Linguistic Z-Number Bonferroni Mean and Linguistic Z-Number Geometric Bonferroni Mean Operators: Their Applications in Portfolio Selection Problems

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ABSTRACT The optimal combination of assets can be selected by the traditional portfolio theory which uses historical quantitative data to represent the future return of assets. However, quantitative information is inaccessible in most cases and experts can help investors and fund managers by providing qualitative information. According to above discussion, a new multi-stage qualitative approach is proposed to select the optimal portfolio under linguistic Z-number environment. To achieve this aim, this study firstly develops the Bonferroni mean (BM) operator and the geometric Bonferroni mean (GBM) operator under the linguistic Z-number environment, and introduces linguistic Z-number Bonferroni mean (LZBM) operator and linguistic Z-number geometric Bonferroni mean (LZGBM) operator to aggregate the qualitative evaluation information. Then, using the developed aggregation operators, two qualitative portfolio selection models are proposed based on the max-score rule and the score-accuracy trade-off rule for the general investors and risky investors, respectively. Finally, to illustrate the validity of the proposed models, a case study including 20 corporations of Tehran stock exchange market in Iran is provided and the obtained results are analyzed. Moreover, the qualitative proposed models are compared with another available model. The obtained results indicate that the qualitative proposed approach can help investors and fund managers to make more credible decisions so that they can select the optimal assets with considering different criteria when experts are assured about their assessments or opinions. Therefore, the qualitative proposed models are superior and more general in comparison with the other ones due to capturing the reliability of information. Also, the obtained results show the influence of reliability measures in investment processes.

INDEX TERMS Portfolio selection, linguistic Z-number, reliability, linguistic scale function, aggregation operator.

I. INTRODUCTION

Decision-making is an inseparable fact in the real world and people always encounter different decisions in daily life. Since complexity of the decision-making environment is daily grown, decision makers (DMs) or managers no longer are satisfied with the classic or deterministic procedures for representing their knowledge about events and they are intending to make more fruitful decisions using more

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general data. Hence, Zadeh [1], [2] introduced the fuzzy set theory to consider uncertainty existing in assessment information. However, the fuzzy set theory has shortcomings in the uncertain information representing. The fuzzy sets devote a single quantity to each element, which is called the membership degree. These single quantities are placed within the interval [0, 1]. But, they are not sufficient to represent comprehensive data because of the loss of complete knowledge [3]. Hence, in order to better represent qualitative data, some authors introduced various types of fuzzy sets such as the interval-valued fuzzy sets [4], the type-2 fuzzy sets [5],

the fuzzy multi-sets [6], the intuitionistic fuzzy sets [7], the hesitant fuzzy sets [8] and the interval-valued hesitant fuzzy sets [9]. Although the fuzzy set theory and its developed forms have created massive solstice in the description of uncertain and ambiguous data, they are unable to capture the reliability of information. Therefore, Zadeh [10] introduced a new concept called Z-number in order to make more general structure for representing uncertain information related to the real world phenomena. Thus, in comparison with fuzzy set theory and its extended forms, Z-numbers are powerful tools for describing and modeling human knowledge. The studies carried out on Z-numbers can be categorized into two aspects. The first aspect includes the basic investigations such as conversion techniques [11], [12], arithmetic operations over Z-numbers [13]–[15], ranking methods [16]–[20] and development researches [21], [22]. The second aspect is related to the application of Z-numbers in decision-making and optimization problems. Yang and Wang [23] presented a stochastic multi-criteria acceptability analysis (SMAA) model to aid decision making problem under discrete Z-numbers. Shen and Wang [24] developed a vlse kriterijumsk optimizacija kompromisno resenjethe method called Z-VIKOR method based on a new comprehensive weighted distance measure under Z-numbers. Kang *et al.* [25] presented a total utility under Z-number information and expressed its mathematical properties to be applied in the decision making problems. Sadi-Nezhad and Sotoudeh-Anvari [26] proposed a new data envelopment analysis called Z-DEA. They utilized the concept of Z-numbers to describe input and output parameters in their model. Yaakob and Gegov [27] introduced a novel technique for order preference by similarity to ideal solution (TOPSIS) method based on the concept of Z-numbers called Z-TOPSIS. Generally, the main advantage of Z-numbers is that they incorporate possibilistic and probabilistic constraints simultaneously. In other words, the concept of Z-numbers can describe the real preferences, qualitative assessments, estimations and opinions of decision-makers. Moreover, they reflect reliability in addition to uncertainty and ambiguity relevant to information simultaneously. This particularity causes Z-numbers to be especially useful for representing evaluation information particularly financial information.

Due to the fuzziness and uncertainty of the real information and the intrinsic ambiguity of expert knowledge, experts usually utilize linguistic terms to express the evaluation information. For instance, to assess the financial performance of a corporation or estimate the future return of an asset, experts can apply linguistic terms such as "low", "medium", and "high". The utilization of linguistic terms increases the credibility and flexibility of the decision-making models [28]. Linguistic variables were widely applied in different areas [29]–[34]. Rodríguez *et al.* [29], [35] and Wang *et al.* [36] introduced hesitant fuzzy linguistic term sets (HFLTSs) and intuitionistic linguistic sets, respectively. The fuzziness and randomness existing in linguistic terms accurately match up the constraint and probability measure

of Z-numbers [37]. Therefore, in order to combine linguistic terms and Z-numbers, Peng and Wang [38] introduced hesitant uncertain linguistic Z-numbers (HULZNs). Furthermore, Wang *et al.* [37] introduced linguistic Z-numbers and extended their operations. The main advantage of the linguistic Z-numbers (LZNs) is that they consider both the fuzziness and randomness existing in qualitative information simultaneously. For instance, to evaluate the financial performance of a firm or estimate the future return of an asset, a linguistic Z-number can apply linguistic terms such as "low", "medium" and "high" to describe fuzzy constraint and employ linguistic terms such as "seldom", "sometimes" and "usually" to represent the probability measure. In other words, a linguistic Z-number such as (low, sometimes) can be applied to assess the financial performance of a firm. As discussed, linguistic Z-numbers are more general, credible and flexible structures for describing qualitative data, especially in financial markets. Therefore, in this study, the linguistic Z-numbers are employed to better describe the qualitative financial data that experts estimate.

Generally, aggregation operators are useful tools to deal with the assessment information under various types of uncertain environments. Compared with traditional decision-making methods, aggregation operators can obtain aggregated values and then rank them. Due to the superior of the aggregation methods in decision-making problems [39], the idea of aggregation operators has become an interesting issue of research. Up to now, there are only five aggregation operators to aggregate evaluation information under linguistic Z-number environment. Peng and Wang [38] developed the hesitant uncertain linguistic Z-number power weighted average operator and the hesitant uncertain linguistic Z-number power weighted geometric operator. Also, Mahmoodi *et al.* [40] proposed three weighted averaging (WA) aggregation operators, which is called linguistic Z-number weighted averaging (LZWA), linguistic Z-number ordered weighted averaging (LZOWA) and linguistic Z-number hybrid weighted averaging (LZHWA). The main deficiency of these aggregation operators is that they do not consider the interrelationship of input data and suppose that all input arguments are independent. Thus, to overcome this limitation, it is essential to develop more general aggregation operators for handling interrelationship between input arguments. Bonferroni mean (BM) operator [3] and geometric Bonferroni mean (GBM) operator [41] capture interrelationship among input arguments. In this study, the BM and GBM operators are developed to aggregate linguistic Z-number information.

One of the initial concerns on any portfolio selection problem is to choose the optimal combination of assets. However, in the real investment processes, investors or fund managers encounter positions in which the financial criteria have conflict together. Markowitz [42] presented a quantitative model called mean-variance model in order to choose the optimal portfolio based on a trade-off between the expected return and the risk. He established a basic framework to optimize port-

folio selection problems. Creating more diversified portfolios is the principal privilege of Markowitz model. In spite of the considerable significance of the Markowitz theory, some scholars believed that diversification does not necessarily lead to diminish the overall risk when the financial markets encounter some political and economic tensions. On the other hand, the Markowitz's model only applied historical data to represent the financial information while there are many non-probability factors in the financial markets that cannot be resolved applying probability theory. Thus, in order to create more realistic models, many researchers developed Markowitz model with consideration of different assumptions, constraints and objectives under various environments. Some accomplished investigations on the Markowitz's mean-variance model under different types of uncertain environments are presented below. For example, Li and Xu [43] presented a multi-objective portfolio selection model with fuzzy random returns. Zhou and Xu [44] suggested a new qualitative approach to choose the optimal portfolio under hesitant fuzzy environment. Mansour *et al.* [45] proposed a multi-objective imprecise programming for financial portfolio selection model under fuzzy environment. However, the mentioned portfolio models have shortcomings. For instance, they are unable to capture expert's reliability in the evaluation information modeling. Therefore, in this study, two qualitative portfolio models are proposed based on aggregation operators under linguistic Z-number environment. On the basis of the mentioned analysis, the linguistic Z-numbers can more credibly and flexibly describe the knowledge, experience and judgment of experts. Moreover, BM and GBM operators aggregate assessment information under linguistic Z-number environment and the aggregated values are applied to formulate portfolio selection problem based on the max-score rule and the score-accuracy trade-off rule. Therefore, the proposed method is very powerful to manage the investment process by the incorporation of asset allocation problem and aggregation operators under linguistic Z-number environment. Thus, the primary aims of this study are briefly highlighted as follows:

1. Uncertainty is an inseparable element of financial markets. Thus, there are different predictions along with various reliabilities about the future performance of an asset. Since linguistic Z-numbers are more general structures to represent the real world information and incorporate possibilistic and probabilistic constraint, we apply the concept of linguistic Z-numbers to better describe and evaluate the future performance of each asset with respect to different criteria.

2. We develop BM and GBM operators under linguistic Z-number environment, and introduce LZBM operator and LZGBM operator. The suggested operators are very useful tools to analyze linguistic Z-number data, and they generalize some available operators.

3. We propose a qualitative approach to model cardinality constrained portfolio selection problem based

on LZBM and LZGBM operators under linguistic Z-number environment, and we use the max-score rule and the score-accuracy trade-off rule to formulate the models.

The remainder of this study is structured as follows: Section 2 includes the necessary prerequisite definitions. In Section 3, we introduce two aggregation operators under linguistic Z-number environment and present their properties. Two new qualitative models are developed in order to select the optimal assets in a portfolio based on linguistic Z-number data in Section 4. Section 5 provides the required actual data as a case study and the obtained results. Finally, concluding remarks and future work suggestions are presented in Section 6.

II. PRELIMINARIES

This section covers the definitions, materials and fundamental operations of linguistic term sets and Z-numbers which are required in this study.

A. THE LINGUISTIC TERM SETS

Definition 1: Suppose $S = \{s_i | i = 0, 1, \ldots, 2p\}$ is a finite set of discrete linguistic terms having odd cardinality where s_i shows a possible value of linguistic variable and p is a nonnegative integer. S possesses the following features [32], [46]:

I. S is ordered: $s_i < s_j$ if and only if $i < j$

II. S conforms negation operator: $neg(s_i) = s_{2p-i}$.

S is a discrete set of linguistic terms. When information is aggregated together, the obtained results are not usually compatible with the elements existing in the language evaluation scale. Hence, a continuous set is required to better analyze real decision making problems. Xu [30], [47]–[49] presented a continuous linguistic term set $\bar{S} = \{s_i | i \in [0, p]\}$ where p is a adequately large positive integer and $s_i < s_j$ if $i < j$. If s_i belongs to S, then s_i is named an original linguistic term, otherwise, s_i is a virtual linguistic term. Often, original linguistic terms are employed to assess alternatives or express expert's opinions. Virtual linguistic terms are only obtained from the operations in order to maintain all resulted information [37], [38], [50].

B. LINGUISTIC SCALE FUNCTIONS

Computation with linguistic terms is placed within the category of computing with words. Therefore, to simplify the computation and define the arithmetic operations under uncertain linguistic environment, Wang *et al.* [51] presented some linguistic scale functions (LSF). LSFs allocate various semantic values to linguistic terms under variant status in order to apply data and to represent the semantics more effectually [51]. As stated by Xu [49], $s_i \in S$ has the strictly monotonically ascending relationship with its subscript i.

Definition 2: Suppose θ_i $(i = 0, 1, \ldots, 2p)$, which belongs to the positive real number set, is a numerical value, then the linguistic scale function f, which is a strictly monotonically ascending function with respect to label i, is described as

follows [51]:

$$f: \quad s_i \rightarrow \theta_i \quad where \ 0 \le \theta_0 \le \theta_1 \le \ldots \le \theta_{2p}$$

When DMs express their opinions by applying the linguistic terms $s_i \epsilon S$, the numeric value θ_i indicates the DM's preference. Hence, the function or value denotes the semantics of the linguistic terms [51].

Some linguistic scale functions, which can be used in the operations of linguistic Z-numbers, are introduced in the following [51].

1) $f_1 (s_i)$ is described according to the subscript function $g (s_i) = i$ as follows [51]:

$$f_1 (s_i) = \theta_i = \frac{i}{2p} \ (i = 0, 1, \ldots, 2p) \quad and \ \theta_i \epsilon \ [0, 1] \tag{1}$$

2) $f_2 (s_i)$ is described according to the exponential scale as follows [51]:

$$f_2 (s_i) = \theta_i = \begin{cases} \dfrac{a^p - a^{p-i}}{2a^p - 2} & 0 \le i \le p \\ \dfrac{a^p + a^{i-p} - 2}{2a^p - 2} & p + 1 \le i \le 2p \end{cases} \tag{2}$$

Two methods are proposed to compute the value of a. The first one that has been suggested by Bao [50] is an experimental method. This method states that the value of a can be placed within the interval of [1.36, 1.4]. The second one is a subjective method. According to this method, the value of a can be obtained as $a^k = mora = \sqrt[k]{m}$ where k is the scale level and m shows the importance ratio [50], [51].

3) The improved linguistic scale function ($f_3 (s_i)$) is described according to the concept of prospect theory as follows [3], [52]:

$$f_3 (s_i) = \theta_i = \begin{cases} \dfrac{p^\alpha - (p - i)^\alpha}{2p^\alpha} & 0 \le i \le p \\ \dfrac{p^\beta + (i - p)^\beta}{2p^\beta} & p + 1 \le i \le 2p \end{cases} \tag{3}$$

where α and β belong to the interval of (0, 1] and $\theta_i = \frac{i}{2p}$ when $\alpha = \beta = 1$.

The mentioned linguistic scale functions (LSFs) can be developed as a strictly monotonically ascending and continuous function in order to maintain all the provided data and simplify the computations. Then, it can easily be stated as follows [51]:

$$f^*: \quad \bar{S} \rightarrow R^+ \quad s.t. f^* (s_i) = \theta_i$$

C. LINGUISTIC Z-NUMBERS AND THEIR OPERATIONS
In this subsection, linguistic Z-numbers (LZNs) are introduced and their arithmetic operations are defined.

D. Z-NUMBERS AND LINGUISTIC Z-NUMBERS
Decision making and selection problems have always been an inseparable fact in the life of human. Usually, the decisions are made based on information, knowledge and experiments of DMs. Naturally, information relevant to real world events is imperfect, uncertain and ambiguous. Hence, fuzziness imposing soft restrictions on the values of uncertain variables is used to describe incomplete information [53]. With increment of complexity and dynamism of decision space, taking into account only fussiness is no longer adequate. Therefore, Zadeh [10] presented the concept of Z-numbers in order to better investigate uncertain variables. Z-numbers associate partial reliability as an intrinsic feature of information with fuzziness.

Definition 3 ([10] Z-Number): A Z-number is formed two components and is indicated as an ordered pair of fuzzy numbers, $Z = \left(\tilde{A}, \tilde{B} \right)$. The first component, $\tilde{A}$, is a fuzzy constraint on the values which can be allocated to an uncertain variable X. The second component, $\tilde{B}$, shows a soft restriction on the measure of reliability of $\tilde{A}$. Often $\tilde{A}$ and $\tilde{B}$ are expressed by using linguistic terms.

Definition 4 ([37] Linguistic Z-Numbers): Suppose V is a universe of discourse and two finite discrete linguistic term sets representing different preference data are defined as $S = \{s_0, s_1, \ldots, s_{2p}\}$ *and* $S' = \{s'_0, s'_1, \ldots, s'_{2q}\}$ where p and q are nonnegative integers. Therefore, a Z-number linguistic set in V is defined as follows:

$$Z = \left\{ \left(v, A_{\emptyset(v)}, B_{\varphi(v)} \right) | v \epsilon V \right\} \tag{4}$$

where $A_{\emptyset(v)}$ is a fuzzy constraint on the values which can be assigned to the uncertain variable and $B_{\varphi(v)}$ characterizes a reliability measure of the first component. $A_{\emptyset(v)}$ and $B_{\varphi(v)}$ are described by using uncertain linguistic terms.

E. LINGUISTIC Z-NUMBER OPERATIONS
Some operations of linguistic Z-numbers were extended by Wang *et al.* [37] with consideration of both components simultaneously. The proposed operations preserve both the flexibility of linguistic term sets and the reliability value of Z-numbers.

Definition 5 [37]: Suppose two linguistic Z-numbers are defined as $z_i = \left(A_{\emptyset(i)}, B_{\varphi(i)} \right)$ and $z_j = \left(A_{\emptyset(j)}, B_{\varphi(j)} \right)$. f^* and g^* functions can be chosen among $f_1 (s_i)$, $f_2 (s_i)$ and $f_3 (s_i)$. Hence, some operations of linguistic Z-numbers are defined as follows:

$neg (z_i)$

$$= \left(f^{*-1} \left(f^* \left(A_{2p} \right) - f^* \left(A_{\emptyset(i)} \right) \right), g^{*-1} \left(g^* \left(B_{2q} \right) \right. \right.$$
$$\left. \left. - g^* \left(B_{\varphi(i)} \right) \right) \right) \tag{5}$$

$z_i + z_j$

$$= \left(f^{*-1} \left(f^* \left(A_{\emptyset(i)} \right) + f^* \left(A_{\emptyset(j)} \right) \right), \right.$$
$$\left. g^{*-1} \left(\frac{f^* \left(A_{\emptyset(i)} \right) \times g^* \left(B_{\varphi(i)} \right) + f^* \left(A_{\emptyset(j)} \right) \times g^* \left(B_{\varphi(j)} \right)}{f^* \left(A_{\emptyset(i)} \right) + f^* \left(A_{\emptyset(j)} \right)} \right) \right) \tag{6}$$

$$\rho z_i = \left(f^{*-1} \left(\rho f^* \left(A_{\emptyset(i)} \right) \right), B_{\varphi(i)} \right), \quad \rho \geq 0 \tag{7}$$

$$z_i \times z_j = \left(f^{*-1} \left(f^* \left(A_{\emptyset(i)} \right) f^* \left(A_{\emptyset(j)} \right) \right), g^{*-1} \left(g^* \left(B_{\varphi(i)} \right) g^* \left(B_{\varphi(j)} \right) \right) \right) \tag{8}$$

$$z_i^\rho = \left(f^{*-1} \left(f^* \left(A_{\emptyset(i)} \right)^\rho \right), g^{*-1} \left(g^* \left(B_{\varphi(i)} \right)^\rho \right) \right), \quad \rho \geq 0 \tag{9}$$

Definition 6 [37]: Suppose $z_i = \left(A_{\emptyset(i)}, B_{\varphi(i)} \right)$ is a linguistic Z-numbers. Then, the score function of linguistic Z-number is equal to:

$$E(z_i) = f^* \left(A_{\emptyset(i)} \right) \times g^* \left(B_{\varphi(i)} \right) \tag{10}$$

The accuracy function of linguistic Z-number is as follows:

$$D(z_i) = f^* \left(A_{\emptyset(i)} \right) \times \left(1 - g^* \left(B_{\varphi(i)} \right) \right) \tag{11}$$

By using the score and accuracy functions, a comparison technique is defined for two LZNs as follows [37]:

 I. If $E(z_i) > E(z_j)$, then $z_i > z_j$
 II. If $E(z_i) = E(z_j)$, then
    If $D(z_i) > D(z_j)$, then $z_i > z_j$
    If $D(z_i) = D(z_j)$, then $z_i > z_j$

III. LINGUISTIC Z-NUMBER AGGREGATION OPERATORS
This section presents two aggregation operators for linguistic Z-numbers and states their properties.

A. BONFERRONI MEAN (BM) OPERATORS AND GEOMETRIC BONFERRONI MEAN OPERATORS
Definition 7 ([3] Bonferroni Mean Operators): Suppose c_i $(i = 1, \ldots, m)$ is a set of non-negative real numbers and $r, t \geq 0$. Therefore, the Bonferroni mean (BM) operator is defined as follows:

$$BM^{r,t}(c_1, c_2, \ldots, c_m) = \left(\frac{1}{m(m-1)} \sum_{\substack{i=1 \\ j=1 \\ i \neq j}}^{m} c_i^r c_j^t \right)^{\frac{1}{r+t}} \tag{12}$$

Definition 8 ([41] Geometric Bonferroni Mean Operators): Suppose c_i $(i = 1, \ldots, m)$ is a set of non-negative real numbers and $r, t \geq 0$. Therefore, the geometric Bonferroni

mean (GBM) operator is defined as follows:

$$GBM^{r,t}(c_1, c_2, \ldots, c_m) = \frac{1}{r+t} \left(\prod_{\substack{i=1 \\ j=1 \\ i \neq j}}^{m} \left(rc_i + tc_j \right) \right)^{\frac{1}{m(m-1)}} \tag{13}$$

The Bonferroni mean (BM) and geometric Bonferroni mean (GBM) operators introduced by Bonferroni [3] and Zhu *et al.* [41], respectively, are only proper when the input parameters are non-negative exact values. Following to Bonferroni [3] and Zhu *et al.* [41], some researchers extended BM and GBM operators under uncertain information. For example, Wei *et al.* [54] developed two aggregation operators called the uncertain linguistic Bonferroni mean (ULBM) and uncertain linguistic geometric Bonferroni mean (ULGBM) operators under uncertain linguistic information. Liu and Jin [55] presented some BM operators based on trapezoid fuzzy linguistic variables. Liu *et al.* [56] developed some partitioned Bonferroni mean operators for intuitionistic fuzzy numbers. In the following subsections, we develop the BM and GBM operators under linguistic Z-numbers environment.

B. LINGUISTIC Z-NUMBER BONFERRONI MEAN (LZBM) OPERATORS
Definition 9: Suppose $Z = \left\{ z_i = \left(A_{\emptyset(i)}, B_{\varphi(i)} \right) | i = 1, \ldots, m \right\}$ is a set of LZNs and $r, t \geq 0$, then the LZBM operator can be defined as follows:

$$LZBM^{r,t}(z_1, z_2, \ldots, z_m) = \left(\frac{1}{m(m-1)} \sum_{\substack{i=1 \\ j=1 \\ i \neq j}}^{m} z_i^r z_j^t \right)^{\frac{1}{r+t}} \tag{14}$$

Based on the operations of linguistic Z-numbers represented in Definition 5, we can get the following result.

Theorem 1: Suppose $z_i = \left(A_{\emptyset(i)}, B_{\varphi(i)} \right)$ $(i = 1, \ldots, m)$ is a set of LZNs. Then the aggregated value acquired according to LZBM operator is also a LZN and it is computed as follows (15), shown at the bottom of this page.

Proof: According to Definition 5, the aggregated value is also a LZN. Now, by using the mathematical induction method, Eq. (15) will easily be proven in the following.

At first, assume $m = 2$ and $z_1 = \left(A_{\emptyset(1)}, B_{\varphi(1)} \right)$ and $z_2 = \left(A_{\emptyset(2)}, B_{\varphi(2)} \right)$. Hence, according to Definition 5,

$$LZBM^{r,t}(z_1, z_2, \ldots, z_m) = \left(f^{*-1} \left(\left(\frac{1}{m(m-1)} \sum_{\substack{i=1 \\ j=1 \\ i \neq j}}^{m} \left(f^* \left(A_{\emptyset(i)} \right)^r f^* \left(A_{\emptyset(j)} \right)^t \right) \right)^{\frac{1}{r+t}} \right), \right.$$
$$\left. g^{*-1} \left(\left(\frac{\sum_{\substack{i=1 \\ j=1 \\ i \neq j}}^{m} \left(\left(f^* \left(A_{\emptyset(i)} \right)^r f^* \left(A_{\emptyset(j)} \right)^t \right) \times \left(g^* \left(B_{\varphi(i)} \right)^r g^* \left(B_{\varphi(j)} \right)^t \right) \right)}{\sum_{\substack{i=1 \\ j=1 \\ i \neq j}}^{m} \left(f^* \left(A_{\emptyset(i)} \right)^r f^* \left(A_{\emptyset(j)} \right)^t \right)} \right)^{\frac{1}{r+t}} \right) \right) \tag{15}$$

z_1^r, as shown at the bottom of this page. Then, $LZBM^{r,t}(z_1, z_2)$, as shown at the bottom of this page.

It is obvious that Theorem 1 is true for the case of $m = 2$. Now, it is assumed that this theorem is true for $m = k$, therefore, we have $LZBM^{r,t}(z_1, z_2, \ldots, z_k)$, as shown at the bottom of this page.

Finally, for the case of $m = k + 1$, we can acquire the following expression $LZBM^{r,t}(z_1, z_2, \ldots, z_{k+1})$, as shown at the bottom of this page.

Since this theorem is true for the case of $m = k$, it will also be true for the case of $m = k + 1$. Consequently, according to the mathematical induction, Eq. (15) is true for all m. ∎

$$z_1^r$$
$$= \left(f^{*-1}\left(\left(f^*\left(A_{\emptyset(1)} \right) \right)^r \right), g^{*-1}\left(\left(g^*\left(B_{\varphi(1)} \right) \right)^r \right) \right)$$

$$z_2^t$$
$$= \left(f^{*-1}\left(\left(f^*\left(A_{\emptyset(2)} \right) \right)^t \right), g^{*-1}\left(\left(g^*\left(B_{\varphi(2)} \right) \right)^t \right) \right)$$

$$z_1^r \times z_2^t$$
$$= \left(f^{*-1}\left(\left(f^*\left(A_{\emptyset(1)} \right) \right)^r \times \left(f^*\left(A_{\emptyset(2)} \right) \right)^t \right), g^{*-1}\left(\left(g^*\left(B_{\varphi(1)} \right) \right)^r \times \left(g^*\left(B_{\varphi(2)} \right) \right)^t \right) \right)$$

$$z_2^r \times z_1^t$$
$$= \left(f^{*-1}\left(\left(f^*\left(A_{\emptyset(2)} \right) \right)^r \times \left(f^*\left(A_{\emptyset(1)} \right) \right)^t \right), g^{*-1}\left(\left(g^*\left(B_{\varphi(2)} \right) \right)^r \times \left(g^*\left(B_{\varphi(1)} \right) \right)^t \right) \right)$$

$$(z_1^r \times z_2^t) + (z_2^r \times z_1^t)$$
$$= \left(f^{*-1}\left(\left(\left(f^*\left(A_{\emptyset(1)} \right) \right)^r \times \left(f^*\left(A_{\emptyset(2)} \right) \right)^t \right) + \left(\left(f^*\left(A_{\emptyset(2)} \right) \right)^r \times \left(f^*\left(A_{\emptyset(1)} \right) \right)^t \right) \right), \right.$$
$$\left. g^{*-1}\left(\frac{\left(\left(f^*\left(A_{\emptyset(1)} \right) \right)^r \times \left(f^*\left(A_{\emptyset(2)} \right) \right)^t \right) \times \left(\left(g^*\left(B_{\varphi(1)} \right) \right)^r \times \left(g^*\left(B_{\varphi(2)} \right) \right)^t \right) + \left(\left(f^*\left(A_{\emptyset(2)} \right) \right)^r \times \left(f^*\left(A_{\emptyset(1)} \right) \right)^t \times \left(\left(g^*\left(B_{\varphi(2)} \right) \right)^r \times \left(g^*\left(B_{\varphi(1)} \right) \right)^t \right) \right) \right)}{\left(\left(f^*\left(A_{\emptyset(1)} \right) \right)^r \times \left(f^*\left(A_{\emptyset(2)} \right) \right)^t \right) + \left(\left(f^*\left(A_{\emptyset(2)} \right) \right)^r \times \left(f^*\left(A_{\emptyset(1)} \right) \right)^t \right)} \right) \right)$$

$$\frac{1}{m(m-1)}\left((z_1^r \times z_2^t) + (z_2^r \times z_1^t) \right)$$
$$= \left(f^{*-1}\left(\frac{1}{m(m-1)}\left(\left(\left(f^*\left(A_{\emptyset(1)} \right) \right)^r \times \left(f^*\left(A_{\emptyset(2)} \right) \right)^t \right) + \left(\left(f^*\left(A_{\emptyset(2)} \right) \right)^r \times \left(f^*\left(A_{\emptyset(1)} \right) \right)^t \right) \right) \right), \right.$$
$$\left. g^{*-1}\left(\frac{\left(\left(f^*(A_{\emptyset(1)}) \right)^r \times \left(f^*(A_{\emptyset(2)}) \right)^t \right) \times \left(\left(g^*(B_{\varphi(1)}) \right)^r \times \left(g^*(B_{\varphi(2)}) \right)^t \right) + \left(\left(f^*(A_{\emptyset(2)}) \right)^r \times \left(f^*(A_{\emptyset(1)}) \right)^t \times \left(\left(g^*(B_{\varphi(2)}) \right)^r \times \left(g^*(B_{\varphi(1)}) \right)^t \right) \right) \right)}{\left(f^*(A_{\emptyset(1)}) \right)^r \times \left(f^*(A_{\emptyset(2)}) \right)^t + \left(\left(f^*(A_{\emptyset(2)}) \right)^r \times \left(f^*(A_{\emptyset(1)}) \right)^t \right)} \right) \right)$$

$$LZBM^{r,t}(z_1, z_2) = \left(f^{*-1}\left(\left(\frac{1}{m(m-1)}\left(\left(\left(f^*\left(A_{\emptyset(1)} \right) \right)^r \times \left(f^*\left(A_{\emptyset(2)} \right) \right)^t \right) + \left(\left(f^*\left(A_{\emptyset(2)} \right) \right)^r \times \left(f^*\left(A_{\emptyset(1)} \right) \right)^t \right) \right) \right)^{\frac{1}{r+t}} \right), \right.$$
$$\left. g^{*-1}\left(\left(\frac{\left(\left(f^*(A_{\emptyset(1)}) \right)^r \times \left(f^*(A_{\emptyset(2)}) \right)^t \right) \times \left(\left(g^*(B_{\varphi(1)}) \right)^r \times \left(g^*(B_{\varphi(2)}) \right)^t \right) + \left(\left(f^*(A_{\emptyset(2)}) \right)^r \times \left(f^*(A_{\emptyset(1)}) \right)^t \times \left(\left(g^*(B_{\varphi(2)}) \right)^r \times \left(g^*(B_{\varphi(1)}) \right)^t \right) \right) \right)}{\left(f^*(A_{\emptyset(1)}) \right)^r \times \left(f^*(A_{\emptyset(2)}) \right)^t + \left(\left(f^*(A_{\emptyset(2)}) \right)^r \times \left(f^*(A_{\emptyset(1)}) \right)^t \right)} \right)^{\frac{1}{r+t}} \right) \right)$$

$$LZBM^{r,t}(z_1, z_2, \ldots, z_k) = \left(f^{*-1}\left(\left(\frac{1}{m(m-1)} \sum_{\substack{i=1 \\ j=1 \\ i \neq j}}^{k} \left(\left(f^*\left(A_{\emptyset(i)} \right) \right)^r \times \left(f^*\left(A_{\emptyset(j)} \right) \right)^t \right) \right)^{\frac{1}{r+t}} \right), \right.$$
$$\left. g^{*-1}\left(\left(\frac{\sum_{\substack{i=1 \\ j=1 \\ i \neq j}}^{k} \left(\left(\left(f^*\left(A_{\emptyset(i)} \right) \right)^r \times \left(f^*\left(A_{\emptyset(j)} \right) \right)^t \right) \times \left(\left(g^*\left(B_{\varphi(i)} \right) \right)^r \times \left(g^*\left(B_{\varphi(j)} \right) \right)^t \right) \right)}{\sum_{\substack{i=1 \\ j=1 \\ i \neq j}}^{k} \left(\left(f^*\left(A_{\emptyset(i)} \right) \right)^r \times \left(f^*\left(A_{\emptyset(j)} \right) \right)^t \right)} \right)^{\frac{1}{r+t}} \right) \right)$$

$$LZBM^{r,t}(z_1, z_2, \ldots, z_{k+1}) = \left(f^{*-1}\left(\left(\frac{1}{m(m-1)} \sum_{\substack{i=1 \\ j=1 \\ i \neq j}}^{k+1} \left(\left(f^*\left(A_{\emptyset(i)} \right) \right)^r \times \left(f^*\left(A_{\emptyset(j)} \right) \right)^t \right) \right)^{\frac{1}{r+t}} \right), \right.$$
$$\left. g^{*-1}\left(\left(\frac{\sum_{\substack{i=1 \\ j=1 \\ i \neq j}}^{k+1} \left(\left(\left(f^*\left(A_{\emptyset(i)} \right) \right)^r \times \left(f^*\left(A_{\emptyset(j)} \right) \right)^t \right) \times \left(\left(g^*\left(B_{\varphi(i)} \right) \right)^r \times \left(g^*\left(B_{\varphi(j)} \right) \right)^t \right) \right)}{\sum_{\substack{i=1 \\ j=1 \\ i \neq j}}^{k} \left(\left(f^*\left(A_{\emptyset(i)} \right) \right)^r \times \left(f^*\left(A_{\emptyset(j)} \right) \right)^t \right)} \right)^{\frac{1}{r+t}} \right) \right)$$

It can easily be proven that the LZBM operator has the following properties.

Theorem 2 (Idempotency): Suppose $z_i = \left(A_{\emptyset(i)}, B_{\varphi(i)}\right)$ $(i = 1, \ldots, m)$ is a collection of linguistic Z-numbers. If all z_i are equal, i.e. $z_i = \left(A_{\emptyset(i)}, B_{\varphi(i)}\right) = \bar{z} = \left(A_{\emptyset}, B_{\varphi}\right)$ for all i, then

$$\text{LZBM}^{r,t}(z_1, z_2, \ldots, z_m) = \bar{z}$$

Proof: $\text{LZBM}^{r,t}(z_1, z_2, \ldots, z_m)$, as shown at the bottom of this page.

Theorem 3 (Boundedness): Suppose $z_i = \left(A_{\emptyset(i)}, B_{\varphi(i)}\right)$ $(i = 1, \ldots, m)$ is a collection of linguistic Z-numbers, and let

$$z^- = \min_i z_i = \left(A_{z^-}, B_{z^-}\right) = \left(\min_i \left(A_{\emptyset(i)}\right), \min_i \left(B_{\varphi(i)}\right)\right)$$

$$z^+ = \max_i z_i = \left(A_{z^+}, B_{z^+}\right) = \left(\max_i \left(A_{\emptyset(i)}\right), \max_i \left(B_{\varphi(i)}\right)\right)$$

Then

$$z^- \leq \text{LZBM}^{r,t}(z_1, z_2, \ldots, z_m) \leq z^+$$

Proof: Let $\text{LZBM}^{r,t}(z_1, z_2, \ldots, z_m) = z_T = \left(A_{\emptyset(T)}, B_{\varphi(T)}\right)$, since $A_{z^-} \leq A_{\emptyset(i)}$ and $B_{z^-} \leq B_{\varphi(i)}$. Thus,

$f^*\left(A_{z^-}\right) \leq f^*\left(A_{\emptyset(i)}\right)$ and $g^*\left(B_{z^-}\right) \leq g^*\left(B_{\varphi(i)}\right)$ and $z^- < z_i$. Then, we have:

$$E\left(z^-\right) = f^*\left(A_{z^-}\right) \times g^*\left(B_{z^-}\right) \leq E\left(z_T\right)$$
$$= f^*\left(A_{\emptyset(T)}\right) \times g^*\left(B_{\varphi(T)}\right)$$

where z_T, as shown at the bottom of the this page.

Therefore, $z^- \leq \text{LZBM}^{r,t}(z_1, z_2, \ldots, z_m)$ can be obtained based on the comparison method of linguistic Z-numbers presented in Definition 6. Similarly, $\text{LZBM}^{r,t}(z_1, z_2, \ldots, z_m) \leq z^+$ can also be obtained. Thus, $z^- \leq \text{LZBM}^{r,t}(z_1, z_2, \ldots, z_m) \leq z^+$. ∎

Theorem 4 (Monotonicity): Suppose $z_i = \left(A_{\emptyset(i)}, B_{\varphi(i)}\right)$ and $z'_i = \left(A'_{\emptyset(i)}, B'_{\varphi(i)}\right)$ $(i = 1, \ldots, m)$ are two sets of LZNs. If $\forall i$: $z_i \leq z'_i$, then

$$\text{LZBM}^{r,t}(z_1, z_2, \ldots, z_m) \leq \text{LZBM}^{r,t}(z'_1, z'_2, \ldots, z'_m).$$

Proof: Since $z_i \leq z'_i$, according to Definition 5, it can be resulted that $A_{\emptyset(i)} \leq A'_{\emptyset(i)}$ and $B_{\varphi(i)} \leq B'_{\varphi(i)}$, for all i. Therefore, $f^*\left(A_{\emptyset(i)}\right) \leq f^*\left(A'_{\emptyset(i)}\right)$ and $g^*\left(B_{\varphi(i)}\right) \leq g^*\left(B'_{\varphi(i)}\right)$. Consequently, we have $\text{LZBM}^{r,t}(z_1, z_2, \ldots, z_m)$, as shown at the bottom of the next page.

$$\text{LZBM}^{r,t}(z_1, z_2, \ldots, z_m) = \text{LZBM}^{r,t}(\bar{z}, \bar{z}, \ldots, \bar{z})$$

$$= \left(f^{*-1}\left(\left(\frac{1}{m(m-1)} \sum_{\substack{i=1 \\ j=1 \\ i \neq j}}^{m} \left(f^*(A_\emptyset)^r f^*(A_\emptyset)^t\right)\right)^{\frac{1}{r+t}}\right),\right.$$

$$\left. g^{*-1}\left(\left(\frac{\sum_{\substack{i=1 \\ j=1 \\ i \neq j}}^{m} \left(\left(f^*(A_\emptyset)^r f^*(A_\emptyset)^t\right) \times \left(g^*(B_\varphi)^r g^*(B_\varphi)^t\right)\right)}{\sum_{\substack{i=1 \\ j=1 \\ i \neq j}}^{m} \left(f^*(A_\emptyset)^r f^*(A_\emptyset)^t\right)}\right)^{\frac{1}{r+t}}\right)\right)$$

$$= \left(f^{*-1}\left(\left(\left(f^*(A_\emptyset)^r f^*(A_\emptyset)^t\right)\right)^{\frac{1}{r+t}}\right), g^{*-1}\left(\left(\frac{\left(f^*(A_\emptyset)^r f^*(A_\emptyset)^t\right) \times \left(g^*(B_\varphi)^r g^*(B_\varphi)^t\right)}{f^*(A_\emptyset)^r f^*(A_\emptyset)^t}\right)^{\frac{1}{r+t}}\right)\right)$$

$$= \left(f^{*-1}\left(f^*(A_\emptyset)\right), g^{*-1}\left(g^*(B_\varphi)\right)\right) = \left(A_\emptyset, B_\varphi\right) = \bar{z}$$

$$z_T = \left(A_{\emptyset(T)}, B_{\varphi(T)}\right)$$

$$= \left(f^{*-1}\left(\left(\frac{1}{m(m-1)} \sum_{\substack{i=1 \\ j=1 \\ i \neq j}}^{m} \left(f^*\left(A_{\emptyset(i)}\right)^r f^*\left(A_{\emptyset(j)}\right)^t\right)\right)^{\frac{1}{r+t}}\right),\right.$$

$$\left. g^{*-1}\left(\left(\frac{\sum_{\substack{i=1 \\ j=1 \\ i \neq j}}^{m} \left(\left(f^*\left(A_{\emptyset(i)}\right)^r f^*\left(A_{\emptyset(j)}\right)^t\right) \times \left(g^*\left(B_{\varphi(i)}\right)^r g^*\left(B_{\varphi(j)}\right)^t\right)\right)}{\sum_{\substack{i=1 \\ j=1 \\ i \neq j}}^{m} \left(f^*\left(A_{\emptyset(i)}\right)^r f^*\left(A_{\emptyset(j)}\right)^t\right)}\right)^{\frac{1}{r+t}}\right)\right)$$

Theorem 5 (Commutativity): Suppose $z_i = \left(A_{\emptyset(i)}, B_{\varphi(i)}\right)$ and $z_i' = \left(A_{\emptyset(i)}', B_{\varphi(i)}'\right)$ $(i = 1, \ldots, m)$ are two sets of LZNs, where $z_i' = \left(A_{\emptyset(i)}', B_{\varphi(i)}'\right)$ $(i = 1, \ldots, m)$ is any permutation of $z_i = \left(A_{\emptyset(i)}, B_{\varphi(i)}\right)$ $(i = 1, \ldots, m)$, then

$$\text{LZBM}^{r,t}\left(z_1, z_2, \ldots, z_m\right) = \text{LZBM}^{r,t}\left(z_1', z_2', \ldots, z_m'\right).$$

Proof: Let $z_i' = \left(A_{\emptyset(i)}', B_{\varphi(i)}'\right)$, then $\text{LZBM}^{r,t}\left(z_1', z_2', \ldots, z_m'\right)$, as shown at the bottom of this page.

Since $z_i' = \left(A_{\emptyset(i)}', B_{\varphi(i)}'\right)$ $(i = 1, \ldots, m)$ is any permutation of $z_i = \left(A_{\emptyset(i)}, B_{\varphi(i)}\right)$ $(i = 1, \ldots, m)$, then by Eq. 15, we have $\text{LZBM}^{r,t}\left(z_1', z_2', \ldots, z_m'\right)$, as shown at the bottom of the next page. ∎

Now some special cases of the LZBM operator are discussed with respect to the parameters r and t.

Case 1: If $t \to 0$, then the LZBM operator is reduced to the linguistic Z-number generalized mean (LZGM) operator. $\lim_{t \to 0} \text{LZBM}^{r,t}\left(z_1, z_2, \ldots, z_m\right)$, as shown at the bottom of the next page.

Case 2: If $r \to 2$ and $t \to 0$, then the LZBM operator is reduced to the linguistic Z-number square mean (LZSM) operator.

$$\begin{aligned}LZBM^{2,0}\left(z_1, z_2, \ldots, z_m\right) &= \left(\frac{1}{m}\sum_{i=1}^{m} z_i^2\right)^{\frac{1}{2}} \\ &= LZSM\left(z_1, z_2, \ldots, z_m\right)\end{aligned}$$

Case 3: If $r \to 1$ and $t \to 0$, then the LZBM operator is reduced to the linguistic Z-number mean (LZM) operator.

$$\begin{aligned}LZBM^{1,0}\left(z_1, z_2, \ldots, z_m\right) &= \frac{1}{m}\sum_{i=1}^{m} z_i \\ &= LZM\left(z_1, z_2, \ldots, z_m\right)\end{aligned}$$

$$
\begin{aligned}
&\text{LZBM}^{r,t}\left(z_1, z_2, \ldots, z_m\right) \\
&= \left(f^{*-1}\left(\left(\frac{1}{m(m-1)}\sum_{\substack{i=1\\j=1\\i\neq j}}^{m}\left(f^*\left(A_{\emptyset(i)}\right)^r f^*\left(A_{\emptyset(j)}\right)^t\right)\right)^{\frac{1}{r+t}}\right),\right.\\
&\qquad \left. g^{*-1}\left(\left(\frac{\sum_{\substack{i=1\\j=1\\i\neq j}}^{m}\left(\left(f^*\left(A_{\emptyset(i)}\right)^r f^*\left(A_{\emptyset(j)}\right)^t\right) \times \left(g^*\left(B_{\varphi(i)}\right)^r g^*\left(B_{\varphi(j)}\right)^t\right)\right)}{\sum_{\substack{i=1\\j=1\\i\neq j}}^{m}\left(f^*\left(A_{\emptyset(i)}\right)^r f^*\left(A_{\emptyset(j)}\right)^t\right)}\right)^{\frac{1}{r+t}}\right)\right)\\
&\leq \left(f^{*-1}\left(\left(\frac{1}{m(m-1)}\sum_{\substack{i=1\\j=1\\i\neq j}}^{m}\left(f^*\left(A_{\emptyset(i)}'\right)^r f^*\left(A_{\emptyset(j)}'\right)^t\right)\right)^{\frac{1}{r+t}}\right),\right.\\
&\qquad \left. g^{*-1}\left(\left(\frac{\sum_{\substack{i=1\\j=1\\i\neq j}}^{m}\left(\left(f^*\left(A_{\emptyset(i)}'\right)^r f^*\left(A_{\emptyset(j)}'\right)^t\right) \times \left(g^*\left(B_{\varphi(i)}'\right)^r g^*\left(B_{\varphi(j)}'\right)^t\right)\right)}{\sum_{\substack{i=1\\j=1\\i\neq j}}^{m}\left(f^*\left(A_{\emptyset(i)}'\right)^r f^*\left(A_{\emptyset(j)}'\right)^t\right)}\right)^{\frac{1}{r+t}}\right)\right)\\
&= \text{LZBM}^{r,t}\left(z_1', z_2', \ldots, z_m'\right)
\end{aligned}
$$

$$
\begin{aligned}
&\text{LZBM}^{r,t}\left(z_1', z_2', \ldots, z_m'\right) = \left(f^{*-1}\left(\left(\frac{1}{m(m-1)}\sum_{\substack{i=1\\j=1\\i\neq j}}^{m}\left(f^*\left(A_{\emptyset(i)}'\right)^r f^*\left(A_{\emptyset(j)}'\right)^t\right)\right)^{\frac{1}{r+t}}\right),\right.\\
&\qquad \left. g^{*-1}\left(\left(\frac{\sum_{\substack{i=1\\j=1\\i\neq j}}^{m}\left(\left(f^*\left(A_{\emptyset(i)}'\right)^r f^*\left(A_{\emptyset(j)}'\right)^t\right) \times \left(g^*\left(B_{\varphi(i)}'\right)^r g^*\left(B_{\varphi(j)}'\right)^t\right)\right)}{\sum_{\substack{i=1\\j=1\\i\neq j}}^{m}\left(f^*\left(A_{\emptyset(i)}'\right)^r f^*\left(A_{\emptyset(j)}'\right)^t\right)}\right)^{\frac{1}{r+t}}\right)\right)
\end{aligned}
$$

Case 4: If $r \rightarrow 1$ *and* $t \rightarrow 1$, then the LZBM operator is reduced to the linguistic Z-number interrelated square mean (LZISM) operator.

$$LZBM^{1,1}(z_1, z_2, \ldots, z_m) = \left(\frac{1}{m(m-1)} \sum_{\substack{i=1 \\ j=1 \\ i \neq j}}^{m} z_i z_j \right)^{\frac{1}{2}}$$
$$= LZISM(z_1, z_2, \ldots, z_m)$$

C. LINGUISTIC Z-NUMBER GEOMETRIC BONFERRONI MEAN (LZGBM) OPERATORS

Definition 10: Suppose $Z = \{ z_i = (A_{\emptyset(i)}, B_{\varphi(i)}) | i = 1, \ldots, m \}$ is a set of LZNs and $r, t \geq 0$, then the LZGBM operator can be defined as follows:

$$LZGBM^{r,t}(z_1, z_2, \ldots, z_m)$$
$$= \frac{1}{r+t} \left(\prod_{\substack{i=1 \\ j=1 \\ i \neq j}}^{m} (rz_i + tz_j) \right)^{\frac{1}{m(m-1)}} \quad (16)$$

$$LZBM^{r,t}(z'_1, z'_2, \ldots, z'_m) = \left(f^{*-1} \left(\left(\frac{1}{m(m-1)} \sum_{\substack{i=1 \\ j=1 \\ i \neq j}}^{m} \left(f^*\left(A'_{\emptyset(i)}\right)^r f^*\left(A'_{\emptyset(j)}\right)^t \right) \right)^{\frac{1}{r+t}} \right), \right.$$
$$\left. g^{*-1} \left(\left(\frac{\sum_{\substack{i=1 \\ j=1 \\ i \neq j}}^{m} \left(\left(f^*\left(A'_{\emptyset(i)}\right)^r f^*\left(A'_{\emptyset(j)}\right)^t \right) \times \left(g^*\left(B'_{\varphi(i)}\right)^r g^*\left(B'_{\varphi(j)}\right)^t \right) \right)}{\sum_{\substack{i=1 \\ j=1 \\ i \neq j}}^{m} \left(f^*\left(A'_{\emptyset(i)}\right)^r f^*\left(A'_{\emptyset(j)}\right)^t \right)} \right)^{\frac{1}{r+t}} \right) \right)$$

$$= \left(f^{*-1} \left(\left(\frac{1}{m(m-1)} \sum_{\substack{i=1 \\ j=1 \\ i \neq j}}^{m} \left(f^*\left(A_{\emptyset(i)}\right)^r f^*\left(A_{\emptyset(j)}\right)^t \right) \right)^{\frac{1}{r+t}} \right), \right.$$
$$\left. g^{*-1} \left(\left(\frac{\sum_{\substack{i=1 \\ j=1 \\ i \neq j}}^{m} \left(\left(f^*\left(A_{\emptyset(i)}\right)^r f^*\left(A_{\emptyset(j)}\right)^t \right) \times \left(g^*\left(B_{\varphi(i)}\right)^r g^*\left(B_{\varphi(j)}\right)^t \right) \right)}{\sum_{\substack{i=1 \\ j=1 \\ i \neq j}}^{m} \left(f^*\left(A_{\emptyset(i)}\right)^r f^*\left(A_{\emptyset(j)}\right)^t \right)} \right)^{\frac{1}{r+t}} \right) \right)$$
$$= LZBM^{r,t}(z_1, z_2, \ldots, z_m).$$

$$\lim_{t \to 0} LZBM^{r,t}(z_1, z_2, \ldots, z_m) = \lim_{t \to 0} \left(\frac{1}{m(m-1)} \sum_{\substack{i=1 \\ j=1 \\ i \neq j}}^{m} z_i^r z_j^t \right)^{\frac{1}{r+t}} = \lim_{t \to 0} \left(f^{*-1} \left(\left(\frac{1}{m(m-1)} \sum_{\substack{i=1 \\ j=1 \\ i \neq j}}^{m} \left(f^*\left(A_{\emptyset(i)}\right)^r f^*\left(A_{\emptyset(j)}\right)^t \right) \right)^{\frac{1}{r+t}} \right), \right.$$
$$\left. g^{*-1} \left(\left(\frac{\sum_{\substack{i=1 \\ j=1 \\ i \neq j}}^{m} \left(\left(f^*\left(A_{\emptyset(i)}\right)^r f^*\left(A_{\emptyset(j)}\right)^t \right) \times \left(g^*\left(B_{\varphi(i)}\right)^r g^*\left(B_{\varphi(j)}\right)^t \right) \right)}{\sum_{\substack{i=1 \\ j=1 \\ i \neq j}}^{m} \left(f^*\left(A_{\emptyset(i)}\right)^r f^*\left(A_{\emptyset(j)}\right)^t \right)} \right)^{\frac{1}{r+t}} \right) \right)$$

$$= \left(f^{*-1} \left(\left(\frac{1}{m} \sum_{i=1}^{m} \left(f^*\left(A_{\emptyset(i)}\right)^r \right) \right)^{\frac{1}{r}} \right), \right.$$
$$\left. g^{*-1} \left(\left(\frac{\sum_{i=1}^{m} \left(\left(f^*\left(A_{\emptyset(i)}\right)^r \right) \times \left(g^*\left(B_{\varphi(i)}\right)^r \right) \right)}{\sum_{i=1}^{m} \left(f^*\left(A_{\emptyset(i)}\right)^r \right)} \right)^{\frac{1}{r}} \right) \right)$$

$$= \left(\frac{1}{m} \sum_{i=1}^{m} z_i^r \right)^{\frac{1}{r}} = LZGM^{r,0}(z_1, z_2, \ldots, z_m)$$

Based on the operations of linguistic Z-numbers represented in Definition 5, we can get the following result.

Theorem 6: Suppose $z_i = \left(A_{\emptyset(i)}, B_{\varphi(i)}\right)$ $(i = 1, \ldots, m)$ is a set of LZNs. Then the aggregated value acquired according to LZGBM operator is also a LZN and it is computed as follows (17), as shown at the bottom of this page.

Similarly, Theorem 1 can easily be proven by using the mathematical induction method.

It can easily be proven that the LZBM operator has the following properties.

Theorem 7 (Idempotency): Suppose $z_i = \left(A_{\emptyset(i)}, B_{\varphi(i)}\right)$ $(i = 1, \ldots, m)$ is a collection of linguistic Z-numbers. If all z_i are equal, i.e. $z_i = \left(A_{\emptyset(i)}, B_{\varphi(i)}\right) = \bar{z} = \left(A_{\emptyset}, B_{\varphi}\right)$ for all i, then

$$\text{LZGBM}^{\text{r,t}} (z_1, z_2, \ldots, z_m) = \bar{z}$$

Theorem 8 (Boundedness): Suppose $z_i = \left(A_{\emptyset(i)}, B_{\varphi(i)}\right)$ $(i = 1, \ldots, m)$ is a collection of linguistic Z-numbers, and let

$$z^- = \min_i z_i = \left(A_{z^-}, B_{z^-}\right) = \left(\min_i \left(A_{\emptyset(i)}\right), \min_i \left(B_{\varphi(i)}\right)\right)$$

$$z^+ = \max_i z_i = \left(A_{z^+}, B_{z^+}\right) = \left(\max_i \left(A_{\emptyset(i)}\right), \max_i \left(B_{\varphi(i)}\right)\right)$$

Then

$$z^- \leq \text{LZGBM}^{\text{r,t}} (z_1, z_2, \ldots, z_m) \leq z^+$$

Theorem 9 (Monotonicity): Suppose $z_i = \left(A_{\emptyset(i)}, B_{\varphi(i)}\right)$ and $z'_i = \left(A'_{\emptyset(i)}, B'_{\varphi(i)}\right)$ $(i = 1, \ldots, m)$ are two sets of LZNs. If $\forall i : z_i \leq z'_i$, then

$$\text{LZGBM}^{\text{r,t}} (z_1, z_2, \ldots, z_m) \leq \text{LZGBM}^{\text{r,t}} \left(z'_1, z'_2, \ldots, z'_m\right).$$

Theorem 10 (Commutativity): Suppose $z_i = \left(A_{\emptyset(i)}, B_{\varphi(i)}\right)$ and $z'_i = \left(A'_{\emptyset(i)}, B'_{\varphi(i)}\right)$ $(i = 1, \ldots, m)$ are two sets of LZNs, where $z'_i = \left(A'_{\emptyset(i)}, B'_{\varphi(i)}\right)$ $(i = 1, \ldots, m)$ is any permutation of $z_i = \left(A_{\emptyset(i)}, B_{\varphi(i)}\right)$ $(i = 1, \ldots, m)$, then

$$\text{LZGBM}^{\text{r,t}} (z_1, z_2, \ldots, z_m) = \text{LZGBM}^{\text{r,t}} \left(z'_1, z'_2, \ldots, z'_m\right).$$

Now some special cases of the LZGBM operator are discussed with respect to the parameters r and t.

Case 1: If $t \to 0$, then the LZGBM operator is reduced to the linguistic Z-number generalized geometric mean (LZGGM) operator. $\lim\limits_{t \to 0} \text{LZGBM}^{\text{r,t}} (z_1, z_2, \ldots, z_m)$, as shown at the bottom of this page.

Case 2: If $r \to 2$ *and* $t \to 0$, then the LZGBM operator is reduced to the linguistic Z-number square geometric mean (LZSGM) operator.

$$LZGBM^{2,0} (z_1, z_2, \ldots, z_m) = \frac{1}{2} \left(\prod_{i=1}^{m} (2z_i)\right)^{\frac{1}{m}}$$
$$= LZSGM (z_1, z_2, \ldots, z_m)$$

$$LZGBM^{r,t} (z_1, z_2, \ldots, z_m) = \left(f^{*-1} \left(\frac{1}{r+t} \left(\left(\prod_{\substack{i=1 \\ j=1 \\ i \neq j}}^{m} \left(rf^* \left(A_{\emptyset(i)}\right) + tf^* \left(A_{\emptyset(j)}\right)\right)\right)^{\frac{1}{m(m-1)}}\right)\right),\right.$$
$$\left. g^{*-1} \left(\left(\prod_{\substack{i=1 \\ j=1 \\ i \neq j}}^{m} \left(\frac{\left(rf^* \left(A_{\emptyset(i)}\right) \times g^* \left(B_{\varphi(i)}\right)\right) + \left(tf^* \left(A_{\emptyset(j)}\right) \times g^* \left(B_{\varphi(j)}\right)\right)}{rf^* \left(A_{\emptyset(i)}\right) + tf^* \left(A_{\emptyset(j)}\right)}\right)\right)^{\frac{1}{m(m-1)}}\right)\right) \quad (17)$$

$$\lim_{t \to 0} \text{LZGBM}^{\text{r,t}} (z_1, z_2, \ldots, z_m) = \left(\frac{1}{r+t} \left(\prod_{\substack{i=1 \\ j=1 \\ i \neq j}}^{m} \left(rz_i + tz_j\right)\right)^{\frac{1}{m(m-1)}}\right)$$

$$= \lim_{t \to 0} \left(f^{*-1} \left(\frac{1}{r+t} \left(\left(\prod_{\substack{i=1 \\ j=1 \\ i \neq j}}^{m} \left(rf^* \left(A_{\emptyset(i)}\right) + tf^* \left(A_{\emptyset(j)}\right)\right)\right)^{\frac{1}{m(m-1)}}\right)\right),\right.$$

$$\left. g^{*-1} \left(\left(\prod_{\substack{i=1 \\ j=1 \\ i \neq j}}^{m} \left(\frac{\left(rf^* \left(A_{\emptyset(i)}\right) \times g^* \left(B_{\varphi(i)}\right)\right) + \left(tf^* \left(A_{\emptyset(j)}\right) \times g^* \left(B_{\varphi(j)}\right)\right)}{rf^* \left(A_{\emptyset(i)}\right) + tf^* \left(A_{\emptyset(j)}\right)}\right)\right)^{\frac{1}{m(m-1)}}\right)\right)$$

$$= \left(f^{*-1} \left(\frac{1}{r} \left(\left(\prod_{i=1}^{m} \left(rf^* \left(A_{\emptyset(i)}\right)\right)\right)^{\frac{1}{m}}\right)\right), g^{*-1} \left(\left(\prod_{i=1}^{m} \left(\frac{\left(rf^* \left(A_{\emptyset(i)}\right) \times g^* \left(B_{\varphi(i)}\right)\right)}{rf^* \left(A_{\emptyset(i)}\right)}\right)\right)^{\frac{1}{m}}\right)\right)$$

$$= \frac{1}{r} \left(\prod_{i=1}^{m} (rz_i)\right)^{\frac{1}{m}} = LZGGM^{r,0} (z_1, z_2, \ldots, z_m)$$

Case 3: If $r \to 1$ *and* $t \to 0$, then the LZGBM operator is reduced to the linguistic Z-number geometric mean (LZGM) operator.

$$LZGBM^{1,0}(z_1, z_2, \ldots, z_m) = \left(\prod_{i=1}^{m} (z_i)\right)^{\frac{1}{m}}$$
$$= LZGM(z_1, z_2, \ldots, z_m)$$

Case 4: If $r \to 1$ *and* $t \to 1$, then the LZGBM operator is reduced to the linguistic Z-number interrelated square geometric mean (LZISGM) operator.

$$LZGBM^{1,1}(z_1, z_2, \ldots, z_m) = \frac{1}{2}\left(\prod_{\substack{i=1 \\ j=1 \\ i \neq j}}^{m} (z_i + z_j)\right)^{\frac{1}{m(m-1)}}$$
$$= LZISGM(z_1, z_2, \ldots, z_m)$$

IV. THE PORTFOLIO SELECTION PROBLEMS BASED ON THE PROPOSED AGGREGATION OPERATORS UNDER LINGUISTIC Z-NUMBER ENVIRONMENT

In this section, a qualitative framework is proposed to construct the portfolios based on the proposed aggregation operators under linguistic Z-number environment. Expert's knowledge and opinions are the most important source of data to evaluate the performance of assets that can be applied to describe uncertainty and reliability of information, simultaneously. The proposed aggregation operators are powerful tools to incorporate expert's opinions under linguistic Z-number environment. Figure 1 shows a total schematic of the proposed method.
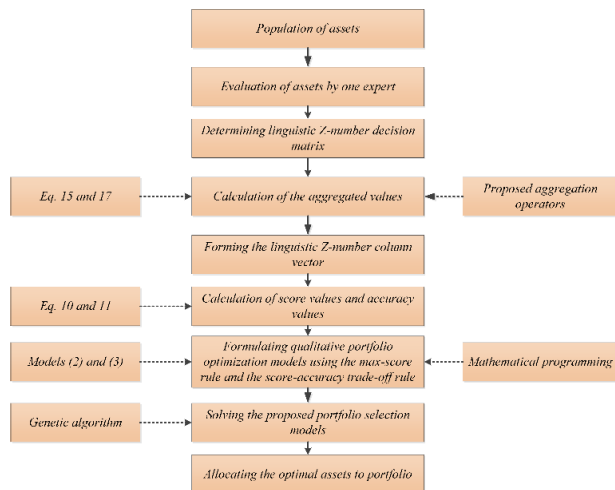
FIGURE 1. Typical flowchart for the portfolio selection based on the linguistic Z-number aggregation operators.

Therefore, let us investigate n risky assets with uncertain rate of return $\tilde{R}_i$ ($i = 1, \ldots, n$). Suppose l_i (≥ 0) is the minimum fraction of total capital which can be invested in the ith asset and u_i ($0 \leq l_i \leq u_i$) is the maximum fraction of total investment which can be assigned to the ith asset. Let x_i is the weight of the ith asset in the portfolio and y_i is a binary variable which is equal to one when the corresponding asset is allocated to portfolio, otherwise it is zero. Hence, the standard

form of portfolio selection model is presented as follows [57]:

Model 1:

$$Max \sum_{i=1}^{n} \tilde{R}_i x_i \tag{18}$$

$$\text{S.t.} \sum_{i=1}^{n} \sum_{k=1}^{n} Cov(R_i, R_k) x_i x_k \leq \sigma_g^2 \tag{19}$$

$$\sum_{i=1}^{n} x_i = 1 \tag{20}$$

$$\sum_{i=1}^{n} y_i = h \tag{21}$$

$$l_i y_i \leq x_i \leq u_i y_i, \quad i = 1, \ldots, n \tag{22}$$

$$y_i \in \{0, 1\}, \quad i = 1, \ldots, n \tag{23}$$

$$x_i \geq 0, \quad i = 1, \ldots, n \tag{24}$$

where $Cov(R_i, R_k)$ is the covariance between returns of the ith and k^{th} assets. Constraint (20) is the budget constraint. Constraint (21) called cardinality constraint guarantees that the portfolio is confined to preserve a predetermined number of assets such as h. Eventually, constraint (24) shows the prohibition of short selling.

A. THE PROPOSED QUALITATIVE APPROACH FOR CARDINALITY CONSTRAINED PORTFOLIO SELECTION PROBLEM UNDER LINGUISTIC Z-NUMBER ENVIRONMENT

In this subsection, two scenarios are considered to model the portfolio selection problem under the linguistic Z-number environment. In the first scenario, which is proper for the general investors, a qualitative portfolio model is proposed based on the max-score rule. In the second scenario, which can be used by the risky investors, a qualitative model is developed based on score-accuracy trade-off rule.

Suppose an investor or a fund manager is interested in investing on n risky assets $\{y_1, y_2, \ldots, y_n\}$. He/she selects the asset based on m criteria $\{c_1, c_2, \ldots, c_m\}$. To make more credible decisions, they use experts' opinions. Thus, experts evaluate the performance of each asset with respect to different criteria. The evaluation information is represented by the linguistic Z-numbers as $Z_{ij} = \left(A_{\emptyset(\alpha)}^{ij}, B_{\varphi(\alpha)}^{ij}\right)$ ($i = 1, \ldots, n; j = 1, \ldots, m$) where $A_{\emptyset(\alpha)}^{ij} \in S$ is a fuzzy constraint on the values that experts devote to the ith asset with respect to the jth criterion, and $B_{\varphi(\alpha)}^{ij} \in S'$ is a reliability measure of the first component. In other words, experts apply a linguistic term selected the set of $S = \{s_0, \ldots, s_{2p}\}$ to express their assessments about the performance of each asset and then use a linguistic term from the set of $S' = \{s'_0, \ldots, s'_{2q}\}$ to describe their reliability in the assessments. This qualitative data can be indicated as a decision matrix with linguistic Z-number elements as $Z = [z_{ij}]_{n \times m}$.

In this step, the aggregation techniques described in Section 3 are applied to combine the linguistic Z-number information, and the comprehensive assessment values $v_i = \left(A_{\emptyset(i)}^{v_i}, B_{\varphi(i)}^{v_i}\right)$ ($i = 1, \ldots, n$) are calculated for each asset. Therefore, the decision matrix $Z = [z_{ij}]_{n \times m}$ is converted into a column vector $V = [v_i]_{n \times 1}$. The aggregated values v_i can be obtained based on the proposed aggregation

operators as follows:

$$v_i = LZBM\,(z_{i1}, \ldots, z_{im})$$
$$\text{Or} \quad v_i = LZGBM\,(z_{i1}, \ldots, z_{im})$$

Now, the score and accuracy values of the aggregated results are obtained, respectively, as follows:

$$E\,(v_i) = f^*\left(A_{\emptyset(i)}^{V_i}\right) \times g^*\left(B_{\varphi(i)}^{V_i}\right)$$
$$D\,(v_i) = f^*\left(A_{\emptyset(i)}^{V_i}\right) \times \left(1 - g^*\left(B_{\varphi(i)}^{V_i}\right)\right)$$

According to Definition 6, the score function shows the grade of adaptability to which the asset i satisfies the investor's requirement. The greater the score value, the more adaptability the asset i satisfies the investor's requirement. Also, according to Definition 6, the accuracy function evaluates the grade of accuracy of linguistic Z-numbers. Thus, the greater the accuracy value, the more grade of accuracy of linguistic Z-number. Since the relationship among the score and accuracy functions is equivalent to the relationship among the mean and variance of quantitative information under deterministic environment [58], we can apply the score and accuracy values to evaluate the expected return and risk of portfolio under the linguistic Z-number environment. In the following, similar to the traditional asset allocation models, two qualitative portfolio selection models are proposed to assign the optimal assets and obtain the optimal investment ratios.

Based on Definition 6, the best alternatives are selected according to higher score value [37]. Therefore, the first scenario is formulated by using the max-score rule to obtain the optimal portfolio as follows:

Model 2: $\quad Max \sum_{i=1}^{n} \left(\left(f_i^*\left(A_{\emptyset(i)}^{V_i}\right) \times g_i^*\left(B_{\varphi(i)}^{V_i}\right)\right) x_i\right)$
s.t. Constraints (20) – (24)

It should be noted that Model 2 is appropriate for general investors who only seek to maximize expected return. Since the linguistic Z-number sets cannot be discerned when they have the equal score values, the accuracy values are used to compare these alternatives such that the best alternative has the highest accuracy value [37]. Therefore, by using this issue, the second scenario is modeled based on a trade-off between the score and accuracy values. Model 3 is proper for the risky investors who want to attain the maximum expected return along with the desirable level of risk. Finally, this qualitative model is proposed based on the score-accuracy trade-off rule to allocate the optimal assets as follows:

Model 3:
$$Max \sum_{i=1}^{n} \left(\left(f_i^*\left(A_{\emptyset(i)}^{V_i}\right) \times g_i^*\left(B_{\varphi(i)}^{V_i}\right)\right) x_i\right)$$
$$\text{s.t.} \sum_{i=1}^{n} \left(\left(f_i^*\left(A_{\emptyset(i)}^{V_i}\right) \times \left(1 - g_i^*\left(B_{\varphi(i)}^{V_i}\right)\right)\right) x_i\right) \geq \gamma$$
Constraints (20) - (24).

where $\gamma \in \left[0, \max_{1\leq i\leq n}\text{accuracy value}\right]$ is considered as investor's preference for a minimum admissible risk level of the portfolio. The following cases may occur:

1- if $\gamma > \max_{1\leq i\leq n}$accuracy value, then, no feasible solution can be detected.
2- if $\gamma = \max_{1\leq i\leq n}$accuracy value, then, only some of assets having the maximum accuracy value can be selected.
3- if $0 \leq \gamma < \max_{1\leq i\leq n}$ accuracy value, then, the higher the γ-value is the higher the effect of admissible risk level in the portfolio selection. The lower the γ-value is the lower the effect of admissible risk level in the portfolio selection.

The proposed models are mixed integer linear convex optimization problems. The linearity structure of these models maintains a very significant feature that every local optimal point is also a global optimal point. This feature guarantees that the obtained solutions by the proposed models are optimal. However, computational complexity of mixed integer linear optimization problems is associated with the number of binary and integer variables [59]. Speranza [59] indicated that finding the proper solutions for portfolio selection models (as a MILP model) in a rational time is impossible when the number of assets is greater than 15. In the following, Mansini and Speranza [60] proved that solving portfolio selection model with round lots is NP-hard. In the literature, some authors such as Mashayekhi and Omrani [57] and Li and Xu [43] developed genetic algorithm to look for more suitable solutions in cardinality constrained portfolio models. Therefore, in this study, genetic algorithm (GA) is applied in order to achieve high quality solutions.

V. CASE STUDY AND COMPUTATIONAL RESULTS
To illustrate the validation of the proposed qualitative approach, the proposed models (Model 2 and Model 3) are applied in a real case. Tehran Stock Exchange (TSE) Market in Iran is considered as the resource of information. The necessary data is available through the site of Tehran stock exchange market.[1] TSE is Iran's largest stock exchange, which first opened in 1967. In May 2012, TSE listed 339 companies with a combined market capitalization of US\$104.21 billion. There are 37 industries such as the automotive, telecommunications, petrochemical, mining, steel iron, copper, banking, and financial mediation at the stock market in TSE. At the end of each season, the department of information of the Tehran stock exchange market reveals the name of 50 best corporations. These corporations are selected by certain criteria. We choose 20 firms $(y_1, \ldots, y_{20})$ with the best performance in the latest financial statement from 20 January 2019 to 20 November 2019 to validate the proposed qualitative approach. Then, based on the fundamental analysis, one expert expressed her/his qualitative opinions about the future performance of each asset with respect to the mentioned criteria by using linguistic terms.

As discussed above, the investment decisions are generally made based on the assessments and opinions of experts. Obtaining exact quantitative information about the

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[1]www.TSE.ir

TABLE 1. The performance of each asset with respect to criteria which is described by using LZNs.

ID	Criteria			ID	Criteria		
	c_1	c_2	c_3		c_1	c_2	c_3
1	(s_5, s'_3)	(s_4, s'_4)	(s_5, s'_4)	11	(s_5, s'_4)	(s_4, s'_4)	(s_4, s'_4)
2	(s_6, s'_4)	(s_4, s'_4)	(s_6, s'_3)	12	(s_4, s'_3)	(s_3, s'_3)	(s_5, s'_0)
3	(s_6, s'_0)	(s_4, s'_1)	(s_3, s'_4)	13	(s_6, s'_3)	(s_6, s'_1)	(s_6, s'_1)
4	(s_7, s'_4)	(s_6, s'_4)	(s_6, s'_4)	14	(s_3, s'_1)	(s_2, s'_2)	(s_1, s'_1)
5	(s_5, s'_4)	(s_4, s'_3)	(s_5, s'_2)	15	(s_7, s'_0)	(s_5, s'_1)	(s_5, s'_1)
6	(s_6, s'_4)	(s_4, s'_3)	(s_4, s'_2)	16	(s_3, s'_4)	(s_2, s'_4)	(s_1, s'_4)
7	(s_4, s'_2)	(s_2, s'_3)	(s_2, s'_3)	17	(s_6, s'_3)	(s_4, s'_3)	(s_5, s'_3)
8	(s_6, s'_1)	(s_4, s'_2)	(s_2, s'_4)	18	(s_4, s'_2)	(s_3, s'_1)	(s_4, s'_0)
9	(s_5, s'_2)	(s_4, s'_2)	(s_4, s'_2)	19	(s_3, s'_3)	(s_2, s'_4)	(s_2, s'_4)
10	(s_7, s'_4)	(s_5, s'_4)	(s_7, s'_4)	20	(s_7, s'_1)	(s_5, s'_2)	(s_6, s'_1)

TABLE 2. The aggregated results based on the proposed operators for $f^*(\theta_i) = f_1(\theta_i)$ and $g^*(\theta_i) = f_1(\theta_i)$.

ID	LZBM	LZGBM	ID	LZBM	LZGBM
1	$(s_{4.64}, s'_{3.63})$	$(s_{4.66}, s'_{3.64})$	11	$(s_{4.31}, s'_4)$	$(s_{4.32}, s'_4)$
2	$(s_{5.26}, s'_{3.63})$	$(s_{5.31}, s'_{3.62})$	12	$(s_{3.93}, s'_{0.88})$	$(s_{3.97}, s'_{1.65})$
3	$(s_{4.2}, s'_{0.55})$	$(s_{4.28}, s'_{1.06})$	13	$(s_6, s'_{1.48})$	$(s_6, s'_{1.58})$
4	$(s_{6.32}, s'_4)$	$(s_{6.32}, s'_4)$	14	$(s_{1.86}, s'_{1.28})$	$(s_{1.95}, s'_{1.32})$
5	$(s_{4.64}, s'_{2.91})$	$(s_{4.66}, s'_{2.96})$	15	$(s_{5.61}, s'_{0.29})$	$(s_{5.64}, s'_{0.55})$
6	$(s_{4.59}, s'_{2.94})$	$(s_{4.64}, s'_{3.065})$	16	$(s_{1.86}, s'_4)$	$(s_{1.95}, s'_4)$
7	$(s_{2.55}, s'_{2.59})$	$(s_{2.62}, s'_{2.53})$	17	$(s_{4.94}, s'_3)$	$(s_{4.98}, s'_3)$
8	$(s_{3.73}, s'_{1.95})$	$(s_{3.91}, s'_{1.86})$	18	$(s_{3.64}, s'_{0.44})$	$(s_{3.65}, s'_{0.87})$
9	$(s_{4.31}, s'_2)$	$(s_{4.32}, s'_2)$	19	$(s_{2.3}, s'_{3.61})$	$(s_{2.32}, s'_{3.58})$
10	$(s_{6.27}, s'_4)$	$(s_{6.31}, s'_4)$	20	$(s_{5.95}, s'_{1.26})$	$(s_{5.98}, s'_{1.27})$

TABLE 3. The score and accuracy values of aggregated results.

ID	LZBM		LZGBM		ID	LZBM		LZGBM	
	Sc.	Ac.	Sc.	Ac.		Sc.	Ac.	Sc.	Ac.
1	0.53	0.053	0.53	0.05	11	0.54	0	0.54	0
2	0.59	0.06	0.61	0.06	12	0.11	0.38	0.21	0.29
3	0.07	0.45	0.14	0.39	13	0.28	0.47	0.29	0.45
4	0.79	0	0.79	0	14	0.07	0.16	0.08	0.16
5	0.42	0.16	0.43	0.15	15	0.05	0.65	0.09	0.61
6	0.42	0.15	0.44	0.14	16	0.23	0	0.24	0
7	0.21	0.11	0.21	0.12	17	0.46	0.15	0.47	0.16
8	0.23	0.24	0.23	0.26	18	0.05	0.41	0.1	0.36
9	0.27	0.27	0.27	0.27	19	0.26	0.03	0.26	0.03
10	0.78	0	0.79	0	20	0.23	0.51	0.24	0.51

performance of each asset with respect to different criteria is often difficult and sometimes impossible because there are some uncertain and unpredictable factors such as political and economic tensions, which can influence the worthiness of securities. In addition, because there are some newly added firms to Tehran Stock Exchange Market in Iran, the quantitative financial information about them are not adequately accessible. Hence, investors cannot completely obtain the financial information about these firms. In this situation, the proposed qualitative approaches, which have been extended under linguistic Z-number environment, are more suitable in order to select the optimal portfolio and obtain the optimal investment ratios. Consequently, the portfolio selection approach proposed in the previous sections is used in this case study.

Generally, investors or fund managers use different criteria to assess the available assets ($y_1, \ldots, y_{20}$). Thus, it is assumed that they want to utilize three following criteria to evaluate the performance of each asset: the profitability of investment c_1, the reputation of corporation c_2 and liquidity c_3. Then, experts evaluate the performance of each asset with respect to these three criteria and express their opinions in the form of linguistic terms. All the assessment information is represented by the linguistic Z-numbers as Z_{ij} ($i = 1, \ldots, 20; j = 1, 2, 3$). For this aim, expert applies a linguistic term from the set of $S = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$ = {very very low, very low, low, almost medium, medium, almost high, high, very high, perfect} to express his/her assessment about each asset and then he/she uses a linguistic term from the set of $S' = \{s'_0, s'_1, s'_2, s'_3, s'_4\}$ = {seldom, occasionally, frequently, regularly, usually} to represent his/her reliability in the assessments. Consequently, the Z-number linguistic matrix $Z = [z_{ij}]_{20 \times 3}$ is constructed, which is shown in Table 1.

Step 1: The comprehensive values for each asset are calculated by using LZBM and LZGBM operators (for $r = t = 0.5$) and the Z-number linguistic matrix

$Z = [z_{ij}]_{20 \times 3}$ is transformed into the column vector of aggregated values $V = [v_i]_{20 \times 1}$. The aggregated results, which are calculated based on the proposed operators, are listed in Table 2. Then, according to Definition 6, the score and accuracy values of each asset are obtained, and their results are reported in Table 3.

Step 2: The qualitative portfolio selection approach is modeled based on Model 2 and Model 3 for general investors and risky investors, respectively.

Step 3: The models resulted based on two proposed operators are solved by GA. The input parameters of the proposed models are as follows: $h = 6$, $l_i = 0.05$ and $u_i = 0.6$. The parameters of genetic algorithm are adjusted by applying Taguchi experimental design method [61]. The Taguchi toolbox in Minitab software is employed to tune parameters. The adjusted parameters of GA are as follows: POP_{size} : 150; crossover rate: 0.75; mutation rate: 0.2; maximum iteration: 400. GA is run 10 times for each case in MATLAB R2014a on a PC with Pentium(R) Dual core-CPU 2.0 GHz Processor and 2 GB of RAM memory.

The selected assets and their investment ratios along with the performance of the selected portfolios resulted by using Model 2 and Model 3 are indicated in Tables 4 and 5, respectively. Moreover, Figures 2 and 3 show the investment ratios of the selected assets based on Model 2 and Model 3, respectively.

As mentioned above, three criteria (the profitability of investment c_1, the reputation of corporation c_2 and

TABLE 4. The selected portfolios obtained by using Model 2.

Operator	Goal	Selected asset and investment ratio					
LZBM	0.7	1	2	4	10	11	17
		0.085	0.084	0.330	0.330	0.086	0.084
LZGBM	0.7052	1	2	4	10	11	17
		0.08	0.08	0.33	0.33	0.08	0.08

TABLE 5. The selected portfolios obtained by using Model 3 for $\gamma = 0.3$.

Operator	Goal	Selected asset and investment ratio					
LZBM	0.4393	2	4	10	13	15	20
		0.136	0.163	0.171	0.147	0.195	0.188
LZGBM	0.438	4	10	13	15	17	20
		0.144	0.189	0.174	0.181	0.134	0.178
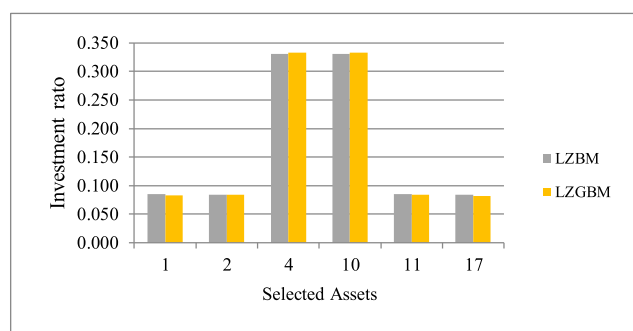
FIGURE 2. The investment ratios of the selected assets which are obtained by Model 2.
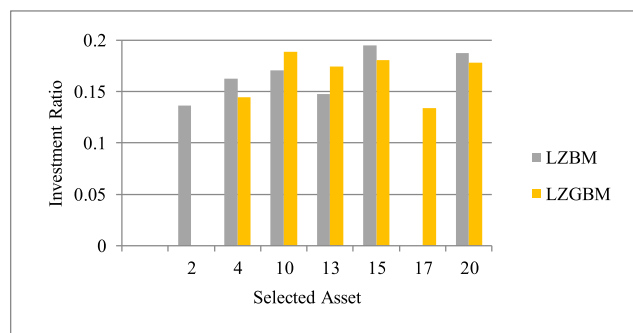
FIGURE 3. The investment ratios of the selected assets which are obtained by Model 3 for $\gamma = 0.3$.

liquidity c_3) have been considered in order to evaluate the assets and allocate the appropriate securities by using the proposed qualitative approach. Here, the computational results, which are obtained by Model 2 and Model 3, are separately discussed as follows:

Portfolio Selection Using Model 2: As shown in Figure 2, $f^*(\theta_i) = f_1(\theta_i)$ and $g^*(\theta_i) = f_1(\theta_i)$ are applied to deal with linguistic Z-number information, and the optimal assets devote to portfolio using Model 2 under the proposed operators. Therefore, if LZBM and LZGBM operators are used to calculate the comprehensive values of assets, then the performance of assets 4 and 10 become better than the remaining assets. As clear in Table 4, some assets such as 1, 11 and 17,

which are selected by Model 2 under the proposed operators, have lower values of profitability, reputation and liquidity in comparison with the assets 15 and 20, but their reliability measures are high. This issue reflects the influence of reliability measures in the evaluation information modeling.

Moreover, since two components of linguistic Z-numbers demonstrate experts' opinions and assessments about the performance of each asset, different LSFs can be used to describe them. Various LSFs can be devoted to the first component in order to represent experts' opinions and evaluations. Also, different LSFs can be allocated to the second component to characterize the confidence level related to information. Therefore, investors or decision-makers can effectively and flexibly select different LSFs according to their priorities in order to achieve more accurate results. Moreover, it can be noted that the main core of portfolio optimization problem is diversification. Therefore, this study applies the cardinality constraint to handle the portfolio diversification. Using cardinality constraint, investors can determine the number of assets (h) that they can manage in their portfolio. Furthermore, the maximal and minimal portions (u_i *and* l_i) of the total budget, which can be invested in each asset, help investors to assign the capital according to their preferences. Also, the lower bound constraint prevents a great number of very slight investment and upper bound constraint guarantees the diversification portfolio. Consequently, if the diversification resulted by Model 2 does not satisfy investor, more diversified portfolios can be constructed by changing h, l_i and u_i.

Portfolio Selection Using Model 3: As it can be seen in Figure 3, the performance of some assets such as 4, 10, 13, 15 and 20 becomes better than other assets when LZBM and LZGBM operators are used to calculate the aggregated values of assets. Moreover, it is clear in Figure 3 that Model 3 can select the optimal portfolio based on a trade-off between the score value and the accuracy value.

A. SENSITIVITY ANALYSIS

In this subsection, the results of sensitivity analysis are discussed under two situations:

1. The effects of the critical parameters t and r on the portfolio selection problem based on LZBM and LZGBM operators.
2. The influences of alterations of the desirable risk level of the portfolio (γ) in Model 3.

B. THE INFLUENCE OF PARAMETERS *t* AND *r* ON THE PORTFOLIO SELECTION PROBLEM BASED ON LZBM AND LZGBM OPERATORS

To analyze the effect of variation of input parameters t and r in LZBM and LZGBM operators, the various values of parameters t and r are used to investigate the obtained results of portfolio selection.

First, the role of parameters t and r is analyzed on LZBM operator. As seen in Table 6, we find that the selected assets based on Model 2 using LZMB operator with various values of t and r are slightly different, but their investment

TABLE 6. Portfolio selection for different values of parameters *t* and *r* based on LZBM in Model 2.

t	r	Goal	Selected asset and investment ratios					
0.5	0.5	0.712	1	2	4	10	11	17
			0.073	0.072	0.352	0.352	0.074	0.075
	5	0.725	1	2	4	6	10	11
			0.088	0.089	0.324	0.086	0.324	0.086
	100	0.82	1	2	4	6	10	11
			0.075	0.076	0.348	0.076	0.348	0.075
5	0.5	0.736	1	2	4	6	10	11
			0.074	0.074	0.348	0.079	0.348	0.074
	5	0.747	1	2	4	6	10	11
			0.073	0.071	0.357	0.070	0.357	0.071
	100	0.828	1	2	4	6	10	11
			0.082	0.083	0.334	0.083	0.334	0.083
100	0.5	0.786	2	4	5	6	10	11
			0.067	0.364	0.068	0.067	0.364	0.068
	5	0.797	1	2	4	6	10	11
			0.084	0.081	0.333	0.082	0.333	0.085
	100	0.857	1	2	4	6	10	11
			0.078	0.078	0.342	0.078	0.342	0.08

TABLE 7. Portfolio selection for different values of parameters *t* and *r* based on LZGBM in Model 2.

t	r	Goal	Selected asset and investment ratio					
0.5	0.5	0.707	1	2	4	10	11	17
			0.082	0.079	0.338	0.337	0.081	0.08
	5	0.715	1	2	4	10	11	17
			0.073	0.067	0.359	0.359	0.071	0.06
	100	0.704	1	2	4	10	11	17
			0.081	0.079	0.341	0.341	0.079	0.08
5	0.5	0.707	1	2	4	10	11	17
			0.079	0.081	0.343	0.343	0.079	0.07
	5	0.708	1	2	4	10	11	17
			0.082	0.082	0.339	0.338	0.083	0.07
	100	0.704	1	2	4	10	11	17
			0.081	0.083	0.337	0.337	0.082	0.08
100	0.5	0.707	1	2	4	10	11	17
			0.077	0.079	0.344	0.344	0.083	0.07
	5	0.705	1	2	4	10	11	17
			0.079	0.082	0.337	0.337	0.082	0.08
	100	0.703	1	2	4	10	11	17
			0.085	0.086	0.329	0.329	0.084	0.09

ratios are different. It is clear that both selected assets and their investment ratios have been diversified when the values of parameters *t* and *r* are changed. Also, the score value of alternatives (assets) becomes greater when the values of parameters t and r are increased. Here, there is a remarkable point that with increase of the values of parameters *t* and *r*, the assets with high score value are selected. For example, if *t* = 0.5 and *r* = 0.5, the score values of some assets such as 4, 10 and 11 are equal to 0.79, 0.785, and 0.539, respectively. Now, if *t* = 100 and *r* = 100, the assets 4, 10 and 11 with the score values 0.808, 0.87 and 0.558 are assigned to the optimal portfolio. On the basis of these discussions, the parameters t and r can reflect the preferences of investors or DMs. In the practical applications, the pessimistic or circumspect investors can devote higher values to the parameters *t* and *r*. Therefore, every investor or fund manager can choose the proper values of the parameters *t* and *r* according to their preferences.

Figure 4 shows the sensitivity of the portfolio performance with respect to the parameters *t* and *r*. It is obvious that the portfolio performance becomes greater when the values of parameters *t* and *r* are increased.

selected based on Model 2 using LZGMB operator with various values of *t* and *r* are slightly different, but their investment ratios are different. As it is obvious in Table 7, investors or fund managers can select more diversified portfolio with variation of the parameter values *t* and *r*. Moreover, unlike LZBM operator, in this situation, the score values of each asset become lower when the values of parameters *t* and *r* are increased. For instance, if *t* = 0.5 and *r* = 0.5, the score values of some assets such as 1, 2, and 4 are equal to 0.53, 0.61, and 0.143, respectively. Now, if *t* = 100 and *r* = 0.5, the assets 1, 2 and 4 with the score values 0.52, 0.59 and 0.037 are assigned to the optimal portfolio. Based on this, the parameters *t* and *r* can reflect the mentality of investors or DMs. In the practical applications, the pessimistic or circumspect investors can allocate lower values to the parameters *t* and *r* to deal with LZGBM operator. Therefore, every investor or fund manager can choose the proper values of the parameters *t* and *r* according to their preferences.

Figure 5 indicates the sensitivity of the portfolio performance with respect to the parameters *t* and *r* in Model 2 based on LZGBM operator. It is obvious that the portfolio performance becomes lower when the values of parameters *t* and *r* are increased.
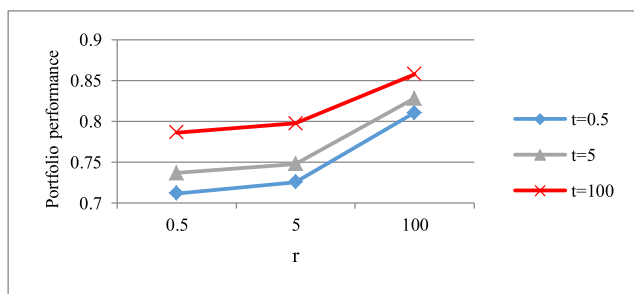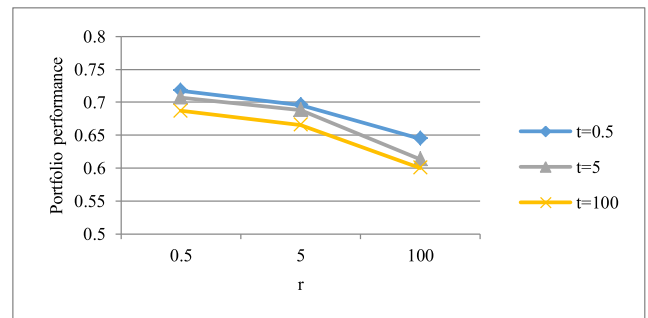

FIGURE 4. The sensitivity of the portfolio performance with respect to the parameters *t* and *r* in Model 2 based on LZBM operator.

Now, the effect of the parameters *t* and *r* is investigated to choose portfolio based on Model 2 under LZGBM operator. Table 7 shows the results of sensitivity analysis. Similar to the obtained results for LZBM operator, we find that the assets

FIGURE 5. The sensitivity of the portfolio performance with respect to the parameters t and r in Model 2 based on LZGBM operator.

TABLE 8. Portfolio selection based on the various values of admissible risk level γ in Model 3.

Operator	γ	Goal	Selected assets and investment ratio					
LZBM	0	0.695	1	2	4	10	11	17
			0.072	0.070	0.358	0.358	0.072	0.071
	0.3	0.439	2	4	10	13	15	20
			0.136	0.163	0.171	0.147	0.195	0.188
	0.5	0.178	3	12	13	15	18	20
			0.075	0.080	0.293	0.179	0.080	0.293
LZGBM	0	0.678	1	2	4	10	11	17
			0.083	0.080	0.338	0.338	0.081	0.080
	0.3	0.4235	4	10	13	15	17	20
			0.165	0.148	0.178	0.182	0.163	0.164
	0.5	0.168	3	12	13	15	18	20
			0.077	0.088	0.177	0.297	0.064	0.297

C. THE INFLUENCE OF PARAMETER γ ON THE PORTFOLIO SELECTION PROBLEM IN MODEL 3

In this subsection, sensitivity analysis is implemented by altering the desirable risk level of the portfolio (γ). It can be derived from the results reported in Table 8 that the more diversified portfolios are constructed at in a given risk level (γ) when the desirable level of risk is changed. This can help investors or DMs better manage investments corresponding to their preferences. In addition, as shown in Figure 6, the attainment level of the max-score value is reduced when the γ-value is increased. This issue matches up with diversification axiom and reflects a trade-off between the score value and accuracy value. Moreover, it can be noted that the determination of γ-value may depend on the mentality of investor. A conservative investor can select higher γ-value, but a risk-seeker investor can select lower γ-value. Therefore, this issue can provide additional useful information to help the investors efficiently for making more fruitful decisions.
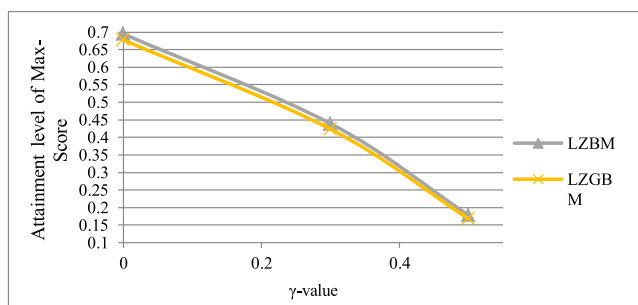

FIGURE 6. Trade-off among the max-score value and risk level using Model 3.

D. COMPARISON WITH BONFERRONI MEAN (BM) OPERATOR [3]

BM operator was introduced by Bonferroni [3], which can only be used in situations where input arguments are crisp numbers. However, in financial markets, exact values cannot play a role in representing qualitative information because of the growing complexity and diversity of socio-economic environment. DMs always express their opinions in the form of various possible linguistic terms with different reliabilities.

TABLE 9. The selected assets result by different methods.

Operators	Selected assets					
LZWA [40]	2	4	6	10	11	19
LZOWA [40]	2	4	6	10	11	19
LZHWA [40]	2	4	6	10	11	19
LZBM	1	2	4	10	11	17
LZGBM	1	2	4	10	11	17

Linguistic Z-numbers properly satisfy this requirement. But the BM operator is unable to solve portfolio selection problems under Linguistic Z-number environment. Therefore, it is essential to develop some linguistic Z-numbers BM operators in order to solve these problems. To sum up, the proposed operators can be used to aggregate evaluation information under linguistic Z-number environment.

E. COMPARISON WITH MAHMOODI et al. METHOD [40]

In order to further investigate the validity of the proposed aggregation operators, we compare them with Mahmoodi *et al.* methods [40] built on some linguistic Z-number weighted averaging operators. The selected assets obtained by Model 2 are shown in Table 9.

From Table 9, it can be found that the assets selected by LZBM and LZGBM operators are slightly different from the assets obtained by LZWA, LZOWA and LZHWA operators [40]. This difference displays an advantage of the proposed aggregation operators which capture the interrelationship between input arguments, while Mahmoodi *et al.* methods [40] suppose the input arguments are independent.

Therefore, the proposed aggregation operators can be more general and suitable for handling real portfolio selection problems where the input arguments have interacting relationships.

F. MANAGERIAL RESULTS

On the basis of the above analysis, the proposed qualitative approach for portfolio selection problems has the following advantages:

1. Linguistic Z-numbers can be applied more flexibly and effectively in order to describe the required financial information. Since the expression of financial data in the form of LZNs is far more convenient, experts' assessment data can easily be represented as linguistic Z-numbers.

2. Since the utilization of linguistic scale functions in the arithmetic operations of LZNs lead to the generation of various results, investors or fund managers can flexibly employ different LSFs according to their preferences and priorities. Therefore, the proposed qualitative approach can present more diversified portfolios.

3. In this study, two aggregation operators are proposed to combine the linguistic Z-number information. The LZBM and LZGBM operators consider the

interrelationship between the input arguments, which is necessary in the real investment processes. Although there are some aggregation operators to aggregate information under different kinds of fuzzy environment, the reliability of data is not considered. The proposed aggregation operators not only consider the interrelationship of input arguments, but also capture the requirement of reliability. Therefore, the proposed operators are more general than other existing operators and deal with uncertain information.

4. The proposed models are developed to select the optimal portfolio with consideration of uncertainty and reliability of linguistic assessment information, simultaneously. These models are superior than others because of three main reasons:

- Preventing data loss in the asset management.
- Considering the data reliability in addition to the interrelationship of input arguments.
- Matching with the preferences of investors or DMs.

VI. CONCLUSION

A qualitative and holistic methodology is presented to construct more diversified portfolios with the consideration of investor's preferences. The main steps of the extended methodology are: (i) extending two aggregation operators under linguistic Z-number environment and introducing LZBM operator and LZGBM operator to combine the linguistic Z-number information; (ii) proposing two qualitative hybrid portfolio optimization models under linguistic Z-number environment to assist investors for more flexible and more credible selecting the optimal portfolios according to their preferences. A qualitative portfolio optimization model has been proposed based on the max-score rule which is suitable for the general investors, and another qualitative portfolio optimization model has been presented according to the score-accuracy trade-off rule which is proper for risky investors. These proposed models can distinguish the risk seeker investors and the risk averter investors with the variation of the risk level. Moreover, to illustrate the effectiveness, the efficient frontiers of the qualitative proposed models have been analyzed. The results indicate that investors and fund managers can more flexibly and more reliably construct more diversified portfolio with the determination of various values of input parameters or different LSFs based on their preferences. For future research, the proposed approach can be applied to model group portfolio selection problems based on different aggregation operators under linguistic Z-number environment. Moreover, the proposed models can be developed according to other assumptions, constraints and objectives such as ethical goals, and entropy constraints.

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