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A Fully Complex-Valued and Robust ZNN Model for Dynamic Complex Matrix Inversion Under External Noises

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ABSTRACT Dynamic complex-valued matrix inversion is often used in the field of mathematics and engineering. Over the past years, many recurrent neural network models have been designed and researched to analyse the process of matrix inversion without noises interference. However, there are many types of uncertain noises in actual model design and analysis. In this article, a novel fully complex-valued and robust Zeroing neural network (CVRZNN) is firstly proposed for calculating the dynamic complex matrix inversion under the interference of external noise environment, and its robustness is analysed and demonstrated in the presence of various types of external noises. Compared with the previous zeroing neural network (ZNN) and the gradient neural network (GNN) for dynamic complex matrix inversion, this novel CVRZNN model has good robustness under three kinds of external noises. Besides, the theoretical analysis shows that the CVRZNN model can globally converge to zero under constant noise. Through comparative simulation results, the excellent performance of the proposed CVRZNN model is obviously demonstrated, which is much better than that of the previous GNN and ZNN models.

INDEX TERMS Zeroing neural network (ZNN), gradient neural network, dynamic complex-valued matrix inversion, robustness, external noise.

I. INTRODUCTION

The dynamic complex matrix inversion is often emerged in scientific research and widely applied in big data processing [1], MIMO system [2]–[5], robotic arm tracking [6]–[9], control engineering [10], [11], etc. For solving the problem of matrix inversion, several scholars found many numerical methods, such as Newton iterative method [12], Greville recursive method [13], etc. These numerical iteration methods are the process of using previous data to produce large amounts of new data. Nonetheless, it is easy to encounter a matrix with a large number of dimensions due to the development of big data, and thus the spatial complexity of numerical iterative algorithms will increase greatly. To solve the high spatial complexity, neural network methods are proposed and studied because of its parallel-computation and distributed-storage ability. For example, many neural network models are well developed in the actual calculation process, such as optimal parallel computing [14]–[16], power electronics and motor drives [17], which show better performance than numerical iterative methods.

Recently, the gradient neural network (GNN) model is used to solve the static matrix inversion. For example, in [18], the GNN model with the activation function is used to find static matrix inversion. Besides, in [19], a novel GNN model is proposed to solve static complex matrix inversion, which can achieve a superior performance. Thus we can know that the GNN model has wide applications in solving matrix inversion. But the research of GNN is often applied under time invariant conditions. However, the static matrix cannot satisfy the requirement of the increasingly developed engineering techniques, so the time-varying matrix is proposed to be studied.

To solve time-varying problems effectively, zeroing neural network (ZNN) [20], [21] has been proposed. This neural network fully exploits time derivation information of

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time-varying problems, and is able to find the exact solutions. In [22], Lv *et al.* propose a ZNN model for solving dynamic linear equations, and this model can reach finite time convergence if the sign-bi-power function is added. In [23], Xiao *et al.* propose a complex-valued nonlinear recurrent neural network model for solving matrix inversion, which can get more accurate solution than the ZNN model in solving the same problems. Those mentioned neural models can get accurate solutions under the ideal conditions. Besides, the neural network model is also applied in the fuzzy system that are related with time-varying parameter matrix. For example, in [24], [25], it is concerned with nonfragile H_∞ filtering of continuous-time and fuzzy peak-to-peak filtering with multipath data packet dropouts in fuzzy systems, which are based on dynamic matrix inversion.

It is worth pointing out that the above mentioned neural models do not consider noise interference in the process of solving matrix inversion, but in actual application scenarios, there are various uncertain noises. Hence, many researchers start to consider the design of noise-tolerant models in the study of neural networks [26]–[30]. For example, in [29], an integral-enhanced ZNN model is designed to solve the matrix inversion in the real domain under the interference of various external noises. The theoretical analysis and experimental results show that the state solution can converge to the theoretical solution under external noises. It must point out that [29] focuses on matrix inversion in the real domain, but the study on neural networks for matrix inversion in the complex domain is also necessary. The main reason is to expand the research fields of neural networks from the real to complex domains, and this systematic approach will be applied to solve a lot of scientific and engineering problems encountered in the complex domain.

Generally speaking, when we encounter the complex-valued problems, there are usually two approaches to solve them. The first method is to separate the real part and imaginary part, which divides a complex problem into two real problems and it will increase the computation complexity. For example, in [32], this method is adopted to transform the complex matrix into two real matrices. The second method is to adopt complex modulus conversion technique, which can solve directly complex problems in the complex domain without transformation. For example, in [33], this method is adopted to solve complex Sylvester matrix equation in the complex domain. Furthermore, external noises are considered in the complex-valued problems. For example, in [34], Xiao *et al.* propose a noise tolerant ZNN model for solving time-varying complex matrix problem under various kinds of noises, which adopts the first method to solve complex-valued problem and time complexity and space complexity are increased. Therefore, in order to reduce the computation complexity and suppress external disturbances, a fully complex-valued and robust zeroing neural network (CVRZNN) model is designed by using the second method to solve the complex-valued matrix inversion problem under various external noises. Those external noises mainly include constant noise, linear noise, random noise, and hybrid noise, etc. In addition, the convergence and robustness of the CVRZNN model are theoretically analyzed in the presence of constant noise, linear noise, and random noise. In simulation examples, the GNN model [35], [36] and the ZNN model [37], [38] are applied to complex-valued matrix inversion for comparative purpose. The simulation results show that the state solution of the CVRZNN model can converge to the theoretical inversion of dynamic complex-valued matrix under external noises, while the ZNN model and the GNN model can hardly converge to theoretical inversion under the same conditions.

In the last part of this introduction, we list the contributions of this work as follows.

- A novel fully complex-valued and robust zeroing neural network (CVRZNN) model is proposed to solve dynamic matrix inversion with various types of noises. The CVRZNN model can directly find the accurate inversion of complex matrix without extra transformation, and greatly increase the computational efficiency, as compared to the handling process of [34].
- The strict theoretical analysis is provided to ensure the convergence and robustness of the CVRZNN model simultaneously under noise interferences. It is proved that the steady-state error of the CVRZNN model can converge to 0 under constant noise, and can be bounded by upper bounds under linear and random noises.
- Two different dimensional numerical examples are presented to illustrate the effectiveness and advantage of the CVRZNN model, in comparison with existing neural-network models.

II. PROBLEM FORMULATION AND PRELIMINARIES

In this part, the solving problem and preliminary preparations are described to lay the foundation for further investigation. Matrix inversion in the complex domain is often formulated as follows:

$$\mathrm{A}(t)\mathrm{Z}(t) - I = 0 \in \mathbb{C}^{n\times n}, \tag{1}$$

where $\mathrm{A}(t) \in \mathbb{C}^{n\times n}$ is a known dynamic complex matrix, $\mathrm{Z}(t) \in \mathbb{C}^{n\times n}$ is a matrix needed to be worked out in complex domain, and $I \in \mathbb{C}^{n\times n}$ is the unit matrix of the complex domain. It is worth noting that $\mathrm{A}^{-1}(t) \in \mathbb{C}^{n\times n}$ is used to denote the theoretical solution of (1). In this article, the goal is to solve $\mathrm{Z}(t)$ from (1) by proposing a fully complex-valued and robust zeroing neural network (CVRZNN) model under various kinds of external noises. Compared with the neural model proposed in [34], the CVRZNN model does not process the real and imaginary parts of the complex matrix separately, while the complex matrix is treated as a whole to be discussed and analyzed.

III. DESIN OF CVRZNN MODEL

In this part, there will provide the design steps of the proposed CVRZNN model. For comparison, the ZNN model and GNN model are also introduced. The following matrix-valued error function $\mathrm{S}(t)$ is first defined as

$$\mathrm{S}(t) = \mathrm{A}(t)\mathrm{Z}(t) - \mathrm{I} \in \mathbb{C}^{n\times n}. \quad (2)$$

The advantage of adopting matrix-valued error function is to primely monitor the process of solving (1) in complex domain.

Then, inspired by [28]–[30], the following design formula with an integral term is designed in the complex domain to ensure that $S(t)$ can converge to zero under various kinds of external noises:

$$\dot{\mathrm{S}}(t) = -\lambda S(t) - \xi \int_0^t S(x)\mathrm{d}x \in \mathbb{C}^{n\times n}, \quad (3)$$

where $\lambda > 0$ and $\xi > 0$ are scale factors which are used to gauge the convergence rate of the CVRZNN model. Considering Eq. (2) and new design formula (3), the CVRZNN model can be obtained as follows:

$$\mathrm{A}(t)\dot{\mathrm{Z}}(t) = -\lambda(\mathrm{A}(t)\mathrm{Z}(t) - \mathrm{I}) - \dot{\mathrm{A}}(t)\mathrm{Z}(t) - \xi \int_0^t (\mathrm{A}(x)\mathrm{Z}(x) - \mathrm{I})\mathrm{d}x. \quad (4)$$

When it comes to the robustness of CVRZNN model (4), the model disturbed by various external noise interference is directly presented as follows:

$$\mathrm{A}(t)\dot{\mathrm{Z}}(t) = -\lambda(\mathrm{A}(t)\mathrm{Z}(t) - \mathrm{I}) - \dot{\mathrm{A}}(t)\mathrm{Z}(t) - \xi \int_0^t (\mathrm{A}(x)\mathrm{Z}(x) - \mathrm{I})\mathrm{d}x + \phi(t), \quad (5)$$

where $\phi(t) \in \mathbb{C}^{n\times n}$ represents uncertain noises with matrix-form, and generally includes constant noise, linear noise, random noise, and their hybrid noise.

In order to compare with CVRZNN model (4), the existing ZNN and GNN models for complex matrix inversion (1) can be developed by considering external noises. Specifically, the ZNN model for computing dynamic complex matrix inversion disturbed by external noise is directly given as below:

$$\mathrm{A}(t)\dot{\mathrm{Z}}(t) = -\lambda(\mathrm{A}(t)\mathrm{Z}(t) - \mathrm{I}) - \dot{\mathrm{A}}(t)\mathrm{Z}(t) + \phi(t), \quad (6)$$

and the GNN model for computing dynamic complex matrix inversion disturbed by external noise is

$$\dot{\mathrm{Z}}(t) = -\lambda \mathrm{A}^{\mathrm{T}}(x)(\mathrm{A}(t)\mathrm{Z}(t) - \mathrm{I}) + \phi(t), \quad (7)$$

where $\lambda > 0$ and $\phi(t) \in \mathbb{C}^{n\times n}$ are defined the same as the CVRZNN model.

IV. THEORETICAL ANALYSIS AND RESULTS

In this section, the detailed theoretical analysis of the newly proposed CVRZNN model will be divided into the following two parts. The first is to analyze the global exponential convergence of CVRZNN model (4), and the second is to analyze the robustness of CVRZNN model (4) when facing with various kinds of external noises.

A. CONVERGENCE OF CVRZNN MODEL

In this subsection, the convergence performance of CVRZNN model (4) in a noiseless environment is proved through two theories.

Theorem 1: Given a fully complex-valued time-varying invertible matrix $\mathrm{A}(t) \in \mathbb{C}^{n\times n}$ defined in (1), the state matrix $Z(t) \in \mathbb{C}^{n\times n}$ generated by CVRZNN model (4), from any initial value $\mathrm{Z}(0) \in \mathbb{C}^{n\times n}$, is globally convergent to the theoretical inversion $\mathrm{A}^{-1}(t)$ of (1).

Proof 1: As for CVRZNN model (4), it can be derived from the simplified design formula $\dot{S}(t) = -\lambda S(t) - \xi \int_0^t S(x)\mathrm{d}x$, which can be equivalently rewritten as the following set of $n * n$ neuron's dynamic systems:

$$\dot{s}_{pq}(t) = -\lambda s_{pq}(t) - \xi \int_0^t s_{pq}(x)\mathrm{d}x \quad \forall p \in \{1, 2, \cdots, n\}, \quad \forall q \in \{1, 2, \cdots, n\}, \quad (8)$$

where $s_{pq}(t) \in \mathbb{C}$ represents the pqth element of $S(t)$. For showing the stability of Eq. (8), the Lyapunov function can be defined as follows:

$$\mathrm{V}_{pq}(t) = |s_{pq}(t)|^2 + \xi \left| \int_0^t s_{pq}(x)\mathrm{d}x \right|^2, \quad (9)$$

where $|s_{pq}(t)|$ represents the modulus of the complex variable $s_{pq}(t)$. In complex domain, we have $|s_{pq}(t)|^2 = s_{pq}(t)\overline{s_{pq}(t)}$ with $\overline{s_{pq}(t)}$ denoting the conjugate of $s_{pq}(t)$. As the same reason, we can know $|\int_0^t s_{pq}(x)\mathrm{d}x|^2 = \int_0^t s_{pq}(x)\mathrm{d}x \overline{\int_0^t s_{pq}(x)\mathrm{d}x}$ with $\overline{\int_0^t s_{pq}(x)\mathrm{d}x}$ denoting the conjugate of $\int_0^t s_{pq}(x)\mathrm{d}x$. Additionally, it is evident to see the Lyapunov function $\mathrm{V}_{pq}(t)$ is constantly positive, i.e., $\mathrm{V}_{pq}(t) > 0$ for any $s_{pq}(t) \neq 0$ or $\int_0^t s_{pq}(x)\mathrm{d}x \neq 0$, and $\mathrm{V}_{pq}(t) = 0$ just for $s_{pq}(t) = \int_0^t s_{pq}(x)\mathrm{d}x = 0$. Based on Eqs. (8) and (9), we can obtain

$$\begin{aligned}
\dot{V}_{pq}(t) &= \dot{s}_{pq}(t)\overline{s_{pq}(t)} + s_{pq}(t)\dot{\overline{s_{pq}(t)}} \\
&\quad + \xi s_{pq}(t)\overline{\int_0^t s_{pq}(x)\mathrm{d}x} + \xi \overline{s_{pq}(t)} \int_0^t s_{pq}(x)\mathrm{d}x \\
&= -2\lambda s_{pq}(t)\overline{s_{pq}(t)} \\
&= -2\lambda |s_{pq}(t)|^2 \leq 0.
\end{aligned}$$

According to the Lyapunov stability theory, we can come to the conclusion that the pqth subsystem (8) is stable and can globally converge to zero for any $p \in \{1, 2, \cdots, n\}$ and $q \in \{1, 2, \cdots, n\}$. Hence, it can be inferred that $\mathrm{S}(t)$ can globally converge to 0. That is, CVRZNN model (4) converges to the theoretical inversion of dynamic matrix inversion problem (1). The testimony about the newly proposed CVRZNN model on global convergence is accomplished.

In order to further investigate the convergence rate of the newly proposed CVRZNN model (4), we will provide the following theorem.

Theorem 2: Given a fully complex-valued time-varying invertible matrix $\mathrm{A}(t) \in \mathbb{C}^{n\times n}$ defined in (1), the state solution $\mathrm{Z}(t)$ obtained by the CVRZNN model (4), from any initial value $\mathrm{Z}(0)$, will exponentially converge to the theoretical inversion $\mathrm{A}^{-1}(t)$ of (1).

Proof 2: Defining $\varepsilon(t) = \int_0^t S(x)\mathrm{d}x$, its first-order time derivative and second-order time derivative are as $\dot{\varepsilon}(t) = \mathrm{S}(t)$ and $\ddot{\varepsilon}(t) = \dot{S}(t)$. The noise-tolerant formula $\dot{\mathrm{S}}(t) = -\lambda S(t) - \xi \int_0^t S(x)\mathrm{d}x$ can be rewritten as the following second-order dynamic system:

$$\ddot{\varepsilon}(t) = -\lambda\dot{\varepsilon}(t) - \xi\varepsilon(t). \tag{10}$$

Let $\varepsilon_{pq}(t)$, $\dot{\varepsilon}_{pq}(t)$, and $\ddot{\varepsilon}_{pq}(t)$ denote the pqth item of $\varepsilon(t)$, $\dot{\varepsilon}(t)$, and $\ddot{\varepsilon}(t)$ respectively. The pqth subsystem of (10) can be depicted as

$$\ddot{\varepsilon}_{pq}(t) = -\lambda\dot{\varepsilon}_{pq}(t) - \xi\varepsilon_{pq}(t). \tag{11}$$

Obviously, this a second-order linear dynamic system with scalar-valued, and thus we can analyze its characteristic roots to show the stability. Let ζ_1 and ζ_2 be the two solutions of the above equation (11). Then, we can compute $\zeta_1 = 0.5(-\lambda + \sqrt{\lambda^2 - 4\xi})$ and $\zeta_2 = 0.5(-\lambda - \sqrt{\lambda^2 - 4\xi})$. In view of the initial values $\varepsilon_{pq}(0) = 0$, and $\dot{\varepsilon}_{pq}(0) = s_{pq}(0)$, the solution to (11) belongs to one of the following three situations.

- If $\lambda^2 > 4\xi$, ζ_1, ζ_2 are real number and $\zeta_1 \neq \zeta_2$. We have
$$\varepsilon_{pq}(t) = \frac{s_{pq}(0)(\exp(\zeta_1 t) - \exp(\zeta_2 t))}{\sqrt{\lambda^2 - 4\xi}},$$
and further available
$$\begin{aligned} s_{pq}(t) &= \dot{\varepsilon}_{pq}(t) \\ &= \frac{s_{pq}(0)(\zeta_1\exp(\zeta_1 t) - \zeta_2\exp(\zeta_2 t))}{\sqrt{\lambda^2 - 4\xi}}. \end{aligned}$$
Then, the error matrix can be derived as follows:
$$\mathrm{S}(t) = \frac{\mathrm{S}(0)(\zeta_1\exp(\zeta_1 t) - \zeta_2\exp(\zeta_2 t))}{\sqrt{\lambda^2 - 4\xi}}.$$
- If $\lambda^2 = 4\xi$, ζ_1, ζ_2 are real number and $\zeta_1 = \zeta_2$. We have
$$\varepsilon_{pq}(t) = s_{pq}(0)t\exp(\zeta_1 t),$$
and further available
$$\begin{aligned} s_{pq}(t) &= \dot{\varepsilon}_{pq}(t) \\ &= s_{pq}(0)\exp(\zeta_1 t) + s_{pq}(0)\zeta_1 t\exp(\zeta_1 t) \\ &= s_{pq}(0)\exp(\zeta_1 t)(1 + \zeta_1 t). \end{aligned}$$
Then, the error matrix can be derived as follows:
$$\begin{aligned} \mathrm{S}(t) &= \mathrm{S}(0)\exp(\zeta_1 t) + \mathrm{S}(0)\zeta_1 t\exp(\zeta_1 t) \\ &= \mathrm{S}(0)\exp(\zeta_1 t)(1 + \zeta_1 t). \end{aligned}$$
- If $\lambda^2 < 4\xi$, $\zeta_1 = x + yi$ and $\zeta_2 = x - yi$ are conjugate complex numbers. Then we have
$$\varepsilon_{pq}(t) = \frac{s_{pq}(0)\sin(yt)\exp(xt)}{y},$$
and further available
$$s_{pq}(t) = s_{pq}(0)\exp(xt)\left(\frac{x\sin(yt)}{y} + \cos(yt)\right).$$
Then, the error matrix can be derived as follows:
$$\mathrm{S}(t) = \mathrm{S}(0)\exp(xt)\left(\frac{x\sin(yt)}{y} + \cos(yt)\right).$$

According to the above three situations and Theorem 1 in [39], we can come to the conclusion that for a fully complex-valued time-varying invertible matrix $\mathrm{A}(t) \in \mathbb{C}^{n\times n}$ defined in (1), the state solution $\mathrm{Z}(t)$ obtained by the CVRZNN model (4), from any initial value Z(0), will exponentially converge to the theoretical inversion $\mathrm{A}^{-1}(t)$ of (1).

B. ROBUSTNESS OF CVRZNN MODEL WITH NOISES

This part offers three theorems to demonstrate the robustness performance of the newly proposed CVRZNN model (5) under three kinds of external noises (i.e., constant noise, linear time-varying noise and random noise).

1) CONSTANT NOISE

The first type of noise to be discussed is constant noise. In this situation, the steady-state error of CVRZNN model (5) is proved to be convergent to zero.

Theorem 3: Given a fully complex-valued time-varying invertible matrix $\mathrm{A}(t) \in \mathbb{C}^{n\times n}$ defined in (1), while constant noise $\phi(t) = \phi$ is added, the state solution $\mathrm{Z}(t)$ obtained by CVRZNN model (5) from any initial value Z(0) globally converges to the theoretical inversion $\mathrm{A}^{-1}(t)$ of (1).

Proof 3: For the pqth subsystem of the proposed CVRZNN model (5), while the constant noise ϕ is injected, according to the Laplace transform [43], we can obtain

$$\mu s_{pq}(\mu) = s_{pq}(0) - \lambda s_{pq}(\mu) - \frac{\xi}{\mu}s_{pq}(\mu) + \phi_{pq}(\mu),$$

from which we further have

$$s_{pq}(\mu) = \frac{\mu(s_{pq}(0) + \phi_{pq}(\mu))}{\mu^2 + \mu\lambda + \xi}. \tag{12}$$

Obviously, $\mu/(\mu^2 + \mu\lambda + \xi)$ is the transfer function and its poles are $\mu_1 = 0.5(-\lambda + \sqrt{\lambda^2 - 4\xi})$ and $\mu_2 = 0.5(-\lambda - \sqrt{\lambda^2 - 4\xi})$. For $\lambda > 0$ and $\xi > 0$, it is obvious that the system is stable because its two poles are situated in the left half plane. Through the Laplace transform, it can be written as $\phi_{pq}(\mu) = \phi_{pq}/\mu$. For system (12), using the final value theorem [43], we can obtain

$$\begin{aligned} \lim_{t\to\infty} s_{pq}(t) &= \lim_{\mu\to 0} \mu s_{pq}(\mu) \\ &= \lim_{\mu\to 0} \frac{\mu^2(s_{pq}(0) + \phi_{pq}/\mu)}{\mu^2 + \mu\lambda + \xi} \\ &= 0. \end{aligned}$$

Thus we can come to the conclusion that $\lim_{t\to\infty} \|S(t)\|_{\mathrm{F}} = 0$. The testimony about constant noise part is accomplished.

2) LINEAR TIME-VARYING NOISE

Because linear time-varying noise is often emerged in actual applications, it is necessary to discuss the robustness of the newly proposed CVRZNN model (5) under linear time-varying noise. The strict theoretical analysis is given to prove the superior robustness of CVRZNN model (5) under linear time-varying noise interference.

Theorem 4: Given a fully complex-valued time-varying invertible matrix $\mathrm{A}(t) \in \mathbb{C}^{n\times n}$ defined in (1), while linear time-varying noise $\phi(t) = Bt$ is added, the steady-state error generated by the CVRZNN model (5) is bounded by a finite upper bound, which is derived as

$$\lim_{t\to\infty} \|S(t)\|_\mathrm{F} = \frac{\|B\|_\mathrm{F}}{\xi},$$

where B denotes a constant matrix.

Proof 4: For each pqth element of CVRZNN model (5), while linear time-varying noise $\phi(t) = Bt$ is inject, according to the Laplace transform [43], we can obtain

$$\mu s_{pq}(\mu) = s_{pq}(0) - \lambda s_{pq}(\mu) - \frac{\xi}{\mu} s_{pq}(\mu) + \frac{b_{pq}}{\mu^2},$$

where b_{pq} denotes the pq element of constant matrix B. Then, we have

$$s_{pq}(\mu) = \frac{\mu(s_{pq}(0) + b_{pq}/\mu^2)}{\mu^2 + \mu\lambda + \xi}. \tag{13}$$

For (13), applying the final value theorem [43], we can obtain

$$\begin{aligned}\lim_{t\to\infty} s_{pq}(t) &= \lim_{\mu\to 0} \mu s_{pq}(\mu)\\ &= \lim_{\mu\to 0} \frac{\mu^2(s_{pq}(0) + b_{pq}/\mu^2)}{\mu^2 + \mu\lambda + \xi}\\ &= \frac{b_{pq}}{\xi}.\end{aligned}$$

Obviously, we can come to the conclusion that

$$\lim_{t\to\infty} \|S(t)\|_\mathrm{F} = \frac{\|B\|_\mathrm{F}}{\xi}.$$

Especially, we have $\lim_{t\to\infty} \|S(t)\|_\mathrm{F} \to 0$ if $\xi \to \infty$. The testimony with linear time-varying noise is accomplished.

3) BOUNDED RANDOM NOISE

Compared with the previous constant noise and linear time-varying noise, in this part, the robustness of the newly proposed CVRZNN model (5) in the case of bounded random noise is discussed and analyzed.

Theorem 5: Given a fully complex-valued time-varying invertible matrix $\mathrm{A}(t) \in \mathbb{C}^{n\times n}$ defined in (1), while bounded random noise $\phi(t) = \sigma(t) < +\infty$ is added, the steady-state error generated by CVRZNN model (5) is bounded by $2\sqrt{2n}\sup_{0\le x\le t} |\sigma_{ij}(x)|/\sqrt{\lambda^2 - 4\xi}$ for $\lambda^2 > 4\xi$, or $4\sqrt{2n}\xi \sup_{0\le x\le t} |\sigma_{ij}(x)|/(\lambda\sqrt{4\xi - \lambda^2})$ for $\lambda^2 < 4\xi$, where $\sigma_{ij}(t)$ represents the ijth item of $\sigma(t)$. Besides, $\lim_{t\to\infty} \|S(t)\|_\mathrm{F}$ can be random small for large enough λ and an appropriate value of ξ.

Proof 5: While a bounded random noise $\phi(t) = \sigma(t)$ is added, CVRZNN model (5) can be transformed as follows:

$$\dot{\mathrm{S}}(t) = -\lambda S(t) - \xi \int_0^t S(x)\mathrm{d}x + \sigma(t),$$

of which its pqth subsystem can be depicted as

$$\dot{s}_{pq}(t) = -\lambda s_{pq}(t) - \xi \int_0^t s_{pq}(x)\mathrm{d}x + \sigma_{pq}(t). \tag{14}$$

For different λ and ξ, we can consider three situations to be discussed.

- If $\lambda^2 > 4\xi$, for (14), we can obtain

$$\begin{aligned}s_{pq}(t) = &\frac{s_{pq}(0)(\zeta_1 \exp(\zeta_1 t) - \zeta_2 \exp(\zeta_2 t))}{(\zeta_1 - \zeta_2)}\\ &+ \Big(\int_0^t \big(\zeta_1 \exp(\zeta_1(t-x)) - \zeta_2 \exp(\zeta_2(t-x))\big)\\ &\times \sigma_{pq}(x)\mathrm{d}x \Big) \frac{1}{(\zeta_1 - \zeta_2)},\end{aligned}$$

where ζ_1 and ζ_2 have been defined as before, i.e., $\zeta_{1,2} = (-\lambda \pm \sqrt{\lambda^2 - 4\xi})/2$. Through the triangle inequality, it can obtain

$$\begin{aligned}|s_{pq}(t)| \le &\frac{|s_{pq}(0)(\zeta_1 \exp(\zeta_1 t) - \zeta_2 \exp(\zeta_2 t))|}{(\zeta_1 - \zeta_2)}\\ &+ \frac{\int_0^t |\zeta_1 \exp(\zeta_1(t-x))| \cdot |\sigma_{pq}(x)|\mathrm{d}x}{(\zeta_1 - \zeta_2)}\\ &+ \frac{\int_0^t |\zeta_2 \exp(\zeta_2(t-x))| \cdot |\sigma_{pq}(x)|\mathrm{d}x}{(\zeta_1 - \zeta_2)}.\end{aligned}$$

Furthermore, it can obtain

$$\begin{aligned}|s_{pq}(t)| &\le \frac{|s_{pq}(0)(\zeta_1 \exp(\zeta_1 t) - \zeta_2 \exp(\zeta_2 t))|}{(\zeta_1 - \zeta_2)}\\ &\quad + \frac{2}{(\zeta_1 - \zeta_2)} \max_{0\le x\le t} |\sigma_{pq}(x)|\\ &= \frac{|s_{pq}(0)(\zeta_1 \exp(\zeta_1\ t) - \zeta_2 \exp(\zeta_2 t))|}{(\zeta_1 - \zeta_2)}\\ &\quad + \frac{2}{\sqrt{\lambda^2 - 4\xi}} \max_{0\le x\le t} |\sigma_{pq}(x)|.\end{aligned}$$

Finally,

$$\lim_{t\to\infty} \|S(t)\|_\mathrm{F} \le \frac{2\sqrt{2}n}{\sqrt{\lambda^2 - 4\xi}} \sup_{0\le x\le t} |\sigma_{pq}(x)|.$$

- When $\lambda^2 = 4\xi$, for (14), we can obtain

$$\begin{aligned}s_{pq}(t) = &s_{pq}(0)t\zeta_1 \exp(\zeta_1 t) + s_{pq}(0)\exp(\zeta_1 t)\\ &+ \int_0^t ((t-x)\zeta_1 \exp(\zeta_1(t-x)))\sigma_{pq}(x)\mathrm{d}x\\ &+ \int_0^t \exp(\zeta_1(t-x))\sigma_{pq}(x)\mathrm{d}x,\end{aligned}$$

where ζ_1 is defined as before, i.e., $\zeta_1 = (\lambda + \sqrt{\lambda^2 - 4\xi})/2 = -\lambda/2$. For $\zeta_1 = -\lambda/2$, $\lambda > 0$, $|\zeta_1| > 0$ and $t >= 0$, there exist $\tau > 0$ and $\nu > 0$ such that

$$|\zeta_1| t \exp(\zeta_1 t) \le \tau \exp(-\nu t). \tag{15}$$

Based on the foundations of the inequality (15) and the triangle inequality, we have

$$\begin{aligned}|s_{pq}(t)| &\leq |s_{pq}(0)(\zeta_1 t\exp(\zeta_1 t)+\exp(\zeta_1 t))|\\&\quad+\int_0^t|\tau\exp(-\nu(t-x))|\cdot|\sigma_{pq}(x)|\mathrm{d}x\\&\quad+\int_0^t|\exp(\zeta_1(t-x))|\cdot|\sigma_{pq}(x)|\mathrm{d}x\\&\leq|s_{pq}(0)(\zeta_1 t\exp(\zeta_1 t)+\exp(\zeta_1 t))|\\&\quad+(\frac{\tau}{\nu}-\frac{1}{\zeta_1})\max_{0\leq x\leq t}|\sigma_{pq}(x)|.\end{aligned}$$

Finally,

$$\lim_{t\to\infty}\|S(t)\|_{\mathrm{F}}\leq(\frac{\tau}{\nu}-\frac{1}{\zeta_1})\sqrt{2}n\sup_{0\leq x\leq t}|\sigma_{pq}(x)|.$$

- When $\lambda^2<4\xi$, for (14), we can obtain

$$\begin{aligned}s_{pq}(t)&=s_{pq}(0)\exp(\alpha t)(\alpha\sin(\beta t)/\beta+\cos(\beta t))\\&\quad+\int_0^t\big(\alpha\sin(\beta(t-x))\exp(\alpha(t-x))/\beta\\&\quad+\cos(\beta(t-x))\exp(\alpha(t-x))\big)\sigma_{pq}(x)\mathrm{d}x,\end{aligned}$$

where α and β are defined as $\alpha=-\lambda/2$ and $\beta=\sqrt{4\xi-\lambda^2})/2$. According to the foundations of the triangle inequality, in the similar way, it can obtain

$$\begin{aligned}|s_{pq}(t)|&\leq|s_{pq}(0)\exp(\alpha t)(\alpha\sin(\beta t)/\beta+\cos(\beta t)))|\\&\quad-\frac{\sqrt{\alpha^2+\beta^2}}{\alpha\beta}\max_{0\leq x\leq t}|\sigma_{pq}(x)|\\&=|s_{pq}(0)\exp(\alpha t)(\alpha\sin(\beta t)/\beta+\cos(\beta t))|\\&\quad+\frac{4\xi}{\lambda\sqrt{4\xi-\lambda^2}}\max_{0\leq x\leq t}|\sigma_{pq}(x)|.\end{aligned}$$

Finally,

$$\lim_{t\to\infty}\|S(t)\|_{\mathrm{F}}\leq\frac{4\sqrt{2}\xi n}{\lambda\sqrt{4\xi-\lambda^2}}\sup_{0\leq x\leq t}|\sigma_{pq}(x)|.$$

Thus the testimony with bounded random noise is accomplished.

Remark 1: It is worth noting that the proof of (15) can be found in Appendix 1 of [39], and the main derivative process is shown as below. From the given conditions with the inequality (15), it can be converted as

$$0\leq t\leq 2\tau(\exp((\lambda/2)-\nu)t)/\lambda. \tag{16}$$

When $t=0$, for any $\tau>0$ and $\nu>0$, the inequality (16) as well as (15) are true. Then, we just need to prove the situation of $t>0$. For insuring the inequality (15) being always true, we should keep $(\lambda/2-\nu)>0$. By taking the logarithm of the inequality (16), we can obtain

$$\ln t\leq\ln(2\tau/\lambda)+((\lambda/2)-\nu)t.$$

That is,

$$\ln(2\tau/\lambda)\geq\ln t-((\lambda/2)-\nu)t. \tag{17}$$

It is evident that if $\ln(2\tau/\lambda)$ is lager than or equal to the maximum value of $\ln t-((\lambda/2)-\nu)t$, the equality of (17) will always be true. That is, the equalities (16) and (15) will always be true. The following steps will find the maximum value of $\ln t-((\lambda/2)-\nu)t$. We can find the maximum value of $\ln t-((\lambda/2)-\nu)t$ by taking the derivative of the function, finding the stationary point of the function and then determining the maximum value. Through the computation, we can obtain the stationary point $t=2/(\lambda-2\nu)$. Taking $t=2/(\lambda-2\nu)$ into $\ln(2\tau/\lambda)=\ln t-((\lambda/2)-\nu)t$, we can obtain $(2\tau/\lambda)(\lambda/2)-\nu\exp(1)=1$, which is the condition for the quality in (15). Then, we can discuss the right-hand side of (17) with the maximum value for insuring the inequality (15) always true in any others cases. Taking $t=2/(\lambda-2\nu)$ into (15), we can obtain

$$(2\tau/\lambda)(\lambda/2-\nu)\exp(1)\geq 1. \tag{18}$$

Setting $\tau=\lambda/2$ in (18), we can get $0\leq\nu\leq\lambda/2-\exp(-1)$. And then the equality of (15) can be proved by discussing formula $\lambda/2-\exp(-1)$.

V. ILLUSTRATIVE EXAMPLES

In the previous sections, it has been analysed and demonstrated to possess the convergence and robustness of CVRZNN model (5) under different kinds of external noises. In this section, two examples on complex-valued matrix inversion will be presented to illustrate the superior performance of CVRZNN model (5) as compared with existing GNN and ZNN models.

A. EXAMPLE I

In order to further verify the globally convergence and the robustness of CVRZNN model(5), we can consider the following example for complex-valued time-varying matrix A(t):

$$\mathrm{A}(t)=\begin{bmatrix}\cos(8t)+i & -\sin(8t)+i\\ \sin(8t)+i & \cos(8t)+i\end{bmatrix}\in\mathbb{C}^{2\times 2}. \tag{19}$$

For verifying the accuracy of the state solutions generated by CVRZNN model (5), ZNN model (6), and GNN model (7), the theoretical inversion of this time-varying complex matrix (19) is computed as follows:

$$\mathrm{A}^{-1}(t)=\begin{bmatrix}\cos(8t)+i & \sin(8t)+i\\ -\sin(8t)+i & \cos(8t)+i\end{bmatrix}\in\mathbb{C}^{2\times 2}.$$

Besides, four different noise-disturbed cases are respectively discussed when CVRZNN model (5), ZNN model (6), and GNN model (7) are applied to find complex-valued time-varying matrix inversion of (19).

1) Noiseless: By setting $\lambda=\xi=10$, and from any initial states $Z(0)$ generated from $[-1,1]^{2\times 2}$, simulation output results of CVRZNN model (5) for solving the complex-valued matrix inverse problem (19) are shown in Figs. 1 and 2. As depicted in Fig. 1 where the red curves represent the theoretical solution $\mathrm{A}^{-1}(t)$ of (19), and the blue curves represent the

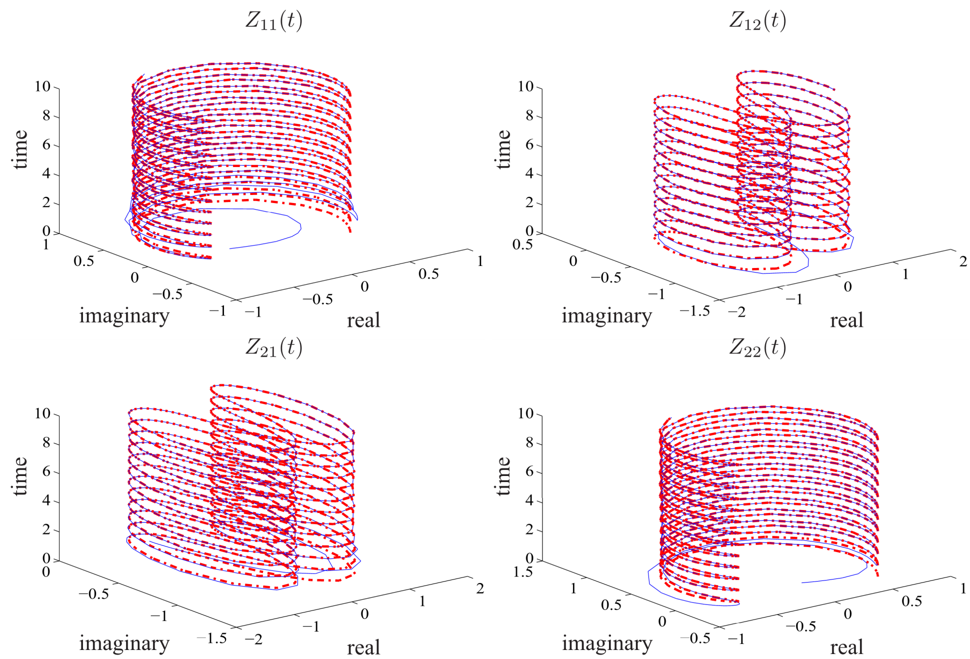

FIGURE 1. State trajectory of the newly proposed CVRZNN model (5) for complex matrix inversion (19) with $\lambda = \xi = 10$ under no noise.

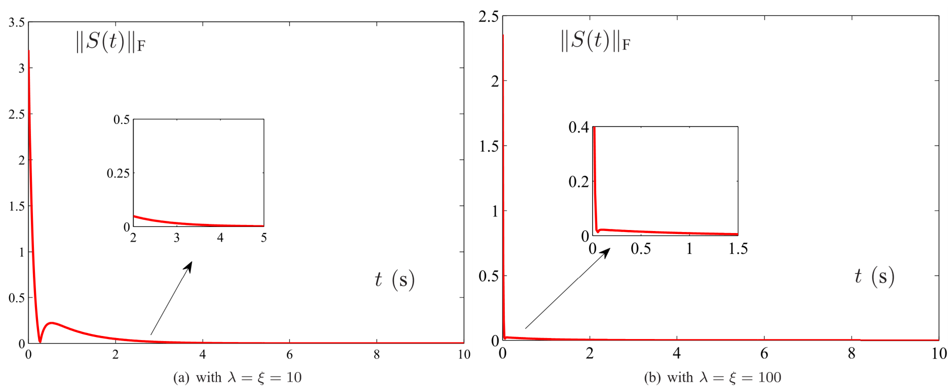

FIGURE 2. Error norm results of the newly proposed CVRZNN model (5) for complex matrix inversion (19) with different values of λ and ξ under no noise.

state solution of CVRZNN model (5), the state matrix $Z(t) \in \mathbb{C}^{2\times 2}$ obtained by CVRZNN model (5) can converge to the time-varying theoretical inversion $A^{-1}(t)$ accurately. The corresponding error norm $\|S(t)\|_F$ is shown in Fig. 2(a) for clearly demonstrating the convergence process of CVRZNN model (5). As seen in Fig. 2(a), the error norm $\|S(t)\|_F$ synthesized by CVRZNN model (5) can descend to zero within about 2.5 s. In addition, through expanding the values of design parameters λ and ξ from $\lambda = \xi = 10$ to $\lambda = \xi = 100$, the corresponding error norm $\|S(t)\|_F$ is shown in Fig. 2(b), and it only takes about 1.0 s to decrease to 0. The above computer simulation results verify the previous theoretical analysis results in Theorem 1.

2) Constant noise: In actually applications, there will exist noise interference. When constant noise $\phi(t) = 10$ is injected into CVRZNN model (5), with $\lambda = \xi = 10$, the computer simulation results are depicted in Fig. 3. It is easy to conclude that the state matrix $Z(t) \in \mathbb{C}^{2\times 2}$ obtained by CVRZNN model (5) converges to the time-varying theoretical inversion $A^{-1}(t)$ accurately. The corresponding error norm $\|S(t)\|_F$ shown in Fig. 4(a) needs about 4.5 s to decrease to zero even in the presence of constant noise $\phi(t) = 10$. For comparative purposes, ZNN model (6) and GNN model (7) are applied to compute the complex-valued matrix inverse of (19) under the same noise-disturbed environment. Due to similarity, state solution trajectories of ZNN model (6) and GNN model (7) are deleted,

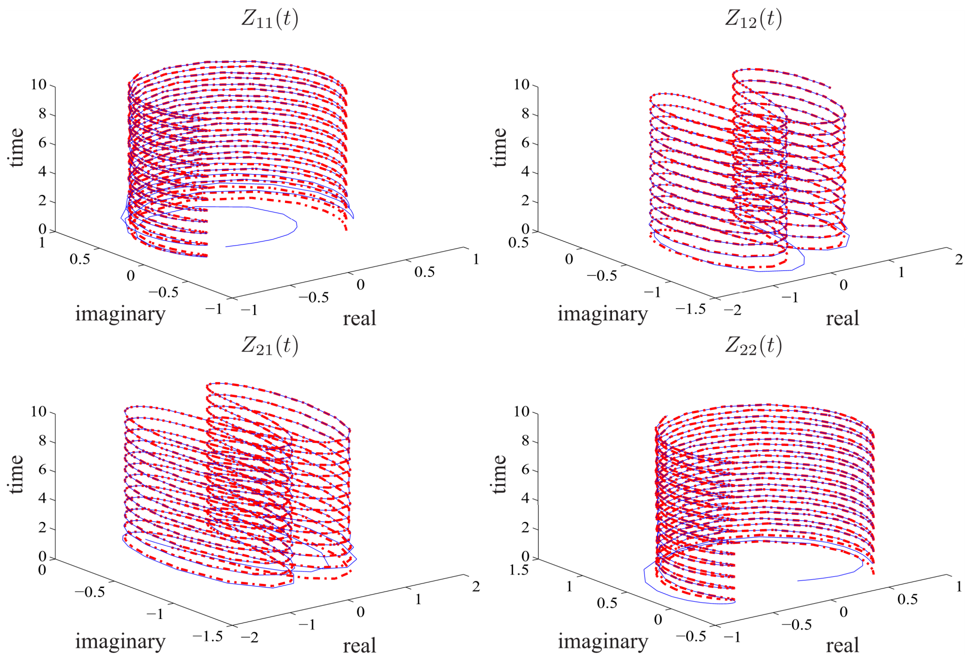

FIGURE 3. State trajectory of the newly proposed CVRZNN model (5) for complex matrix inversion (19) with $\lambda = \xi = 10$ under constant noise $\phi(t) = 10$.

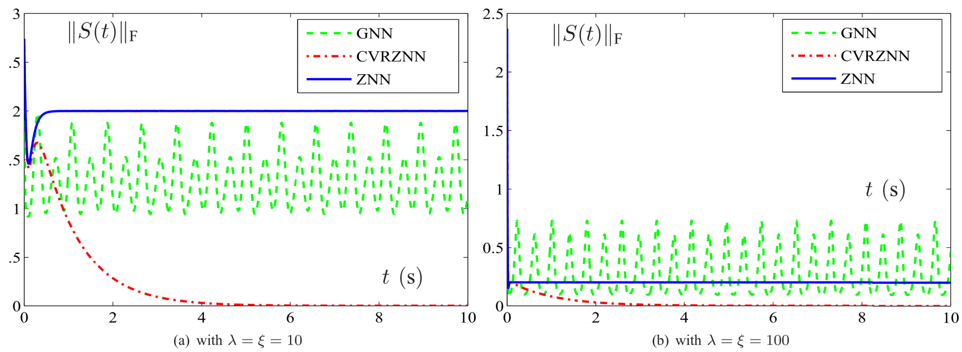

FIGURE 4. Error norm results of the newly proposed CVRZNN model (5) for complex matrix inversion (19) with different values of λ and ξ under constant noise $\phi(t) = 10$.

and the error norm results are shown in Fig. 4(a). From this figure, we can conclude that the error norms of ZNN model (6) and GNN model (7) cannot converge to zero in the presence of constant noise $\phi(t) = 10$. Through expanding the values of design parameters λ and ξ from $\lambda = \xi = 10$ to $\lambda = \xi = 100$, it can be conferred from Fig. 4(b) that the error norms of ZNN (6) and GNN model (7) become smaller, but still cannot converge to zero. As for CVRZNN model (5), the convergence time of the error norm is decreased from about 4.5 s to about 2.0 s. Through these comparative simulation results, we can come to the conclusion that CVRZNN model (5) has the excellent robustness performance for counting complex-valued time-varying matrix inverse in constant-noise situation.

3) Linear noise: The linear time-varying noise in this item is discussed and computer simulation results are depicted in Fig. 5. With $\lambda = \xi = 10$, it can be seen from Fig. 5(a) that the error norm $\|S(t)\|_F$ of CVRZNN model (5) needs about 2.5 s to decrease to zero even in the presence of linear time-varying noise $\phi(t) = 0.5t$. For comparative purposes, ZNN model (6) and GNN model (7) are applied to compute the complex-valued matrix inverse of (19) under the same noise-disturbed environment. From Fig. 5(a), we can conclude that the error norms of ZNN model (6) and GNN model (7) cannot converge to zero. Through expanding the values of design parameters λ and ξ from $\lambda = \xi = 10$ to $\lambda = \xi = 100$, it can be conferred from Fig. 5(b) that error norms of ZNN model (6) and GNN model (7) become smaller, but still cannot converge

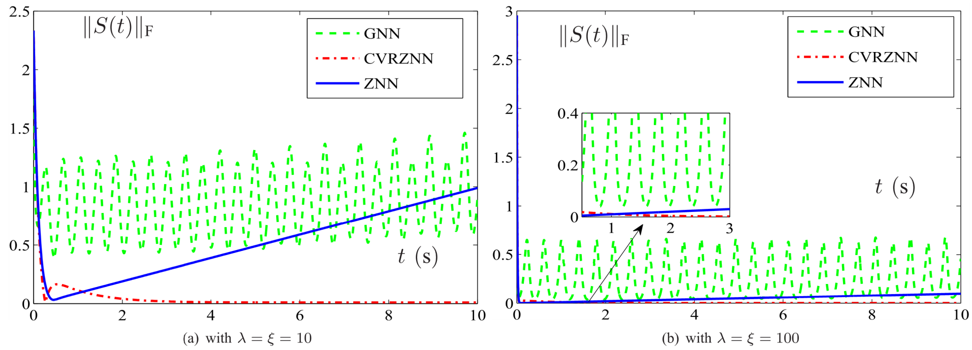

FIGURE 5. Error norm results of CVRZNN model (5), ZNN model (6), and GNN model (7) for complex matrix inversion (19) with different values of λ and ξ under linear time-varying noise $\phi(t) = 0.5t$.

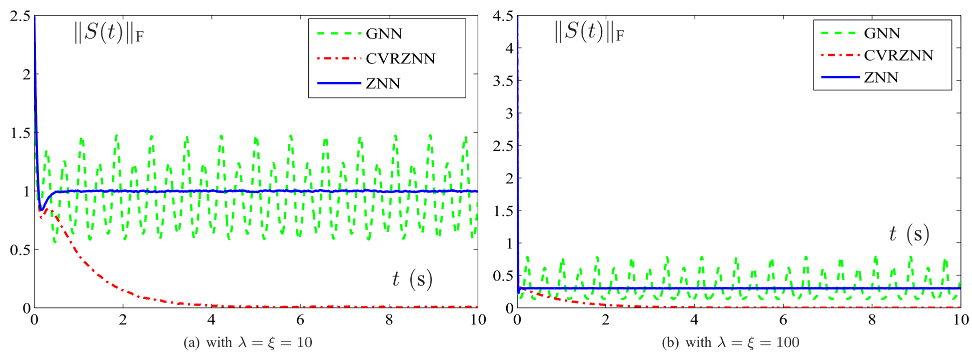

FIGURE 6. Error norm results of CVRZNN model (5), ZNN model (6), and GNN model (7) for complex matrix inversion (19) with different values of λ and ξ under random noise $\sigma_{pq}(t) = 0.5(\mathrm{rand}(4, 1) - 0.5\mathrm{ones}(4, 1))$.

to zero. As for CVRZNN model (5), the convergence time of the error norm is decreased from about 2.5 s to about 1.5 s. Through these comparative simulation results, we can come to the conclusion that CVRZNN model (5) has the excellent robustness performance for counting complex-valued time-varying matrix inverse under linear time-varying noise situation.

4) Bounded random noise: In this item, the bounded random noise $\sigma_{pq}(t) = 0.5(\mathrm{rand}(4, 1) - 0.5\mathrm{ones}(4, 1))$ is injected into such three neural models, the error norms $\|S(t)\|_{\mathrm{F}}$ are shown in Fig. 6 under different values of design parameters. In Fig. 6(a), with $\lambda = \xi = 10$, it can be seen that the error norm $\|S(t)\|_{\mathrm{F}}$ of CVRZNN model (5) needs about 4.0 s to decrease to zero under random noise $\sigma_{pq}(t) = 0.5(\mathrm{rand}(4, 1) - 0.5\mathrm{ones}(4, 1))$. In contrast, the error norms of ZNN model (6) and GNN model (7) cannot converge to zero under the same noise-disturbed environment. Through expanding the values of design parameters λ and ξ from $\lambda = \xi = 10$ to $\lambda = \xi = 100$, it can be conferred from Fig. 6(b) that the error norms of ZNN model (6) and GNN model (7) still cannot converge to zero. As for CVRZNN model (5), the convergence time of the error norm is decreased from about 4.0 s to about 2.0 s. Through these comparative simulation results, we can come to the conclusion that CVRZNN model (5) has the excellent robustness performance for counting complex-valued time-varying matrix inverse in the random noise situation.

5) In addition to the above three kinds of noise, we have added another time-varying noise $\sigma_{pq}(t) = 0.5(\exp(0.2t))$ into such three neural models. The error norms $\|S(t)\|_{\mathrm{F}}$ are shown in Fig. 7 under different values of design parameters. From Fig. 7(a) and Fig. 7(b), they have the same results as the previous noises. Through expanding the values of design parameters λ and ξ from $\lambda = \xi = 10$ to $\lambda = \xi = 100$, it can accelerate the convergence rate of CVRZNN model (5). However, ZNN model (6) and GNN model (7) still cannot converge to 0. Additional experimental results also suggest that CVRZNN model (5) has the excellent robustness performance for counting complex-valued time-varying matrix inverse with the time-varying noise interference.

In general, experimental results show that the CVRZNN model has superior performance for solving the inverse of dynamic complex matrix with noise interference.

B. EXAMPLE II

In order to further verify the global convergence and the robustness of CVRZNN model (5), we can transform the 2-dimensional complex matrix into a 4-dimensional complex

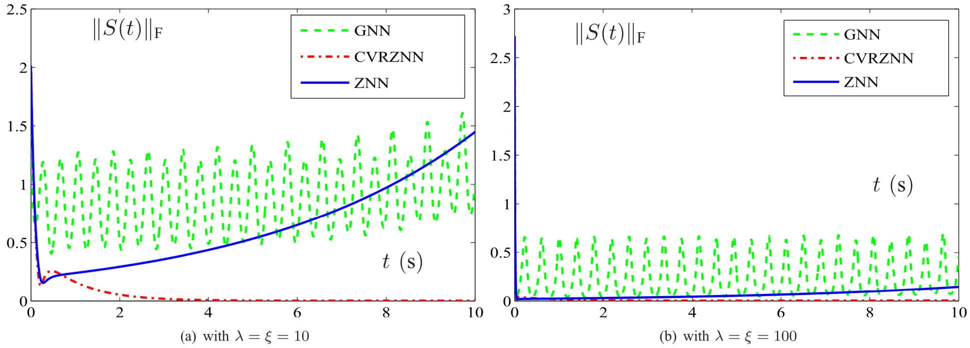

FIGURE 7. Error norm results of CVRZNN model (5), ZNN model (6), and GNN model (7) for complex matrix inversion (19) with different values of λ and ξ under noise $\sigma_{pq}(t) = 0.5(\exp(0.2t))$.

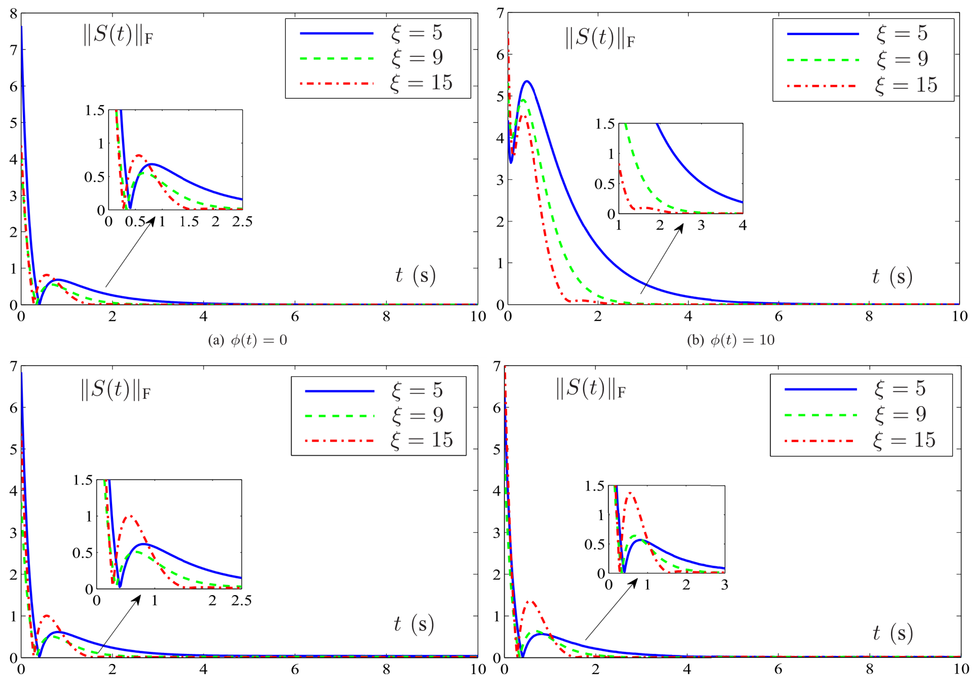

FIGURE 8. Error norm results of CVRZNN model (5) for complex matrix inversion (20) with the same λ [*i.e.*, $\lambda = 6$] and diverse ξ [*i.e.*, $\xi = 5, 9, 15$] under different external noises.

matrix as follows:

$$\mathrm{B}(t) = \begin{bmatrix} \cos(8t) & -\sin(8t)+i & -\sin(8t) & \cos(8t)+i \\ \sin(8t) & \cos(8t)+i & \cos(8t) & \sin(8t)+i \\ \sin(8t) & -\cos(8t)+i & \cos(8t) & -\sin(8t)+i \\ -\cos(8t) & -\sin(8t)+i & \sin(8t) & \cos(8t)+i \end{bmatrix}$$

$$\mathrm{C}(t) = \begin{bmatrix} i & 0 & i & 0 \\ i & 0 & i & 0 \\ i & 0 & i & 0 \\ i & 0 & i & 0 \end{bmatrix} \in \mathbb{C}^{4\times4},$$

$$\mathrm{A}(t) = \mathrm{B}(t) + \mathrm{C}(t) \in \mathbb{C}^{4\times4}. \quad (20)$$

In this example, we study different values of design parameters λ and ξ to discuss CVRZNN model (5) in the presence of four kinds of external noises. Specifically, λ is always set as 6, and ξ is set as 5, 9 and 15, which represent the situations of $\lambda^2 - 4\xi > 0$, $\lambda^2 - 4\xi = 0$ and $\lambda^2 - 4\xi < 0$ separately. The simulation results of error norm $\|S(t)\|_F = \|A(t)Z(t) - I\|_F$ generated by CVRZNN model (5) are demonstrated in Fig. 8. Due to the noise interference, the comparison results of ZNN model (6) and GNN model (7) will lead to large errors, and thus their simulations are ignored in this example. As depicted in Fig. 8(a), the steady-state error norm of CVRZNN model (5) can converge to zero accurately with no noise interference. In addition, as the value of ξ is increased from 5 to 9 and to 15, the convergence speed of CVRZNN model (5) is continuously accelerated. At the same time, when a constant noise, time-varying noise or random noise is injected into CVRZNN model (5), as shown in Fig. 8(b), (c) and (d) respectively, the steady-state error norm of CVRZNN model (5) can still converge to zero quickly. In addition, the convergence speed of CVRZNN model (5) can be accelerated with the value of ξ is increased, which illustrates efficacy of CVRZNN model (5) for solving dynamic complex-valued matrix inversion problem under external noise interference.

VI. CONCLUSION

For solving dynamic complex-valued matrix inversion problem under external noise interference, a novel fully complex-valued and robust zeroing neural network (CVRZNN) model (5) has been represented and analysed. Through theoretical proof, it can be seen that the steady-state error of CVRZNN model (5) can globally and exponentially converge to 0 with three kinds of external noises, which has demonstrated the solution obtained by CVRZNN model (5) can converge to the theoretical solution of the dynamic matrix inversion problem. For experimental comparison, ZNN model (6) and GNN model (7) have been introduced to solve the same problem in the complex domain. The comprehensive results show that CVRZNN model (5) has superior performance to ZNN model (6) and GNN model (7) for solving the dynamic complex-valued matrix inversion problem. Besides, due to the uncertainty of noise in industry, the design formula with integral term cannot satisfy the finite-time convergence. So in the future work, several activation functions will be considered to accelerate the convergence rate of the model to achieve finite-time convergence and improve the noise-disturbance capacity of the model.

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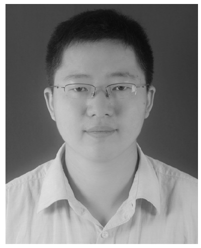

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