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# Linear Codes Over a Non-Chain Ring and the MacWilliams Identities

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**ABSTRACT** This paper is concerned with the linear codes over the non-chain ring  $R = \mathbb{F}_2[v]/\langle v^4 - v \rangle$ . First, several weight enumerators over  $R$  are defined. Then the MacWilliams identity is obtained, which can establish an important relation respect to the complete weight enumerators. Meanwhile, the symmetric weight enumerators between linear code and its dual over  $R$  are established by the Gray map from  $R^n$  to  $\mathbb{F}_2^{4n}$ . Finally, several examples are given to illustrate our main results and some open problems are also proposed.

**INDEX TERMS** Generator matrix, linear code, MacWilliams identity, weight enumerator.

## I. INTRODUCTION

Codes over rings are very important in algebraic coding theory and applications to combined coding and modulation. So many researchers pay much attention to the study of codes over rings, and a lot results are obtained [1, 3, 11]. In [4], Dougherty *et al.* determined the Type II codes, where all self-dual codes over the ring were classified for length up to 8. Then Betsumiya [2] and Qian [9] generalized the results to the ring  $\mathbb{F}_2^m + u\mathbb{F}_2^m$  and  $\mathbb{F}_2 + u\mathbb{F}_2 + u^2\mathbb{F}_2$ , respectively. In addition, it is worth noting that self-duality is preserved under Gray maps.

Furthermore,  $(1 + u)$  constacyclic and cyclic codes over the ring  $\mathbb{F}_2 + u\mathbb{F}_2$  were introduced in [10]. It was showed that every Gray image of a linear cyclic code over  $\mathbb{F}_2 + u\mathbb{F}_2$  is equivalent to a cyclic code with odd length. In [16], Zhu, Wang and Shi investigated the structure and properties of cyclic codes over ring  $\mathbb{F}_2 + u\mathbb{F}_2$ . Cyclic codes over the ring were principally generated and generator polynomial was obtained in [16].

The weight distribution of a code with length  $n$  specifies the number of codewords of each possible weight  $0, 1, \dots, n$ . When weight distribution does not uniquely determine a code in general, it plays an important role in calculating the error rate of decoding. As we know the weight distribution of  $C$

is uniquely determined by the weight distribution of  $C^\perp$  and vice versa. The linear relation between the weight distribution of  $C$  and  $C^\perp$  was first developed by MacWilliams [8]. Since then there have been many different weight enumerators for codes over finite fields and finite rings with respect to the MacWilliams identity.

The MacWilliams identities of various weight enumerators for linear codes over  $\mathbb{Z}_4$  were studied in [7]. The Lee and Euclidean weight enumerators for linear codes over the ring  $\mathbb{Z}_\ell$  were discussed in [11] and [15], sufficient conditions for the existence of the MacWilliams type identities with respect to the Lee and Euclidean weight enumerators for linear codes were given. Meanwhile, the MacWilliams identities of various weight enumerators for linear codes over non-principal rings were also widely studied. For instance, Shi *et al.* determined the MacWilliams identities for linear codes with respect to Lee weight enumerator over the rings  $\mathbb{F}_2 + v\mathbb{F}_2 + v^2\mathbb{F}_2$  [12] and  $\mathbb{F}_p + v\mathbb{F}_p$  [13], respectively. On the other hand, Gao studied the linear codes over the ring  $\mathbb{F}_p + u\mathbb{F}_p + u^2\mathbb{F}_p$  with  $u^3 = u$ , which was an open problem of [12]. Later, the MacDonald codes over this non-chain ring and the applications in constructing secret sharing schemes were studied in [6].

Inspired by the work listed above, we investigate several different weight enumerators of linear codes over a non-chain ring  $R = \mathbb{F}_2[v]/\langle v^4 - v \rangle$ , and we also give the MacWilliams identity on linear codes over  $R$  with respect to these weight

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enumerators. The remainder of this paper is organized as follows. In section 2, we define the Lee weight of the elements in the ring  $R$  and give the generator matrices for a linear code  $C$  and its dual  $C^\perp$ . In section 3, we define various weight enumerators over the ring  $R$  and obtain the MacWilliams identities for these different weight enumerators. In section 4, two examples are computed to describe the application. Section 5 contains the conclusion of the paper and puts the obtained results into perspective.

**II. PRELIMINARY**

In this section, we introduce some preliminary results as follows. Assume that  $R$  is the non-principal ideal ring  $\mathbb{F}_2[v]/\langle v^4 - v \rangle$  with characteristic 2, the ring is endowed with addition and multiplication with  $v^4 = v$ . The elements in  $R$  can be written as  $x = a + bv + cv^2 + dv^3$ ,  $a, b, c, d \in \mathbb{F}_2$ , where  $1, 1 + v + v^3, 1 + v^2 + v^3$  are units of  $R$ . A linear code over  $R$  of length  $n$  is an  $R$ -submodule of  $R^n$ . For a code  $C$  over  $R$ , a matrix  $G$  is a generator matrix for  $C$  if its row vectors generate  $C$ .

Similar to the discussion of the generator matrix of a linear code over  $\mathbb{F}_2 + v\mathbb{F}_2 + v^2\mathbb{F}_2$  in [13], every  $R$ -linear code  $C$  containing some nonzero codewords is equivalent to an  $R$ -linear code with a generator matrix  $G$  in the form of

$$G = \begin{bmatrix} I_{k_1} & A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & vI_{k_2} & 0 & vA_{21} & vA_{22} \\ 0 & 0 & (1+v)I_{k_3} & (1+v)A_{31} & 0 \\ 0 & 0 & 0 & (v+v^2)I_{k_4} & 0 \\ 0 & 0 & 0 & 0 & (1+v+v^2)I_{k_5} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ A_{15} & A_{16} & A_{17} \\ vA_{23} & vA_{24} & vA_{25} \\ (1+v)A_{32} & (1+v)A_{33} & (1+v)A_{34} \\ 0 & 0 & (v+v^2)A_{41} \\ (1+v+v^2)A_{51} & (1+v+v^2)A_{52} & (1+v+v^2)A_{53} \\ (1+v^3)I_{k_6} & 0 & (1+v^3)A_{61} \\ v & (v+v^2+v^3)I_{k_7} & (v+v^2+v^3)A_{71} \end{bmatrix}$$

where  $I_{k_i}$  are  $k_i \times k_i$  identity matrices and  $C$  contains  $2^{4k_1+3k_2+3k_3+2k_4+2k_5+k_6+k_7}$  codewords. Similarly, every  $R$ -linear code containing some nonzero codewords is equivalent to an  $R$ -linear code with a parity check matrix  $H$  as follows.

$$H = \begin{bmatrix} J_{0,1} & J_{0,2} & J_{0,3} \\ J_{7,1} & J_{7,2} & (1+v)A_{33}^T \\ J_{6,1} & J_{6,2} & vA_{32}^T \\ J_{5,1} & (v+v^2)A_{22}^T & 0 \\ J_{4,1} & (1+v+v^2)A_{21}^T & 0 \\ (v+v^2+v^3)A_{12}^T & 0 & (v+v^2+v^3)I_{k_3} \\ (1+v^3)A_{11}^T & (1+v^3)I_{k_2} & 0 \end{bmatrix}$$

$$\begin{bmatrix} J_{0,4} & J_{0,5} & J_{0,6} & J_{0,7} & I_{n-\sum_{i=1}^7 k_i} \\ 0 & (1+v)A_{52}^T & 0 & (1+v)I_{k_7} & 0 \\ 0 & vA_{51}^T & vI_{k_6} & 0 & 0 \\ 0 & (v+v^2)I_{k_5} & 0 & 0 & 0 \\ (1+v+v^2)A_{31}^T & (1+v+v^2)I_{k_4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where  $J_{0,i}, J_{7,1}, J_{7,2}, J_{6,1}, J_{6,2}, J_{5,1}, J_{4,1}$  are  $R$ -matrix for  $1 \leq i \leq 7$ . Furthermore, the code  $C^\perp$  contains  $2^{4n-4k_1-3k_2-3k_3-2k_4-2k_5-k_6-k_7}$  codewords.

*Definition 1:* The Lee weight  $W_L(r)$  of each element  $r \in R$  is defined as

$$W_L(r) = \begin{cases} 0, & \text{if } r \in D_0 = \{0\}; \\ 1, & \text{if } r \in D_1 = \{1 + v^3, v, v^2, v^3\}; \\ 2, & \text{if } r \in D_2 = \{1, 1 + v + v^3, 1 + v^2 + v^3, \\ & v + v^2, v + v^3, v^2 + v^3\}; \\ 3, & \text{if } r \in D_3 = \{1 + v, 1 + v + v^2 + v^3, 1 + v^2, \\ & v + v^2 + v^3\}; \\ 4, & \text{if } r \in D_4 = \{1 + v + v^2\}. \end{cases}$$

According to the definition of the Lee weight of elements in  $R$ , for  $x = (x_1, x_2, \dots, x_n)$ , we get  $W_{D_j}(x) = \sum_{i=0}^{n-1} \delta_{D_j, x_i}$ , where

$$\delta_{D_j, x_i} = \begin{cases} 1, & \text{if } x_i \in D_j; \\ 0, & \text{if } x_i \notin D_j. \end{cases}$$

The Lee weight of  $C$  is defined to be the rational sum of the Lee weight of its components  $W_L(c) = \sum_{i=0}^{n-1} W_L(c_i)$ . Then  $W_L(c)$  is defined by  $W_L(c) = W_{D_1}(c) + 2W_{D_2}(c) + 3W_{D_3}(c) + 4W_{D_4}(c)$ . The Hamming weight  $W_H(c)$  of a codeword  $c$  is the number of nonzero components in  $C$ , i.e.  $W_H(c) = W_{D_1}(c) + W_{D_2}(c) + W_{D_3}(c) + W_{D_4}(c)$ . For any  $x_1, x_2 \in C$ , the Lee distance of  $C$  is given by  $d_L(x_1, x_2) = W_L(x_1 - x_2)$ . The Lee weight of  $C$  is the smallest nonzero Lee weight among all codewords of  $C$ , and the Lee distance of  $C$  is the smallest nonzero Lee distance between all pairs of distinct codewords of  $C$ .

*Definition 2:* The Gray map  $\Phi$  from  $R^n$  to  $\mathbb{F}_2^{4n}$  is defined as  $\Phi(a + bv + cv^2 + dv^3) = (a, b, c, a + d)$  for  $a, b, c, d \in \mathbb{F}_2$ .

Obviously,  $\Phi$  is an isometry from  $(R^n, \text{Lee distance})$  to  $(\mathbb{F}_2^{4n}, \text{Hamming distance})$ . It is easy to verify that the Lee weight of  $C$  is the Hamming weight of  $\Phi(C)$ . Moreover, we have the following theorem.

*Theorem 1:* If  $C^\perp$  is the dual of  $C$ , then  $\Phi(C)^\perp = \Phi(C^\perp)$ . Furthermore, if  $C$  is a self-dual code, so is  $\Phi(C)$ .

*Proof:* Let  $c_1 = a_1 + b_1v + c_1v^2 + d_1v^3 \in C$ ,  $c_2 = a_2 + b_2v + c_2v^2 + d_2v^3 \in C^\perp$ , where  $a_i, b_i, c_i, d_i \in \mathbb{F}_2$ . If  $c_1 \cdot c_2 = 0$  in  $R$ , which entails  $\Phi(c_1) \cdot \Phi(c_2) = 0$ , so we have  $\Phi(C)^\perp \subseteq \Phi(C^\perp)$ . Since  $\Phi(C)$  is a  $[4n, 4n - 4k_1 - 3k_2 - 3k_3 - 2k_4 - 2k_5 - k_6 - k_7]$  code, then  $\Phi(C)^\perp$  is a  $[4n, 4n - 4k_1 - 3k_2 - 3k_3 - 2k_4 - 2k_5 - k_6 - k_7]$  code. Therefore, we can obtain  $|\Phi(C)^\perp| = 2^{4n-4k_1-3k_2-3k_3-2k_4-2k_5-k_6-k_7}$ . However,  $|\Phi(C^\perp)| = |C^\perp| = 2^{4n-4k_1-3k_2-3k_3-2k_4-2k_5-k_6-k_7}$ , hence,  $\Phi(C)$  and  $\Phi(C^\perp)$  are dual  $\mathbb{F}_2$ -linear codes.

According to the above theorem, we have the following statement.

*Remark 1:* For  $X, Y \in R^n$ , let  $W_H(\Phi(X))$  be the Hamming weight of  $\Phi(X)$  over  $\mathbb{F}_2^{4n}$  and  $W_L(X)$  be the Lee weight of  $X$ . Then  $W_L(X) = W_H(\Phi(X))$ ,  $d_L(X, Y) = d_H(\Phi(X), \Phi(Y))$ . Moreover, the minimum Lee weight of  $C$  is the same as the minimal Hamming weight of  $\Phi(C)$ .

### III. SEVERAL WEIGHT ENUMERATORS OF LINEAR CODES AND THEIR MacWilliams IDENTITIES OVER R

MacWilliams identity is one of the most important coding theory that describes how the weight enumerators of a linear code and of its dual code relate to each other. Let  $C$  be a code over  $R$  with length  $n$ , and  $B_i$  denote the number of codewords of Lee weight  $i$  in  $C$ . Then  $B_0, B_1, \dots, B_{4n}$  is called the Lee weight distribution of  $C$ . The Lee weight enumerator of  $C$  is defined as

$$Lee_C(X, Y) = \sum_{i=0}^{4n} B_i X^{4n-i} Y^i,$$

or

$$Lee_C(X, Y) = \sum_{c \in C} X^{4n-W_L(c)} Y^{W_L(c)},$$

and the Hamming weight enumerator of  $C$  as  $Ham_C(X, Y) = \sum_{c \in C} X^{4n-W_H(c)} Y^{W_H(c)}$ . Similarly, we define the symmetric weight enumerator

$$swe_C(X_0, X_1, X_2, X_3, X_4) = \sum_{c \in C} X_0^{W_{D_0}(c)} X_1^{W_{D_1}(c)} X_2^{W_{D_2}(c)} X_3^{W_{D_3}(c)} X_4^{W_{D_4}(c)}.$$

Based on the definitions of three weight enumerators above, we get four relational expressions below.

*Theorem 2:* Let  $C$  be a linear code over  $R$ . Then

- (a)  $Lee_C(X, Y) = swe_C(X^4, X^3Y, X^2Y^2, XY^3, Y^4)$ ;
- (b)  $Ham_C(X, Y) = swe_C(X, Y, Y, Y, Y)$ ;
- (c)  $Lee_C(X, Y) = W_{\Phi(C)}(X, Y)$ ;
- (d)  $Lee_{C^\perp}(X, Y) = \frac{1}{|C|} Lee_C(X + Y, X - Y)$ .

*Proof:* According to the definition of the symmetrized weight enumerator, we get (a) and (b) immediately. In addition, we can obtain (c) from the definition of Lee weight enumerator. Combining  $\Phi(C^\perp) = \Phi(C)^\perp$  with Theorem 1, we have

$$W_{\Phi(C^\perp)}(X, Y) = \frac{1}{|\Phi(C)|} W_{\Phi(C)}(X + Y, X - Y).$$

By  $|C| = |\Phi(C)|$  and (c), we obtain

$$\begin{aligned} Lee_{C^\perp}(X, Y) &= W_{\Phi(C^\perp)}(X, Y) \\ &= \frac{1}{|\Phi(C)|} W_{\Phi(C)}(X + Y, X - Y) \\ &= \frac{1}{|C|} Lee_C(X + Y, X - Y), \end{aligned}$$

which completes the proof of (d).

Similar to the discussion of Theorem 3.1 in [12], we conclude the following theorem.

*Theorem 3:* Let  $C$  be a linear code of length  $n$  over  $R$ , and  $B_0, B_1, \dots, B_{4n}$  be its Lee weight distribution. Let  $B'_0, B'_1, \dots, B'_{4n}$  be the Lee weight distribution of  $C^\perp$ . Then

$$B'_k = \frac{1}{|C|} \sum_{i=0}^{4n} B_i P_k(i, 4n),$$

where  $P_k(i, 4n) = \sum_{j=0}^k (-1)^j C_i^j C_{4n-i}^{k-j}$  is the krawtchouk polynomial.

Now we introduce another weight enumerator of  $C$ , that is complete weight enumerator

$$cwe_C(X_0, X_1, \dots, X_{1+v+v^2+v^3}) = \sum_{c \in C} X_0^{w_0(c)} X_1^{w_1(c)} \dots X_{1+v+v^2+v^3}^{w_{1+v+v^2+v^3}(c)}.$$

Before the study of MacWilliams identity with respect to complete weight enumerator, we give some notes as follows.

The weight of  $x$  at  $a$  is defined to be  $W_a(x) = \sum_{i=1}^n \delta_{a,x_i}$  for  $x = (x_1, x_2, \dots, x_n)$ , where

$$\delta_{a,x_i} = \begin{cases} 1, & \text{if } x_i = a; \\ 0, & \text{if } x_i \neq a. \end{cases}$$

Assume that  $\lambda$  is a function from  $R$  to the complex domain  $C$  for all  $t = a + bv + cv^2 + dv^3 \in R$ . Then  $\lambda$  is a character with  $\lambda(t) = (-1)^d$ .

*Lemma 1:* Let  $C$  be a linear code of length  $n$  over  $R$ . Then  $\sum_{x \in C} f(x) = \frac{1}{|C|} \sum_{x \in C} f'(x)$  holds for all  $x, y \in C$ , where  $f'(x) = \sum_{y \in R^n} \lambda\langle x, y \rangle f(y)$ ,  $f$  is a function from  $R^n$  to complex  $C$ , and  $\langle x, y \rangle$  is the inner product of  $x$  and  $y$ .

*Theorem 4:* Let  $C$  be a linear code of length  $n$  over  $R$ . Then

$$\begin{aligned} cwe_{C^\perp}(X_0, X_1, \dots, X_{1+v+v^2+v^3}) &= \frac{1}{|C|} cwe_C\left(\sum_{r \in R} \lambda(0r) X_r, \sum_{r \in R} \lambda(1r) X_r, \dots, \sum_{r \in R} \lambda((1+v+v^2+v^3)r) X_r\right) \end{aligned}$$

*Proof:* Let  $f(x) = \prod_r X_r^{W_r(x)}$ . Then  $f'(x) = \sum_{y \in R^n} \lambda\langle x, y \rangle X_0^{W_0(y)} X_1^{W_1(y)} \dots X_{1+v+v^2+v^3}^{W_{1+v+v^2+v^3}(y)}$  for any  $a \in R$ ,  $W_a(y) = \sum_{i=0}^n \delta_{a,y_i}$ , and

$$\begin{aligned} f'(x) &= \sum_{y \in R^n} (\lambda(x_1 y_1) X_0^{\delta_{0,y_1}} \dots X_{1+v+v^2+v^3}^{\delta_{1+v+v^2+v^3,y_1}}) \\ &\quad \dots (\lambda(x_n y_n) X_0^{\delta_{0,y_n}} \dots X_{1+v+v^2+v^3}^{\delta_{1+v+v^2+v^3,y_n}}) \\ &= \sum_{y_1 \in R} \lambda(x_1 y_1) X_0^{\delta_{0,y_1}} X_1^{\delta_{1,y_1}} \dots X_{1+v+v^2+v^3}^{\delta_{1+v+v^2+v^3,y_1}} \end{aligned}$$

$$\begin{aligned} & \dots \sum_{y_n \in R} \lambda(x_n y_n) X_0^{\delta_{0,y_n}} X_1^{\delta_{1,y_n}} \dots X_{1+v+v^2+v^3}^{\delta_{1+v+v^2+v^3,y_n}} \\ &= \left( \sum_{r \in R} \lambda(x_1 r) X_r \right) \dots \left( \sum_{r \in R} \lambda(x_n r) X_r \right). \end{aligned}$$

Since the number of the components of  $X$  that equals  $j$  is  $W_j(x)$ , we have that

$$\begin{aligned} f'(x) &= \prod \left( \sum \lambda(jr) X_r \right)^{W_j(x)} cwe_{C^\perp}(X_0, \dots, X_{1+v+v^2+v^3}) \\ &= \sum \prod X_r^{W_r(c)} = \sum f'(x) = \frac{1}{|C|} \sum f'(x) \\ &= \frac{1}{|C|} cwe_C \left( \sum_{r \in R} \lambda(0r) X_r, \sum_{r \in R} \lambda(1r) X_r, \dots, \sum_{r \in R} \lambda((1+v+v^2+v^3)r) X_r \right) \end{aligned}$$

which completes the proof.

The complete weight enumerator of  $R$ -linear code multiplied by equal value may be different, while the symmetric weight enumerator of it is identical. We get another definition of the symmetric weight enumerator by the complete weight enumerator below.

**Definition 3:** For any  $a \in D_i$ , it has  $I(a) = i$ , where  $i \in \{0, 1, 2, 3, 4\}$ . Then the symmetric weight enumerator is defined as  $swe_C(X_0, X_1, X_2, X_3, X_4) = cwe_C(X_{I(0)}, X_{I(1)}, \dots, X_{I(1+v+v^2+v^3)})$ .

**Lemma 2:** (a) If  $i \in D_0$ , then  $\sum_{r \in D_0} \lambda(ir) =$

$$1, \sum_{r \in D_1} \lambda(ir) = 4, \sum_{r \in D_2} \lambda(ir) = 6, \sum_{r \in D_3} \lambda(ir) = 4 \text{ and } \sum_{r \in D_4} \lambda(ir) = 1;$$

(b) If  $i \in D_1$ , then  $\sum_{r \in D_0} \lambda(ir) = 1, \sum_{r \in D_1} \lambda(ir) =$

$$2, \sum_{r \in D_2} \lambda(ir) = 0, \sum_{r \in D_3} \lambda(ir) = -2 \text{ and } \sum_{r \in D_4} \lambda(ir) = -1;$$

(c) If  $i \in D_2$ , then  $\sum_{r \in D_0} \lambda(ir) = 1, \sum_{r \in D_1} \lambda(ir) =$

$$0, \sum_{r \in D_2} \lambda(ir) = -2, \sum_{r \in D_3} \lambda(ir) = 0 \text{ and } \sum_{r \in D_4} \lambda(ir) = 1;$$

(d) If  $i \in D_3$ , then  $\sum_{r \in D_0} \lambda(ir) = 1, \sum_{r \in D_1} \lambda(ir) =$

$$-2, \sum_{r \in D_2} \lambda(ir) = 0, \sum_{r \in D_3} \lambda(ir) = 2 \text{ and } \sum_{r \in D_4} \lambda(ir) = -1;$$

(e) If  $i \in D_4$ , then  $\sum_{r \in D_0} \lambda(ir) = 1, \sum_{r \in D_1} \lambda(ir) =$

$$-4, \sum_{r \in D_2} \lambda(ir) = 6, \sum_{r \in D_3} \lambda(ir) = -4 \text{ and } \sum_{r \in D_4} \lambda(ir) = 1.$$

**Lemma 3:**  $\sum_{r \in R} \lambda(ir) = \sum_{r \in R} \lambda(jr)$ , ( $i, j \in D_l, l = 0, 1, \dots, 4$ ).

**Proof:** According to Lemma 2, we get  $\sum_{r \in R} \lambda(ir) = \sum_{r \in R} \lambda(jr) = 16$ , for  $i, j \in D_0$ , and  $\sum_{r \in R} \lambda(ir) = \sum_{r \in R} \lambda(jr) = 0$ , for  $i, j \in D_t, t \in \{1, 2, 3, 4\}$ .

By the introduction of symmetric weight enumerator, we establish the MacWilliams identity with respect to it between  $R$ -linear code and its dual code.

**Theorem 5:** Let  $C$  be a linear code of length  $n$  over  $R$ . Then

$$\begin{aligned} swe_{C^\perp}(X_0, X_1, X_2, X_3, X_4) &= \frac{1}{|C|} swe_C(X_0 + 4X_1 + 6X_2 + 4X_3 + X_4, X_0 \\ &+ 2X_1 - 2X_3 - X_4, X_0 - 2X_2 + X_4, X_0 - 2X_1 \\ &+ 2X_3 - X_4, X_0 - 4X_1 + 6X_2 - 4X_3 + X_4). \end{aligned}$$

**Proof:** Applying Theorem 4, we have that

$$\begin{aligned} swe_{C^\perp}(X_0, X_1, X_2, X_3, X_4) &= cwe_{C^\perp}(X_{I(0)}, X_{I(1)}, \dots, X_{I(1+v+v^2+v^3)}) \\ &= \frac{1}{|C|} cwe_C \left( \sum_{r \in R} \lambda(0r) X_r, \sum_{r \in R} \lambda(1r) X_r, \dots, \sum_{r \in R} \lambda((1+v+v^2+v^3)r) X_r \right) \\ &= \frac{1}{|C|} cwe_C \left( \sum_{s=0}^4 \sum_{r \in D_s} \lambda(0r) X_s, \sum_{s=0}^4 \sum_{r \in D_s} \lambda(1r) X_s, \dots, \sum_{s=0}^4 \sum_{r \in D_s} \lambda((1+v+v^2+v^3)r) X_s \right). \end{aligned}$$

Meanwhile, for  $i, j \in D_t$ , by Lemma 3, we get

$$\sum_{s=0}^3 \sum_{r \in D_s} \lambda(ir) X_s = \sum_{s=0}^3 \sum_{r \in D_s} \lambda(jr) X_s.$$

Then

$$\begin{aligned} swe_{C^\perp}(X_0, X_1, X_2, X_3, X_4) &= \frac{1}{|C|} swe_C \left( \sum_{s=0}^4 \sum_{r \in D_s} \lambda(0r) X_s, \sum_{s=0}^4 \sum_{r \in D_s} \lambda(1r) X_s, \sum_{s=0}^4 \sum_{r \in D_s} \lambda((1+v^3)r) X_s, \sum_{s=0}^4 \sum_{r \in D_s} \lambda(v^3 r) X_s, \dots, \sum_{s=0}^4 \sum_{r \in D_s} \lambda((1+v+v^3)r) X_s \right). \end{aligned}$$

By taking  $0, 1+v^3, v^3, 1, 1+v+v^3$  from  $D_0, D_1, D_2, D_3, D_4$ , we apply Lemma 2 and obtain that

- $\sum_{s=0}^4 \sum_{r \in D_s} \lambda(0r) X_s = X_0 + 4X_1 + 6X_2 + 4X_3 + X_4;$
- $\sum_{s=0}^4 \sum_{r \in D_s} \lambda(v^3 r) X_s = X_0 - 2X_2 + X_4;$
- $\sum_{s=0}^4 \sum_{r \in D_s} \lambda((1+v^3)r) X_s = X_0 + 2X_1 - 2X_3 - X_4;$

- $\sum_{s=0}^4 \sum_{r \in D_s} \lambda(1r)X_s = X_0 - 2X_1 + 2X_3 - X_4;$
- $\sum_{s=0}^4 \sum_{r \in D_s} \lambda((1+\nu+\nu^3)r)X_s = X_0 - 4X_1 + 6X_2 - 4X_3 + X_4.$

**IV. APPLICATION**

As applications we provide two examples to illustrate our results. Let  $C$  be a linear code of length 2 over  $R$  with generator matrix

$$G = \begin{pmatrix} \nu & 0 \\ 0 & \nu + 1 \end{pmatrix}.$$

Then the code  $C$  has 64 codewords. According to Section 2, the dual code  $C^\perp$  is a linear code with generator matrix

$$H = \begin{pmatrix} 0 & 1 + \nu^3 \\ \nu^3 + \nu^2 + \nu & 0 \end{pmatrix}.$$

Then  $C^\perp$  has 4 codewords, i.e.  $C^\perp = \{(0, 0), (0, \nu + \nu^2 + \nu^3), (1 + \nu^3, 0), (1 + \nu^3, \nu + \nu^2 + \nu^3)\}$ . Therefore we get three weight enumerators below,

$$\begin{aligned} swe_{C^\perp}(X_0, X_1, X_2, X_3, X_4) &= X_1^2 + 2X_1X_3 + X_1X_0, \\ Lee_{C^\perp}(X, Y) &= X^6Y^2 + 2X^4Y^4 + X^7Y, \\ Ham_{C^\perp}(X, Y) &= X^2 + XY + 2Y^2. \end{aligned}$$

Applying Theorem 5, we obtain the MacWilliams identity with respect to symmetric enumerator between  $R$ -linear code and its dual code

$$\begin{aligned} &swe_C(X_0, X_1, X_2, X_3, X_4) \\ &= \frac{1}{|C^\perp|} swe_{C^\perp}(X_0 + 4X_1 + 6X_2 + 4X_3 + X_4, X_0 + 2X_1 \\ &\quad - 2X_3 - X_4, X_0 - 2X_2 + X_4, X_0 - 2X_1 + 2X_3 - X_4, \\ &\quad X_0 - 4X_1 + 6X_2 - 4X_3 + X_4) \\ &= X_0^2 + 9X_1^2 + 9X_2^2 + X_3^2 + 6X_0X_1 + 36X_0X_2 \\ &\quad + 5X_0X_3 + 18X_1X_2 + 6X_1X_3 + 6X_2X_3. \end{aligned}$$

Meanwhile, it follows by Theorem 2,

$$\begin{aligned} Ham_C(X, Y) &= swe_C(X, Y, Y, Y, Y) = X^2 + 49Y^2 \\ &\quad + 14XY, \\ Lee_C(X, Y) &= \frac{1}{4}((X + Y)^8 + 2(X + Y)^7(X - Y) \\ &\quad + (X + Y)^6(X - Y)^2) \end{aligned}$$

By listing the codewords of  $C$  literally, the conclusion is proved to be completely correct under direct verification. The Lee-weight distribution of  $C$  can also be computed by using Theorem 3. By the relation of  $B'_i$  and  $B_i$ , we get  $B_0 = 4, B_1 = 8, B_3 = 10, B_4 = 6, B_5 = 16$  and  $B_6 = 18, B_7 = 2, B_8 = 2$ , which is consistent with the results computed by Theorem 2. In addition, if  $|C|$  is far greater than  $|C^\perp|$ , then we can get the weight distribution of  $C$  directly without requiring the specific codewords. The following example is a good illustration, assume  $C$  is a linear

code of length 4 over  $R$  with generator matrix

$$G = \begin{pmatrix} \nu & 0 & 0 & 0 \\ 0 & \nu & 0 & 0 \\ 0 & 0 & 1 + \nu & 0 \\ 0 & 0 & 0 & 1 + \nu \end{pmatrix}.$$

The code  $C$  has  $2^{12}$  codewords, the dual code  $C^\perp$  is a linear code with generator matrix

$$H = \begin{pmatrix} 0 & 0 & \nu + \nu^2 + \nu^3 & 0 \\ 0 & 0 & 0 & \nu + \nu^2 + \nu^3 \\ 1 + \nu^3 & 0 & 0 & 0 \\ 0 & 1 + \nu^3 & 0 & 0 \end{pmatrix}.$$

$C^\perp$  has 16 codewords, i.e.  $C^\perp = \{(0, 0, 0, 0), (0, 1 + \nu^3, 0, 0), (0, 0, \nu + \nu^2 + \nu^3, 0), (0, 0, 0, \nu + \nu^2 + \nu^3), (1 + \nu^3, 0, 0, 0), (1 + \nu^3, 1 + \nu^3, 0, 0), (1 + \nu^3, 0, 0, \nu + \nu^2 + \nu^3), (1 + \nu^3, 0, \nu + \nu^2 + \nu^3, 0), (0, 1 + \nu^3, \nu + \nu^2 + \nu^3, 0), (0, 0, \nu + \nu^2 + \nu^3, \nu + \nu^2 + \nu^3), (0, 1 + \nu^3, 0, \nu + \nu^2 + \nu^3), (1 + \nu^3, 1 + \nu^3, \nu + \nu^2 + \nu^3, 0), (1 + \nu^3, 1 + \nu^3, 0, \nu + \nu^2 + \nu^3), (1 + \nu^3, 0, \nu + \nu^2 + \nu^3, \nu + \nu^2 + \nu^3), (0, 1 + \nu^3, \nu + \nu^2 + \nu^3, \nu + \nu^2 + \nu^3), (1 + \nu^3, 1 + \nu^3, \nu + \nu^2 + \nu^3, \nu + \nu^2 + \nu^3)\}$ .

Therefore we get three weight enumerators below,

$$\begin{aligned} swe_{C^\perp}(X_0, X_1, X_2, X_3, X_4) &= X_0^4 + 2X_1X_0^3 + 2X_3X_0^3 + X_1^2X_0^2 + 4X_1X_3X_0^2 \\ &\quad + X_3^2X_0^2 + 2X_1^2X_3X_0 + 2X_1X_3^2X_0 + X_1^2X_3^2, \\ Lee_{C^\perp}(X, Y) &= X^{16} + 2X^{15}Y + X^{14}Y^2 + 2X^{13}Y^3 + 4X^{12}Y^4 \\ &\quad + 2X^{11}Y^5 + X^{10}Y^6 + 2X^9Y^7 + X^8Y^8, \\ Ham_{C^\perp}(X, Y) &= X^4 + 4X^3Y + 6X^2Y^2 + 4XY^3 + Y^4. \end{aligned}$$

Applying Theorem 5, we obtain the MacWilliams identity with respect to symmetric enumerator between  $R$ -linear code and its dual code

$$\begin{aligned} &swe_C(X_0, X_1, X_2, X_3, X_4) \\ &= \frac{1}{|C^\perp|} swe_{C^\perp}(X_0 + 4X_1 + 6X_2 + 4X_3 + X_4, X_0 + 2X_1 \\ &\quad - 2X_3 - X_4, X_0 - 2X_2 + X_4, X_0 - 2X_1 + 2X_3 - X_4, \\ &\quad X_0 - 4X_1 + 6X_2 - 4X_3 + X_4) \\ &= X_0^4 + 8X_0^3X_1 + 12X_0^3X_2 + 8X_0^3X_3 + 22X_0^2X_1^2 \\ &\quad + 72X_0^2X_1X_2 + 52X_0^2X_1X_3 + 54X_0^2X_2^2 + 72X_0^2X_2X_3 \\ &\quad + 22X_0^2X_3^2 + 24X_0X_1^3 + 132X_0X_1^2X_2 + 104X_0X_1^2X_3 \\ &\quad + 216X_0X_1X_2^2 + 312X_0X_1X_2X_3 + 104X_0X_1X_3^2 \\ &\quad + 108X_0X_2^3 + 216X_0X_2^2X_3 + 132X_0X_2X_3^2 + 24X_0X_3^3 \\ &\quad + 9X_1^4 + 72X_1^3X_2 + 60X_1^3X_3 + 198X_1^2X_2^2 \\ &\quad + 312X_1^2X_2X_3 + 118X_1^2X_3^2 + 216X_1X_2^3 \\ &\quad + 468X_1X_2^2X_3 + 312X_1X_2X_3^2 + 60X_1X_3^3 \\ &\quad + 81X_2^4 + 216X_2^3X_3 + 198X_2^2X_3^2 + 72X_2X_3^3 \\ &\quad + 144X_3^4. \end{aligned}$$

In addition, it follows by Theorem 2,

$$\begin{aligned}
 Ham_C(X, Y) &= swe_c(X, Y, Y, Y, Y) \\
 &= X^4 + 28X^3Y + 294X^2Y^2 \\
 &\quad + 1372XY^3 + 2536Y^4 \\
 Lee_C(X, Y) &= X^{16} + 8X^{15}Y + 34X^{14}Y^2 + 104X^{13}Y^3 \\
 &\quad + 247X^{12}Y^4 + 464X^{11}Y^5 + 700X^{10}Y^6 \\
 &\quad + 848X^9Y^7 + 799X^8Y^8 + 552X^7Y^9 \\
 &\quad + 258X^6Y^{10} + 72X^5Y^{11} + 9X^4Y^{12}.
 \end{aligned}$$

As  $B'_1 = 1, B'_2 = 1$  and  $B'_4 = 2$ , by the relation of  $B'_i$  and  $B_i$ , we get  $B_0 = 1, B_1 = 8, B_2 = 34, B_3 = 104, B_4 = 247, B_5 = 464, B_6 = 700, B_7 = 848, B_8 = 799, B_9 = 552, B_{10} = 258, B_{11} = 72, B_{12} = 9$ , which is consistent with the results computed by Theorem 2.

### V. CONCLUSION

In this work, we investigate several weight enumerators of the linear codes over the non-principle ideal ring  $\mathbb{F}_2[v]/\langle v^4 - v \rangle$ . The MacWilliams identity respect to the complete weight enumerator and symmetric weight enumerator are determined. It is meaningful to consider the Type II codes over this non-chain ring, and there is a natural problem to study the existence of a mass formula for self-dual codes over  $R$ . Furthermore, the construction of linear codes over  $R$  (image codes) with few weight distribution is another research direction.

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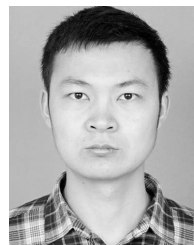
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