IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received April 19, 2020, accepted May 1, 2020, date of publication May 11, 2020, date of current version May 28, 2020.

Digital Object Identifier 10.1109/ACCESS.2020.2993784

Synchronization Methods for the Degn-Harrison Reaction-Diffusion Systems

FATIHA MESDOUI[1], ADEL OUANNAS[2], NABIL SHAWAGFEH[1], GIUSEPPE GRASSI[3], (Senior Member, IEEE), AND VIET-THANH PHAM[4,5], (Member, IEEE)

[1]Department of Mathematics, The University of Jordan, Amman 11942, Jordan
[2]Laboratory of Mathematics, Informatics and Systems (LAMIS), University of Larbi Tebessi, Tebessa 12002, Algeria
[3]Dipartimento di Ingegneria dell'Innovazione, Università del Salento, 73100 Lecce, Italy
[4]Faculty of Electrical and Electronic Engineering, Phenikaa Institute for Advanced Study (PIAS), Phenikaa University, Hanoi 100000, Vietnam
[5]Phenikaa Research and Technology Institute (PRATI), A&A Green Phoenix Group, Hanoi 100000, Vietnam

Corresponding author: Viet-Thanh Pham (thanh.phamviet@phenikaa-uni.edu.vn)

ABSTRACT The paper concerns the problem of synchronization-control in nonlinear bacterial cultures reaction-diffusion model, linear and nonlinear controllers have been proposed to study the complete synchronization of couples of the Degn-Harrison system with identical and non-identical coefficients. Throughout the paper, we use numerical simulation to show the effectiveness of the proposed results.

INDEX TERMS Degn-Harrison, complete synchronization, Lyapunov method, reaction-diffusion.

I. INTRODUCTION

Synchronization is a process of controlling the output of the response system (slave system) to force its behavior to follow that of the corresponding drive system (master system) asymptotically. Since the pioneering study of Pecora and Carroll [1], various control schemes have been introduced to synchronize dynamical systems due to its applications in image processing, cryptography, ecological system, combinatorial optimization, lasers technology, and secure communications [2].

Considerable research has been devoted to study the synchronization in low dimensional systems represented by unidimensional ordinary differential equations or maps [3]–[6]. Nevertheless, synchronizing in high-dimensional systems modeled in the spatial-temporal domain and described by nonlinear reaction-diffusion systems which state variables depend on the time and spatial position stills in its initial stage.

Reaction-diffusion systems models act a central role in describing the phenomena that exist in neuronal networks, chemical reaction systems, image processing and ecosystems. Due to the spatial component, this kind of model is extensively used to understand a wide range of complex dynamical structures and spatiotemporal patterns as well as rotating spirals, circulating pulses on a ring, target waves and oscillating spots [7]–[10]. For this reason, the study of synchronization in this kind of model is important to our comprehension of a wide variety of phenomena in the real-life.

Recently, significant efforts are made to investigate the synchronization in reaction-diffusion systems. For instance, the backstopping synchronization method [11], the hybrid adaptive synchronization approach [12], the graph-theoretic synchronization technique [13], impulsive type synchronization approach [14] and pinning impulsive synchronization [15] for PDEs have been proposed. Moreover, novel control synchronization schemes have been designed to achieve synchronization of the FitzHugh-Nagumo model [16], a three-component autocatalytic model [17], multi-layered natural and media networks [18], [19], the Newton-Leipnik chaotic system [20]. Also, linear and nonlinear control are suggested to synchronize a class of reaction diffusion systems [20], [21].

The present paper deals with the analysis of control synchronization for bacterial culture model introduced by Degn and Harrison [23] as

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1\Delta u_1 + a - u_1 - u_2 g_k(u_1), & (x,t)\in\Omega\times\mathbb{R}^+, \\ \dfrac{\partial u_2}{\partial t} = d_2\Delta u_2 + b - u_2 g_k(u_1), & (x,t)\in\Omega\times\mathbb{R}^+, \\ \dfrac{\partial u_i}{\partial \eta} = 0,\ i=1,2, & (x,t)\in\partial\Omega\times\mathbb{R}^+, \\ 0\le u_i(x,0) = u_{i0}(x),\ i=1,2, & x\in\Omega, \end{cases} \tag{1}$$

The associate editor coordinating the review of this manuscript and approving it for publication was Yilun Shang.

where $u_1(x,t)$, $u_2(x,t)$ represent the oxygen and the nutrient respectively, a, b, d_1, d_2, k are positive constants and $g_k(u_1) = \frac{u_1}{1+ku_1^2}$. $\Omega \subset \mathbb{R}^n, (n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, η is unit vector normal to $\partial\Omega$.

The Degn-Harrison model (1) is used to describe the effect of the oxygen concentration in the Klebsiella aerogenes bacteria culture, for a more detailed background of chemical reaction scheme and significance of system (1) we refer interested reader to [24], [25].

The Degn-Harrison system (1) has been studied extensively in the literature, but most of the researches focus on the dynamics of this model including the local and global asymptotic stability of the steady-state solutions [25], [26], Turing instability [27], [28] and Hopf bifurcation [29], [30]. However, as far as we know, this is the first work deal with control synchronization of the model (1).

The contribution of this paper is the development of novel methods for synchronization of Degn-Harrison reaction-diffusion system with identical or non-identical coefficients.

This work is organized as follows. In section 2 and 3 along with the proposed control laws and proofs of their convergence based on the Lyapunov approach and Green identity. In section 4, we consider numerical applications to illustrate the effectiveness of the development schemes. Finally, In section 5 we give the conclusion of our work.

II. IDENTICAL SYSTEMS

To analyze the synchronization between two identical Degn-Harrison models, we use the master-slave (drive-response) formalism, where the two Degn-Harrison reaction-diffusion systems are coupled, in such a manner that the output of the second (slave) system tracks the output of the first (master) system asymptotically. In this case, we design appropriate functions called controllers to force the difference of states of synchronized systems converge to zero. This process is called complete synchronization.

The following result provides the existence, uniqueness and the boundedness of the solution of the Degn-Harrison system (1).

Lemma 1: [25] Suppose that $b < a$ and $u_{i0}(x) \in C(\bar{\Omega}) \cap C^2(\Omega)$ then

1) The model (1) possesses a global unique solution $u_i(x,t) \in C^{1,2}(\Omega \times \mathbb{R}_*^+) \cap C(\bar{\Omega} \times \mathbb{R}^+)$, this solution is positive for all $(x,t) \in \bar{\Omega} \times \mathbb{R}^+$.
2) $\mathcal{R} = [\bar{u}_1, a] \times [2b\sqrt{k}, \bar{u}_2]$ where $\bar{u}_1 = \frac{b(a-b)}{a(1+a^2k)}$ and $\bar{u}_2 = \frac{a-\bar{u}_1}{g_k(\bar{u}_1)}$ is an invariant rectangle for the system (1).

First of all, we assume that the master and the slave systems are identical in all coefficients except in the associate initial condition. Therefore, the slave system associated with the master system (1) can be written as

$$\begin{cases} \dfrac{\partial v_1}{\partial t} = d_1\Delta v_1 + a - v_1 - v_2 g_k(v_1) + L_1, \\ \quad (x,t) \in \Omega \times \mathbb{R}^+, \\ \dfrac{\partial v_2}{\partial t} = d_2\Delta v_2 + b - v_2 g_k(v_1) + L_2, \\ \quad (x,t) \in \Omega \times \mathbb{R}^+, \\ \dfrac{\partial v_i}{\partial \eta} = 0, \ i = 1, 2, \\ \quad (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ 0 \leq v_i(x,0) = v_i(x), i = 1, 2, \\ \quad x \in \Omega, \end{cases} \tag{2}$$

where $v_i = v_i(x,t)$, $(i = 1, 2)$ are states of the slave system (2) and L_i are controllers to be designed.

The aim this section is to determine a suitable controls L_i to force the synchronization errors $e(x,t) = (e_1(x,t), e_2(x,t))$, defined by

$$e(x,t) = v(x,t) - u(x,t), \tag{3}$$

where $u(x,t) = (u_1(x,t), u_2(x,t))$ and $v(x,t) = (v_1(x,t), v_2(x,t))$ are the solutions of systems (1) and (2) respectively, to converge towards zero as t goes to infinity.

Definition 2: The master system (1) and the slave system (2) are said to be completely synchronized if

$$\lim_{t\to\infty} \|e(x,t)\| = 0.$$

Differentiate the errors given (3) with respect to time to get

$$\begin{cases} \dfrac{\partial e_1(x,t)}{\partial t} = d_1\Delta e_1 - e_1 + u_2 g_k(u_1) - v_2 g_k(v_1) + L_1, \\ \dfrac{\partial e_2(x,t)}{\partial t} = d_2\Delta e_2 + u_2 g_k(u_1) - v_2 g_k(v_1) + L_2. \end{cases} \tag{4}$$

One can observe that the error system (4) satisfies the homogeneous Neumann boundary conditions

$$\frac{\partial e_1}{\partial \eta} = \frac{\partial e_2}{\partial \eta} = 0, \quad \text{for all } (x,t) \in \partial\Omega \times \mathbb{R}^+. \tag{5}$$

The following lemma is needed in the proofs of results of this paper.

Lemma 3: There exists a positive constant K depending on a, b and k such that

$$|u_2 g_k(u_1) - v_2 g_k(v_1)| \leq K\left(|v_1 - u_1| + |v_2 - u_2|\right). \tag{6}$$

Proof:

$$\begin{aligned} |u_2 g_k(u_1) - v_2 g_k(v_1)| &\leq |u_2 g_k(u_1) - v_2 g_k(u_1)| \\ &\quad + |v_2 g_k(u_1) - v_2 g_k(v_1)| \\ &\leq |v_2 - u_2|\,|g_k(u_1)| + |v_2|\,|u_1 - v_1| \\ &\quad + k\,|v_2|\,|u_1 - v_1|\,|g_k(u_1)|\,|g_k(v_1)|. \end{aligned}$$

The function $g_k(.)$ has a maximum $\frac{1}{2\sqrt{k}}$, then

$$|u_2 g_k(u_1) - v_2 g_k(v_1)| \leq \frac{5}{4}|v_2||u_1 - v_1| + \frac{1}{2\sqrt{k}}|v_2 - u_2|.$$

Due to Lemma 1 $|v_2| < \bar{v}_2$, thus, we can choose the constant K as

$$K \geq \max\{\frac{5}{4}\bar{v}_2, \frac{1}{2\sqrt{k}}\}.$$

The synchronization error defined in (3) goes to 0, as t goes to $+\infty$ if and only if the zero solution of the synchronization error system (4) is globally asymptotically stable. That is, in the following Theorem, we determine the controllers L_1 and L_2, in linear forms to achieve synchronization between systems given in Eq. (1) and Eq (2).

Theorem 4: The master system (1) and the slave system (2) are completely synchronized under the following linear control law

$$\begin{aligned} L_1 &= (1 - 2K)(v_1 - u_1), \\ L_2 &= -2K(v_2 - u_2). \end{aligned} \tag{7}$$

Proof: Merging the Eq. (7) and Eq. (4), we obtain

$$\begin{cases} \dfrac{\partial e_1(x,t)}{\partial t} = d_1\Delta e_1 - 2Ke_1 + u_2 g_k(u_1) - v_2 g_k(v_1), \\ \dfrac{\partial e_2(x,t)}{\partial t} = d_2\Delta e_2 - 2Ke_2 + u_2 g_k(u_1) - v_2 g_k(v_1). \end{cases} \tag{8}$$

Now, we construct a Lyapunov functional as

$$V = \frac{1}{2}\int_\Omega \left(e_1^2 + e_2^2\right) dx,$$

then

$$\begin{aligned} \frac{\partial V}{\partial t} &= \int_\Omega \left(e_1 \frac{\partial e_1}{\partial t} + e_2 \frac{\partial e_2}{\partial t}\right) dx \\ &= \int_\Omega e_1 \left(d_1\Delta e_1 - 2Ke_1 + u_2 g_k(u_1) - v_2 g_k(v_1)\right) dx \\ &\quad + \int_\Omega e_2 \left(d_2\Delta e_2 - 2Ke_2 + u_2 g_k(u_1) - v_2 g_k(v_1)\right) dx \\ &= \int_\Omega (d_1 e_1 \Delta e_1)\, dx + \int_\Omega (d_2 e_2 \Delta e_2)\, dx \\ &\quad - 2K \int_\Omega \left(e_1^2 + e_2^2\right) dx \\ &\quad + \int_\Omega \left[(u_2 g_k(u_1) - v_2 g_k(v_1))(e_1 + e_2)\right] dx. \end{aligned}$$

Using Green identity, we can deduce

$$\begin{aligned} \frac{\partial V}{\partial t} &= -\int_\Omega d_1 |\nabla e_1|^2 dx + \int_{\partial\Omega} d_1 e_1 \frac{\partial e_1}{\partial \eta} d\sigma - \int_\Omega d_2 |\nabla e_2|^2 dx \\ &\quad + \int_{\partial\Omega} d_2 e_2 \frac{\partial e_2}{\partial \eta} d\sigma - 2K \int_\Omega \left(e_1^2 + e_2^2\right) dx \\ &\quad + \int_\Omega \left[(u_2 g_k(u_1) - v_2 g_k(v_1))(e_1 + e_2)\right] dx. \end{aligned}$$

By using the homogeneous Neumann boundary conditions (5) and Lemma 3, we get

$$\begin{aligned} &\frac{\partial V}{\partial t} \\ &= -\int_\Omega \left[d_1 |\nabla e_1|^2 + d_2 |\nabla e_2|^2\right] dx - 2K \int_\Omega \left(e_1^2 + e_2^2\right) dx \\ &\quad + \int_\Omega \left[(u_2 g_k(u_1) - v_2 g_k(v_1))(e_1 + e_2)\right] dx \\ &\leq -\int_\Omega \left[d_1 (\nabla e_1)^2 + d_2 (\nabla e_2)^2\right] dx - 2K \int_\Omega \left(e_1^2 + e_2^2\right) dx \\ &\quad + \int_\Omega \left[|u_2 g_k(u_1) - v_2 g_k(v_1)|(|e_1| + |e_2|)\right] dx \\ &\leq -\int_\Omega \left[d_1 |\nabla e_1|^2 + d_2 |\nabla e_2|^2\right] dx - 2K \int_\Omega \left(e_1^2 + e_2^2\right) dx \\ &\quad + \int_\Omega K(|e_1| + |e_2|)^2 dx \\ &= -\int_\Omega \left[d_1 |\nabla e_1|^2 + d_2 |\nabla e_2|^2\right] dx - 2K \int_\Omega \left(e_1^2 + e_2^2\right) dx \\ &\quad + \int_\Omega K\left(e_1^2 + 2|e_1||e_2| + e_2^2\right) dx \\ &= -\int_\Omega \left[d_1 |\nabla e_1|^2 + d_2 |\nabla e_2|^2\right] dx - K\int_\Omega (|e_1| - |e_2|)^2 dx \\ &< 0. \end{aligned}$$

Based on the Lyapunov stability theory, we conclude that the zero solution of the synchronization error system (4) is globally asymptotically stable. Therefore, the master-slave systems (1) and (2) are completely synchronized.

III. NON IDENTICAL SYSTEMS

In this section, we consider master-slave of the Degn-Harrison reaction-diffusion systems which are not identical. In this case, the slave system associated with the master system (1) is given by

$$\begin{cases} \dfrac{\partial v_1}{\partial t} = \hat{d}_1 \Delta v_1 + \hat{a} - v_1 - v_2 g_k(v_1) + U_1, \\ \quad (x,t) \in \Omega \times \mathbb{R}^+ \\ \dfrac{\partial v_2}{\partial t} = \hat{d}_2 \Delta v_2 + \hat{b} - v_2 g_k(v_1) + U_2, \\ \quad (x,t) \in \Omega \times \mathbb{R}^+ \\ \dfrac{\partial v_i}{\partial \eta} = 0, \; i = 1,2, \quad (x,t) \in \partial\Omega \times \mathbb{R}^+ \\ 0 \leq v_i(x,0) = v_i(x), i = 1,2, \quad x \in \Omega \end{cases} \tag{9}$$

where $v_i = v_i(x,t)$, $(i = 1,2)$ are states of the slave system (9), $\hat{d}_1, \hat{d}_2, \hat{a}, \hat{b}$ are positive constants and U_i are controllers to be designed.

The synchronization error system for $(x,t) \in \Omega \times \mathbb{R}^+$, can be derived as

$$\begin{cases} \dfrac{\partial e_1(x,t)}{\partial t} = \dfrac{\partial v_1(x,t)}{\partial t} - \dfrac{\partial u_1(x,t)}{\partial t} \\ \quad = \hat{d}_1 \Delta v_1 - d_1 \Delta u_1 + (\hat{a} - a) - e_1 \\ \quad + (u_2 g_k(u_1) - v_2 g_k(v_1)) + U_1, \\ \dfrac{\partial e_2(x,t)}{\partial t} = \dfrac{\partial v_2(x,t)}{\partial t} - \dfrac{\partial u_2(x,t)}{\partial t} \\ \quad = \hat{d}_2 \Delta v_2 - d_2 \Delta u_2 + (\hat{b} - b) \\ \quad + (u_2 g_k(u_1) - v_2 g_k(v_1)) + U_2. \end{cases} \tag{10}$$

Theorem 5: If there exists control constants $(l_i), i = 1 \ldots 4$ such that

$$\begin{cases} \hat{d}_1 - d_1 - l_1 \geq 0, \\ \hat{d}_2 - d_2 - l_3 \geq 0, \\ l_2 = \hat{a} - a, \\ l_4 = \hat{b} - b, \end{cases} \tag{11}$$

then, the non identical master-slave systems given in (1) and (9) are completely synchronized under the following nonlinear control low

$$\begin{aligned} U_1 &= -\left((\hat{d}_1 - 2d_1)\Delta u_1 + d_1 \Delta v_1\right) \\ &\quad + (1-2K)e_1 - l_1 \Delta e_1 - l_2, \\ U_2 &= -\left((\hat{d}_2 - 2d_2)\Delta u_2 + d_2 \Delta v_2\right) - 2Ke_2 - l_3 \Delta e_2 - l_4. \end{aligned} \tag{12}$$

Proof: By using (12) the error system (10) can be written as

$$\begin{cases} \dfrac{\partial e_1(x,t)}{\partial t} = \left(\hat{d}_1 - d_1 - l_1\right) \Delta e_1 + (\hat{a} - a - l_2) \\ \qquad -2Ke_1 + (u_2 g_k(u_1) - v_2 g_k(v_1)), \\ \dfrac{\partial e_2(x,t)}{\partial t} = \left(\hat{d}_2 - d_2 - l_3\right) \Delta e_2 + \left(\hat{b} - b - l_4\right) \\ \qquad -2Ke_2 + (u_2 g_k(u_1) - v_2 g_k(v_1)). \end{cases} \tag{13}$$

Let us introduce the following Lyapunov functional

$$V = \frac{1}{2} \int_\Omega \left(e_1^2 + e_2^2\right) dx.$$

Differentiate the above function with respect to time, yields

$$\begin{aligned} \frac{\partial V}{\partial t} &= \int_\Omega \left(e_1 \frac{\partial e_1}{\partial t} + e_2 \frac{\partial e_2}{\partial t}\right) dx \\ &= \int_\Omega \left(\hat{d}_1 - d_1 - l_1\right) e_1 \Delta e_1 dx \\ &\quad + \int_\Omega \left(\hat{d}_2 - d_2 - l_3\right) e_2 \Delta e_2 dx \\ &\quad + \int_\Omega \Big[(\hat{a} - a - l_2)\, e_1 + \left(\hat{b} - b - l_4\right) e_2 \\ &\quad -\ 2K(e_1^2 + e_2^2)\Big]\, dx \\ &\quad + \int_\Omega [(u_2 g_k(u_1) - v_2 g_k(v_1))(e_1 + e_2)]\, dx \\ &\leq -\int_\Omega \left(\hat{d}_1 - d_1 - l_1\right) |\nabla e_1|^2 dx \\ &\quad - \int_\Omega \left(\hat{d}_2 - d_2 - l_3\right) |\nabla e_2|^2 dx \\ &\quad + \int_\Omega \Big[(\hat{a} - a - l_2)\, e_1 + \left(\hat{b} - b - l_4\right) e_2 \\ &\quad - 2K(e_1^2 + e_2^2)\Big]\, dx \\ &\quad + \int_\Omega K \left(e_1^2 + 2\,|e_1|\,|e_2| + e_2^2\right) dx \\ &= -\int_\Omega \left(\hat{d}_1 - d_1 - l_1\right) |\nabla e_1|^2 dx \\ &\quad - \int_\Omega \left(\hat{d}_2 - d_2 - l_3\right) |\nabla e_2|^2 dx \\ &\quad + \int_\Omega \left[(\hat{a} - a - l_2)\, e_1 + \left(\hat{b} - b - l_4\right) e_2\right] dx \\ &\quad - K \int_\Omega (|e_1| - |e_2|)^2\, dx. \end{aligned}$$

Using the conditions given in (11), we obtain

$$\begin{aligned} \frac{\partial V}{\partial t} &= -\int_\Omega \left(\hat{d}_1 - d_1 - l_1\right) |\nabla e_1|^2 dx \\ &\quad - \int_\Omega \left(\hat{d}_2 - d_2 - l_3\right) |\nabla e_2|^2 dx \\ &\quad - K \int_\Omega (|e_1| - |e_2|)^2\, dx < 0. \end{aligned}$$

Based on the Lyapunov stability theory, we conclude that the zero solution of the synchronization error system (13) is globally asymptotically stable. Hence, the master system (1) and the slave system (9) are globally completely synchronized.

IV. NUMERICAL SIMULATIONS

To test the theoretical findings and to clarify the feasibility of the synchronization schemes introduced in the previous sections, two numerical examples are presented. The first one is to the case of identical coefficients, while the second one is to the nonidentical coefficients case. The numerical simulations are performed in one (two) dimensional space using a three (five) central difference scheme.

Example 1:

Let $(a, b, k, d_1, d_2) = (1.2371, 0.34, 19.974, 3, 2)$ and

$$u_1(0, x) = 0.8\,(1 + 0.3 \sin(0.2x)), \tag{14}$$

$$u_2(0, x) = 0.4\,(1 + 0.3 \cos(0.2x)). \tag{15}$$

and the slave system (2) equiped with the initial conditions

$$\begin{aligned} v_1(0, x) &= \sin(0.3x), \\ v_2(0, x) &= \cos(0.3x). \end{aligned}$$

The spatio-temporal solution of the system (1) (i.e., the system (2) for $L_1 = L_2 = 0$) with zero Neumann boundary conditions are shown in Figures 1 and 2.

According to the Theorem 4, if we choose $K = 158$, then the controller L_1, L_2 can be designed as

$$\begin{aligned} L_1 &= -315(v_1 - u_1), \\ L_2 &= -316(v_2 - u_2). \end{aligned} \tag{16}$$

As a result from the performed numerical simulations, we can observe that with the addition of appropriate linear controllers given by (7), the dynamics of the systems, given in (1) and (2) become synchronized and the zero steady-state of the synchronization error system given in (8) becomes asymptotically stable. Hence, the errors defined in (3) goes to 0, as t goes to $+\infty$, see Figures 3 and 4. In addition, Figures 5, 6, 7 show the pattern formations in two-dimensional space, of synchronization error system (8) in

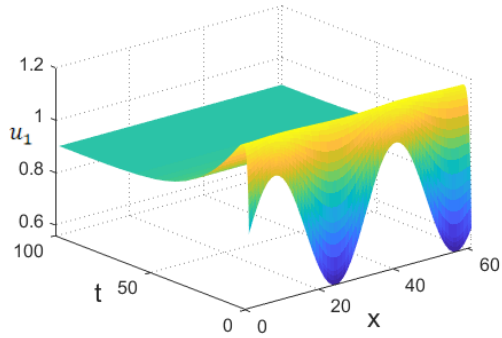


FIGURE 1. Dynamic behavior of u_1 the solution of system (1) for $(a, b, k, d_1, d_2) = (1.2371, 0.34, 19.974, 3, 2)$ and initial condition (14-15).

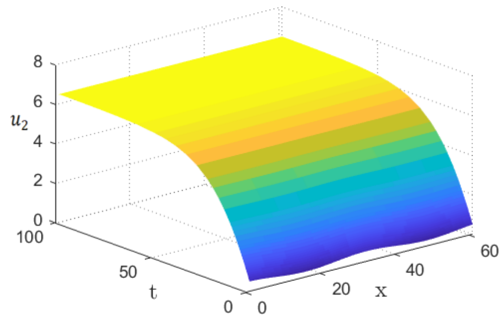


FIGURE 2. Dynamic behavior of u_2 the solution of system (1) for $(a, b, k, d_1, d_2) = (1.2371, 0.34, 19.974, 3, 2)$ and initial condition (14-15).

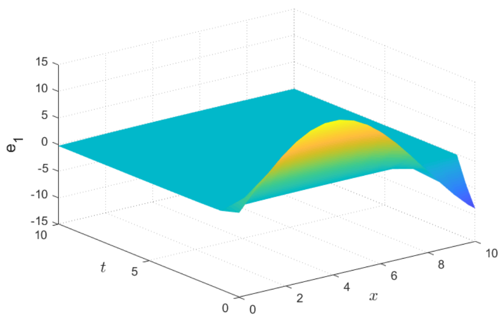


FIGURE 3. Dynamic behavior of e_1 the solution of the synchronization error system (8).

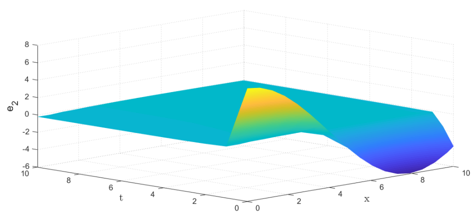


FIGURE 4. Dynamic behavior of e_2 the solution of the synchronization error system (8).

$t = 0$, $t = 5$, and $t = 10$, respectively, these figures indicate that the zero steady-state of the system (8) is asymptotically stable in the 2D spatial domain.

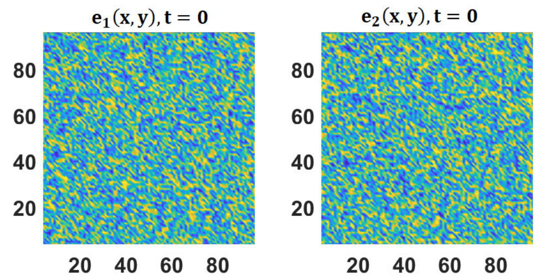


FIGURE 5. Pattern formations of e_1 (left) and e_2 (right) the solutions of the synchronization error system (8) in 2D for $t = 0$.

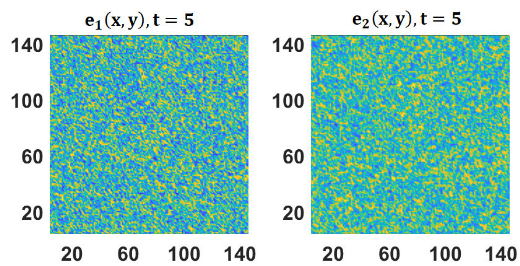


FIGURE 6. Pattern formations of e_1 (left) and e_2 (right) the solutions of the synchronization error system (8) in 2D for $t = 5$.

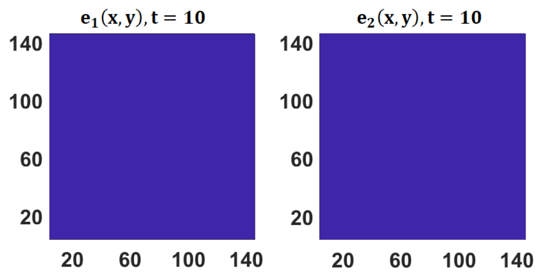


FIGURE 7. Pattern formations of e_1 (left) and e_2 (right) the solutions of the synchronization error system (8) in 2D for $t = 10$.

Example 2: In order to put Theorem 5 to the test, let us consider the parameter set

$$(a, b, k, d_1, d_2) = (10, 6, 1, 5, 3), \tag{17}$$
$$(\hat{a}, \hat{b}, k, \hat{d}_1, \hat{d}_2) = (6, 5, 1, 3, 2), \tag{18}$$

when the initial conditions associated with the system (1) are given by

$$u_1(0, x) = 0.8\,(1 + 0.3\sin(0.2x)), \tag{19}$$
$$u_2(0, x) = 0.4\,(1 + 0.3\cos(0.2x)). \tag{20}$$

Then the solutions u_1 and u_2 of the system (1) with zero Neumann boundary conditions are shown in Figure 8 and 9. For the slave system (9) with $(U_1 = 0, U_2 = 0)$, if the initial conditions given by

$$v_1(0, x) = 0.8(1 + 16.3\cos x), \tag{21}$$
$$v_2(0, x) = 0.4(1 + \cos x). \tag{22}$$

The solutions v_1, v_2 are shown in Figures 10 and 11.

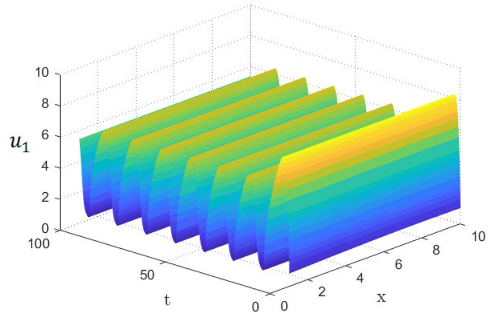


FIGURE 8. Dynamic behavior of u_1 the solution of system (1) for $(a, b, k, d_1, d_2) = (10, 6, 1, 5, 3)$ and initial condition (19-20).

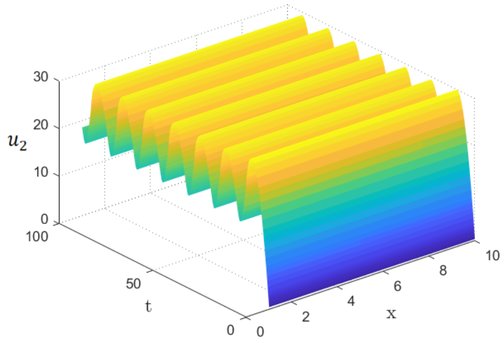


FIGURE 9. Dynamic behavior of u_2 the solution of system (1) for $(a, b, k, d_1, d_2) = (10, 6, 1, 5, 3)$ and initial condition (19-20).

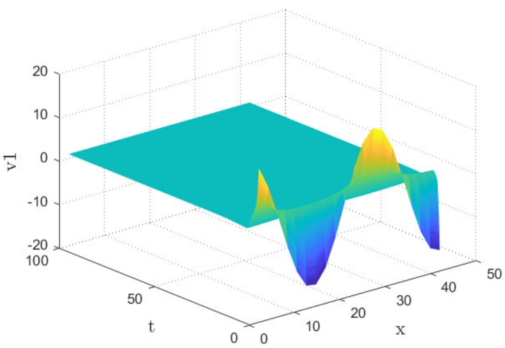


FIGURE 10. Dynamic behavior of v_1 the solution of system (9) for $(\hat{a}, \hat{b}, k, \hat{d}_1, \hat{d}_2) = (6, 5, 1, 3, 2)$, and initial condition (21-22).

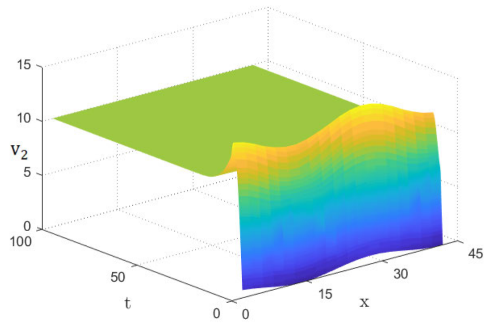


FIGURE 11. Dynamic behavior of v_2 the solution of system (9) for $(\hat{a}, \hat{b}, k, \hat{d}_1, \hat{d}_2) = (6, 5, 1, 3, 2)$, and initial condition (21-22).

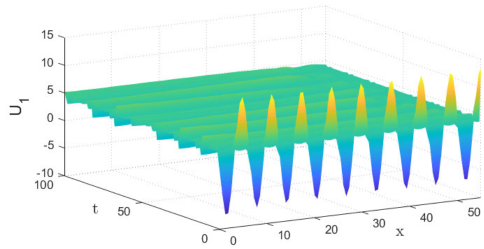


FIGURE 12. The distribution of the controllers U_1 given by Eq (23).

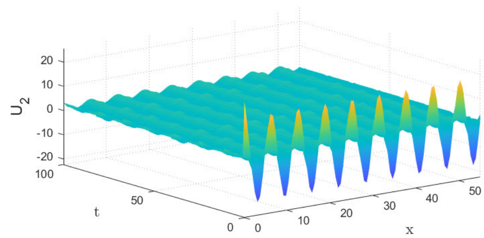


FIGURE 13. The distribution of the controllers U_2 given by Eq (23).

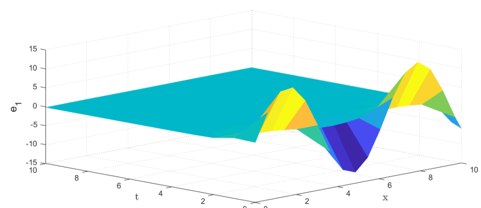


FIGURE 14. Dynamic behavior of e_1 the solution of the synchronization error system (13).

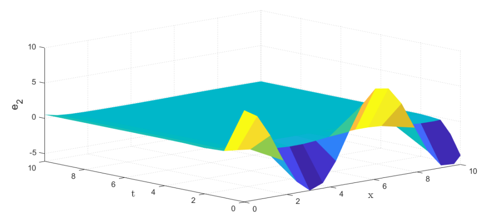


FIGURE 15. Dynamic behavior of e_2 the solution of the synchronization error system (13).

According to the Theorem 5, if we choose the control constants as

$$(l_1, l_2, l_3, l_4) = (-3, -4, -2, -1).$$

and $K = 55$, then the controllers U_1, U_2 can be designed as

$$\begin{aligned} U_1 &= 7\Delta u_1 - 5\Delta v_1 - 109e_1 + 3\Delta e_1 + 4. \\ U_2 &= 4\Delta u_2 - 3\Delta v_2 - 110e_2 + 2\Delta e_2 + 1. \end{aligned} \quad (23)$$

so, from the Theorem 5, with the parameter set (17-18), the master system (1) and the slave (9) are globally synchronized. The distribution of the controllers U_1, U_2 and the spatio-temporal evolution of the synchronization error system (13) states are shown in Figures 12-13 and 14-15 respectively.

Moreover, Figures 16, 17, and 18 show the pattern formations in of error systems in 2D spatial domain for $t = 0$, $t = 5$, and $t = 20$, respectively, this figures indicate that the zero steady

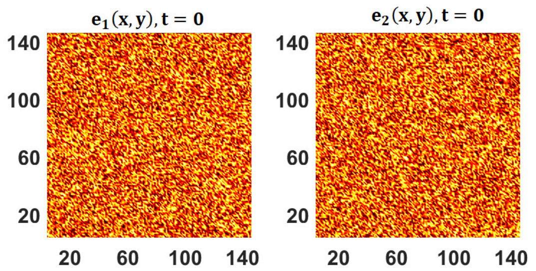


FIGURE 16. Pattern formations of e_1 (left) and e_2 (right) the solutions of the synchronization error system (13) in 2D for $t = 0$.

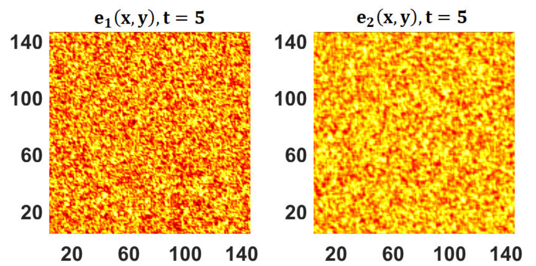


FIGURE 17. Pattern formations of e_1 (left) and e_2 (right) the solutions of the synchronization error system (13) in 2D for $t = 5$.

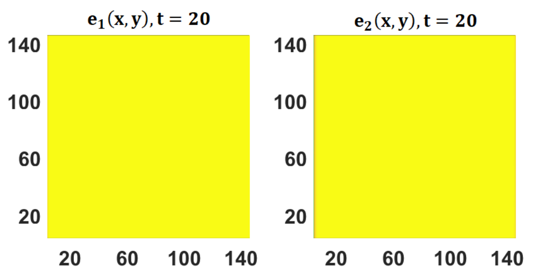


FIGURE 18. Pattern formations of e_1 (left) and e_2 (right) the solutions of the synchronization error system (13) in 2D for $t = 20$.

state of system (10) is asymptotically stable in 2D spatial domain.

V. CONCLUSION

In the present work, we have established novel methods to investigate the synchronization in nonlinear bacterial cultures spatiotemporal model. First, a spatial-time coupling process for the complete synchronization was introduced. Next, suitable linear and nonlinear control schemes are proposed to realize the synchronization for identical and nonidentical cases. The synchronization results are derived based on the Lyapunov theory and master-slave formulation. Numerical simulations, consisting of displaying synchronously behaviors of identical and nonidentical Degn-Harrison systems, are given to show the effectiveness and applicability of the proposed synchronization schemes. Simultaneously, comparing time evolutions of the synchronization errors displayed in Fig. 7 and 18, for the two-dimensional space, we can conclude that the synchronization in nonidentical coefficients case (converges to zero as $t = 20$) is slower than the case of identical coefficients (converges to zero as $t = 10$). In our future research, we plan to study the synchronization behaviors in many types of spatiotemporal models, including the lattice maps and stochastic reaction-diffusion systems.

ACKNOWLEDGMENT

The work of Adel Ouannas was supported by the Directorate General for Scientific Research and Technological Development of Algeria.

REFERENCES

[1] L. M. Pecora and T. L. Carroll, "Synchronization in chaotic systems," *Phys. Rev. Lett.*, vol. 64, no. 8, pp. 821–824, Feb. 1990, doi: 10.1103/PhysRevLett.64.821.

[2] D. Eroglu, J. S. W. Lamb, and T. Pereira, "Synchronisation of chaos and its applications," *Contemp. Phys.*, vol. 58, no. 3, pp. 207–243, Jul. 2017, doi 10.1080/00107514.2017.1345844.

[3] A. C. J. Luo, "A theory for synchronization of dynamical systems," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 14, no. 5, pp. 1901–1951, May 2009, doi: 10.1016/j.cnsns.2008.07.002.

[4] T. Kapitaniak, "Continuous control and synchronization in chaotic systems," *Chaos, Solitons Fractals*, vol. 6, pp. 237–244, Jan. 1995, doi: 10.1016/0960-0779(95)80030-K.

[5] A. Ouannas, Z. Odibat, and N. Shawagfeh, "A new Q–S synchronization results for discrete chaotic systems," *Differ. Equ. Dyn. Syst.*, vol. 27, no. 4, pp. 413–422, Oct. 2019, doi: 10.1007/s12591-016-0278-x.

[6] A. Ouannas, Z. Odibat, N. Shawagfeh, A. Alsaedi, and B. Ahmad, "Universal chaos synchronization control laws for general quadratic discrete systems," *Appl. Math. Model.*, vol. 45, pp. 636–641, May 2017, doi: 10.1016/j.apm.2017.01.012.

[7] A. S. Mikhailov and K. Showalter, "Control of waves, patterns and turbulence in chemical systems," *Phys. Rep.*, vol. 425, nos. 2–3, pp. 79–194, Mar. 2006, doi: 10.1016/j.physrep.2005.11.003.

[8] A. S. Mikhailov and G. Ertl, Eds., *Engineering Of Chemical Complexity II*. Hackensack NJ, USA: Wspc, 2014.

[9] J. Buceta and K. Lindenberg, "Patterns in reaction–diffusion systems generated by global alternation of dynamics," *Phys. A, Stat. Mech. Appl.*, vol. 325, nos. 1–2, pp. 230–242, Jul. 2003, doi: 10.1016/S0378-4371(03)00202-4.

[10] M. Mimura, H. Sakaguchi, and M. Matsushita, "Reaction–diffusion modelling of bacterial colony patterns," *Phys. A, Stat. Mech. Appl.*, vol. 282, nos. 1–2, pp. 283–303, Jul. 2000, doi: 10.1016/S0378-4371(00)00085-6.

[11] K.-N. Wu, T. Tian, and L. Wang, "Synchronization for a class of coupled linear partial differential systems via boundary control," *J. Franklin Inst.*, vol. 353, no. 16, pp. 4062–4073, Nov. 2016, doi: 10.1016/j.jfranklin.2016.07.019.

[12] C. He and J. Li, "Hybrid adaptive synchronization strategy for linearly coupled reaction–diffusion neural networks with time-varying coupling strength," *Neurocomputing*, vol. 275, pp. 1769–1781, Jan. 2018, doi: 10.1016/j.neucom.2017.10.022.

[13] T. Chen, R. Wang, and B. Wu, "Synchronization of multi-group coupled systems on networks with reaction-diffusion terms based on the graph-theoretic approach," *Neurocomputing*, vol. 227, pp. 54–63, Mar. 2017, doi: 10.1016/j.neucom.2016.09.097.

[14] L. Liu, W.-H. Chen, and X. Lu, "Impulsive H_∞ synchronization for reaction–diffusion neural networks with mixed delays," *Neurocomputing*, vol. 272, pp. 481–494, Jan. 2018, doi: 10.1016/j.neucom.2017.07.023.

[15] H. Chen, P. Shi, and C.-C. Lim, "Pinning impulsive synchronization for stochastic reaction–diffusion dynamical networks with delay," *Neural Netw.*, vol. 106, pp. 281–293, Oct. 2018, doi: 10.1016/j.neunet.2018.07.009.

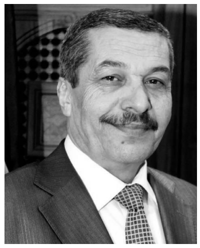
[16] B. Ambrosio and M. A. Aziz-Alaoui, "Synchronization and control of coupled reaction–diffusion systems of the FitzHugh–Nagumo type," *Comput. Math. Appl.*, vol. 64, no. 5, pp. 934–943, Sep. 2012, doi: 10.1016/j.camwa.2012.01.056.

[17] N. Parekh, V. R. Kumar, and B. D. Kulkarni, "Control of spatiotemporal chaos: A study with an autocatalytic reaction-diffusion system," *Pramana*, vol. 48, no. 1, pp. 303–323, Jan. 1997, doi: 10.1007/BF02845637.

[18] C. Wang and J. Ma, "A review and guidance for pattern selection in spatiotemporal system," *Int. J. Mod. Phys. B*, vol. 32, no. 6, Mar. 2018, Art. no. 1830003, doi: 10.1142/S0217979218300037.
[19] F. Wu, Y. Wang, J. Ma, W. Jin, and A. Hobiny, "Multi-channels coupling-induced pattern transition in a tri-layer neuronal network," *Phys. A, Stat. Mech. Appl.*, vol. 493, pp. 54–68, Mar. 2018, doi: 10.1016/j.physa.2017.10.041.
[20] D. Mansouri, S. Bendoukha, S. Abdelmalek, and A. Youkana, "On the complete synchronization of a time-fractional reaction–diffusion system with the Newton–Leipnik nonlinearity," *Applicable Anal.*, pp. 1–20, May 2019, doi: 10.1080/00036811.2019.1616694.
[21] A. Ouannas, M. Abdelli, Z. Odibat, X. Wang, V.-T. Pham, G. Grassi, and A. Alsaedi, "Synchronization control in reaction-diffusion systems: Application to Lengyel-Epstein system," *Complexity*, vol. 2019, Feb. 2019, Art. no. 2832781, doi: 10.1155/2019/2832781.
[22] A. Ouannas, X. Wang, V.-T. Pham, G. Grassi, and V. V. Huynh, "Synchronization results for a class of fractional-order spatiotemporal partial differential systems based on fractional Lyapunov approach," *Boundary Value Problems*, vol. 2019, no. 1, p. 74, Dec. 2019, doi: 10.1186/s13661-019-1188-y.
[23] H. Degn and D. E. F. Harrisson, "Theory of oscillations of respiration rate in continuous culture of Klebsiella aerogenes," *J. Theor. Biol.*, vol. 22, no. 2, pp. 238–248, Feb. 1969, doi: 10.1016/0022-5193(69)90003-4.
[24] V. Fairén and M. G. Velarde, "Time-periodic oscillations in a model for the respiratory process of a bacterial culture," *J. Math. Biol.*, vol. 8, no. 2, pp. 147–157, 1979, doi: 10.1007/BF00279718.
[25] B. Lisena, "Some global results for the Degn–Harrison system with diffusion," *Medit. J. Math.*, vol. 14, no. 2, p. 91, Apr. 2017, doi: 10.1007/s00009-017-0894-x.
[26] A. Abbad, S. Bendoukha, and S. Abdelmalek, "On the local and global asymptotic stability of the Degn–Harrison reaction-diffusion model," *Math. Methods Appl. Sci.*, vol. 42, no. 2, pp. 567–577, Jan. 2019, doi: 10.1002/mma.5362.
[27] S. Li, J. Wu, and Y. Dong, "Turing patterns in a reaction–diffusion model with the Degn–Harrison reaction scheme," *J. Differ. Equ.*, vol. 259, no. 5, pp. 1990–2029, Sep. 2015, doi: 10.1016/j.jde.2015.03.017.
[28] J. Zhou, "Pattern formation in a general Degn-Harrison reaction model," *Bull. Korean Math. Soc.*, vol. 54, no. 2, pp. 655–666, Mar. 2017, doi: 10.4134/BKMS.b160249.
[29] Y. Dong, S. Li, and S. Zhang, "Hopf bifurcation in a reaction–diffusion model with Degn–Harrison reaction scheme," *Nonlinear Anal., Real World Appl.*, vol. 33, pp. 284–297, Feb. 2017, doi: 10.1016/j.nonrwa.2016.07.002.
[30] R. Peng, F.-Q. Yi, and X.-Q. Zhao, "Spatiotemporal patterns in a reaction–diffusion model with the Degn–Harrison reaction scheme," *J. Differ. Equ.*, vol. 254, no. 6, pp. 2465–2498, Mar. 2013, doi: 10.1016/j.jde.2012.12.009.

FATIHA MESDOUI received the master's degree in dynamical system and geometry from the University of Sciences and Technology Houari Boumediene, Algeria. She is currently pursuing the Ph.D. degree with the Department of Mathematics, The University of Jordan. Her research interests include classification of bursting mappings, modeling the dynamic of a neuron, and synchronization in integer-order or fractional-order ordinary and partial differential equations.

ADEL OUANNAS received the B.S. degree in mathematics from Larbi Ben M'hidi University, in 2003, and the M.Sc. and Ph.D. degrees from the University of Constantine, in 2006 and 2015, respectively. His research interests include synchronization of integer and fractional order chaotic systems, stability theory, and applications of chaos in engineering.



NABIL SHAWAGFEH received the bachelor's and master's degrees in mathematics from The University of Jordan, in 1977 and 1979, respectively, and the Ph.D. Dissertation in mathematics from Clarkson University, USA, in 1983. In the last 25 years, he held a positions, such as the Chairman of the Mathematics Department, the Dean of Academic Research, and the Vice-President for Academic Affairs with The University of Jordan. From 2008 to 2012, he was the President of Al al-Bayt University, Jordan. He was also the President of Al-Balqa' Applied University, from 2012 to 2016. He is currently a Professor of mathematics with The University of Jordan. He has supervised 13 Ph.D. and 21 M.Sc. theses. He has published more than 50 publications. He was involved as a chair of member of several, administrative, accreditation, academic, and scientific research committee councils on the university and on the national levels.

GIUSEPPE GRASSI (Senior Member, IEEE) received the Laurea degree (Hons.) in electronic engineering from the Università di Bari, Bari, Italy, and the Ph.D. degree in electrical engineering from the Politecnico di Bari, Bari, in 1994. From January 2016 to Juanuary 2019, he was the Head of the Dipartimento di Ingegneria dell'Innovazione, Università del Salento, Lecce, Italy. In 1994, he joined the Dipartimento di Ingegneria dell'Innovazione, Università del Salento, where he is currently a Professor of electrical engineering. He has published 90 articles in international journals (Web of Science) and 95 papers in proceedings of international conferences. His research interests include chaotic and hyperchaotic circuits, continuous- and discrete-time fractional-order systems, synchronization properties and chaos-based cryptography, cellular neural networks theory and applications, and stability of nonlinear systems. He is a member of the IEEE Technical Committee on Nonlinear Circuits and Systems. From 2004 to 2007, he served as an Associate Editor for *Dynamics of Impulsive Continuous and Discrete Time Systems-Series B*. From 2008 to 2011, he served as an Associate Editor for the IEEE TRANSACTION ON CIRCUITS AND SYSTEMS II-EXPRESS BRIEFS.

VIET-THANH PHAM (Member, IEEE) graduated in electronics and telecommunications from the Hanoi University of Technology, Vietnam, in 2005. He received the Ph.D. degree in electronics, automation and control of complex systems engineering from the University of Catania, Italy, in 2013. He is currently the Director of Research with the Faculty of Electrical and Electronic Engineering, Phenikaa Institute for Advanced Study (PIAS), Phenikaa University. His scientific research interests include chaos, nonlinear control, applications of nonlinear systems, analysis and design of analog circuits, and FPGA-based digital circuits.