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Fixed-Time Leader-Following Flocking for Nonlinear Second-Order Multi-Agent Systems

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ABSTRACT This work focuses on the fixed-time leader-following flocking for multi-agent systems. Different from the previous continuous inherent dynamic $f(x_i, v_i, t)$ for agent i , a new nonlinear discontinuous one is firstly proposed in this paper. Employing non-smooth techniques, graph theory and fixed-time stability theory, the fixed-time leader-following flocking is achieved. Moreover, an upper bound of the settling time is independent of initial states. In addition, two illustrative examples are given to verify the effectiveness of the theoretical results.

INDEX TERMS Fixed-time flocking, multi-agent system, discontinuous.

I. INTRODUCTION

The study of cooperative control of autonomous agents has recently attracted significant interest, due to it is a basic research of multi-agent systems and it has potential applications, such as formation flying of unmanned aerial vehicles (short for UAVs) [3], distributed sensor networks [17] and cooperation of multi-robot teams [2]. Particularly, flocking, which means that agents in a group organize into an ordered motion by interacting with their local neighbors, is one fundamental research problem of cooperative control. Recently, some impressive results of flocking have been obtained for second-order multi-agent systems, such as the communication function with time delay [36], the straight line formation [13], flocking with deterministic or stochastic additional forces [15], [29], flocking with leaders [19] and references therein. The other newest developments are captured in [6], [18], [34], [37].

It must be noticed that the settling time is an important criterion to appraise the designed protocols as studying the cooperative control problems, because the fast convergence is always pursued to achieve better performance and robustness [21], [41]. As far as we observe, most of the reported literatures dealt with the asymptotic behaviors,

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which means that the flocking phenomenon occurs only if time approaches to infinity. From a practical point of view, however, it is more practical significance that the consensus behavior can be achieved in a finite-time or fixed-time. For this reason, the issue of the finite-time cooperative control for multi-agent systems has attracted much attention recently [9], [44], [46], [46]. Recently, a significant amount of works have concentrated on the finite-time or fixed-time consensus of multi-agent systems. For examples, Khanzadeh and Pourgholi [2] used non-singular terminal sliding mode technique to investigate the fixed-time leader-follower consensus tracking of second-order multi-agent systems with bounded input uncertainties. Ning *et al.* [4], [5] considered the finite-time and fixed-time leader-following consensus for multi-agent systems with discontinuous inherent dynamics. Defoort *et al.* [22] dealt with the leader-follower fixed-time consensus for multi-agent systems with unknown non-linear inherent dynamics. Although there are a lot of existing papers dealing with the finite-time or fixed-time consensus issue of multi-agent systems, there are very few papers considering the finite-time or fixed-time flocking problems up to now. As far as we know, there are fewer results on the finite-time or fixed-time flocking for multi-agent systems. By using nonsmooth stability analysis and graph theory, the author [16] proposed a discontinuous protocol and derived a general condition of the initial configurations to study the finite-time

flocking problem for a discontinuous Cucker-Smale type model under a long-range interaction. Employing a finite time stability theory and some inequalities, the result in [35] shows that when the communication weight function for Cucker-Smale systems has a positive infimum, the flocking can be formed within a finite-time. By combining a fixed time stability theory and graph theory, the results of [24] illustrate that the bipartite flocking for nonlinear multi-agent systems can be arose within a fixed-time. Liu *et al.* [8] focused on the problem of the finite-time flocking with uniform minimal distance for second-order multi-agent systems. Nevertheless, there are still many challenges remaining in studying the finite-time or fixed-time flocking issues for the multi-agent systems. Take the following problems for example:

(I). Can the flocking occur in the leader-following multi-agent systems with discontinuous inherent dynamics in a fixed-time?

(II). If it is a case, under what conditions?

(III). How to estimate the upper bound of the settling time?

Actually, these issues are inspired partially by the consensus problem of the multi-agent systems and partially by the self-organizing behavior of the animals, such as flocks of sheep, schools of wolves and herds of geese. The results of the existing works show that the flocking is essentially related to the consensus, which aims at designing distributed control protocols to drive a group of agents to reach an agreement on system states [4]. Roughly speaking, we say that a system has a consensus meaning that the positions of agents will converge as time approaches to a finite/fixed-time T or ∞ . However, a system has a flocking requesting that the velocities of agents converge as time approaches to a finite/fixed-time T or ∞ and the positions have a boundedness for all $t \geq 0$. These requirements satisfy the three heuristic rules: separation, alignment and cohesion, which was first introduced by Reynolds [7]. In this respect, it is more difficult to study flocking problem than to study consensus problem.

Motivated by the above analysis, the main purpose of this work is to resolve the aforementioned problems (I), (II) and (III). The main contributions in the present work are summarized as follows:

1) An inherent dynamics which is the first time we proposed here is discontinuous and it is quite different from a Lipschitz-type function [22], [39], [40]. It is well known that, for systems with the discontinuous inherent dynamics, the right-hand side for the dynamics is no longer continuous. This leads the problem more difficulty than continuous one and implies the necessity to invent new analytical methods. Motivated by [4], [5] in which the authors studied the finite-time consensus problem, we apply the non-smooth techniques to overcome this drawback. Consequently, our exploration is more general and it can broaden the scope of practical applications of the multi-agent systems by employing this more generalized form.

2) Unlike the results in [16], [35], where the flocking is achieved in a finite-time depending on initial states of the systems, we investigate the fixed-time leader-following flocking and estimate the upper bound of the settling time which is independence of initial states by the new protocol. To the best of authors' knowledge, the results of this paper are novel in the aspect of the finite-time and fixed-time leader-following flocking without collision.

The rest of this paper is organized as follows. In Section II, the preliminaries and problem formulation are given. Also, some necessary lemmas are presented in this section. The detail proof of the fixed-time flocking results are presented in Section III. In Section IV, the numerical examples are carried out to validate the effectiveness of theoretical results. Finally, the brief summary of our main results is given in Section V.

II. PRELIMINARIES AND PROBLEM FORMULATION

A. SOME NOTATIONS

Notations \mathbb{R} , \mathbb{R}^+ , \mathbb{R}^d , $\mathbb{R}^{n \times n}$ mean the one-dimensional real space, one-dimensional positive real space, d -dimensional real vector space and $n \times n$ real matrix space, respectively. Let $\mathbf{1}_d$ be a $d \times 1$ column vector with all entries are 1, and I_d be a identity matrix of dimension d . Note $x = [x_1^T, x_2^T, \dots, x_n^T]^T$, where $x_i \in \mathbb{R}^d$, and the power of the vector x_i is defined as $x_i^r = [(x_i^{(1)})^r, (x_i^{(2)})^r, \dots, (x_i^{(d)})^r]^T \in \mathbb{R}^d$, where $r \in \mathbb{R}$. For simplicity, we denote $\text{sgn}(x_i) = [\text{sgn}(x_i^{(1)}), \text{sgn}(x_i^{(2)}), \dots, \text{sgn}(x_i^{(d)})]^T$, where $\text{sgn}(\cdot)$ is the signum function

$$\text{sgn}(s) = \begin{cases} 1, & s > 0, \\ 0, & s = 0, \\ -1, & s < 0. \end{cases}$$

As usual, p -norm is defined as

$$\|x_i\|_p^p = (|x_i^{(1)}|^p + |x_i^{(2)}|^p + \dots + |x_i^{(d)}|^p), \quad p > 0.$$

B. GRAPH THEORY

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a weighted graph, where $\mathcal{V} = \{v_1, v_2, \dots, v_N\}$ means a set of nodes, $\mathcal{E} = \{(v_i, v_j) | v_i, v_j \in \mathcal{V}\}$ denotes a set of edges. A weighted directed graph $\mathcal{G}(\mathcal{A})$ is a graph \mathcal{G} with a nonnegative adjacency matrix $\mathcal{A} = [a_{ij}]_{N \times N}$ with weights $a_{ij} > 0$ if $(v_j, v_i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. If $a_{ij} = a_{ji}$, it means that \mathcal{G} is an undirected graph. In this paper, we concentrate on the undirected $\mathcal{G}(\mathcal{A})$ and assume that no self-loops exist, i.e., $(v_j, v_i) \notin \mathcal{E}$, thus $a_{ii} = 0$. The Laplacian matrix $L = [l_{ij}]_{N \times N}$ of $\mathcal{G}(\mathcal{A})$ is defined by $l_{ij} = -a_{ij}$ for $i \neq j$, and $l_{ii} = \sum_{j=1}^N a_{ij}$. As shown in [4], [5], each agent interacts with its neighbouring set $\mathcal{N}_i = \{v_j \in \mathcal{V} | (v_i, v_j) \in \mathcal{V}\}$, $i \in \mathcal{N}$. Similar to [18], in the present paper, if two agents interact with each other, the weight between them is set to be 1, and be 0 otherwise. Besides \mathcal{G} , another graph $\bar{\mathcal{G}}$ consists of the graph \mathcal{G} , the node 0 and edges between the leader and its neighbours. In particular, only when

the i th agent has the access to the information of the leader 0, a notation, d_i is set to be 1, and be 0 otherwise. Then a new matrix H is defined as $H = L + \text{diag}(d_1, d_2, \dots, d_N)$.

C. NON-SMOOTH PRELIMINARIES

Definition 1: [1] Consider an equation or a system in vector notation

$$\dot{x} = F(x(t), t), \tag{1}$$

with a piecewise continuous function F in a $(n + 1)$ -dimensional domain $G \subseteq \mathbb{R}^{n+1}$, $\dot{x} = dx/dt$, M is a set (of measure zero) of points of discontinuity of the function F , then $x(\cdot)$ is called a Filippov solution of (1) on $[t_0, t_1] \subset \mathbb{R}$ if it is absolutely continuous and for almost all $t \in [t_0, t_1]$ it satisfies the differential inclusion $\dot{x} \in \mathcal{K}[F](x(t), t)$, where $\mathcal{K}[F](x(t), t)$ is a set-valued map given by

$$\mathcal{K}[F](x(t), t) = \overline{\text{co}}\{\lim F(x^*(t), t) | x^*(t) \rightarrow x(t), (x^*(t), t) \notin M, t = \text{const}\}, \tag{2}$$

where $\overline{\text{co}}$ denotes the convex closure and M denotes the set of measure zero. For simplicity, $\mathcal{K}[F](x(t), t)$ is occasionally denoted as $\mathcal{K}[F(x(t), t)]$ in this paper.

Definition 2: [17] Let $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ be a locally Lipschitz function, then the set-valued Lie derivative of f with respect to F at x is

$$\tilde{\mathcal{L}}_F f(x(t), t) = \{z \in \mathbb{R}^n | \exists w \in \mathcal{K}[F](x(t), t) \text{ such that } \xi^T w = z, \forall \xi \in \partial f\}, \tag{3}$$

where ∂f is the Clarke’s generalized gradient of f .

D. PROBLEM FORMULATION

Consider a system consisting of one leader (indexed as agent 0) and a group of N followers (indexed as agents $1, 2, \dots, N$). As described in [4], [5], the status i th agent can be described by

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = f(x_i(t), v_i(t), t) + u_i(t), \end{cases} \tag{4}$$

subject to the initial data

$$(x_i, v_i)(0) =: (x_{i0}, v_{i0}), \quad i = 1, \dots, N, \tag{5}$$

where $t \geq 0$, $x_i = (x_i^{(1)}, x_i^{(2)}, \dots, x_i^{(d)}) \in \mathbb{R}^d$ and $v_i = (v_i^{(1)}, v_i^{(2)}, \dots, v_i^{(d)}) \in \mathbb{R}^d$ denote the position and velocity of i th agent at time t , respectively. The function $f : \mathbb{R}^d \times \mathbb{R}^d \times [0, +\infty) \rightarrow \mathbb{R}^d$ is the inherent nonlinear dynamics for agent i , and $u_i(t)$ is the control acceleration of i th agent, called protocol, to be designed. Let $f(x_i(t), v_i(t), t) = (f_1(x_i(t), v_i(t), t), \dots, f_n(x_i(t), v_i(t), t))^T$, $i = \{1, 2, \dots, N\}$. The leader can be described by

$$\begin{cases} \dot{x}_0(t) = v_0(t), \\ \dot{v}_0(t) = f(x_0(t), v_0(t), t) + u_0(t), \end{cases} \tag{6}$$

where $t \geq 0$, $x_0 = (x_0^{(1)}, x_0^{(2)}, \dots, x_0^{(d)}) \in \mathbb{R}^d$ and $v_0 = (v_0^{(1)}, v_0^{(2)}, \dots, v_0^{(d)}) \in \mathbb{R}^d$ denote the position and velocity of

the leader, f_0 has the same definition as that of the followers, and $u_0(t) \in \mathbb{R}^d$ is the control input for the leader.

It is worth pointing out that for studying the consensus problem in some existing papers, the nonlinear dynamics $f(\cdot)$ is either assumed to satisfy the Lipschitz continuous condition (see e.g. [12], [20], [33]), i.e., for all $x_i(t), x_j(t) \in \mathbb{R}^d$ and c is a non-negative constant such that

$$\|f(x_1(t), t) - f(x_2(t), t)\|_2 \leq c \|x_1(t) - x_2(t)\|_2, \tag{7}$$

or satisfy that (see e.g. [31]), for $\forall x, y \in \mathbb{R}^d$, there exists a positive constant ρ such that

$$(x - y)(f(t, x) - f(t, y)) \leq \rho(x - y)^T(x - y), \tag{8}$$

which implies that f is a nonlinear continuous function. However, in the real life, the continuity of f is not always guaranteed due to unexpected disturbances or noises. Based on this fact, we consider the discontinuous case in this paper, that is, $f(\cdot)$ is a discontinuous function and satisfies the following hypothesis:

Assumption 1: For all $i = 1, 2, \dots, d$, $f_i(\cdot)$ is continuously differentiable, except on a countable set of isolated points $\{\rho_k^i\} = \{(x_k^i, v_k^i)\}$, where there exist finite right and left limits $f_i^+(\rho_k^i)$ and $f_i^-(\rho_k^i)$, respectively, $k = 1, 2, \dots$;

Assumption 2: There exist $\varepsilon_1 > 0$, $\varepsilon_2 \geq 0$, $\nu > 0$ and $\mu > 0$ such that the following condition holds

$$\|g(t)\|_2 \leq -\frac{1}{\mu + \varepsilon_1 \|v_i(t) - v_j(t)\|_2} - \varepsilon_2 \|x_i(t) - x_j(t)\|_2 + \nu,$$

for all $g(t) \in \mathcal{K}[f(x_i(t), v_i(t), t) - f(x_j(t), v_j(t), t)]$.

Remark 1: Assumption 1 is usually assumed to study the consensus problem of discontinuous multi-agent system (e.g. [4], [5], [38]). If Assumptions 1 and 2 hold, the solution of system (4)-(6) is understood in the Filippov sense due to Definition 1. Comparing to the continuous inherent dynamics (e.g. (7) and (8)), the discontinuous $f(\cdot)$ have wider practical applications by considering a more generalize form. The Assumption 2 is a sufficient condition we first propose to study the fixed-time flocking of multi-agent systems.

Assumption 3: The control input of the leader to its followers is bounded, i.e., $\exists l > 0$, such that

$$\|u_0(t)\|_2 \leq l < \infty.$$

Assumption 4: The undirected graph \mathcal{G} is connected, and there exists at least one $d_i > 0$.

Remark 2: If Assumption 4 holds, then it follows from [38] that H is positive definite. So, we can denote $\lambda_{\min}(H)$ and $\lambda_{\max}(H)$ to be the smallest and the largest eigenvalue of H , respectively. Moreover, although Assumptions 3 and 4 are frequently used to investigate the leader-following consensus issue (see e.g. [4], [5], [38]), there are very few results on the flocking by employing them.

For better legibility, we use the following handy notations:

$$d_X(t) := \max_{0 \leq i, j \leq N} \|x_i(t) - x_j(t)\| \quad \text{for all } t \geq 0$$

and

$$d_V(t) := \max_{0 \leq i, j \leq N} \|v_i(t) - v_j(t)\| \quad \text{for all } t \geq 0,$$

where $\|\cdot\|$ denotes the 2-norm in \mathbb{R}^d . Note that the functions $d_X(t)$, $d_V(t)$ and $d_{X_{\min}}(t)$ are not C_1 smooth in general.

Definition 3: We say that the system (4)-(6) has a finite-time (fixed-time) leader-following flocking if the solutions of the system (4)-(6) satisfy the following conditions

$$\begin{cases} \lim_{t \rightarrow T} d_V(t) = 0, & d_V(t) = 0, \forall t \geq T, \quad \text{and} \\ \sup_{0 \leq t < \infty} d_X(t) < C, \end{cases} \quad (9)$$

where C is a constant and T called the settling time which is depending (non-depending) on initial states.

Now, we provide some lemmas that will be used in the following paragraphs.

Lemma 3: [10] For any vector $x \in \mathbb{R}^n$, and if $p > r > 0$, where p and r are scalar constants, then

$$\|x\|_p \leq \|x\|_r \leq n^{\frac{1}{r} - \frac{1}{p}} \|x\|_p.$$

Consider the following systems

$$\dot{x}(t) = g(x(t), t), \quad x(0) = x_0, \quad (10)$$

where $x \in \mathbb{R}^d$, $g(x) : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$ is Lebesgue measurable and locally essential bounded (but may be discontinuous with respect to x).

Lemma 4: [11] For system (10), if there exists a regular, positive definite and radially unbounded function $V(x(t)) : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfying the following inequality

$$\frac{d}{dt} V(x(t)) \leq -r - \varepsilon V^k(x(t)), \quad x(t) \in \mathbb{R}^d \setminus \{0\},$$

where $r > 0$, $\varepsilon > 0$, $k \geq 0$. Then, the following statements hold:

(i) If $k > 1$, then the origin of system (10) is fixed-time stable and the settling time T is estimated by

$$T \leq T^* := \frac{1}{r} \left(\frac{r}{\varepsilon}\right)^{\frac{1}{k}} \left(1 + \frac{1}{k-1}\right).$$

(ii) If $k = 1$, then the origin of system (10) is finite-time stable and the settling time T is estimated by

$$T \leq T^* := \frac{1}{\varepsilon} \ln \frac{r + \varepsilon V(x_0)}{r}.$$

III. FIXED-TIME FLOCKING

As put it before, the aim of this paper is to study the fixed-time flocking for system (4)-(6). To this end, motivated by [4], [5], we design the control protocol $u_i(t)$ for each follower as

$$\begin{aligned} u_i(t) &= -\alpha \left(\sum_{j=1}^N a_{ij}(v_i(t) - v_j(t)) + d_i(v_i(t) - v_0(t)) \right)^{1+\frac{b}{a}} \\ &\quad - \beta \text{sign} \left(\sum_{j=1}^N a_{ij}(v_i(t) - v_j(t)) + d_i(v_i(t) - v_0(t)) \right), \end{aligned} \quad (11)$$

where $\alpha > 0$, $\beta > 0$, $a > 0$ and $b \geq 0$.

Now, we state our main results as follows:

Theorem 5: Under the Assumption 1, 2, 3, 4 and employing (11) with $b > 0$, if $\beta \geq \frac{\sqrt{N}}{\sqrt{d}} + l$, then systems (4)-(6) has a fixed-time leader-following flocking. Moreover, an upper bound of the settling time can be estimated by

$$T^* = \frac{(1+\varepsilon_1)(b+2a)}{b\lambda_{\min}(H)} \left(\frac{\lambda_{\min}(H)}{[\alpha(dN)^{-\frac{b}{2a}}(2\lambda_{\min}(H))^{\frac{2a+b}{2a}}]} \right)^{\frac{2a}{2a+b}}. \quad (12)$$

proof:

Step 1: We try to find out $T^* > 0$ such that $v_i(t) = v_0(t)$ for all $t \geq T^*$. Let $e_i(t) = v_i(t) - v_0(t)$, $i \in \{1, 2, \dots, N\}$. Then combining (11), (4) with (6) implies that

$$\begin{aligned} \dot{e}_i &= \dot{v}_i - \dot{v}_0 \\ &= -\alpha \left(\sum_{j=0}^N a_{ij}(e_i - e_j) \right)^{1+\frac{b}{a}} - \beta \text{sign} \sum_{j=0}^N a_{ij}(e_i - e_j) \\ &\quad + f(x_i, v_i, t) - f(x_0, v_0, t) - u_0(t). \end{aligned}$$

Now, we denote

$$\begin{aligned} e &= [e_1^T, e_2^T, \dots, e_N^T]^T, \\ F(x, v) &= [f^T(x_1(t), v_1(t), t), \dots, f^T(x_N(t), v_N(t), t)]^T, \\ F(x_0, v_0) &= [f^T(x_0(t), v_0(t), t), \dots, f^T(x_0(t), v_0(t), t)]^T, \end{aligned}$$

then

$$\begin{aligned} \dot{e} &= -\alpha((H \otimes I_d)e)^{1+\frac{b}{a}} - \beta \text{sign}((H \otimes I_d)e) \\ &\quad + F(x, v) - F(x_0, v_0) - \mathbf{1}_d \otimes u_0, \end{aligned} \quad (13)$$

where \otimes be the Kronecker product. In addition, let $\dot{e} = F(e)$ and choose the following Lyapunov functions

$$V(e) = \frac{1}{2} e^T (H \otimes I_d) e. \quad (14)$$

Since the right hand side of (13) is discontinuous, it deduces from Definition 2 that the set-valued Lie derivative of $V(e)$ is given by

$$\begin{aligned} \tilde{\mathcal{L}}_F V(e) &= e^T (H \otimes I_d) \mathcal{K}[F](e) \\ &= \mathcal{K}[-\alpha e^T (H \otimes I_d) ((H \otimes I_d) e)^{1+\frac{b}{a}} \\ &\quad - \beta e^T (H \otimes I_d) \text{sign}((H \otimes I_d) e) \\ &\quad + e^T (H \otimes I_d) F(x, v) - e^T (H \otimes I_d) F(x_0, v_0) \\ &\quad - e^T (H \otimes I_d) (\mathbf{1}_d \otimes u_0)] \\ &= -\alpha \| (H \otimes I_d) e \|_{2+\frac{b}{a}}^{2+\frac{b}{a}} - \beta \| (H \otimes I_d) e \|_1 \\ &\quad + e^T (H \otimes I_d) g - e^T (H \otimes I_d) (\mathbf{1}_d \otimes u_0), \end{aligned}$$

where $g = [g_1^T, g_2^T, \dots, g_N^T]^T$, $g_i \in \mathcal{K}[f(x_i(t), v_i(t), t) - f(x_0(t), v_0(t), t)]$ are measurable selections,

$i = \{1, 2, \dots, N\}$. Thus, from Assumption 3, one can easily has

$$\begin{aligned} \tilde{\mathcal{L}}_F V(e) &\leq -\alpha \|(H \otimes I_d)e\|_{2+\frac{b}{a}}^{2+\frac{b}{a}} - \beta \|(H \otimes I_d)e\|_1 \\ &\quad + e^T (H \otimes I_d)g + l \|(H \otimes I_d)e\|_1. \end{aligned} \quad (15)$$

It follows from Remark 2 that H is a positive definite matrix. So there exists a unique upper triangular matrix $A \in \mathbb{R}^{d \times d}$ satisfying $H = A^T A$ (see [26] for example). Thus, employing Assumption 2 and Lemma 3 shows that

$$\begin{aligned} e^T (H \otimes I_d)g &= \langle (H \otimes I_d)e, g \rangle \leq \|(H \otimes I_d)e\|_2 \|g\|_2 \\ &\leq -\frac{\|(H \otimes I_d)e\|_2}{\mu + \varepsilon_1 \|v_i - v_j\|_2} \\ &\quad + \|(H \otimes I_d)e\|_2 (-\varepsilon_2 \|x_i - x_j\|_2 + \nu) \\ &\leq \frac{\nu N}{\sqrt{d}} \|(H \otimes I_d)e\|_1, \end{aligned}$$

which substituting into (15) yields that

$$\begin{aligned} \tilde{\mathcal{L}}_F V(e) &\leq -\alpha \|(H \otimes I_d)e\|_{2+\frac{b}{a}}^{2+\frac{b}{a}} \\ &\quad - (\beta - \frac{\nu N}{\sqrt{d}} - l) \|(H \otimes I_d)e\|_1. \end{aligned}$$

Noting that $\beta \geq \frac{\nu N}{\sqrt{d}} + l$. Then $\tilde{\mathcal{L}}_F V(e) \leq 0$. Therefore, $V(e)$ is a decreasing function, which combining with the fact $\frac{1}{2} \lambda_{\min}(H) \|e\|_2^2 \leq V(e)$ implies that $\|e\|_2$ is bounded for all $t \geq 0$. So, without loss of generality, there exists $\eta > 1$ such that

$$\|e\|_2 \leq \eta \mu, \quad \text{for all } t \geq 0. \quad (16)$$

From the properties of algebraic connectivity of digraphs (see Theorem 3 in [25]), we have

$$\begin{aligned} \|(H \otimes I_d)e\|_2^2 &= e^T [(H \otimes I_d)e]^T (H \otimes I_d)e \\ &= e^T (A^T A \otimes I_d) (A^T A \otimes I_d) e \\ &= e^T (A^T A A^T A \otimes I_d) e \\ &= e^T (A^T \otimes I_d) (A A^T \otimes I_d) (A \otimes I_d) e \\ &\geq \lambda_{\min}(H) e^T (A^T \otimes I_d) (A \otimes I_d) e \\ &= \lambda_{\min}(H) e^T (H \otimes I_d) e \\ &= 2\lambda_{\min}(H) V(e). \end{aligned} \quad (17)$$

By Lemma 3, one can easily get that

$$\|(H \otimes I_d)e\|_2 \leq \left((dN)^{\frac{1}{2} - \frac{1}{2+\frac{b}{a}}} \right) \|(H \otimes I_d)e\|_{2+\frac{b}{a}}.$$

Then

$$\|(H \otimes I_d)e\|_{2+\frac{b}{a}} \geq \left((dN)^{\frac{1}{2} - \frac{1}{2+\frac{b}{a}}} \right)^{-1} \|(H \otimes I_d)e\|_2,$$

which combining with (17) implies that

$$\|(H \otimes I_d)e\|_{2+\frac{b}{a}}^{2+\frac{b}{a}} \geq (dN)^{-\frac{b}{2a}} (2\lambda_{\min}(H))^{1+\frac{b}{2a}} (V(e))^{1+\frac{b}{2a}}.$$

Therefore,

$$\begin{aligned} -\alpha \|(H \otimes I_d)e\|_{2+\frac{b}{a}}^{2+\frac{b}{a}} &\leq -\alpha (dN)^{-\frac{b}{2a}} (2\lambda_{\min}(H))^{1+\frac{b}{2a}} (V(e))^{1+\frac{b}{2a}} \end{aligned} \quad (18)$$

Employing Assumption 2, Lemma 3 and (16) again, we obtain that

$$\begin{aligned} e^T (H \otimes I_d)g &\leq \|(H \otimes I_d)e\|_2 \left(-\frac{1}{\mu + \varepsilon_1 \|v_i - v_j\|_2} - \varepsilon_2 \|x_i - x_j\|_2 + \nu \right) \\ &\leq \|(H \otimes I_d)e\|_2 \\ &\quad \times \left(-\frac{1}{\eta \mu + \varepsilon_1 \|v_i - v_j\|_2} - \varepsilon_2 \|x_i - x_j\|_2 + \nu \right) \\ &\leq \|(H \otimes I_d)e\|_2 \\ &\quad \times \left(-\frac{1}{\eta \mu + \varepsilon_1 \sum_{i=1}^N \|e_i\|_2} - \varepsilon_2 \|x_i - x_j\|_2 + \nu N \right) \\ &= \|(H \otimes I_d)e\|_2 \left(-\frac{1}{\eta \mu + \varepsilon_1 \|e\|_2} - \varepsilon_2 \|x_i - x_j\|_2 + \nu N \right) \\ &\leq -\frac{\lambda_{\min}(H) \|e\|_2}{\eta \mu + \varepsilon_1 \|e\|_2} - \varepsilon_2 \|(H \otimes I_d)e\|_2 \|x_i - x_j\|_2 \\ &\quad + \nu N \|(H \otimes I_d)e\|_2 \\ &\leq -\frac{\lambda_{\min}(H)}{1 + \varepsilon_1} + \frac{\nu N}{\sqrt{d}} \|(H \otimes I_d)e\|_1. \end{aligned} \quad (19)$$

Thus, substituting (18) and (19) into (15) yields that

$$\begin{aligned} \tilde{\mathcal{L}}_F V(e) &\leq -\alpha (dN)^{-\frac{b}{2a}} (2\lambda_{\min}(H))^{1+\frac{b}{2a}} (V(e))^{1+\frac{b}{2a}} \\ &\quad - \beta \|(H \otimes I_n)e\|_1 - \frac{\lambda_{\min}(H)}{1 + \varepsilon_1} \\ &\quad + \frac{\nu N}{\sqrt{d}} \|(H \otimes I_n)e\|_1 + l \|(H \otimes I_n)e\|_1 \\ &= -\frac{\lambda_{\min}(H)}{1 + \varepsilon_1} - \alpha (dN)^{-\frac{b}{2a}} (2\lambda_{\min}(H))^{1+\frac{b}{2a}} (V(e))^{1+\frac{b}{2a}} \\ &\quad - (\beta - \frac{\nu N}{\sqrt{d}} - l) \|(H \otimes I_n)e\|_1 \\ &\leq -\frac{\lambda_{\min}(H)}{1 + \varepsilon_1} - \alpha (dN)^{-\frac{b}{2a}} (2\lambda_{\min}(H))^{1+\frac{b}{2a}} (V(e))^{1+\frac{b}{2a}}. \end{aligned}$$

Therefore, by Lemma 4 (1), the equilibrium of (13) is fixed-time stable, and

$$\begin{aligned} T^* &= T_{\max} \\ &= \frac{(1 + \varepsilon_1)(b + 2a)}{b \lambda_{\min}(H)} \left(\frac{\lambda_{\min}(H)}{[\alpha (dN)^{-\frac{b}{2a}} (2\lambda_{\min}(H))^{\frac{2a+b}{2a}}]} \right)^{\frac{2a}{2a+b}}. \end{aligned}$$

Then, for

$$\dot{V}(e) \leq -\frac{\lambda_{\min}(H)}{1 + \varepsilon_1} - \alpha (dN)^{-\frac{b}{2a}} (2\lambda_{\min}(H))^{1+\frac{b}{2a}} (V(e))^{1+\frac{b}{2a}},$$

by applying the comparison principle [23], the equilibrium is still fixed-time stable and an upper bound of the settling time is estimated as T^* . So, $v_i(t) = v_0(t)$, $i = \{1, 2, \dots, N\}$ for $t \geq T^*$.

Step 2: Since $v_i(t) = v_0(t)$, $i = \{1, 2, \dots, N\}$ for $t \geq T^*$, then for all $i, j \in \{1, 2, \dots, N\}$, applying the definition of norm and triangle inequality to yield that

$$0 \leq \|v_i(t) - v_j(t)\|_2 \leq \|v_i(t) - v_0(t)\|_2 + \|v_j(t) - v_0(t)\|_2 \leq 0,$$

for $t \geq T^*$, which means that

$$d_V(t) = 0 \quad \text{for } t \geq T^*. \quad (20)$$

Furthermore, there exists at most countable number of increasing time t_k such that we can choose indices i and j satisfying $d_V(t) = \|v_i(t) - v_j(t)\|_2$ on any time interval (t_k, t_{k+1}) with $t_{k+1} \leq T^*$ because the number of particles is finite and the continuity of the velocity trajectories. This combines with (20) to imply that $d_V(t)$ is a bounded function for all $t \geq 0$, i.e., there exists a constant $\lambda > 0$ such that

$$|d_V(t)| \leq \lambda d_V(0). \quad (21)$$

Then for any $i, j \in \{1, 2, \dots, N\}$, it is easy to get from the definition of $d_V(t)$ and (21) that

$$\begin{aligned} \|v_i(t) - v_j(t)\|_2 &= \left[\sum_{m=1}^d (v_i^{(m)} - v_j^{(m)})^2 \right]^{\frac{1}{2}} \\ &\leq \left[\sum_{m=1}^d |d_V(t)|^2 \right]^{\frac{1}{2}} \leq \left[\sum_{m=1}^d |\lambda d_V(0)|^2 \right]^{\frac{1}{2}} \\ &\leq \sqrt{d} \lambda d_V(0), \end{aligned}$$

which implies that

$$d_V(t) \leq \sqrt{d} \lambda d_V(0) \quad \text{for } t \geq 0. \quad (22)$$

To verify $d_X(t) < \infty$ for all $t > 0$, we first claim

$$D^+[d_X(t)] \leq d_V(t) \quad \text{for all } t > 0, \quad (23)$$

where D^+ is the upper Dini derivative. In fact, for all $i, j \in \{1, 2, \dots, N\}$, a direct calculation gives that

$$\begin{aligned} \frac{d}{dt} \|x_i(t) - x_j(t)\|_2^2 &= 2\langle x_i(t) - x_j(t), v_i(t) - v_j(t) \rangle \\ &\leq 2\|x_i(t) - x_j(t)\|_2 d_V(t). \end{aligned}$$

On the other hand, we have

$$\frac{d}{dt} \|x_i(t) - x_j(t)\|_2^2 = 2\|x_i(t) - x_j(t)\|_2 \frac{d}{dt} \|x_i(t) - x_j(t)\|_2,$$

from which it follows that

$$\frac{d}{dt} \|x_i(t) - x_j(t)\|_2 \leq d_V(t).$$

Thus, the claim follows since we can select i and j such that $d_X(t) = \|x_i(t) - x_j(t)\|_2$ on any time interval (t_k, t_{k+1}) . Now, integrating both sides of (23) from 0 to t yields that

$$d_X(t) = \int_0^t d_V(s) ds + d_X(0). \quad (24)$$

In what follows, there are two possible cases to consider. One case is $0 < t \leq T^*$. Employing (22) and (24) to yield that

$$\begin{aligned} d_X(t) &\leq \int_0^t \sqrt{d} \lambda d_V(0) ds + d_X(0) \\ &\leq \sqrt{d} \lambda d_V(0) T^* + d_X(0). \end{aligned}$$

The other case is $t > T^*$. It follows from (20),(22) and (24) that

$$\begin{aligned} d_X(t) &\leq \int_0^{T^*} \sqrt{d} \lambda d_V(0) ds + \int_{T^*}^t d_V(s) ds + d_X(0) \\ &= \sqrt{d} \lambda d_V(0) T^* + d_X(0). \end{aligned}$$

Hence,

$$d_X(t) \leq \sqrt{d} \lambda d_V(0) T^* + d_X(0) < \infty \quad \text{for all } t > 0.$$

Therefore, the proof is completed by applying Definition 3.

It is worthy remarking that if $b = 0$ in (11), that is,

$$\begin{aligned} u_i(t) &= -\alpha \left(\sum_{j=1}^N a_{ij}(v_i - v_j) + d_i(v_i - v_0) \right) \\ &\quad -\beta \text{sign} \left(\sum_{j=1}^N a_{ij}(v_i - v_j) + d_i(v_i - v_0) \right). \quad (25) \end{aligned}$$

Then employing the similar analysis of Theorem 5 and applying Lemma 4 (ii), one can easily get the following finite-time flocking result.

Corollary 6: Under the Assumption 1, 2, 3, 4 and employing (25), then system (4)-(6) has a finite-time leader-following flocking and the settling time is estimated as

$$T_0 = \frac{1}{2\alpha \lambda_{\min}(H)} \ln \frac{\lambda_{\min}(H) + 2\alpha \lambda_{\min}(H) V(e(0))(1 + \varepsilon_1)}{\lambda_{\min}(H)}.$$

Remark 7: Theorem 5 shows that under some suitable conditions, the leader-following fixed-time flocking of a second order multi-agent system can be established. Moreover, the upper bound of the settling time can be estimated only by the parameters, which is quite different from the finite-time flocking [8], [35] whose upper bound of settling time heavily depends on the initial data of the systems. In [8], [35], the authors considered the finite-time flocking of a C-S type model (without leaders) by using a finite-time stability theorem. However, in this paper, combining the graph theory with a fixed-time stable theorem, we verify that the fixed-time flocking of a second order leader-following multi-agent systems can be occurred. Moreover, the methods in [8], [35] can not directly apply in the present paper.

Remark 8: Although the method we apply to study the speed convergence is similar to [4], [5], the inherent dynamics in the present paper is quite different from [4], [5], in which the authors investigated the fixed-time consensus of a leader-following multi-agent systems. Moreover, under this new inherent dynamics, we show that the fixed-time flocking can be formed. It should be emphasized that the results

and the methods in the present paper are distinct from [16], in which the authors studied the finite-time flocking.

IV. SIMULATIONS

In this section, two numerical examples inherent nonlinear dynamics are given to show the effectiveness of the theoretical analysis obtained in the previous section. A multi-agent system consisting $N = 7$ followers (denoting by 1,2,3,4,5,6,7) and one leader (denoting by 0) in one-dimensional space ($d = 1$) is considered. Assume that the control input of the leader to its followers is set as

$$u_0(t) = 16 \sin(13\pi t).$$

Then it is bounded by $l = 16$. Furthermore, assume that the interaction topology be shown in Figure 1.

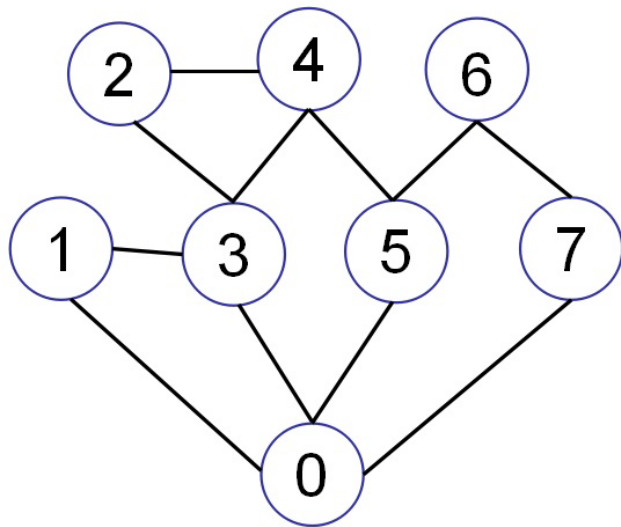


FIGURE 1. Interaction topology connected between followers and leader.

Since the weight of the two interacting agents is set to be 1, and be 0 otherwise (see Part II, B. GRAPH THEORY), the associated matrix \mathcal{A} and matrix H are

$$\mathcal{A} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$H = \begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 4 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 3 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix},$$

where $diag(d_1, \dots, d_7) = diag(1, 0, 1, 0, 1, 0, 1)$ is used. Then the Assumptions 3 and 4 are satisfied.

Example 1: The discontinuous inherent nonlinear dynamics is given by

$$f(x_i(t), v_i(t), t) = 0.03v_i - 0.18 \operatorname{sign}(v_i),$$

where $i = 0, 1, \dots, N$. $\epsilon_1 = 0.03$, $\epsilon_2 = 0$, $\nu = 2$ and $\mu = 1$ such that $2\epsilon_1 \leq \nu - \frac{1}{\mu}$ which makes sure the establishment of Assumptions 1 and 2. Set the initial states of the followers as $x(0) = [0, 0.05, 0.10, 0.15, 0.20, 0.25, 0.30]^T$ and $v(0) = [1, -2, -2, 1, 1, 1, -1]^T$ and those of the leader as $x_0(0) = 1$ and $v_0(0) = 0.5$. Take $\alpha = 2.2$ and $\beta = 32$, $a = 4$ and $b = 9$, then it is can be calculated from (12) that $T^* = 2.78$. By applying Matlab, the following simulation results (Figures 2–4) are obtained.

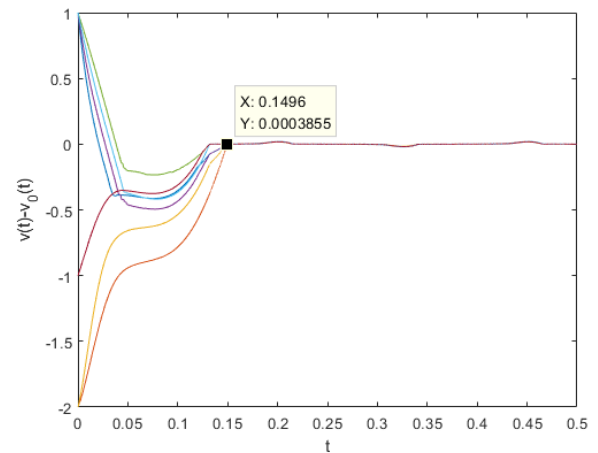


FIGURE 2. The velocities of agents.

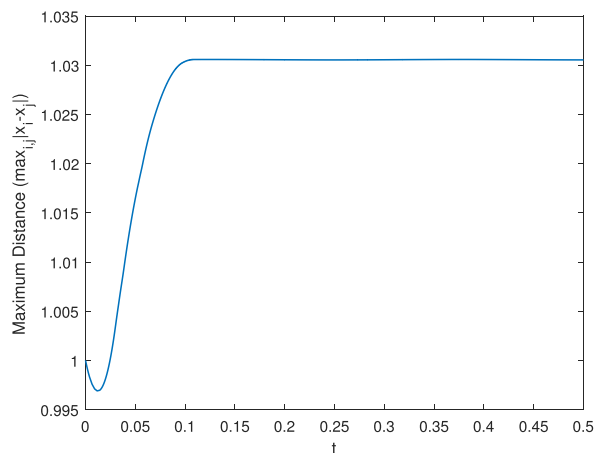


FIGURE 3. The maximum distance of agents.

Figure 2 shows that the relative velocities are convergent to zero after about $t = 0.15$. The result in Figure 3 presents that the maximum positions of all agents are not changing after about $t = 0.15$, which means that the size of the group is remaining the value about 1.03 after $t = 0.15$. Moreover, it can be seen in the Figure 4 that a gradual decline appears in the value of the minimum distance of the agents

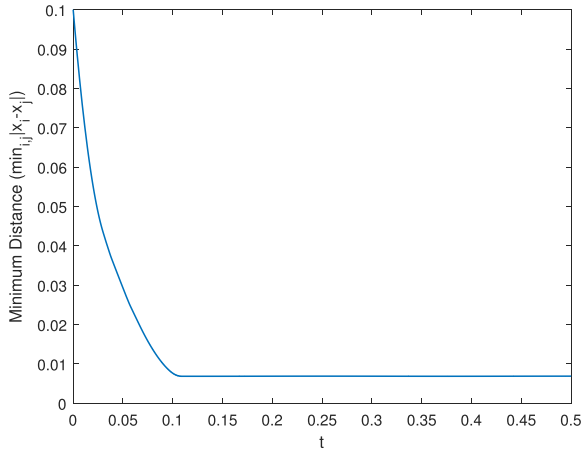


FIGURE 4. The minimum distance of agents (collision avoidance).

before $t = 0.10$, but this trend stops and the value of the minimum distance keeps about 0.008 after $t = 0.15$, which means that the agents do not collide during the process of flocking. Hence, the flocking is achieved before the settling time $T^* = 8.78$. However, we find that when the followers are too close each other in the initial states, it will lead to the collision. For example, the initial position states of the followers are changed to be $x(0) = [0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6]^T$, the agents collides at about $t = 0.025$ during the process of the flocking which shown in the Figure 5. So, the simulative results illustrate the effectiveness of Theorem 5.

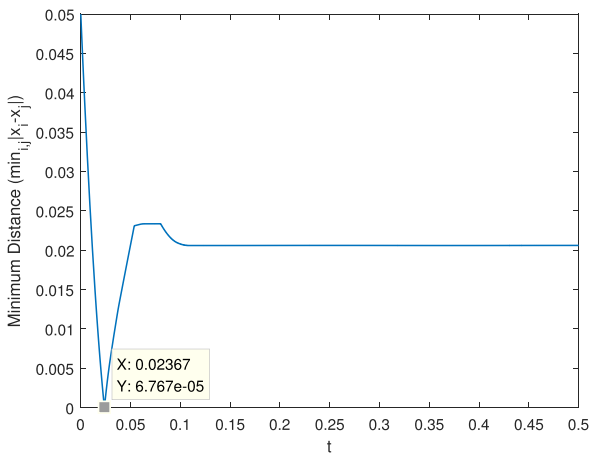


FIGURE 5. The minimum distance of agents (collision).

To better illustrate the effects of different inherent dynamics on the flocking, we add the following Example 2.

Example 2: In this example, the discontinuous inherent nonlinear dynamics is given by

$$f(x_i(t), v_i(t), t) = 0.03 \sin(v_i) - 0.18 \operatorname{sign}(v_i),$$

where $i = 0, 1, \dots, N$. To better illustrate the effects of different inherent dynamics on the flocking, we choose the same parameters as in Example 1, that is, $\epsilon_1 = 0.03$, $\epsilon_2 = 0$, $\nu = 2$ and $\mu = 1$ such that $2\epsilon_1 \leq \nu - \frac{1}{\mu}$ which guarantees the establishment of Assumptions 1 and 2. The initial states

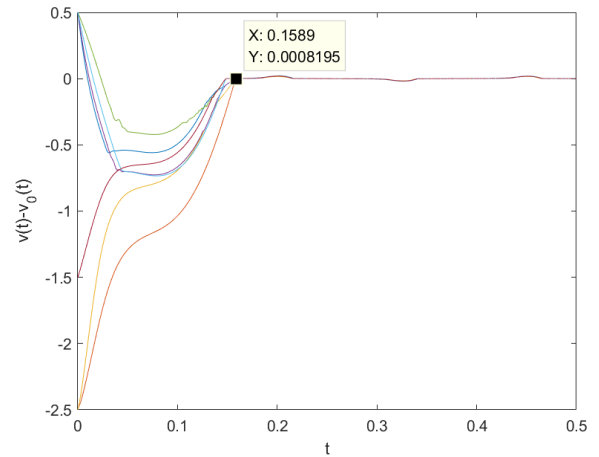


FIGURE 6. The velocities of agents.

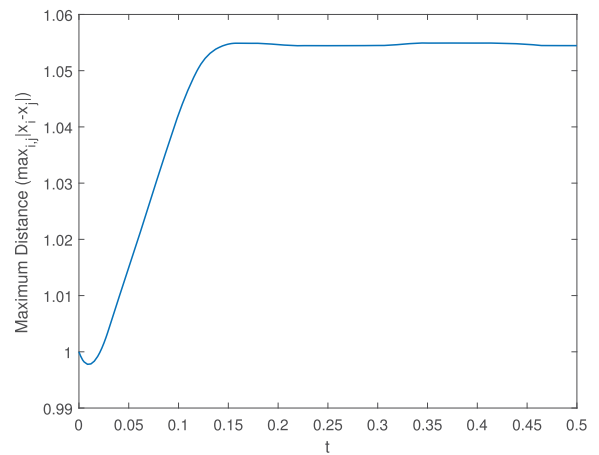


FIGURE 7. The maximum distance of agents.

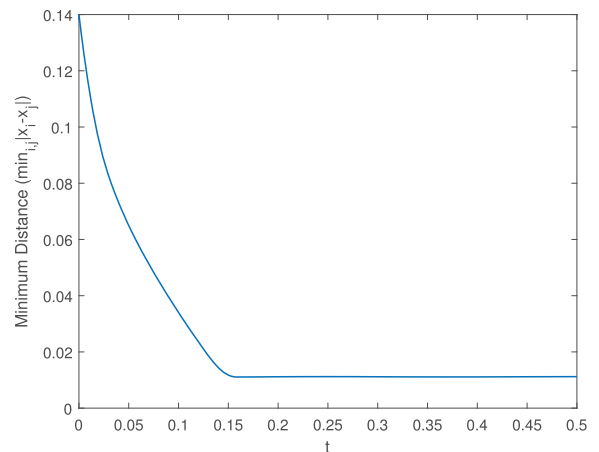


FIGURE 8. The minimum distance of agents (collision avoidance).

of the followers as $x(0) = [0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6]^T$ and $v(0) = [1, -2, -2, 1, 1, 1, -1]^T$ and those of the leader as $x_0(0) = 1$ and $v_0(0) = 0.5$. Set $\alpha = 8$, $\beta = 15$, $a = 4$ $b = 9$ in (12). It follows from (12) that $T^* = 2.60$. Moreover, the following simulation results (Figures 6–8) are obtained by Matlab. As analysis in Example 1, we can see from Figures 6–8 that the collision-avoidance flocking occurs

at about $t = 0.2$. In addition, when the followers are too close each other in the initial states, it will lead to the collision. For example, when the initial position states of the followers are set as $x(0) = [0, 0.14, 0.28, 0.42, 0.56, 0.70, 0.84]^T$, the collision appears at about $t = 0.08$ which shown in Figure 9. Hence, the simulative results effectively demonstrate the theoretical analysis of Theorem 5.

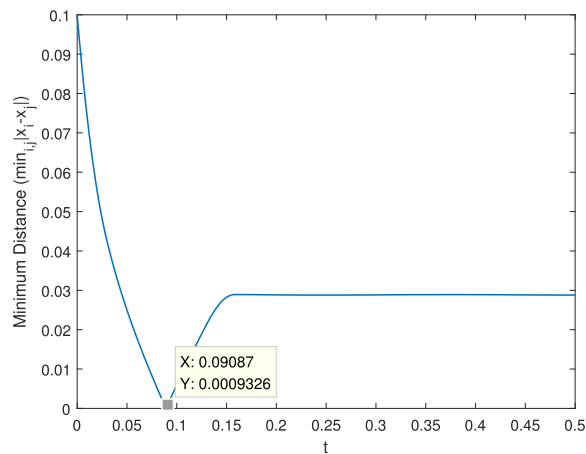


FIGURE 9. The minimum distance of agents (collision).

V. CONCLUSION

The fixed-time leader-following flocking problem of second-order multi-agent systems with the discontinuous inherent dynamics is investigated in the present paper. By using non-smooth techniques and fixed-time stability theory, a new class of nonlinear inherent dynamics has been proposed to ensure that the followers reach the flocking with a leader within a fixed-time. The main results show that the control input of the leader to its followers is bounded and at least one follower obtained the information from the leader directly, the flocking can be occurred in a fixed time. Moreover, an upper bound of the settling time is directly estimated by the system parameters. Comparing with the finite-time flocking, this approach has wider practical applications because the knowledge of the initial conditions is always unavailable in practical scenarios. Although the theoretical analysis result shows that the fixed-time flocking can be achieved under same suitable conditions supposed on the parameters, these conditions can not guarantee the collision avoidance. To indicate this fact, two examples are provided. The simulation results demonstrate that the followers can track the time-varying state of the leader in a fixed-time when the conditions of Theorem 5 satisfied. And it also shows that when the initial position of agents are large sufficiently, the collision does not appear during the process of flocking (Figures 4 and 8). On the contrary, the agents may collide (Figures 5 and 9). These simulated results effectively illustrate the theoretical analysis of the main theorems.

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