^L(t))^2\\
&\quad+\xi_p^L\sum_{q=1}^{n}\sum_{(l,w)\in M^L}|a_{pq}^l|\lambda_q^w(u_q^w(t))^2+\xi_p^L\sum_{q=1}^{n}\\
&\quad\sum_{(l,w)\in M^L}(1+\eta_{pq})e^{\varsigma\tau_{pq}}|b_{pq}^l|\gamma_q^w(u_q^w(t-\tau_{pq}(t)))^2
\end{aligned}$$

$$
\begin{aligned}
&+\xi_p^L\sum_{q=1}^{n}\sum_{(l,w)\in M^L}|c_{pq}^l|\gamma_q^w\int_0^{\infty}k_{pq}(s)e^{\varsigma s}(u_q^w(t-s))^2ds\\
&+\xi_p^L\sum_{q=1}^{n}\sum_{(l,w)\in M^L}(1+\eta_{pq})e^{\varsigma\tau_{pq}}|b_{pq}^l|\gamma_q^w\Big(\frac{(u_q^w(t))^2}{1-\eta_{pq}}\\
&-(u_q^w(t-\tau_{pq}(t)))^2\Big)\\
&+\xi_p^L\sum_{q=1}^{n}\sum_{(l,w)\in M^L}|c_{pq}^l|\gamma_q^w\int_0^{\infty}k_{pq}(s)e^{\varsigma s}((u_q^w(t))^2\\
&-(u_q^w(t-s))^2)ds\Big)\\
=&\sum_{p=1}^{n}\xi_p^L\Big(2(-d_p+\varsigma)+\sum_{q=1}^{n}\sum_{(l,w)\in M^L}|a_{pq}^l|\lambda_q^w\\
&+\sum_{q=1}^{n}\sum_{(l,w)\in M^L}(1+\eta_{pq})e^{\varsigma\tau_{pq}}|b_{pq}^l|\gamma_q^w\\
&+\sum_{q=1}^{n}\sum_{(l,w)\in M^L}|c_{pq}^l|\gamma_q^w\int_0^{\infty}k_{pq}(s)e^{\varsigma s}ds\Big)(u_p^L(t))^2\\
&+\sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{(l,w)\in M^L}\xi_p^L\Big(|a_{pq}^l|\lambda_q^w+\frac{1+\eta_{pq}}{1-\eta_{pq}}e^{\varsigma\tau_{pq}}|b_{pq}^l|\gamma_q^w\\
&+|c_{pq}^l|\gamma_q^w\int_0^{\infty}k_{pq}(s)e^{\varsigma s}ds\Big)(u_q^w(t))^2\\
=&\sum_{q=1}^{n}\xi_q^L\Big(2(-d_q+\varsigma)+\sum_{p=1}^{n}\sum_{(l,w)\in M^L}|a_{qp}^l|\lambda_p^w\\
&+\sum_{p=1}^{n}\sum_{(l,w)\in M^L}(1+\eta_{qp})e^{\varsigma\tau_{qp}}|b_{qp}^l|\gamma_p^w\\
&+\sum_{p=1}^{n}\sum_{(l,w)\in M^L}|c_{qp}^l|\gamma_p^w\int_0^{\infty}k_{qp}(s)e^{\varsigma s}ds\Big)(u_q^L(t))^2\\
&+\sum_{q=1}^{n}\sum_{p=1}^{n}\sum_{(l,w)\in M^L}\xi_p^L\Big(|a_{pq}^l|\lambda_q^w+\frac{1+\eta_{pq}}{1-\eta_{pq}}e^{\varsigma\tau_{pq}}|b_{pq}^l|\gamma_q^w\\
&+|c_{pq}^l|\gamma_q^w\int_0^{\infty}k_{pq}(s)e^{\varsigma s}ds\Big)(u_q^w(t))^2\\
\le&\sum_{q=1}^{n}\Big\{2\xi_q^L(-d_q+\varsigma)+\sum_{p=1}^{n}\sum_{(l,w)\in M^L}\xi_q^L\Big(|a_{qp}^l|\lambda_p^w\\
&+(1+\eta_{qp})e^{\varsigma\tau_{qp}}|b_{qp}^l|\gamma_p^w+|c_{qp}^l|\gamma_p^w\int_0^{\infty}k_{qp}(s)e^{\varsigma s}ds\Big)\\
&+\sum_{p=1}^{n}\sum_{(l,w)\in M^L}\xi_p^L\Big(|a_{pq}^l|\lambda_q^w+\frac{1+\eta_{pq}}{1-\eta_{pq}}e^{\varsigma\tau_{pq}}|b_{pq}^l|\gamma_q^w\\
&+|c_{pq}^l|\gamma_q^w\int_0^{\infty}k_{pq}(s)e^{\varsigma s}ds\Big)\Big\}(u_q^2(t))\\
\le&\ 0.
\end{aligned}
$$

Therefore, $V^L(t)$ is non-increasing, which implies that $\sum_{p=1}^n\xi_p^L(u_p^L(t))^2$ is bounded, i.e. $\|u_p^L(t)\|_{\{\xi,2\}}=O(1)$. Let $V(t)=\sum_{L\in M}V^L(t)$, then we can conclude that $\sum_{p=1}^n\sum_{L\in M}\xi_p^L(u_p^L(t))^2$ is bounded, i.e. $\|u_p(t)\|_{\{\xi,2\}}=O(1)$, which implies that $\|\dot{x}_p(t)\|_{\{\xi,2\}}=O(e^{-\varsigma t})$. On the other hand, suppose that $x^1(t)=x(t,\varphi)$ and $x^2(t)=x(t,\psi)$ are any two solutions of (1), let $y(t)=x^1(t)-x^2(t)$, Then, it can be obtained that

$$
\begin{cases}
\dot{y}_p(t)=-d_py_p(t)+\sum_{q=1}^n a_{pq}f_q'(\alpha_q)y_q(t)\\
\quad+\sum_{q=1}^n b_{pq}g_q'(\beta_q)y_q(t-\tau_{pq}(t))\\
\quad+\sum_{q=1}^n c_{pq}\int_0^{\infty}k_{pq}(s)g_q'(\gamma_q)y_q(t-s)ds,\quad t\ge 0,\\
y_p(s)=\varphi_p(s)-\psi_p(s),\quad s\in(-\infty,0],
\end{cases}
$$

where α_q is between $x_q^1(t)$ and $x_q^2(t)$, β_q is between $x_q^1(t-\tau_{pq}(t))$ and $x_q^2(t-\tau_{pq}(t))$, and γ_q is between $x_q^1(t-s)$ and $x_q^2(t-s)$. By the means of the methods of $\dot{x}(t)$, we can obtain that $y(t)=O(e^{-\varsigma t})$. As a result, according to Cauchy's test for convergence, their exists an equilibrium point $x^*=(x_1^*,x_2^*,\cdots,x_n^*)^T$, such that $\|x(t)-x^*\|_{\{\xi,2\}}=O(e^{-\varsigma t})$ for the discrete-distribute delayed system (1). In the next, we will prove that the equilibrium point of system (1) is unique.

Suppose there are two equilibrium points $x^{*1}(t)$ and $x^{*2}(t)$, let $\omega(t)=x^{*1}(t)-x^{*2}(t)$, then we can easily obtain $\|\omega(t)\|_{\{\xi,2\}}=O(e^{-\varsigma t})$ by means of the same argument of $\dot{x}(t)$, which implies that the discrete-distribute delayed system (1) has an unique global exponential stable equilibrium point. □

Remark 3: For this theorem, we discuss the existence and exponential stability criteria of QVNNs (1) with discrete time-varying delays and distributed delays by the definition of $\{\xi,2\}$-norm and Cauchy convergence principle. It is clearly observed that discrete time-varying delays and distributed delays have important effects on the convergence conditions of QVNNs. If the discrete delays are time-invariant, i.e. $\tau_{pq}(t)=\tau_{pq}$, then the expression $\frac{1+\eta_{pq}}{1-\eta_{pq}}e^{\varsigma\tau_{pq}}|b_{pq}^l|\gamma_q^w$ of (5) will be changed into $e^{\varsigma\tau_{pq}}|b_{pq}^l|\gamma_q^w$. If the distributed delays are nonexistent, then the inequality (5) will be much simpler.

Remark 4: The difficulty of this theorem is how to deal with the discrete time-varying delays and distributed delays via $\{\xi,2\}$-norm. By constructing special $\{\xi,2\}$-norm-type Lyapunov functional, this problem is worked out. In addition, for the sake of discussion, we suppose that the activate functions $f_q^l(x_q^l(t))$ and $g_q^l(x_q^l(t))$ belong to class L_2 for any $l\in M$ and $q=1,2,\cdots,n$. If $f_q^l(x_q^l(t))$ belongs to class L_1 and $g_q^l(x_q^l(t))$ belongs to class L_2 for any $l\in M$ and $q=1,2,\cdots,n$, then the more complex results can be obtained, which can't be discussed in this paper. Furthermore, according to this theorem, we can give the exponential stability criteria of QVNNs (1) with time-invariant asynchronous delays as follows.

Corollary 1: Under assumptions (H1) and (H3), if there exist real constants $\varsigma>0$, and $\xi_p^l>0(p=1,2,\cdots,n,l\in M)$, such that

$$
2\xi_q^L(-d_q+\varsigma)+\sum_{p=1}^{n}\sum_{(l,w)\in M^L}\xi_q^L\Big(|a_{qp}^l|\lambda_p^w
$$

$$+e^{\varsigma\tau_{qp}}|b_{qp}^l|\gamma_p^w+|c_{qp}^l|\gamma_p^w\int_0^\infty k_{qp}(s)e^{\varsigma s}ds\Big)$$
$$+\sum_{p=1}^n\sum_{(l,w)\in M^L}\xi_p^L\Big(|a_{pq}^l|\lambda_q^w+e^{\varsigma\tau_{pq}}|b_{pq}^l|\gamma_q^w$$
$$+|c_{pq}^l|\gamma_q^w\int_0^\infty k_{pq}(s)e^{\varsigma s}ds\Big)\le 0$$

holds for every $L\in M$ and $q=1,2,\cdots,n$. Then, the dynamical system (1) with time-invariant asynchronous discrete and distribute delays has an unique equilibrium point x^*, which is globally exponentially stable.

Proof: Let

$$V^L(t)=\sum_{p=1}^n\Big(\xi_p^L(u_p^L(t))^2+\xi_p^L\sum_{q=1}^n\sum_{(l,w)\in M^L}$$
$$e^{\varsigma\tau_{pq}}|b_{pq}^l|\gamma_q^w\int_{t-\tau_{pq}}^t(u_q^w(s))^2ds$$
$$+\xi_p^L\sum_{q=1}^n\sum_{(l,w)\in M^L}|c_{pq}^l|\gamma_q^w\int_0^\infty k_{pq}(s)$$
$$e^{\varsigma s}(\int_{t-s}^t(u_q^w(r))^2dr)ds\Big),$$

then we can prove this corollary by the above similar process, the details are omitted. □

From the above theorem and corollary, we can easily obtain the following two corollaries, respectively.

Corollary 2: Under assumptions (H1), (H2) and (H3), if there exist real constants $\xi_p^l>0(p=1,2,\cdots,n,l\in M)$, such that

$$-2d_q\xi_q^L+\sum_{p=1}^n\sum_{(l,w)\in M^L}\xi_q^L\big(|a_{qp}^l|\lambda_p^w+(1+\eta_{qp})$$
$$|b_{qp}^l|\gamma_p^w+|c_{qp}^l|\gamma_p^w\big)+\sum_{p=1}^n\sum_{(l,w)\in M^L}\xi_p^L\big(|a_{pq}^l|\lambda_q^w$$
$$+\frac{1+\eta_{pq}}{1-\eta_{pq}}|b_{pq}^l|\gamma_q^w+|c_{pq}^l|\gamma_q^w\big)<0$$

holds for every $L\in M$ and $q=1,2,\cdots,n$. Then, the dynamical system (1) with time-varying discrete and distribute delays has an unique equilibrium point x^*, which is globally exponentially stable.

Corollary 3: Under assumptions (H1) and (H3), if there exist real constants $\xi_p^l>0(p=1,2,\cdots,n,l\in M)$, such that

$$-2d_q\xi_q^L+\sum_{p=1}^n\sum_{(l,w)\in M^L}\xi_q^L\big(|a_{qp}^l|\lambda_p^w+|b_{qp}^l|\gamma_p^w+|c_{qp}^l|\gamma_p^w\big)$$
$$+\sum_{p=1}^n\sum_{(l,w)\in M^L}\xi_p^L\big(|a_{pq}^l|\lambda_q^w+|b_{pq}^l|\gamma_q^w+|c_{pq}^l|\gamma_q^w\big)<0$$

holds for every $L\in M$ and $q=1,2,\cdots,n$. Then, the dynamical system (1) with time-invariant asynchronous discrete and distribute delays has an unique equilibrium point x^*, which is globally exponentially stable.

Remark 5: In this section, the existence and exponential stability sufficient criteria are derived by the definition of $\{\xi,2\}$-norm and Cauchy convergence principle. If the known conditions are changed, more corollaries can be obtained, which are omitted. In the process of discussing, in order to deal with discrete and distributed delays, the construction of special Lyapunov-type functional is very important. It is worthy noting that the discrete time-varying $\tau_{pq}(t)$ can be changed with p,q and t, which is asynchronous time-delay [25].

IV. NUMERICAL EXAMPLE

Example 4.1: Consider the following two-dimensional QVNNs with discrete time-varying delays and distributed delays:

$$\dot{x}_p(t)=-d_px_p(t)+\sum_{q=1}^2a_{pq}f_q(x_q(t))$$
$$+\sum_{q=1}^2b_{pq}g_q(x_q(t-\tau_{pq}(t)))$$
$$+\sum_{q=1}^2c_{pq}\int_0^\infty k_{pq}(s)g_q(x_q(t-s))ds+u_p,\quad(6)$$

where $x_p(t)=x_p^R(t)+\imath x_p^I(t)+\jmath x_p^J(t)+\kappa x_p^K(t)\in\mathbb{Q}$, $f_q(x_q(t))=\tanh(x_q^R(t))+\imath\tanh(x_q^I(t))+\jmath\tanh(x_q^J(t))+\kappa\tanh(x_q^K(t))$, $g_q(x_q(t))=\frac{1}{2}(|x_q^R(t)+1|-|x_q^R(t)-1|)+\imath\frac{1}{2}(|x_q^I(t)+1|-|x_q^I(t)-1|)+\jmath\frac{1}{2}(|x_q^J(t)+1|-|x_q^J(t)-1|)+\kappa\frac{1}{2}(|x_q^K(t)+1|-|x_q^K(t)-1|)$, and $\tau_{pq}(t)=\frac{1}{p}+\frac{1}{50q\pi}\sin(\frac{\pi}{2}t)$ holds for $p,q=1,2$. $d_1=8$, $d_2=8$, $a_{11}=0.3-0.1\imath-0.3\jmath+0.1\kappa$, $a_{12}=-0.1+0.3\imath+0.2\jmath-0.2\kappa$, $a_{21}=-0.2+0.2\imath+0.1\jmath-0.3\kappa$, $a_{22}=0.1-0.1\imath-0.2\jmath+0.1\kappa$, $b_{11}=0.1-0.2\imath+0.2\jmath-0.2\kappa$, $b_{12}=-0.2+0.2\imath-0.2\jmath+0.1\kappa$, $b_{21}=-0.3+0.2\imath-0.2\jmath+0.2\kappa$, $b_{22}=0.1-0.3\imath+0.2\jmath-0.2\kappa$, $c_{11}=0.1-0.1\imath-0.2\jmath+0.2\kappa$, $c_{12}=-0.2-0.2\imath+0.1\jmath+0.1\kappa$, $c_{21}=0.1+0.2\imath-0.2\jmath-0.1\kappa$, $c_{22}=-0.2-0.1\imath+0.1\jmath+0.2\kappa$, $k_{11}(s)=k_{22}(s)=e^{-s}$, $k_{12}(s)=k_{21}(s)=e^{-2s}$ $u_1=-1+\imath+\jmath+2\kappa$, $u_2=1-2\imath+3\jmath-2\kappa$.

Obviously, for any $p,q=1,2$, $f_q(x_q(t))$ and $g_q(x_q(t))$ satisfy the assumption (H1), $k_{pq}(s)$ satisfies the assumption (H3), $\tau_{pq}(t)\le 1+\frac{1}{50\pi}\le 1.01$ is bounded, and $|\dot{\tau}_{pq}(t)|=|\frac{1}{100q}\cos(\frac{\pi}{2}t)|\le\frac{1}{100}<1$. Therefore, assumptions (H1), (H2) and (H3) are all satisfied. Let $\xi_p^L=0.1$ and $\varsigma=0.53$, it can be calculated that

$$2\xi_q^L(-d_q+\varsigma)+\sum_{p=1}^2\sum_{(l,w)\in M^L}\xi_q^L\Big(|a_{qp}^l|\lambda_p^w$$
$$+(1+\eta_{qp})e^{\varsigma\tau_{qp}}|b_{qp}^l|\gamma_p^w+|c_{qp}^l|\gamma_p^w\int_0^\infty k_{qp}(s)e^{\varsigma s}ds\Big)$$

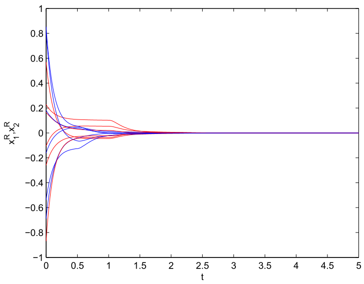


FIGURE 1. State trajectories of $x_1^R(t)$ and $x_2^R(t)$ for Example 1.

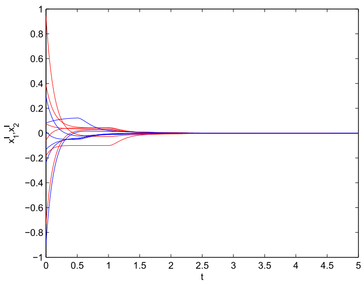


FIGURE 2. State trajectories of $x_1^I(t)$ and $x_2^I(t)$ for Example 1.

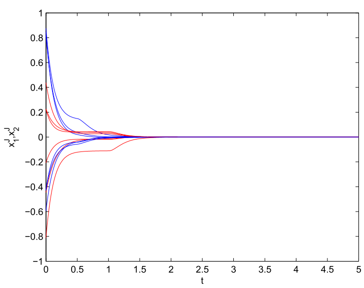


FIGURE 3. State trajectories of $x_1^J(t)$ and $x_2^J(t)$ for Example 1.

$$+\sum_{p=1}^{2}\sum_{(l,w)\in M^L}\xi_p^L\left(|a_{pq}^l|\lambda_q^w+\frac{1+\eta_{pq}}{1-\eta_{pq}}e^{\varsigma\tau_{pq}}|b_{pq}^l|\gamma_q^w\right.$$
$$\left.+|c_{pq}^l|\gamma_q^w\int_0^{\infty}k_{pq}(s)e^{\varsigma s}ds\right)\leq 0$$

holds for any $L\in M$ and $p=1,2$. Thus, the conditions of Theorem 1 have been satisfied, and the dynamical system (6) with discrete time-varying delays and distributed delays has an unique global exponential stable equilibrium point, which can be shown by Figures 1-5.

Remark 6: In this example, discrete time-varying delays and distributed delays are considered for QVNNs with external input $u = 0$. By calculation, the conditions of Theorem 1 are satisfied and the conclusions can be obtained. Figures 1-4 show the state trajectories of every real part and image parts,respectively. Figure 5 shows the state trajectories of all real part and image parts. From these figures, it is observed that discrete time-varying asynchronous delays and distributed delays have crucial effects on the convergence process of QVNNs.

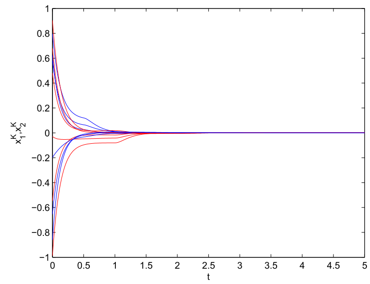


FIGURE 4. State trajectories of $x_1^K(t)$ and $x_2^K(t)$ for Example 1.

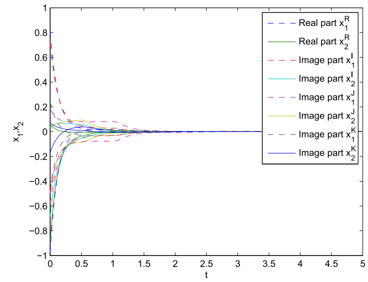


FIGURE 5. State trajectories of $x_1(t)$ and $x_2(t)$ for Example 1.

V. CONCLUSIONS

In the past several decades, the existence and stability of solutions have been the evergreen important topics since various NN models were constructed. Recently, with the development of quaternion application in technology, QVNNs, as an important side of quaternion, have been presented and studied by many scholars. Similar to other NNs, the existence and stability of solutions is one of the most important research contents of QVNNs. Based on the facts, this paper has focused on the existence and exponential stability of solutions of the QVNNs with discrete time-varying delays and distributed delays by means of $\{\xi, 2\}$-norm. Due to the noncommutativity of quaternion multiplication, the delayed QVNNs system has been firstly decomposed into four real-number systems by Hamilton rules. Then the novel stability definition about delayed QVNNs has been introduced by the definition of $\{\xi, 2\}$-norm, and some assumptions of discrete time-varying delays and distributed delays have been given. By constructing special $\{\xi, 2\}$-norm-type Lyapunov functional and taking advantage of Cauchy convergence principle and monotone function, the existence, uniqueness and exponential stability sufficient criteria of solutions have been obtained. Several corollaries have been derived from the main theory. Although the stability of QVNNs with discrete time-varying delays and distributed delays has been studied by $\{\xi, 2\}$-norm, it can also be used to investigate the synchronization of NNs, which is our future research content. Finally, one numerical example and its simulated figures have been given to illustrate the effectiveness of the obtained conclusions in this paper.

REFERENCES

[1] J. Hopfield, "Neural networks and physical systems with emergent collective computational abilities," *Proc. Nat. Acad. Sci. USA*, vol. 79, no. 8, pp. 2554–2558, 1982.

[2] M. Cohen and S. Grossberg, "Absolute stability of global pattern formation and parallel memory storage by competitive neural networks," *IEEE Trans. Syst., Man, Cybern.*, vol. SMC-13, no. 5, pp. 815–826, Sep./Oct. 1983.

[3] J. Hu and J. Wang, "Global uniform asymptotic stability of memristor-based recurrent neural networks with time delays," in *Proc. Int. Joint Conf. Neural Netw. (IJCNN)*, Barcelona, Spain, Jul. 2010, pp. 2127–2134.

[4] T. Chen, "Global exponential stability of delayed hopfield neural networks," *Neural Netw.*, vol. 14, no. 8, pp. 977–980, Oct. 2001.

[5] J.-L. Wang, H.-N. Wu, T. Huang, and S.-Y. Ren, "Passivity and synchronization of linearly coupled reaction-diffusion neural networks with adaptive coupling," *IEEE Trans. Cybern.*, vol. 45, no. 9, pp. 1942–1952, Sep. 2015.

[6] X. Li, R. Rakkiyappan, and G. Velmurugan, "Dissipativity analysis of memristor-based complex-valued neural networks with time-varying delays," *Inf. Sci.*, vol. 294, pp. 645–665, Feb. 2015.

[7] S. Wen, Z. Zeng, T. Huang, and Y. Zhang, "Exponential adaptive lag synchronization of memristive neural networks via fuzzy method and applications in pseudorandom number generators," *IEEE Trans. Fuzzy Syst.*, vol. 22, no. 6, pp. 1704–1713, Dec. 2014.

[8] J. Lu, C. Ding, J. Lou, and J. Cao, "Outer synchronization of partially coupled dynamical networks via pinning impulsive controllers," *J. Franklin Inst.*, vol. 352, pp. 5024–5041, Nov. 2015.

[9] H. Wang, S. Duan, T. Huang, L. Wang, and C. Li, "Exponential stability of complex-valued memristive recurrent neural networks," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 28, no. 3, pp. 766–771, Mar. 2017.

[10] Y. Xu, C. Liu, R. Lu, and C.-Y. Su, "Remote estimator design for time-delay neural networks using communication state information," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 29, no. 10, pp. 5149–5158, Oct. 2018.

[11] Y. Cao, S. Wang, Z. Guo, T. Huang, and S. Wen, "Synchronization of memristive neural networks with leakage delay and parameters mismatch via event-triggered control," *Neural Netw.*, vol. 119, pp. 178–189, Nov. 2019.

[12] S. Wang, Y. Cao, T. Huang, Y. Chen, P. Li, and S. Wen, "Sliding mode control of neural networks via continuous or periodic sampling event-triggering algorithm," *Neural Netw.*, vol. 121, pp. 140–147, Jan. 2020.

[13] T. Liang, Y. Yang, Y. Liu, and L. Li, "Existence and global exponential stability of almost periodic solutions to Cohen–Grossberg neural networks with distributed delays on time scales," *Neurocomputing*, vol. 123, pp. 207–215, Jan. 2014.

[14] T. Faria and J. J. Oliveira, "General criteria for asymptotic and exponential stabilities of neural network models with unbounded delays," *Appl. Math. Comput.*, vol. 217, no. 23, pp. 9646–9658, Aug. 2011.

[15] X. Li and S. Song, "Impulsive control for existence, uniqueness, and global stability of periodic solutions of recurrent neural networks with discrete and continuously distributed delays," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 24, no. 6, pp. 868–877, Jun. 2013.

[16] H. Wang, J. Tan, T. Huang, and S. Duan, "Impulsive delayed integro-differential inequality and its application on IMNNs with discrete and distributed delays," *Neurocomputing*, vol. 341, pp. 99–106, May 2019.

[17] J. Hu and J. Wang, "Global stability of complex-valued recurrent neural networks with time-delays," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 23, no. 6, pp. 853–865, Jun. 2012.

[18] X. Liu and T. Chen, "Global exponential stability for complex-valued recurrent neural networks with asynchronous time delays," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 27, no. 3, pp. 593–606, Mar. 2016.

[19] X. Chen, Z. Zhao, Q. Song, and J. Hu, "Multistability of complex-valued neural networks with time-varying delays," *Appl. Math. Comput.*, vol. 294, pp. 18–35, Feb. 2017.

[20] Y. Liu, D. Zhang, J. Lou, J. Lu, and J. Cao, "Stability analysis of quaternion-valued neural networks: Decomposition and direct approaches," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 29, no. 9, pp. 4201–4211, Sep. 2018.

[21] X. Chen, Z. Li, Q. Song, J. Hu, and Y. Tan, "Robust stability analysis of quaternion-valued neural networks with time delays and parameter uncertainties," *Neural Netw.*, vol. 91, pp. 55–65, Jul. 2017.

[22] Q. Song and X. Chen, "Multistability analysis of quaternion-valued neural networks with time delays," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 29, no. 11, pp. 5430–5440, Nov. 2018.

[23] X. Chen, Q. Song, Z. Li, Z. Zhao, and Y. Liu, "Stability analysis of continuous-time and discrete-time quaternion-valued neural networks with linear threshold neurons," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 29, no. 7, pp. 2769–2781, Jul. 2018.

[24] Y. Liu, D. Zhang, and J. Lu, "Global exponential stability for quaternion-valued recurrent neural networks with time-varying delays," *Nonlinear Dyn.*, vol. 87, no. 1, pp. 553–565, Jan. 2017.

[25] X. Liu and Z. Li, "Global μ-stability of quaternion-valued neural networks with unbounded and asynchronous time-varying delays," *IEEE Access*, vol. 7, pp. 9128–9141, 2019.

[26] S. M. A. Pahnehkolaei, A. Alfi, and J. A. T. Machado, "Delay-dependent stability analysis of the QUAD vector field fractional order quaternion-valued memristive uncertain neutral type leaky integrator echo state neural networks," *Neural Netw.*, vol. 117, pp. 307–327, Sep. 2019.

[27] T. Isokawa, T. Kusakabe, N. Matsui, and F. Peper, *Quaternion Neural Network and Its Application*. Berlin, Germany: Springer, 2003, pp. 318–324.

[28] M. Yoshida, Y. Kuroe, and T. Mori, "Models of Hopfield-type quaternion neural networks and their energy functions," *Int. J. Neural Syst.*, vol. 15, no. 01n02, pp. 129–135, Feb. 2005.

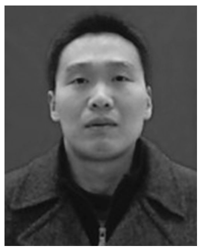
[29] X. Chen, Q. Song, and Z. Li, "Design and analysis of quaternion-valued neural networks for associative memories," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 48, no. 12, pp. 2305–2314, Dec. 2018.

[30] X. Chen and Q. Song, "State estimation for quaternion-valued neural networks with multiple time delays," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 49, no. 11, pp. 2278–2287, Nov. 2019.

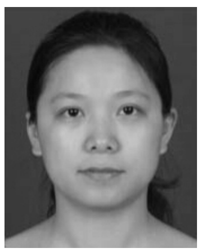
[31] C.-A. Popa and E. Kaslik, "Multistability and multiperiodicity in impulsive hybrid quaternion-valued neural networks with mixed delays," *Neural Netw.*, vol. 99, pp. 1–18, Mar. 2018.

[32] Z. Tu, Y. Zhao, N. Ding, Y. Feng, and W. Zhang, "Stability analysis of quaternion-valued neural networks with both discrete and distributed delays," *Appl. Math. Comput.*, vol. 343, pp. 342–353, Feb. 2019.

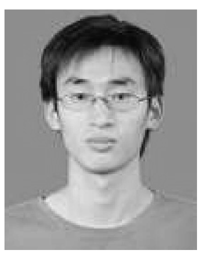
[33] S. M. A. Pahnehkolaei, A. Alfi, and J. A. T. Machado, "Delay independent robust stability analysis of delayed fractional quaternion-valued leaky integrator echo state neural networks with QUAD condition," *Appl. Math. Comput.*, vol. 359, pp. 278–293, Oct. 2019.



HUAMIN WANG received the B.S. degree in applied mathematics from Chongqing University, Chongqing, China, in 2003, the M.S. degree in applied mathematics from Henan Normal University, Xinxiang, China, in 2012, and the Ph.D. degree in applied mathematics from the College of Electronics and Information Engineering, Southwest University, Chongqing, in 2016. He is currently an Associate Professor with the Department of Mathematics, Luoyang Normal University, Luoyang, Henan, China. His current research interests include neural networks, stability and synchronization analysis, impulsive differential dynamical systems, and memristive systems.



JIE TAN received the B.S. and M.S. degrees in applied mathematics and the Ph.D. degree from Southwest University, Chongqing, China, in 2001, 2006, and 2017, respectively. She is currently an Associate Professor with the College of Mathematics, Physics, and Data Science, Chongqing University of Science and Technology, Chongqing. Her current research interest covers impulsive differential systems, stability analysis of neural networks, and stochastic differential systems.



SHIPING WEN received the M.Eng. degree in control science and engineering from the School of Automation, Wuhan University of Technology, Wuhan, China, in 2010, and the Ph.D. degree in control science and engineering from the School of Automation, Huazhong University of Science and Technology, Wuhan, in 2013. He is currently a Professor with the Centre for Artificial Intelligence, University of Technology Sydney, Australia. His current research interests include memristor-based circuits and systems, neural networks, and deep learning. He currently serves as an Associate Editor for IEEE Access.