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Fault Detection for Nonlinear Networked Control Systems With Sensor Saturation and Random Faults

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ABSTRACT A fault detection filter is designed for a class of nonlinear networked control systems with sensor saturation, random packet loss and delay, a unified model is constructed to describe the transmission delay caused by packet loss and network induction. By constructing Lyapunov function and using the convex optimization algorithm, a parameter matrix of the H_∞ fault detection filter is obtained by solving a series of linear matrix inequalities, which guarantees that the error system satisfies a concerned performance index. Finally, the validity of the proposed method is illustrated by a numerical example on the internet-based three-tank system.

INDEX TERMS Fault detection, nonlinear networked control systems, packet losses, random occurring faults, sensor saturation, time-delays.

I. INTRODUCTION

With the rapid requirement of network, much concern has been paid to network based systems, control systems who are using communication network to connect sensors, controllers, actuators and other system components are called networked control systems (NCSs). In classical control methods, all parts of the feedback control system are clustered in the same small space and are connected through point-to-point dedicated lines [1], [2]. However, in NCSs, information will be transferred between components within the network, which will reduce the necessaries of connection lines, and increase the flexibility definitely [3].

Communication network plays a crucial role for system work, however, it relies on the reliability and security of itself, it is known that data package will be transmitted through the network, delay or packet loss may happens [4], [5]. Another attention should be attracted on saturation, as it is a fact that there is physical limitation in almost every sensor [6]–[9].

The scale of modern control systems is becoming larger and more complex, and the possibility of occurrence of system failure is increasing. Research on fault detection (FD) for NCSs has become an important research stream [10]–[12]. For research work of fault detection problem of NCSs, researchers normally construct appropriate filters and state observers as residual generators to generate residual signals. By appropriate transformation, fault detection problems for NCSs can be converted into H_∞ filtering or H_∞ control problems. For instance, the design of the fault detection filter for nonlinear networked systems with data packet losses was discussed and the addressed FD problem is converted into an auxiliary H_∞ filtering problem [13]. Due to the complex of network, random occurring phenomena (ROP) such as random packet losses, random delays and random nonlinearity have attracted people's attention and these phenomena are usually assumed to occur randomly according to a certain probability [14]. If random occurring phenomena are not considered in the FD research, the research results will have limited application possibilities [15]. Nowadays, ROP are widely studied, in [16], the quantitative output feedback control problem of networked control systems with packet losses was studied and the random packet loss was modeled as Markov chains. In [17], the random time-delay was modeled as Markov chains and the H_∞ filtering problem of networked systems with time delay was studied. To deal with the delta operator systems, H-infinity fault detection problem was discussed in [18]. In recent years, the research on random nonlinearity has attracted more attention due to two main reasons. First, in practical systems, the non-linearity generated by the

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external non-linear input or linearization process of a non-linear system is prevalent. On the other hand, in a network environment, random failure, component damage or problems of subsystem connections may lead to nonlinearities [19]. It should be pointed out that in practice, faults often occur randomly, but only little results can be found on that. In [20], the problem of fault detection for network multi-rate systems with random faults is discussed. Assuming that the faults occur randomly, a corresponding fault detection filter is designed. In [21], the problem of finite-time fault detection for networked systems with random faults and random nonlinearity is studied. The occurrence of faults and nonlinearity is described by a random sequence satisfying Bernoulli distribution.

Above all, it can be seen that most of the existing literatures assume that one or two phenomena of data packet loss, delay, nonlinearity and quantization may happen separately. But in NCSs, these phenomena usually exist simultaneously. Most literature studies actuator faults, but sensor saturation also exists commonly. In [22], an adaptive sliding mode controller design problem is investigated for a class of Markov jump systems with actuator faults. In this paper, we will study the FD problems for nonlinear NCSs involving random packet losses, random time-delays, sensor saturation and randomly occurring faults. Our aim is to design a FD filter to establish sufficient conditions for the existence of the desired FD filter in the presence of these faults mentioned.

The rest contents of this paper are as follows, problem statement and preliminaries are introduced in Section 2. Sufficient conditions for the H_∞ performance analysis and FD filter design are given in Section 3. In Section 4, a numerical example is presented. Finally, a conclusion of this article is presented in Section 5.

Notations: The notations used in this paper are as follows. $\mathbb{R}^{\mathrm{n}}$ denotes the n-dimensional Euclidean space. $L_2[0, \infty)$ is the space of square summable vectors. $\mathrm{E}\{\cdot\}$ stands for the mathematical expectation and Prob$\{\cdot\}$ means the occurrence probability of the event. In symmetric block matrices or complex matrix expressions, we use $*$ to represent a term that is induced by symmetry and diag$\{\cdot\}$ denotes the block diagonal matrix. Matrices, if their dimensions are not explicitly stated, they are assumed to be compatible for algebraic operations. The notation $\mathbf{P} > 0 (\geq 0)$ means that $\mathbf{P}$ is a symmetric and positive definite (semidefinite) matrix. $\mathbf{0}$ and $\mathbf{I}$ represent the zero matrix and identity matrix with compatible dimensions.

II. PROBLEM STATEMENT AND PRELIMINARIES

Consider the following nonlinear networked system with randomly occurring faults, random packet losses, random delays and sensor saturation:

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}_u\mathbf{u}(k) + \mathbf{B}_w\mathbf{w}(k) \\ \quad +\alpha(k)\mathbf{E}\mathbf{f}(k) + \delta(k)\mathbf{g}(\mathbf{x}(k)) \\ \tilde{\mathbf{y}}(k) = I_{\{\tau(k)=0\}}\sigma(\mathbf{C}_0\mathbf{x}(k)) \\ \quad + \sum_{j=1}^{q} I_{\{\tau(k)=d_j\}}\sigma(\mathbf{C}_j\mathbf{x}(k-d_j)) + \mathbf{D}_w\mathbf{w}(k) \end{cases}$$
$$\mathbf{x}(k) = \varphi(k), \quad k = -d_q, -d_q+1, \cdots, 0 \tag{1}$$

where $\mathbf{x}(k) \in \mathbb{R}^n$ is the state vector; $\mathbf{u}(k) \in \mathbb{R}^u$ is the control input; $\varphi(k)$ is a given real initial sequence; $\mathbf{w}(k) \in \mathbb{R}^p$ is the unknown input, which belongs to $L_2[0, \infty)$; $\mathbf{f}(k) \in \mathbb{R}^l$ is the fault to be detected; $d_j \in \mathbb{Z}^+ (j = 1, \cdots, q)$ are constant time-delays and $d_1 < d_2 < \cdots < d_q$; $\tilde{\mathbf{y}}(k) \in \mathbb{R}^m$ is the system output subject to saturation, which may contain random delays and random packet losses; saturation function $\sigma(\cdot) : \mathbb{R}^m \rightarrow \mathbb{R}^m$ belongs to $[\mathbf{L}_1, \mathbf{L}_2]$ with given diagonal matrices $\mathbf{L}_1, \mathbf{L}_2$, where $\mathbf{L}_1 \geq 0$, $\mathbf{L}_2 \geq 0$ and $\mathbf{L}_2 > \mathbf{L}_1$; $I_{\{\tau(k)=0\}}$ and $I_{\{\tau(k)=d_j\}}$ are the indicator functions with $\mathrm{E}\left\{I_{\{\tau(k)=0\}}\right\} = \mathrm{Prob}\{\tau(k)=0\} = \beta_0$ and $\mathrm{E}\left\{I_{\{\tau(k)=d_j\}}\right\} = \mathrm{Prob}\left\{\tau(k)=d_j\right\} = \beta_j$, where $\beta_j > 0 (j = 0, \cdots, q)$ are known positive scalars and $\sum_{j=0}^{q}\beta_j \leq 1$; $\tau(k)$ is a stochastic sequence used to determine how large the occurred delay could be and the possibility of data missing at time k; $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B}_u \in \mathbb{R}^{n\times u}$, $\mathbf{B}_w \in \mathbb{R}^{n\times p}$, $\mathbf{E} \in \mathbb{R}^{n\times l}$, $\mathbf{C}_j \in \mathbb{R}^{m\times n} (j = 0, \cdots, q)$ and $\mathbf{D}_w \in \mathbb{R}^{m\times p}$ are known constant matrices with appropriate dimensions.

Equation (1) uses a uniform structure to simultaneously represent random delays and packet losses. $\sum_{j=0}^{q}\beta_j < 1$ means that the measurement output arrives at a certain moment with a probability $\sum_{j=0}^{q}\beta_j$ regardless of the delay, and the probability of the packet loss is $1 - \sum_{j=0}^{q}\beta_j$. If $\sum_{j=0}^{q}\beta_j = 1$, there will be no missing phenomenon.

The stochastic variables $\alpha(k)$ and $\delta(k)$ are mutually uncorrelated Bernoulli distributed white-noise sequences taking values on 0 and 1 with the following assumptions:

$$\begin{cases} \mathrm{Prob}\{\alpha(k)=1\} = \bar{\alpha} \\ \mathrm{Prob}\{\alpha(k)=0\} = 1-\bar{\alpha} \\ \mathrm{Var}\{\alpha(k)\} = \mathrm{E}\left\{(\alpha(k)-\bar{\alpha})^2\right\} = (1-\bar{\alpha})\,\bar{\alpha} \end{cases} \tag{2}$$

$$\begin{cases} \mathrm{Prob}\{\delta(k)=1\} = \bar{\delta} \\ \mathrm{Prob}\{\delta(k)=0\} = 1-\bar{\delta} \\ \mathrm{Var}\{\delta(k)\} = \mathrm{E}\left\{\left(\delta(k)-\bar{\delta}\right)^2\right\} = \left(1-\bar{\delta}\right)\bar{\delta} \end{cases} \tag{3}$$

Remark 1: Markov chains are used in most literature to describe the occurrence of packet loss and delay, but it is difficult to obtain the transition probability accurately. The advantage of using Bernoulli distributed sequences is that the probabilities of delay, packet loss and randomly occurring fault can be known through statistical methods.

Assumption 1 ([23]): The nonlinear function $\mathbf{g}(\cdot)$ *satisfies* $\mathbf{g}(0) = 0$ *and*

$$[\mathbf{g}(x) - \mathbf{g}(y) - \mathbf{R}_1(x-y)]^{\mathrm{T}}[\mathbf{g}(x) - \mathbf{g}(y) - \mathbf{R}_2(x-y)] \leq 0 \tag{4}$$

where $\mathbf{R}_1 \in \mathbb{R}^{n\times n}$ *and* $\mathbf{R}_2 \in \mathbb{R}^{n\times n}$, *and* $\mathbf{R}_2 - \mathbf{R}_1$ *are known positive definite matrices.*

Taking the phenomenon of sensor saturation into account, the saturation function $\sigma(\cdot) : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is defined as

$$\sigma(v) = [\sigma_1^{\mathrm{T}}(v_1)\ \ \sigma_2^{\mathrm{T}}(v_2) \ldots \sigma_m^{\mathrm{T}}(v_m)]^{\mathrm{T}} \tag{5}$$

where $\sigma_i(\mathbf{v}_i) = \mathrm{sign}(\mathbf{v}_i) \cdot \min\{\mathbf{v}_{i,\max}, |\mathbf{v}_i|\}$, $\mathbf{v}_{i,\max} > 0$ is the i-th element of the saturation level vector $\mathbf{v}_{\max}$, the notation of "sign" denotes the signum function.

Definition 1 ([24]): $\sigma(\nu)$ satisfies the following inequality:

$$[\sigma(\nu) - \mathbf{L}_1\nu]^{\mathrm{T}}[\sigma(\nu) - \mathbf{L}_2\nu] \le 0 \tag{6}$$

According to (6), $\sigma(\mathbf{C}_0\mathbf{x}(k))$ and $\sigma(\mathbf{C}_j\mathbf{x}(k-d_j))$ can be divided into linear and nonlinear parts.

$$\sigma(\mathbf{C}_0\mathbf{x}(k)) = \boldsymbol{\phi}(\mathbf{C}_0\mathbf{x}(k)) + \mathbf{L}_1\mathbf{C}_0\mathbf{x}(k) \tag{7}$$

$$\sigma(\mathbf{C}_j\mathbf{x}(k-d_j)) = \boldsymbol{\phi}(\mathbf{C}_j\mathbf{x}(k-d_j)) + \mathbf{L}_1\mathbf{C}_j\mathbf{x}(k-d_j) \tag{8}$$

Then

$$\boldsymbol{\phi}^{\mathrm{T}}(\mathbf{C}_0\mathbf{x}(k))\left[\boldsymbol{\phi}(\mathbf{C}_0\mathbf{x}(k)) - \bar{\mathbf{L}}\mathbf{C}_0\mathbf{x}(k)\right] \le 0 \tag{9}$$

$$\boldsymbol{\phi}^{\mathrm{T}}(\mathbf{C}_j\mathbf{x}(k-d_j))\left[\boldsymbol{\phi}(\mathbf{C}_j\mathbf{x}(k-d_j)) - \bar{\mathbf{L}}\mathbf{C}_j\mathbf{x}(k-d_j)\right] \le 0 \quad (j = 1, 2, \cdots, q) \tag{10}$$

where $\boldsymbol{\phi}(\mathbf{C}_0\mathbf{x}(k))$ and $\boldsymbol{\phi}(\mathbf{C}_j\mathbf{x}(k-d_j))$ are nonlinear vector-valued functions, $\bar{\mathbf{L}} = \mathbf{L}_2 - \mathbf{L}_1$.

Construct the following full-order FD filter:

$$\begin{cases}\hat{\mathbf{x}}(k+1) = \mathbf{A}_f\hat{\mathbf{x}}(k) + \mathbf{B}_f\tilde{\mathbf{y}}(k)\\ \mathbf{r}(k) = \mathbf{C}_f\hat{\mathbf{x}}(k) + \mathbf{D}_f\tilde{\mathbf{y}}(k)\end{cases} \tag{11}$$

where $\hat{\mathbf{x}}(k) \in \mathbb{R}^n$ denotes the state vector of the filter; $\mathbf{r}(k) \in \mathbb{R}^l$ is the residual; $\tilde{\mathbf{y}}(k) \in \mathbb{R}^m$ is the filter input; $\mathbf{A}_f \in \mathbb{R}^{n\times n}$, $\mathbf{B}_f \in \mathbb{R}^{n\times m}$, $\mathbf{C}_f \in \mathbb{R}^{l\times n}$ and $\mathbf{D}_f \in \mathbb{R}^{l\times m}$ are appropriately dimensioned filter matrices to be determined.

Defining

$$\eta(k) = \begin{bmatrix}\mathbf{x}^{\mathrm{T}}(k) & \hat{\mathbf{x}}^{\mathrm{T}}(k)\end{bmatrix}^{\mathrm{T}},$$
$$\theta(k) = \begin{bmatrix}\mathbf{u}^{\mathrm{T}}(k) & \mathbf{w}^{\mathrm{T}}(k) & \mathbf{f}^{\mathrm{T}}(k)\end{bmatrix}^{\mathrm{T}},$$
$$\mathbf{e}(k) = \mathbf{r}(k) - \mathbf{f}(k),$$

from (1) and (11), we have the following filtering error system:

$$\begin{cases}\eta(k+1) = \bar{\mathbf{A}}_1\eta(k) + \tilde{\beta}_0\bar{\mathbf{A}}_2\eta(k) + \sum_{j=1}^{q}\hat{\mathbf{A}}_j\eta(k-d_j)\\ +\sum_{j=1}^{q}\tilde{\beta}_j\tilde{\mathbf{A}}_j\eta(k-d_j) + \bar{\mathbf{B}}_1\theta(k) + \tilde{\alpha}(k)\bar{\mathbf{B}}_2\theta(k)\\ +\tilde{\mathbf{B}}_0\boldsymbol{\phi}(\mathbf{C}_0\mathbf{x}(k))\\ +\tilde{\beta}_0\bar{\mathbf{B}}_3\boldsymbol{\phi}(\mathbf{C}_0\mathbf{x}(k)) + \sum_{j=1}^{q}\tilde{\mathbf{B}}_j\boldsymbol{\phi}(\mathbf{C}_j\mathbf{x}(k-d_j))\\ +\sum_{j=1}^{q}\tilde{\beta}_j\bar{\mathbf{B}}_3\boldsymbol{\phi}(\mathbf{C}_j\mathbf{x}(k-d_j)) + \bar{\mathbf{E}}_1\mathbf{g}(\mathbf{x}(k)) + \tilde{\delta}(k)\bar{\mathbf{E}}_2\mathbf{g}(\mathbf{x}(k))\\ \mathbf{e}(k) = \bar{\mathbf{C}}_1\eta(k)\\ +\tilde{\beta}_0\bar{\mathbf{C}}_2\eta(k) + \sum_{j=1}^{q}\hat{\mathbf{C}}_j\eta(k-d_j)\\ +\sum_{j=1}^{q}\tilde{\beta}_j\tilde{\mathbf{C}}_j\eta(k-d_j) + \bar{\mathbf{D}}_1\theta(k) + \tilde{\mathbf{D}}_0\boldsymbol{\phi}(\mathbf{C}\mathbf{x}(k))\\ +\tilde{\beta}_0\bar{\mathbf{D}}_2\boldsymbol{\phi}(\mathbf{C}_0\mathbf{x}(k))\\ +\sum_{j=1}^{q}\tilde{\mathbf{D}}_j\boldsymbol{\phi}(\mathbf{C}_j\mathbf{x}(k-d_j)) + \sum_{j=1}^{q}\tilde{\beta}_j\bar{\mathbf{D}}_2\boldsymbol{\phi}(\mathbf{C}_j\mathbf{x}(k-d_j))\end{cases} \tag{12}$$

where

$$\tilde{\alpha}(k) = \alpha(k) - \bar{\alpha}, \quad \tilde{\delta}(k) = \delta(k) - \bar{\delta},$$
$$\tilde{\beta}_0 = I_{\{\tau(k)=0\}} - \beta_0, \quad \tilde{\beta}_j = I_{\{\tau(k)=d_j\}} - \beta_j,$$
$$\bar{\mathbf{A}}_1 = \begin{bmatrix}\mathbf{A} & \mathbf{0}\\ \beta_0\mathbf{B}_f\mathbf{L}_1\mathbf{C}_0 & \mathbf{A}_f\end{bmatrix}, \quad \bar{\mathbf{A}}_2 = \begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{B}_f\mathbf{L}_1\mathbf{C}_0 & \mathbf{0}\end{bmatrix},$$
$$\hat{\mathbf{A}}_j = \begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \beta_j\mathbf{B}_f\mathbf{L}_1\mathbf{C}_j & \mathbf{0}\end{bmatrix}, \quad \tilde{\mathbf{A}}_j = \begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{B}_f\mathbf{L}_1\mathbf{C}_j & \mathbf{0}\end{bmatrix},$$
$$\bar{\mathbf{B}}_1 = \begin{bmatrix}\mathbf{B}_u & \mathbf{B}_w & \bar{\alpha}\mathbf{E}\\ \mathbf{0} & \mathbf{B}_f\mathbf{D}_w & \mathbf{0}\end{bmatrix}, \quad \bar{\mathbf{B}}_2 = \begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{E}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix},$$
$$\tilde{\mathbf{B}}_0 = \begin{bmatrix}\mathbf{0}\\ \beta_0\mathbf{B}_f\end{bmatrix}, \quad \bar{\mathbf{B}}_3 = \begin{bmatrix}\mathbf{0}\\ \mathbf{B}_f\end{bmatrix}, \quad \tilde{\mathbf{B}}_j = \begin{bmatrix}\mathbf{0}\\ \beta_j\mathbf{B}_f\end{bmatrix},$$
$$\bar{\mathbf{E}}_1 = \begin{bmatrix}\bar{\delta}\mathbf{I}\\ \mathbf{0}\end{bmatrix}, \quad \bar{\mathbf{E}}_2 = \begin{bmatrix}\mathbf{I}\\ \mathbf{0}\end{bmatrix}, \quad \bar{\mathbf{C}}_1 = \begin{bmatrix}\beta_0\mathbf{D}_f\mathbf{L}_1\mathbf{C}_0 & \mathbf{C}_f\end{bmatrix},$$
$$\bar{\mathbf{C}}_2 = \begin{bmatrix}\mathbf{D}_f\mathbf{L}_1\mathbf{C}_0 & \mathbf{0}\end{bmatrix}, \quad \hat{\mathbf{C}}_j = \begin{bmatrix}\beta_j\mathbf{D}_f\mathbf{L}_1\mathbf{C}_j & \mathbf{0}\end{bmatrix},$$
$$\tilde{\mathbf{C}}_j = \begin{bmatrix}\mathbf{D}_f\mathbf{L}_1\mathbf{C}_j & \mathbf{0}\end{bmatrix}, \quad \bar{\mathbf{D}}_1 = \begin{bmatrix}\mathbf{0} & \mathbf{D}_f\mathbf{D}_w & -\mathbf{I}\end{bmatrix},$$
$$\tilde{\mathbf{D}}_0 = \beta_0\mathbf{D}_f, \bar{\mathbf{D}}_2 = \mathbf{D}_f, \quad \tilde{\mathbf{D}}_j = \beta_j\mathbf{D}_f.$$

Definition 2: Let $\theta(k) = 0$, the filtering error system (12) is said to be exponentially mean-square stable if, for any initial conditions, there exist constant $\alpha > 0$ and $\kappa \in (0, 1)$, such that

$$\mathrm{E}\left\{\|\eta(k)\|^2\right\} \le \alpha\kappa^k \sup_{-d_q \le i \le 0} \mathrm{E}\left\{\|\eta(i)\|^2\right\}, \quad k \in \mathbb{Z}^+ \tag{13}$$

With this definition, the original FD filter design problem of the system (12) can be further converted into an H_∞ filtering problem. The purpose of this paper is to design a fault detection filter (11) so that the filtering error system (12) satisfies the following two requirements (Q1) and (Q2) in cases that delay, sensor saturation, and packet loss are allowed.

(Q1) If $\theta(k) = 0$, the filtering error system (12) is exponentially mean-square stable;

(Q2) If $\theta(k) \ne 0$, under the zero-initial condition, the following H_∞ performance index is satisfied.

$$\sum_{k=0}^{\infty}\mathrm{E}\left\{\|\mathbf{e}(k)\|^2\right\} \le \gamma^2\mathrm{E}\left\{\sum_{k=0}^{\infty}\|\theta(k)\|^2\right\} \tag{14}$$

where γ is made as small as possible in the feasibility of (14).

Next, we introduce a residual evaluation idea where an evaluation function $J(k)$ and a threshold J_{th} are described as below:

$$J(k) = \left[\sum_{s=0}^{k}\mathbf{r}^{\mathrm{T}}(s)\mathbf{r}(s)\right]^{1/2},$$
$$J_{th} = \sup_{\mathbf{w}(k)\in l_2, \mathbf{f}(k)=0} \mathrm{E}\{J(L)\} \tag{15}$$

where L is the finite length of evaluation time, and the threshold J_{th} represents the upper bound of the residual evaluation function with $\mathbf{f}(k) = 0$. We can detect the occurrence of fault by comparing with J_{th} according to the following rule:

$$\begin{cases}J(k) > J_{th} \Rightarrow \text{faults} \Rightarrow \text{alarm}\\ J(k) \le J_{th} \Rightarrow \text{no faults}\end{cases} \tag{16}$$

Remark 2: The H_∞ performance index γ represents the anti-interference ability of the filtering error system (1), which is often used in filter design problems. A smaller γ means stronger anti-interference ability. Therefore, the random fault detection problem studied in this paper can be transformed into an optimization problem.

III. MAIN RESULTS

In this section, we will derive the sufficient conditions under which the filtering error system (12) satisfies (Q1) and (Q2). Then the fault detection filter can be derived by solving a series of LMIs. The following lemmas will be used in the derivation of our main results.

Lemma 1 ([25]) (Schur complement): Given constant matrices $\mathbf{S}_1, \mathbf{S}_2$ and $\mathbf{S}_3$ where $\mathbf{S}_1 = \mathbf{S}_1^T$ and $\mathbf{S}_2 = \mathbf{S}_2^T$, then $\mathbf{S}_1 + \mathbf{S}_3^T\mathbf{S}_2^{-1}\mathbf{S}_3 < 0$ if and only if

$$\begin{bmatrix} \mathbf{S}_1 & \mathbf{S}_3^{\mathrm{T}} \\ \mathbf{S}_3 & -\mathbf{S}_2 \end{bmatrix} < 0 \text{ or } \begin{bmatrix} -\mathbf{S}_2 & \mathbf{S}_3 \\ \mathbf{S}_3^{\mathrm{T}} & \mathbf{S}_1 \end{bmatrix} < 0 \tag{17}$$

Lemma 2 ([26]): For matrices $\mathbf{A}$, $\mathbf{Q} = \mathbf{Q}^T$ and $\mathbf{P} > 0$ such that $\mathbf{A}^T\mathbf{P}\mathbf{A} - \mathbf{Q} < 0$ holds if and only if there exists a matrix $\mathbf{G}$ such that

$$\begin{bmatrix} -\mathbf{Q} & \mathbf{A}^{\mathrm{T}}\mathbf{G} \\ \mathbf{G}^{\mathrm{T}}\mathbf{A} & \mathbf{P}-\mathbf{G}-\mathbf{G}^{\mathrm{T}} \end{bmatrix} < 0 \tag{18}$$

Lemma 3 ([7]) (S-Procedure): Let $\mathbf{T}_0(\cdot), \mathbf{T}_1(\cdot), \cdots, \mathbf{T}_p(\cdot)$ be quadratic functions of variable $x \in \mathbb{R}^n$.

$\mathbf{T}_j(x) = x^{\mathrm{T}}\Phi_j x \geq 0$ $j = (0, 1, \cdots, p)$, where $\Phi_j^{\mathrm{T}} = \Phi_j$. If there exist $\lambda_1 \geq 0, \cdots, \lambda_p \geq 0$ such that $\Phi_0 - \sum_{j=1}^{p} \lambda_j\Phi_j \leq 0$, then

$$\mathbf{T}_1(x) \leq 0, \cdots, \mathbf{T}_p(x) \leq 0 \Rightarrow \mathbf{T}_0(x) \leq 0 \tag{19}$$

A. H_∞ PERFORMANCE ANALYSIS

Theorem 1: For given positive scalars $\bar{\alpha} > 0$, $\beta_j > 0$ $(j = 0, 1, \ldots, q)$, $\bar{\delta} > 0$, $\gamma > 0$ and FD filter parameters $\mathbf{A}_f$, $\mathbf{B}_f$, $\mathbf{C}_f$ and $\mathbf{D}_f$, the filtering error system (12) is exponentially stable in the mean square sense with a guaranteed performance $\gamma > 0$ if there exist positive matrices $\mathbf{P} > 0$, $\mathbf{Q}_j > 0 (j = 1, 2, \ldots q)$ and scalar $\lambda_1 > 0$, such that the following LMI (20) holds.

$$\Omega = \begin{bmatrix} \Omega_{11} & * & * & * & * & * \\ \Omega_{21} & \Omega_{22} & * & * & * & * \\ \Omega_{31} & \Omega_{32} & \Omega_{33} & * & * & * \\ \Omega_{41} & \Omega_{42} & \Omega_{43} & \Omega_{44} & * & * \\ \Pi_{51} & \Pi_{52} & \Pi_{53} & \Pi_{54} & \Pi_{55} & * \\ \Omega_{61} & \Omega_{62} & \Omega_{63} & \Omega_{64} & \Omega_{65} & \Omega_{66} \end{bmatrix} < 0 \tag{20}$$

where

$$\rho_j = \sqrt{\beta_j}(j = 0, 1, \ldots q), \quad f_1 = \sqrt{\bar{\alpha}(1-\bar{\alpha})},$$
$$f_2 = \sqrt{\bar{\delta}(1-\bar{\delta})}, \Omega_{11} = \Pi_{11} + \bar{\mathbf{C}}_1^{\mathrm{T}}\bar{\mathbf{C}}_1 + \rho_0^2\bar{\mathbf{C}}_2^{\mathrm{T}}\bar{\mathbf{C}}_2,$$
$$\Omega_{21} = \Pi_{21} + \hat{\mathbf{C}}_d^{\mathrm{T}}\bar{\mathbf{C}}_1,$$
$$\Omega_{22} = \Pi_{22} + \hat{\mathbf{C}}_d^{\mathrm{T}}\hat{\mathbf{C}}_d + \rho_d^2\tilde{\mathbf{C}}_d^{\mathrm{T}}\tilde{\mathbf{C}}_d,$$
$$\Omega_{31} = \Pi_{31} + \tilde{\mathbf{D}}_0^{\mathrm{T}}\bar{\mathbf{C}}_1 + \rho_0^2\bar{\mathbf{D}}_2^{\mathrm{T}}\bar{\mathbf{C}}_2,$$
$$\Omega_{32} = \Pi_{32} + \tilde{\mathbf{D}}_0^{\mathrm{T}}\hat{\mathbf{C}}_d,$$
$$\Omega_{33} = \Pi_{33} + \tilde{\mathbf{D}}_0^{\mathrm{T}}\tilde{\mathbf{D}}_0 + \rho_0^2\bar{\mathbf{D}}_2^{\mathrm{T}}\bar{\mathbf{D}}_2,$$
$$\Omega_{41} = \Pi_{41} + \tilde{\mathbf{D}}_d^{\mathrm{T}}\bar{\mathbf{C}}_1,$$
$$\Omega_{42} = \Pi_{42} + \tilde{\mathbf{D}}_d^{\mathrm{T}}\hat{\mathbf{C}}_d + \rho_d^2\bar{\mathbf{D}}_2^{\mathrm{T}}\hat{\mathbf{C}}_d,$$
$$\Omega_{43} = \Pi_{43} + \tilde{\mathbf{D}}_d^{\mathrm{T}}\tilde{\mathbf{D}}_0,$$
$$\Omega_{44} = \Pi_{44} + \tilde{\mathbf{D}}_d^{\mathrm{T}}\tilde{\mathbf{D}}_d + \rho_d^2\bar{\mathbf{D}}_2^{\mathrm{T}}\bar{\mathbf{D}}_2,$$
$$\Omega_{61} = \bar{\mathbf{B}}_1^{\mathrm{T}}\mathbf{P}\bar{\mathbf{A}}_1 + \bar{\mathbf{D}}_1^{\mathrm{T}}\bar{\mathbf{C}}_1,$$
$$\Omega_{62} = \bar{\mathbf{B}}_1^{\mathrm{T}}\mathbf{P}\hat{\mathbf{A}}_d + \bar{\mathbf{D}}_1^{\mathrm{T}}\hat{\mathbf{C}}_d,$$
$$\Omega_{63} = \bar{\mathbf{B}}_1^{\mathrm{T}}\mathbf{P}\tilde{\mathbf{B}}_0 + \bar{\mathbf{D}}_1^{\mathrm{T}}\tilde{\mathbf{D}}_0,$$
$$\Omega_{64} = \bar{\mathbf{B}}_1^{\mathrm{T}}\mathbf{P}\tilde{\mathbf{B}}_d + \bar{\mathbf{D}}_1^{\mathrm{T}}\tilde{\mathbf{D}}_d,$$
$$\Omega_{65} = \bar{\mathbf{B}}_1^{\mathrm{T}}\mathbf{P}\tilde{\mathbf{E}}_1,$$
$$\Omega_{66} = \bar{\mathbf{B}}_1^{\mathrm{T}}\mathbf{P}\bar{\mathbf{B}}_1 + f_1^2\bar{\mathbf{B}}_2^{\mathrm{T}}\mathbf{P}\bar{\mathbf{B}}_2 + \bar{\mathbf{D}}_1^{\mathrm{T}}\bar{\mathbf{D}}_1 - \gamma^2\mathbf{I},$$
$$\Pi_{11} = \bar{\mathbf{A}}_1^{\mathrm{T}}\mathbf{P}\bar{\mathbf{A}}_1 + \rho_0^2\bar{\mathbf{A}}_2^{\mathrm{T}}\mathbf{P}\bar{\mathbf{A}}_2 + \sum_{j=1}^{q}\mathbf{Q}_j - \mathbf{P} - \lambda_1\mathbf{M}^{\mathrm{T}}\tilde{\mathbf{R}}_1\mathbf{M},$$
$$\Pi_{21} = \hat{\mathbf{A}}_d^{\mathrm{T}}\mathbf{P}\bar{\mathbf{A}}_1,$$
$$\Pi_{22} = \hat{\mathbf{A}}_d^{\mathrm{T}}\mathbf{P}\hat{\mathbf{A}}_d + \rho_d^2\tilde{\mathbf{A}}_d^{\mathrm{T}}\mathbf{P}\tilde{\mathbf{A}}_d - \mathbf{Q}_d,$$
$$\Pi_{31} = \tilde{\mathbf{B}}_0^{\mathrm{T}}\mathbf{P}\bar{\mathbf{A}}_1 + \rho_0^2\bar{\mathbf{B}}_3^{\mathrm{T}}\mathbf{P}\bar{\mathbf{A}}_2 + \bar{L}\check{C}_0, \quad \check{\mathbf{C}}_0 = \begin{bmatrix} \mathbf{C}_0 & \mathbf{0} \end{bmatrix},$$
$$\Pi_{32} = \tilde{\mathbf{B}}_0^{\mathrm{T}}\mathbf{P}\hat{\mathbf{A}}_d,$$
$$\Pi_{33} = \tilde{\mathbf{B}}_0^{\mathrm{T}}\mathbf{P}\tilde{\mathbf{B}}_0 + \rho_0^2\bar{\mathbf{B}}_3^{\mathrm{T}}\mathbf{P}\bar{\mathbf{B}}_3 - 2\mathbf{I},$$
$$\Pi_{41} = \tilde{\mathbf{B}}_d^{\mathrm{T}}\mathbf{P}\bar{\mathbf{A}}_1,$$
$$\Pi_{42} = \tilde{\mathbf{B}}_d^{\mathrm{T}}\mathbf{P}\hat{\mathbf{A}}_d + \rho_d^2\bar{\mathbf{B}}_3^{\mathrm{T}}\mathbf{P}\tilde{\mathbf{A}}_d + \bar{L}\check{C}_d,$$
$$\check{\mathbf{C}}_d = diag\left\{\check{\mathbf{C}}_1, \check{\mathbf{C}}_2, \cdots, \check{\mathbf{C}}_q\right\}, \check{\mathbf{C}}_j = \begin{bmatrix} \mathbf{C}_j & \mathbf{0} \end{bmatrix},$$
$$\Pi_{43} = \tilde{\mathbf{B}}_d^{\mathrm{T}}\mathbf{P}\tilde{\mathbf{B}}_0,$$
$$\Pi_{44} = \tilde{\mathbf{B}}_d^{\mathrm{T}}\mathbf{P}\tilde{\mathbf{B}}_d + \rho_d^2\bar{\mathbf{B}}_3^{\mathrm{T}}\mathbf{P}\bar{\mathbf{B}}_3 - 2\mathbf{I},$$
$$\Pi_{51} = \bar{\mathbf{E}}_1^{\mathrm{T}}\mathbf{P}\bar{\mathbf{A}}_1 - \lambda_1\tilde{\mathbf{R}}_2^{\mathrm{T}}\mathbf{M},$$
$$\Pi_{52} = \bar{\mathbf{E}}_1^{\mathrm{T}}\mathbf{P}\hat{\mathbf{A}}_d,$$
$$\Pi_{53} = \bar{\mathbf{E}}_1^{\mathrm{T}}\mathbf{P}\tilde{\mathbf{B}}_0,$$
$$\Pi_{54} = \bar{\mathbf{E}}_1^{\mathrm{T}}\mathbf{P}\tilde{\mathbf{B}}_d,$$
$$\Pi_{55} = \bar{\mathbf{E}}_1^{\mathrm{T}}\mathbf{P}\tilde{\mathbf{E}}_1 + f_2^2\bar{\mathbf{E}}_2^{\mathrm{T}}\mathbf{P}\tilde{\mathbf{E}}_2 - \lambda_1\mathbf{I},$$
$$\hat{\mathbf{A}}_d = [\hat{\mathbf{A}}_1, \hat{\mathbf{A}}_2, \cdots, \hat{\mathbf{A}}_q],$$
$$\tilde{\mathbf{B}}_d = [\tilde{\mathbf{B}}_1, \tilde{\mathbf{B}}_2, \cdots, \tilde{\mathbf{B}}_q],$$
$$\tilde{\mathbf{A}}_d = \mathrm{diag}\left\{\tilde{\mathbf{A}}_1, \tilde{\mathbf{A}}_2, \cdots, \tilde{\mathbf{A}}_q\right\},$$
$$\mathbf{Q}_d = diag\left\{\mathbf{Q}_1, \mathbf{Q}_2, \cdots, \mathbf{Q}_q\right\},$$
$$\beta_d = diag\left\{\beta_1, \beta_2, \cdots, \beta_q\right\},$$
$$\hat{\mathbf{C}}_d = [\hat{\mathbf{C}}_1, \hat{\mathbf{C}}_2, \cdots, \hat{\mathbf{C}}_q],$$
$$\tilde{\mathbf{D}}_d = [\tilde{\mathbf{D}}_1, \tilde{\mathbf{D}}_2, \cdots, \tilde{\mathbf{D}}_q],$$
$$\tilde{\mathbf{C}}_d = \mathrm{diag}\left\{\tilde{\mathbf{C}}_1, \tilde{\mathbf{C}}_2, \cdots, \tilde{\mathbf{C}}_q\right\},$$
$$\mathbf{M} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix}, \quad \tilde{\mathbf{R}}_1 = (\mathbf{R}_1^{\mathrm{T}}\mathbf{R}_2 + \mathbf{R}_2^{\mathrm{T}}\mathbf{R}_1)/2,$$
$$\tilde{\mathbf{R}}_2 = -(\mathbf{R}_1^{\mathrm{T}} + \mathbf{R}_2^{\mathrm{T}})/2.$$

Proof: We construct a Lyapunov functional as follows:

$$V(k) = \eta^{\mathrm{T}}(k)\mathbf{P}\eta(k) + \sum_{j=1}^{q}\sum_{i=k-d_j}^{k-1}\eta^{\mathrm{T}}(i)\mathbf{Q}_j\eta(i) \tag{21}$$

Let

$$\Theta(k) = [\eta^{\mathrm{T}}(k)\ \ \eta^{\mathrm{T}}(k-1)\cdots\eta^{\mathrm{T}}(0)]^{\mathrm{T}}$$

and

$$\Delta V(k) = \mathrm{E}\{V(k+1)\} - V(k),$$

Notice that when $i = 0, \cdots, q$,

$$E\left\{\left(I_{\{\tau(k)=d_i\}} - \beta_i\right)\left(I_{\{\tau(k)=d_j\}} - \beta_j\right)\right\} = \begin{cases}\beta_i(1-\beta_i), & i=j\\ -\beta_i\beta_j, & i\neq j\end{cases} \tag{22}$$

When $\theta(k) = 0$, taking mathematical expectation and we have

$$\begin{aligned}
&\mathrm{E}\{\Delta V(k)\}\\
&= \eta^{\mathrm{T}}(k)\left(\bar{\mathbf{A}}_1^{\mathrm{T}}\mathbf{P}\bar{\mathbf{A}}_1 + \rho_0^2\bar{\mathbf{A}}_2^{\mathrm{T}}\mathbf{P}\bar{\mathbf{A}}_2 + \sum_{j=1}^{q}\mathbf{Q}_j - \mathbf{P}\right)\eta(k)\\
&\quad + 2\boldsymbol{\phi}^{\mathrm{T}}(\mathbf{C}_0\mathbf{x}(k))\left(\tilde{\mathbf{B}}_0^{\mathrm{T}}\mathbf{P}\bar{\mathbf{A}}_1 + \rho_0^2\bar{\mathbf{B}}_3^{\mathrm{T}}\mathbf{P}\bar{\mathbf{A}}_2\right)\cdot\eta(k)\\
&\quad + 2\left(\sum_{j=1}^{q}\tilde{\mathbf{B}}_j\boldsymbol{\phi}\left(\mathbf{C}_j\mathbf{x}(k-d_j)\right)\right)^{\mathrm{T}}\mathbf{P}\bar{\mathbf{A}}_1\eta(k)\\
&\quad + 2\left(\sum_{j=1}^{q}\hat{\mathbf{A}}_j\eta(k-d_j)\right)^{\mathrm{T}}\mathbf{P}\bar{\mathbf{A}}_1\eta(k)\\
&\quad + \boldsymbol{\phi}^{\mathrm{T}}(\mathbf{C}_0\mathbf{x}(k))\left(\tilde{\mathbf{B}}_0^{\mathrm{T}}\mathbf{P}\tilde{\mathbf{B}}_0 + \rho_0^2\bar{\mathbf{B}}_3^{\mathrm{T}}\mathbf{P}\bar{\mathbf{B}}_3\right)\boldsymbol{\phi}(\mathbf{C}_0\mathbf{x}(k))\\
&\quad + 2\left(\sum_{j=1}^{q}\tilde{\mathbf{B}}_j\boldsymbol{\phi}\left(\mathbf{C}_j\mathbf{x}(k-d_j)\right)\right)^{\mathrm{T}}\mathbf{P}\\
&\quad \cdot\tilde{\mathbf{B}}_0\boldsymbol{\phi}(\mathbf{C}_0\mathbf{x}(k))\\
&\quad + 2\left(\sum_{j=1}^{q}\hat{\mathbf{A}}_j\eta(k-d_j)\right)^{\mathrm{T}}\mathbf{P}\tilde{\mathbf{B}}_0\boldsymbol{\phi}(\mathbf{C}_0\mathbf{x}(k))\\
&\quad + \left(\sum_{j=1}^{q}\tilde{\mathbf{B}}_j\boldsymbol{\phi}\left(\mathbf{C}_j\mathbf{x}(k-d_j)\right)\right)^{\mathrm{T}}\\
&\quad \cdot\mathbf{P}\left(\sum_{j=1}^{q}\tilde{\mathbf{B}}_j\boldsymbol{\phi}\left(\mathbf{C}_j\mathbf{x}(k-d_j)\right)\right)\\
&\quad + 2\left(\sum_{j=1}^{q}\hat{\mathbf{A}}_j\eta(k-d_j)\right)^{\mathrm{T}}\mathbf{P}\left(\sum_{j=1}^{q}\tilde{\mathbf{B}}_j\boldsymbol{\phi}\left(\mathbf{C}_j\mathbf{x}(k-d_j)\right)\right)\\
&\quad + \left(\sum_{j=1}^{q}\hat{\mathbf{A}}_j\eta(k-d_j)\right)^{\mathrm{T}}\mathbf{P}\left(\sum_{j=1}^{q}\hat{\mathbf{A}}_j\eta(k-d_j)\right)\\
&\quad - \sum_{j=1}^{q}\eta^{\mathrm{T}}(k-d_j)\mathbf{Q}_j\eta(k-d_j)\\
&\quad + \sum_{j=1}^{q}\beta_j\boldsymbol{\phi}^{\mathrm{T}}\left(\mathbf{C}_j\mathbf{x}(k-d_j)\right)\bar{\mathbf{B}}_3^{\mathrm{T}}\mathbf{P}\bar{\mathbf{B}}_3\boldsymbol{\phi}\left(\mathbf{C}_j\mathbf{x}(k-d_j)\right)\\
&\quad + 2\sum_{j=1}^{q}\beta_j\boldsymbol{\phi}^{\mathrm{T}}\left(\mathbf{C}_j\mathbf{x}(k-d_j)\right)\bar{\mathbf{B}}_3^{\mathrm{T}}\cdot\mathbf{P}\tilde{\mathbf{A}}_j\eta(k-d_j)\\
&\quad + \sum_{j=1}^{q}\beta_j\eta^{\mathrm{T}}(k-d_j)\tilde{\mathbf{A}}_j^{\mathrm{T}}\mathbf{P}\tilde{\mathbf{A}}_j\eta(k-d_j)\\
&\quad + 2\eta^{\mathrm{T}}(k)\bar{\mathbf{A}}_1^{\mathrm{T}}\mathbf{P}\bar{\mathbf{E}}_1\mathbf{g}(\mathbf{x}(k)) + 2\phi^{\mathrm{T}}(\mathbf{C}_0\mathbf{x}(k))\tilde{\mathbf{B}}_0^{\mathrm{T}}\mathbf{P}\bar{\mathbf{E}}_1\mathbf{g}(\mathbf{x}(k))\\
&\quad + 2\left(\sum_{j=1}^{q}\tilde{\mathbf{B}}_j\phi\left(\mathbf{C}_j\mathbf{x}(k-d_j)\right)\right)^{\mathrm{T}}\mathbf{P}\bar{\mathbf{E}}_1\mathbf{g}(\mathbf{x}(k))\\
&\quad + 2\left(\sum_{j=1}^{q}\hat{\mathbf{A}}_j\eta(k-d_j)\right)^{\mathrm{T}}\\
&\mathbf{P}\bar{\mathbf{E}}_1\mathbf{g}(\mathbf{x}(k)) + \mathbf{g}^{\mathrm{T}}(\mathbf{x}(k))\bar{\mathbf{E}}_1^{\mathrm{T}}\mathbf{P}\bar{\mathbf{E}}_1\mathbf{g}(\mathbf{x}(k))\\
&\quad + f_2^2\mathbf{g}^{\mathrm{T}}(\mathbf{x}(k))\bar{\mathbf{E}}_2^{\mathrm{T}}\mathbf{P}\bar{\mathbf{E}}_2\mathbf{g}(\mathbf{x}(k))\\
&\quad - \Bigg[\beta_0\bar{\mathbf{A}}_2\eta(k) + \beta_0\bar{\mathbf{B}}_3\boldsymbol{\phi}(\mathbf{C}_0\mathbf{x}(k))\\
&\quad + \sum_{j=1}^{q}\beta_j\bar{\mathbf{B}}_3\boldsymbol{\phi}\left(\mathbf{C}_j\mathbf{x}(k-d_j)\right)\\
&\quad + \sum_{j=1}^{q}\beta_j\tilde{\mathbf{A}}_j\eta(k-d_j)\Bigg]^{\mathrm{T}}\\
&\quad \cdot\mathbf{P}\Bigg[\beta_0\bar{\mathbf{A}}_2\eta(k) + \beta_0\bar{\mathbf{B}}_3\boldsymbol{\phi}(\mathbf{C}_0\mathbf{x}(k))\\
&\quad + \sum_{j=1}^{q}\beta_j\bar{\mathbf{B}}_3\boldsymbol{\phi}\left(\mathbf{C}_j\mathbf{x}(k-d_j)\right)\\
&\quad + \sum_{j=1}^{q}\beta_j\tilde{\mathbf{A}}_j\eta(k-d_j)\Bigg]\\
&\quad - \sum_{j=1}^{q}\eta^{\mathrm{T}}(k-d_j)\mathbf{Q}_j\eta(k-d_j)
\end{aligned} \tag{23}$$

From (4), we have

$$\begin{bmatrix}\eta(k)\\ \mathbf{g}(\mathbf{x}(k))\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\mathbf{M}^{\mathrm{T}}\tilde{\mathbf{R}}_1\mathbf{M} & *\\ \tilde{\mathbf{R}}_2^{\mathrm{T}}\mathbf{M} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\eta(k)\\ \mathbf{g}(\mathbf{x}(k))\end{bmatrix}\leq 0 \tag{24}$$

Then according to (9) and (10), we have

$$-2\boldsymbol{\phi}^{\mathrm{T}}(\mathbf{C}\mathbf{x}(k))\boldsymbol{\phi}(\mathbf{C}\mathbf{x}(k)) + 2\boldsymbol{\phi}^{\mathrm{T}}(\mathbf{C}\mathbf{x}(k))\bar{\mathbf{L}}\check{\mathbf{C}}_0\xi(k)\geq 0 \tag{25}$$

$$\begin{aligned}&-2\boldsymbol{\phi}^{\mathrm{T}}(\mathbf{C}\mathbf{x}(k-d_j))\boldsymbol{\phi}(\mathbf{C}\mathbf{x}(k-d_j))\\ &\quad + 2\boldsymbol{\phi}^{\mathrm{T}}(\mathbf{C}\mathbf{x}(k))\bar{\mathbf{L}}\check{\mathbf{C}}_j\eta(k-d_j)\geq 0,\quad j=1,2,\cdots,q\end{aligned} \tag{26}$$

According to Lemma 1, the combination of (23)-(25) results in

$$\mathrm{E}\{\Delta V(k)\} \leq \varsigma^{\mathrm{T}}(k)\Pi\varsigma(k) \tag{27}$$

where

$$\begin{aligned}\varsigma(k) &= [\eta^{\mathrm{T}}(k)\ \ \eta_\ell^{\mathrm{T}}\ \ \boldsymbol{\phi}^{\mathrm{T}}(\mathbf{C}_0\mathbf{x}(k))\ \ \boldsymbol{\phi}^{\mathrm{T}}(\mathbf{C}_1\mathbf{x}(k-d_1))\\ &\qquad \cdots\ \boldsymbol{\phi}^{\mathrm{T}}(\mathbf{C}_q\mathbf{x}(k-d_q))\ \ \mathbf{g}^{\mathrm{T}}(\mathbf{x}(k))\ ]^{\mathrm{T}},\\ \eta_\ell &= \left[\eta^{\mathrm{T}}(k-d_1)\ \ \eta^{\mathrm{T}}(k-d_2)\cdots\eta^{\mathrm{T}}(k-d_q)\right]^{\mathrm{T}},\\ \Pi &= \begin{bmatrix}\Pi_{11} & * & * & * & *\\ \Pi_{21} & \Pi_{22} & * & * & *\\ \Pi_{31} & \Pi_{32} & \Pi_{33} & * & *\\ \Pi_{41} & \Pi_{42} & \Pi_{43} & \Pi_{44} & *\\ \Pi_{51} & \Pi_{52} & \Pi_{53} & \Pi_{54} & \Pi_{55}\end{bmatrix}.\end{aligned}$$

From Lemma 1, if inequality (20) holds, for non-zero $\varsigma(k)$, $E\{\Delta V(k)\} < 0$ is satisfied. From reference [27], a positive scalar $\chi > 0$ can always be found such that inequality (28) holds.

$$\Pi = \begin{bmatrix}\Pi_{11} & * & * & * & *\\ \Pi_{21} & \Pi_{22} & * & * & *\\ \Pi_{31} & \Pi_{32} & \Pi_{33} & * & *\\ \Pi_{41} & \Pi_{42} & \Pi_{43} & \Pi_{44} & *\\ \Pi_{51} & \Pi_{52} & \Pi_{53} & \Pi_{54} & \Pi_{55}\end{bmatrix} < \begin{bmatrix}-\chi\mathbf{I} & * & * & * & *\\ \mathbf{0} & \mathbf{0} & * & * & *\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & * & *\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & *\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix} \tag{28}$$

Namely

$$E\{\Delta V(k)\} < -\chi\,\|\eta(k)\|^2 \tag{29}$$

Therefore, according to definition 1, we can conclude that **filtering error system** (12) is exponentially mean-square stable.

When $\theta(k) \neq 0$, define $\xi(k) = [\varsigma^{\mathrm{T}}(k)\ \ \theta^{\mathrm{T}}(k)]^{\mathrm{T}}$, the following inequality can be obtained:

$$\begin{aligned}&E\{V(k+1)(\Theta(k+1))\,|\Theta(k)\}\\ &\quad -V(k)(\Theta(k)){+}\mathrm{E}\left\{\mathbf{e}^{\mathrm{T}}(k)\mathbf{e}(k)\right\}{-}\gamma^2\theta^{\mathrm{T}}(k)\theta(k){\leq}\xi^{\mathrm{T}}(k)\Omega\xi(k)\end{aligned} \tag{30}$$

according to inequality (20), we have $\xi^{\mathrm{T}}(k)\Omega\xi(k) < 0$, so that

$$\begin{aligned}\sum_{k=0}^{\infty}\mathrm{E}\left\{\|\mathbf{e}(k)\|^2\right\} \leq \gamma^2\mathrm{E}\left\{\sum_{k=0}^{\infty}\|\theta(k)\|^2\right\} + \mathrm{E}\{\Delta V(0)\}\\ -\mathrm{E}\{\Delta V(\infty)\}\end{aligned} \tag{31}$$

Considering the zero initial condition and filtering error system (12) is exponentially mean-square stable, we have

$$\sum_{k=0}^{\infty}\mathrm{E}\left\{\|\mathbf{e}(k)\|^2\right\} \leq \gamma^2\mathrm{E}\left\{\sum_{k=0}^{\infty}\|\theta(k)\|^2\right\} \tag{32}$$

Thus, it is observed that an H_∞ performance constraint is achieved and the proof is completed.

B. FD FILTER DESIGN

Theorem 2: For given positive scalars $\bar{\alpha} > 0$, $\beta_j > 0 (j = 0, 1, \ldots, q)$, $\bar{\delta} > 0$ and $\gamma > 0$, the filtering error system (12) is exponentially stable in the mean square with a guaranteed performance $\gamma > 0$ if there exist positive matrices $\mathbf{P} > 0$, $\mathbf{Q}_j > 0 (j = 1, 2, \ldots, q)$, matrices $\mathbf{G}, \bar{\mathbf{A}}_f, \bar{\mathbf{B}}_f, \bar{\mathbf{C}}_f, \bar{\mathbf{D}}_f$ such that the following inequality holds:

$$\Psi = \begin{bmatrix}\Psi_1 & *\\ \Psi_2 & \Psi_3\end{bmatrix} < 0 \tag{33}$$

Furthermore, if (33) is feasible, the parameters of the desired FD filter will be given by

$$\begin{bmatrix}\mathbf{A}_f & \mathbf{B}_f\\ \mathbf{C}_f & \mathbf{D}_f\end{bmatrix} = \begin{bmatrix}\mathbf{G}_3^{-\mathrm{T}} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\bar{\mathbf{A}}_f & \bar{\mathbf{B}}_f\\ \bar{\mathbf{C}}_f & \bar{\mathbf{D}}_f\end{bmatrix} \tag{34}$$

where $\mathbf{P}$, $\mathbf{G}$, $\Psi_1 - -\Psi_{286}$, as shown at the bottom of the next page.

Proof: It is observed that (20) can be rewritten as follows:

$$\begin{aligned}&\Psi_1 + \begin{bmatrix}\bar{\mathbf{C}}_1^{\mathrm{T}}\\ \hat{\mathbf{C}}_d^{\mathrm{T}}\\ \tilde{\mathbf{D}}_0^{\mathrm{T}}\\ \tilde{\mathbf{D}}_d^{\mathrm{T}}\\ \mathbf{0}\\ \bar{\mathbf{D}}_1^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\bar{\mathbf{C}}_1^{\mathrm{T}}\\ \hat{\mathbf{C}}_d^{\mathrm{T}}\\ \tilde{\mathbf{D}}_0^{\mathrm{T}}\\ \tilde{\mathbf{D}}_d^{\mathrm{T}}\\ \mathbf{0}\\ \bar{\mathbf{D}}_1^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}} + \begin{bmatrix}\rho_0\bar{\mathbf{C}}_2^{\mathrm{T}}\\ \mathbf{0}\\ \rho_0\bar{\mathbf{D}}_2^{\mathrm{T}}\\ \mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\end{bmatrix}\begin{bmatrix}\rho_0\bar{\mathbf{C}}_2^{\mathrm{T}}\\ \mathbf{0}\\ \rho_0\bar{\mathbf{D}}_2^{\mathrm{T}}\\ \mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\end{bmatrix}^{\mathrm{T}}\\ &+ \begin{bmatrix}\mathbf{0}\\ \rho_d\tilde{\mathbf{C}}_d^{\mathrm{T}}\\ \mathbf{0}\\ \rho_d\bar{\mathbf{D}}_2^{\mathrm{T}}\\ \mathbf{0}\\ \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{0}\\ \rho_d\tilde{\mathbf{C}}_d^{\mathrm{T}}\\ \mathbf{0}\\ \rho_d\bar{\mathbf{D}}_2^{\mathrm{T}}\\ \mathbf{0}\\ \mathbf{0}\end{bmatrix}^{\mathrm{T}}\\ &+ \begin{bmatrix}\bar{\mathbf{A}}_1^{\mathrm{T}}\\ \hat{\mathbf{A}}_d^{\mathrm{T}}\\ \tilde{\mathbf{B}}_0^{\mathrm{T}}\\ \tilde{\mathbf{B}}_d^{\mathrm{T}}\\ \bar{\mathbf{E}}_1^{\mathrm{T}}\\ \bar{\mathbf{B}}_1^{\mathrm{T}}\end{bmatrix}\mathbf{P}\begin{bmatrix}\bar{\mathbf{A}}_1^{\mathrm{T}}\\ \hat{\mathbf{A}}_d^{\mathrm{T}}\\ \tilde{\mathbf{B}}_0^{\mathrm{T}}\\ \tilde{\mathbf{B}}_d^{\mathrm{T}}\\ \bar{\mathbf{E}}_1^{\mathrm{T}}\\ \bar{\mathbf{B}}_1^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}} + \begin{bmatrix}\rho_0\bar{\mathbf{A}}_2^{\mathrm{T}}\\ \mathbf{0}\\ \rho_0\bar{\mathbf{B}}_3^{\mathrm{T}}\\ \mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\end{bmatrix}\mathbf{P}\begin{bmatrix}\rho_0\bar{\mathbf{A}}_2^{\mathrm{T}}\\ \mathbf{0}\\ \rho_0\bar{\mathbf{B}}_3^{\mathrm{T}}\\ \mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\end{bmatrix}^{\mathrm{T}}\\ &+ \begin{bmatrix}\mathbf{0}\\ \rho_d\tilde{\mathbf{A}}_d^{\mathrm{T}}\\ \mathbf{0}\\ \rho_d\bar{\mathbf{B}}_3^{\mathrm{T}}\\ \mathbf{0}\\ \mathbf{0}\end{bmatrix}\mathbf{P}\begin{bmatrix}\mathbf{0}\\ \rho_d\tilde{\mathbf{A}}_d^{\mathrm{T}}\\ \mathbf{0}\\ \rho_d\bar{\mathbf{B}}_3^{\mathrm{T}}\\ \mathbf{0}\\ \mathbf{0}\end{bmatrix}^{\mathrm{T}}\\ &+ \begin{bmatrix}\mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\\ f_2\bar{\mathbf{E}}_2^{\mathrm{T}}\\ \mathbf{0}\end{bmatrix}\mathbf{P}\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\\ f_2\bar{\mathbf{E}}_2^{\mathrm{T}}\\ \mathbf{0}\end{bmatrix}^{\mathrm{T}} + \begin{bmatrix}\mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\\ f_1\bar{\mathbf{B}}_2^{\mathrm{T}}\end{bmatrix}\mathbf{P}\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\\ f_1\bar{\mathbf{B}}_2^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}} < 0\end{aligned} \tag{35}$$

$$
\begin{aligned}
\mathbf{P} &= \begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_2 \\ \mathbf{P}_3 & \mathbf{P}_3 \end{bmatrix}, \\
\mathbf{G} &= \begin{bmatrix} \mathbf{G}_1 & \mathbf{G}_2 \\ \mathbf{G}_3 & \mathbf{G}_3 \end{bmatrix}, \\
\Psi_1 &= \begin{bmatrix} -\mathbf{P} + \sum_{j=1}^{q} \mathbf{Q}_j - \lambda_1 \mathbf{M}^{\mathrm{T}} \tilde{\mathbf{R}}_1 M & * & * & * & * & * \\ \mathbf{0} & -\mathbf{Q}_d & * & * & * & * \\ L\bar{\breve{\mathbf{C}}}_0 & \mathbf{0} & -2\mathbf{I} & * & * & * \\ \mathbf{0} & L\bar{\breve{\mathbf{C}}}_d & \mathbf{0} & -2\mathbf{I} & * & * \\ -\lambda_1 \tilde{\mathbf{R}}_2^{\mathrm{T}} \mathbf{M} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\lambda_1 \mathbf{I} & * \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\gamma^2 \mathbf{I} \end{bmatrix}, \quad
\Psi_2 = \begin{bmatrix} \Psi_{211} & \Psi_{212} & \Psi_{213} & \Psi_{214} & \mathbf{0} & \Psi_{216} \\ \Psi_{221} & \mathbf{0} & \Psi_{223} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Psi_{232} & \mathbf{0} & \Psi_{234} & \mathbf{0} & \mathbf{0} \\ \Psi_{241} & \Psi_{242} & \Psi_{243} & \Psi_{244} & \Psi_{245} & \Psi_{246} \\ \Psi_{251} & \mathbf{0} & \Psi_{253} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Psi_{262} & \mathbf{0} & \Psi_{264} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \Psi_{275} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \Psi_{286} \end{bmatrix}, \\
\Psi_3 &= \mathrm{diag}\{ -\mathbf{I}, -\mathbf{I}, -\mathbf{I}, \mathbf{P} - \mathbf{G} - \mathbf{G}^{\mathrm{T}}, \mathbf{P} - \mathbf{G} - \mathbf{G}^{\mathrm{T}}, \\
&\quad \mathbf{P} - \mathbf{G} - \mathbf{G}^{\mathrm{T}}, \mathbf{P} - \mathbf{G} - \mathbf{G}^{\mathrm{T}}, \mathbf{P} - \mathbf{G} - \mathbf{G}^{\mathrm{T}}\}, \\
\Psi_{211} &= \begin{bmatrix} \beta_0 \bar{\mathbf{D}}_f \mathbf{L}_1 \mathbf{C}_0 & \bar{\mathbf{C}}_f \end{bmatrix}, \\
\Psi_{212} &= [\hat{\mathbf{C}}_1^1, \hat{\mathbf{C}}_2^1, \cdots, \hat{\mathbf{C}}_q^1], \hat{\mathbf{C}}_j^1 = \begin{bmatrix} \beta_j \bar{\mathbf{D}}_f \mathbf{L}_1 \mathbf{C}_j & \mathbf{0} \end{bmatrix} \quad (j = 1, 2, \cdots, q), \\
\Psi_{213} &= \beta_0 \bar{\mathbf{D}}_f, \quad \Psi_{214} = [\tilde{\mathbf{D}}_1^1, \tilde{\mathbf{D}}_2^1, \cdots, \tilde{\mathbf{D}}_q^1], \ \tilde{\mathbf{D}}_j^1 = \beta_j \bar{\mathbf{D}}_f \ (j = 1, 2, \cdots, q), \\
\Psi_{216} &= \begin{bmatrix} \mathbf{0} & \bar{\mathbf{D}}_f \mathbf{D}_w & -\mathbf{I} \end{bmatrix}, \\
\Psi_{221} &= \begin{bmatrix} \rho_0 \bar{\mathbf{D}}_f \mathbf{L}_1 \mathbf{C}_0 & \mathbf{0} \end{bmatrix}, \quad \Psi_{223} = \rho_0 \bar{\mathbf{D}}_f, \\
\Psi_{232} &= diag\left\{ \tilde{\mathbf{C}}_1^1, \tilde{\mathbf{C}}_2^1, \cdots, \tilde{\mathbf{C}}_q^1 \right\}, \quad \tilde{\mathbf{C}}_j^1 = \begin{bmatrix} \rho_j \bar{\mathbf{D}}_f \mathbf{L}_1 \mathbf{C}_j & \mathbf{0} \end{bmatrix} \quad (j = 1, 2, \cdots, q), \\
\Psi_{234} &= diag\left\{ \rho_1 \bar{\mathbf{D}}_f, \rho_2 \bar{\mathbf{D}}_f, \cdots, \rho_q \bar{\mathbf{D}}_f \right\}, \\
\Psi_{241} &= \begin{bmatrix} \mathbf{G}_1^{\mathrm{T}} \mathbf{A} + \beta_0 \bar{\mathbf{B}}_f \mathbf{L}_1 \mathbf{C}_0 & \bar{\mathbf{A}}_f \\ \mathbf{G}_2^{\mathrm{T}} \mathbf{A} + \beta_0 \bar{\mathbf{B}}_f \mathbf{L}_1 \mathbf{C}_0 & \bar{\mathbf{A}}_f \end{bmatrix}, \\
\Psi_{242} &= \left[\hat{\mathbf{A}}_1^1, \hat{\mathbf{A}}_2^1, \cdots, \hat{\mathbf{A}}_q^1 \right], \quad \hat{\mathbf{A}}_j^1 = \begin{bmatrix} \beta_j \bar{\mathbf{B}}_f \mathbf{L}_1 \mathbf{C}_j & \mathbf{0} \\ \beta_j \bar{\mathbf{B}}_f \mathbf{L}_1 \mathbf{C}_j & \mathbf{0} \end{bmatrix} \quad (j = 1, 2, \cdots, q), \\
\Psi_{243} &= \begin{bmatrix} \beta_0 \bar{\mathbf{B}}_f \\ \beta_0 \bar{\mathbf{B}}_f \end{bmatrix}, \\
\Psi_{244} &= \left[\tilde{\mathbf{B}}_1^1, \tilde{\mathbf{B}}_2^1, \cdots, \tilde{\mathbf{B}}_q^1 \right], \quad \tilde{\mathbf{B}}_j^1 = \begin{bmatrix} \beta_j \bar{\mathbf{B}}_f \\ \beta_j \bar{\mathbf{B}}_f \end{bmatrix} \quad (j = 1, 2, \cdots, q), \\
\Psi_{245} &= \begin{bmatrix} \delta \mathbf{G}_1^{\mathrm{T}} \\ \delta \mathbf{G}_2^{\mathrm{T}} \end{bmatrix}, \\
\Psi_{246} &= \begin{bmatrix} \mathbf{G}_1^{\mathrm{T}} \mathbf{B}_u & \mathbf{G}_1^{\mathrm{T}} \mathbf{B}_w + \bar{\mathbf{B}}_f \mathbf{D}_w & \bar{\alpha} \mathbf{G}_1^{\mathrm{T}} \mathbf{E} \\ \mathbf{G}_2^{\mathrm{T}} \mathbf{B}_u & \mathbf{G}_2^{\mathrm{T}} \mathbf{B}_w + \bar{\mathbf{B}}_f \mathbf{D}_w & \bar{\alpha} \mathbf{G}_2^{\mathrm{T}} \mathbf{E} \end{bmatrix}, \\
\Psi_{251} &= \begin{bmatrix} \rho_0 \bar{\mathbf{B}}_f \mathbf{L}_1 \mathbf{C}_0 & \mathbf{0} \\ \rho_0 \bar{\mathbf{B}}_f \mathbf{L}_1 \mathbf{C}_0 & \mathbf{0} \end{bmatrix}, \\
\Psi_{253} &= \begin{bmatrix} \rho_0 \bar{\mathbf{B}}_f \\ \rho_0 \bar{\mathbf{B}}_f \end{bmatrix}, \\
\Psi_{262} &= diag\left\{ \tilde{\mathbf{A}}_1^1, \tilde{\mathbf{A}}_2^1, \cdots, \tilde{\mathbf{A}}_q^1 \right\}, \quad \tilde{\mathbf{A}}_j^1 = \begin{bmatrix} \rho_j \bar{\mathbf{B}}_f \mathbf{L}_1 \mathbf{C}_j & \mathbf{0} \\ \rho_j \bar{\mathbf{B}}_f \mathbf{L}_1 \mathbf{C}_j & \mathbf{0} \end{bmatrix} \quad (j = 1, 2, \cdots, q), \\
\Psi_{264} &= diag\left\{ \bar{\mathbf{B}}_1^1, \bar{\mathbf{B}}_2^1, \cdots, \bar{\mathbf{B}}_q^1 \right\}, \bar{\mathbf{B}}_j^1 = \begin{bmatrix} \rho_j \bar{\mathbf{B}}_f \\ \rho_j \bar{\mathbf{B}}_f \end{bmatrix} \quad (j = 1, 2, \cdots, q), \\
\Psi_{275} &= \begin{bmatrix} f_2 \mathbf{G}_1^{\mathrm{T}} \\ f_2 \mathbf{G}_2^{\mathrm{T}} \end{bmatrix} \\
\Psi_{286} &= \begin{bmatrix} \mathbf{0} & \mathbf{0} & f_1 \mathbf{G}_1^{\mathrm{T}} \mathbf{E} \\ \mathbf{0} & \mathbf{0} & f_1 \mathbf{G}_2^{\mathrm{T}} \mathbf{E} \end{bmatrix}.
\end{aligned}
$$

Based on Lemma 1, we can convert inequality (35) into inequality (36).

$$\Xi = \begin{bmatrix} \Psi_1 & * \\ \Xi_2 & \Xi_3 \end{bmatrix} < 0 \tag{36}$$

where

$$\Xi_2 = \begin{bmatrix} \bar{\mathbf{C}}_1 & \hat{\mathbf{C}}_d & \tilde{\mathbf{D}}_0 & \tilde{\mathbf{D}}_d & \mathbf{0} & \bar{\mathbf{D}}_1 \\ \rho_0\bar{\mathbf{C}}_2 & \mathbf{0} & \rho_0\bar{\mathbf{D}}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \rho_d\tilde{\mathbf{C}}_d & \mathbf{0} & \rho_d\bar{\mathbf{D}}_2 & \mathbf{0} & \mathbf{0} \\ \bar{\mathbf{A}}_1 & \hat{\mathbf{A}}_d & \tilde{\mathbf{B}}_0 & \tilde{\mathbf{B}}_d & \bar{\mathbf{E}}_1 & \bar{\mathbf{B}}_1 \\ \rho_0\bar{\mathbf{A}}_2 & \mathbf{0} & \rho_0\bar{\mathbf{B}}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \rho_d\tilde{\mathbf{A}}_d & \mathbf{0} & \rho_d\bar{\mathbf{B}}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & f_2\bar{\mathbf{E}}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & f_1\bar{\mathbf{B}}_2 \end{bmatrix}$$

$$\Xi_3 = \operatorname{diag}\left\{-\mathbf{I}, -\mathbf{I}, -\mathbf{I}, -\mathbf{P}^{-1}, -\mathbf{P}^{-1}, -\mathbf{P}^{-1}, -\mathbf{P}^{-1}, -\mathbf{P}^{-1}\right\}$$

According to Lemma 2, inequality (36) holds if and only if there exists a real matrix $\mathbf{G}$, such that inequality (37) holds

$$\Gamma = \begin{bmatrix} \Psi_1 & * \\ \Gamma_2 & \Psi_3 \end{bmatrix} < 0 \tag{37}$$

where Γ_2, as shown at the bottom of this page.

Partition $\mathbf{P}$ and $\mathbf{G}$, we have:

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_2 \\ \mathbf{P}_2^{\mathrm{T}} & \mathbf{P}_3 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \mathbf{G}_1 & \mathbf{G}_2 \\ \mathbf{G}_3 & \mathbf{G}_3 \end{bmatrix} \tag{38}$$

Let

$$\bar{\mathbf{A}}_f = \mathbf{G}_3^{\mathrm{T}}\mathbf{A}_f, \quad \bar{\mathbf{B}}_f = \mathbf{G}_3^{\mathrm{T}}\mathbf{B}_f, \; bar\mathbf{C}_f = \mathbf{C}_f, \; \bar{\mathbf{D}}_f = \mathbf{D}_f \tag{39}$$

It is easy to find that (37) is equivalent to (33). Besides, from condition (34), we know that $\mathbf{G}_3$ is invertible, then the parameters of FD filter can be obtained from (39).

The proof is completed.

Problem 1: The best performance scalar $\gamma_{\min} = \sqrt{\gamma^2}$ for the filtering error system (12) and the optimal FD filter can be obtained by solving the following convex optimization problem.

$$\begin{cases} \text{Minimize}: \gamma^2 \\ \text{s.t.}: \; (33) \end{cases} \tag{40}$$

TABLE 1. $\gamma_{\min}$ for different $\bar{\alpha}$ and $\bar{\delta}$.

$\gamma_{\min}$	$\bar{\alpha}=0.2$	$\bar{\alpha}=0.4$	$\bar{\alpha}=0.5$	$\bar{\alpha}=0.7$	$\bar{\alpha}=0.9$
$\bar{\delta}=0.1$	1.3304	1.3306	1.3308	1.3312	1.3318
$\bar{\delta}=0.3$	1.4987	1.5002	1.5015	1.5053	1.5112
$\bar{\delta}=0.5$	1.7112	1.7155	1.7190	1.7299	1.7474
$\bar{\delta}=0.8$	2.1290	2.1401	2.1494	2.1774	2.2218
$\bar{\delta}=0.9$	2.3005	2.3005	2.3266	2.3622	2.4179

IV. NUMERICAL EXAMPLE

In this section, we use the parameters of the internet-based three-tank system in [23]:

$$\mathbf{A} = \begin{bmatrix} 0.1977 & 0 & 0.0023 \\ 0 & 0.1956 & 0.0022 \\ 0.0023 & 0.0022 & 0.1955 \end{bmatrix},$$

$$\mathbf{B}_u = \begin{bmatrix} 1.2154 & 0 \\ 0 & 1.1984 \\ 0.0183 & 0.0183 \end{bmatrix}, \quad \mathbf{B}_w = \begin{bmatrix} 0 \\ 8.1984 \\ 0.0183 \end{bmatrix},$$

$$\mathbf{E} = \begin{bmatrix} 0.0183 \\ 0.0183 \\ 8.1972 \end{bmatrix}, q = 2, d_1 = 1, d_2 = 2,$$

$$\mathbf{C}_0 = \mathbf{C}_1 = \mathbf{C}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{D}_w = \begin{bmatrix} 0.2 \\ -0.1 \end{bmatrix},$$

$$\mathbf{L}_1 = \begin{bmatrix} 0.6 & 0 \\ 0 & 0.7 \end{bmatrix}, \quad \mathbf{L}_2 = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.8 \end{bmatrix},$$

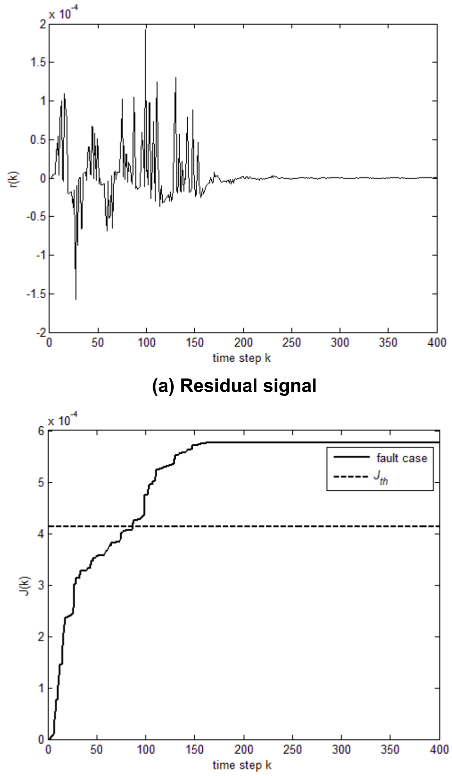
$$u(k) = \mathbf{K}x(k), \quad \mathbf{K} = \begin{bmatrix} 0.1 & 0.2 & 0.3 \\ 0.3 & 0.2 & -0.3 \end{bmatrix},$$

$$\mathbf{R}_1 = \begin{bmatrix} 0.2 & 0.1 & 0.2 \\ 0.1 & 0.3 & 0 \\ -0.1 & 0.1 & 0.3 \end{bmatrix}, \quad \mathbf{R}_2 = \begin{bmatrix} -0.2 & 0.1 & 0 \\ 0.1 & -0.3 & -0.1 \\ -0.1 & 0 & -0.3 \end{bmatrix}.$$

Let $\beta_0 = 0.1, \beta_1 = 0.2, \beta_2 = 0.3$, the H_∞ performance index will be obtained which can be seen from Table 1 under different probabilities of faults and nonlinearity by solving (40). It can be clearly seen that when the probability of faults $\bar{\alpha}$ or probability of the nonlinearity $\bar{\delta}$ increases, the corresponding H_∞ performance index $\gamma_{\min}$ increases. Namely, the disturbance attenuation performance deteriorates.

First, by MATLAB LMI toolbox and let $\bar{\alpha} = 0.3, \beta_0 = 0.1, \beta_1 = 0.2, \beta_2 = 0.3, \bar{\delta} = 0.3$, from system (12), we can obtain the H_∞ performance index $\gamma_{\min} = 1.4993$ and the optimal

$$\Gamma_2 = \begin{bmatrix} \bar{\mathbf{C}}_1 & \hat{\mathbf{C}}_d & \tilde{\mathbf{D}}_0 & \tilde{\mathbf{D}}_d & \mathbf{0} & \bar{\mathbf{D}}_1 \\ \rho_0\bar{\mathbf{C}}_2 & \mathbf{0} & \rho_0\bar{\mathbf{D}}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \rho_d\tilde{\mathbf{C}}_d & \mathbf{0} & \rho_d\bar{\mathbf{D}}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{G}^{\mathrm{T}}\bar{\mathbf{A}}_1 & \mathbf{G}^{\mathrm{T}}\hat{\mathbf{A}}_d & \mathbf{G}^{\mathrm{T}}\tilde{\mathbf{B}}_0 & \mathbf{G}^{\mathrm{T}}\tilde{\mathbf{B}}_d & \mathbf{G}^{\mathrm{T}}\bar{\mathbf{E}}_1 & \mathbf{G}^{\mathrm{T}}\bar{\mathbf{B}}_1 \\ \rho_0\mathbf{G}^{\mathrm{T}}\bar{\mathbf{A}}_2 & \mathbf{0} & \rho_0\mathbf{G}^{\mathrm{T}}\bar{\mathbf{B}}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \rho_d\mathbf{G}^{\mathrm{T}}\tilde{\mathbf{A}}_d & \mathbf{0} & \rho_d\mathbf{G}^{\mathrm{T}}\bar{\mathbf{B}}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & f_2\mathbf{G}^{\mathrm{T}}\bar{\mathbf{E}}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & f_1\mathbf{G}^{\mathrm{T}}\bar{\mathbf{B}}_2 \end{bmatrix}$$



(a) Residual signal

(b) Residual evaluation function

FIGURE 1. Graph of residual signal and residual evaluation function.

FD filter parameters which are shown as follows:

$$\mathbf{A}_f = \begin{bmatrix} 0.1870 & 0.0893 & 0.0173 \\ 0.2729 & 0.8966 & 0.0356 \\ -0.0075 & 0.0311 & 0.1118 \end{bmatrix}$$

$$\mathbf{B}_f = \begin{bmatrix} -0.0076 & 0.0011 \\ 0.0034 & -0.0006 \\ 0.0003 & -0.0000 \end{bmatrix},$$

$$\mathbf{C}_f = \begin{bmatrix} -0.0010 & -0.0023 & -0.0015 \end{bmatrix},$$

$$\mathbf{D}_f = 1 \times 10^{-4} \times \begin{bmatrix} 0.3150 & -0.0492 \end{bmatrix}.$$

Let the nonlinear function $\mathbf{g}(\mathbf{x}(k))$=0.4 $\sin(\mathbf{x}(k))$ and the sensor nonlinearity be as follows

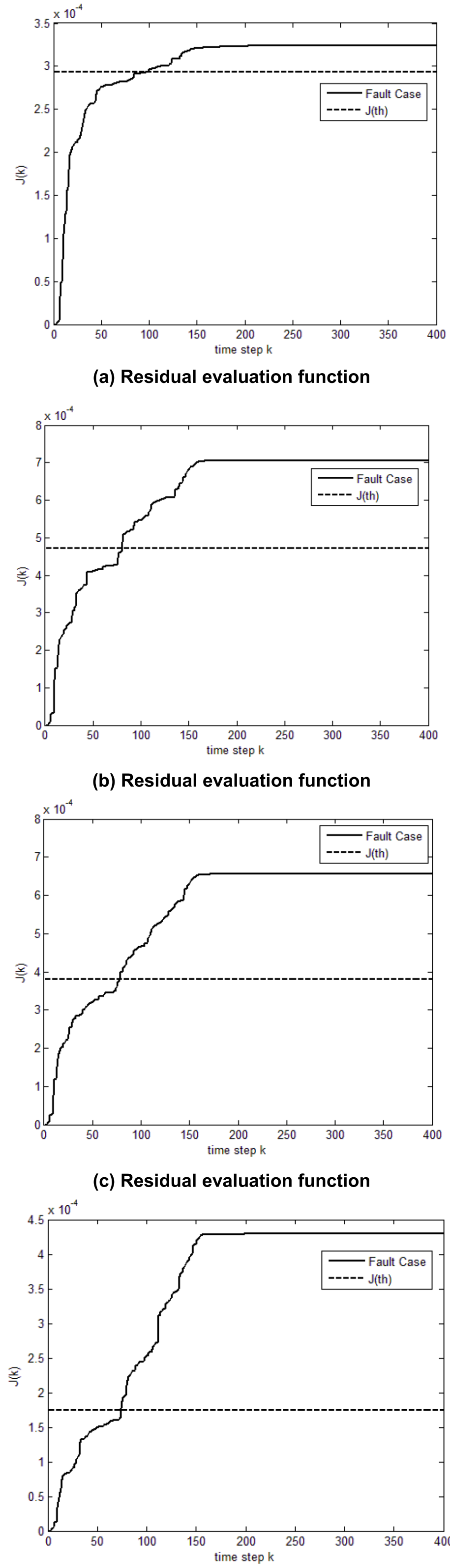
$$\boldsymbol{\phi}(\mathbf{C}_0\mathbf{x}(k)) = \frac{\mathbf{L}_1 + \mathbf{L}_2}{2}\mathbf{C}_0\mathbf{x}(k) + \frac{\mathbf{L}_2 - \mathbf{L}_1}{2}\sin(\mathbf{C}_0\mathbf{x}(k))$$

$$\boldsymbol{\phi}(\mathbf{C}_1\mathbf{x}(k-d_1)) = \frac{\mathbf{L}_1 + \mathbf{L}_2}{2}\mathbf{C}_1\mathbf{x}(k-d_1) + \frac{\mathbf{L}_2 - \mathbf{L}_1}{2}\sin(\mathbf{C}_1\mathbf{x}(k-d_1))$$

$$\boldsymbol{\phi}(\mathbf{C}_2\mathbf{x}(k-d_2)) = \frac{\mathbf{L}_1 + \mathbf{L}_2}{2}\mathbf{C}_2\mathbf{x}(k-d_2) + \frac{\mathbf{L}_2 - \mathbf{L}_1}{2}\sin(\mathbf{C}_2\mathbf{x}(k-d_2))$$

The disturbance input and the fault signal are assumed to be

$$\mathbf{w}(k) = e^{-0.02k}\sin(0.2k),$$



(a) Residual evaluation function

(b) Residual evaluation function

(c) Residual evaluation function

(d) Residual evaluation function

FIGURE 2. Graphs of residual evaluation function.

$$\mathbf{f}(k) = \begin{cases} 0.6 + 0.2\sin(k), & 70 \le k \le 150 \\ 0, & else \end{cases}$$

TABLE 2. Threshold J_{th} and time steps for different $\bar{\alpha}$.

	$\bar{\alpha}=0.1$	$\bar{\alpha}=0.3$	$\bar{\alpha}=0.5$	$\bar{\alpha}=0.7$	$\bar{\alpha}=1$
J_{th}	2.8591×10^{-4}	4.1388×10^{-4}	4.7181×10^{-4}	3.8126×10^{-4}	1.7477×10^{-4}
Time steps	24	15	11	8	4

By selecting the initial states as $\varphi(k) = [\,0\ 0\ 0\,]^{\mathrm{T}}, \bar{\alpha} = 0.3$, the residual signal $\mathbf{r}(k)$ and evolution of residual evaluation function $J(k)$ are shown in Figure 1.

Set the length of the evaluation time as $L = 400$, the threshold is selected as follows:

$$J_{th} = \sup_{\mathbf{f}} = 0\mathrm{E}\left\{\left[\sum_{k=0}^{400} \mathbf{r}^{\mathrm{T}}(k)\,\mathbf{r}(k)\right]^{1/2}\right\}$$

After 200 simulations with $\mathbf{f}(k) = 0$, the average value $J_{th} = 4.1388 \times 10^{-4}$ is obtained and it is considered as the final threshold. Compared with the residual evaluation function (15), it is obvious that

$$4.0704 \times 10^{-4} = J(86) < J_{th} < J(87) = 4.2024 \times 10^{-4}.$$

It is noted that when fault occurs at time $k = 70$, the occurrence of fault can be detected within 17 time steps by the FD filter designed.

Under the conditions $\beta_0 = 0.1, \beta_1 = 0.2, \beta_2 = 0.5, \bar{\delta} = 0.3$, for different values of $\bar{\alpha}$, the values of J_{th} and time steps that fault can be detected are listed in Table 2, and for these different values of $\bar{\alpha}$, which are 0.1, 0.5, 0.7 and 1 respectively, the evolutions of residual evaluation function are shown in Figure 2, respectively.

It can be seen from Table 2 that the occurrence of faults can be effectively detected. The greater probability of a fault happens in the networked system, the shorter time is required to detect the fault.

V. CONCLUSION

The FD problem for nonlinear NCSs along with random packet losses, random delays, sensor saturation and randomly occurred faults is studied in this paper. A unified model is constructed to describe random packet losses and random delays simultaneously. Both sector-bound condition and two independent Bernoulli random sequences are used to describe the occurrence of the fault and nonlinearity. The design problem of FD filter can be turned into an H_∞ filtering problem. Sufficient conditions are derived by Lyapunov stability theory. An example is used to verify the effectiveness of the method proposed in this paper. Furthermore, some interesting research topics are also listed later which will be studied in future. First, sensor saturation phenomenon may also occur randomly, under such condition, fault detection problem for networked control systems will be more practically with actual situations. Second, more general nonlinear systems should be studied, and nonlinear networked systems with randomly occurring sensor saturation will be our next step research topic.

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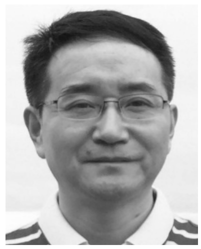

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