

Received March 30, 2020, accepted April 22, 2020, date of publication April 29, 2020, date of current version May 15, 2020.

Digital Object Identifier 10.1109/ACCESS.2020.2991341

# Finite-Time $H_\infty$ Control for Itô-Type Nonlinear Time-Delay Stochastic Systems

ZHIGUO YAN<sup>1,2</sup>, (Member, IEEE), MIN ZHANG<sup>1</sup>, YUNXIA SONG<sup>3</sup>, AND SHIYU ZHONG<sup>4</sup>

<sup>1</sup>School of Electrical Engineering and Automation, Qilu University of Technology(Shandong Academy of Sciences), Jinan 250353, China

<sup>2</sup>School of Control Science and Engineering, Shandong University, Jinan 250061, China

<sup>3</sup>Center for Control Theory and Guidance Technology, Harbin Institute of Technology, Harbin 150001, China

<sup>4</sup>College of Electrical Engineering and Automation, Shandong University of Science and Technology, Qingdao 266590, China

Corresponding author: Zhiguo Yan (yanzg500@sina.com)

This work was supported in part by the National Natural Science Foundation of China under Grant 61877062 and Grant 61977043, and in part by the China Postdoctoral Science Foundation under Grant 2017M610425.

**ABSTRACT** The finite-time  $H_\infty$  control problem for an Itô-type stochastic system with nonlinear perturbation and time delay is investigated. First, the finite-time  $H_\infty$  control problem for a nonlinear time-delay stochastic system is presented taking into consideration both the transient performance and the capability to attenuate the disturbance of a closed-loop system in a given finite-time interval. Second, using the Lyapunov–Krasovskii functional method and the matrix inequality technique, some sufficient conditions for the existence of finite-time  $H_\infty$  state feedback controller and dynamic-output feedback controller for nonlinear time-delay stochastic systems are obtained. These conditions guarantee the mean-square finite-time bounded-ness of the closed-loop systems and determine the  $H_\infty$  control performance index. Third, this problem is transformed into an optimization problem with matrix inequality constraints, and the corresponding algorithms are given to optimize the  $H_\infty$  performance index and obtain the maximum time-delay. Finally, a numerical example is used to illustrate the effectiveness of the proposed method.


**INDEX TERMS** Stochastic systems, nonlinear perturbations, finite-time stability,  $H_\infty$  control, time-delay.

## I. INTRODUCTION

In recent years, control problems of stochastic nonlinear systems have been receiving increased attention because of their extensive applications in many practical systems, such as liquid-level systems [1], chemical reactor systems [2], [3], and industrial and economic systems [4]. In addition, many excellent results have been published. For example, [5] proposed three stochastic nonlinear control schemes to study the global stabilization of stochastic nonlinear systems. In [6], finite-time stability for stochastic nonlinear systems was considered and a general Lyapunov theorem of stochastic finite-time stability was proved. A finite-time tracking problem of switched stochastic nonlinear uncertain systems was studied in [7]. Some other nice results can be referred to [8]–[10]. In considering the influence of time-delays on the system, much of the focus has been on a general model of stochastic nonlinear time-delay systems. To date, numerous results on these systems have been obtained. For instance, [11] studied

the stability of a class of nonlinear uncertain stochastic time-delay systems, and a sufficient delay-dependent criterion was established by constructing a new Lyapunov–Krasovskii function. The output feedback adaptive tracking control problem for a class of stochastic nonlinear time-delay systems was studied in [12], and an observer-based adaptive neural quantization tracking control scheme was proposed. For other excellent results, the reader is referred to [13]–[15] and references therein.

At present, most of the results of stochastic systems are based on the asymptotic stability in the Lyapunov sense, which only concerns the asymptotic behavior of the system in the limit of infinite time. However, the transient behavior is also significant in many practical systems. For example, a large transient voltage can destroy the normal operation of a power system [16]. To deal with this problem, the concept of finite-time stability was proposed, and many interesting results have been published, such as the finite-time stability of stochastic discrete-time-varying systems in [17], stochastic Markov jump systems in [18]–[22], stochastic time-delay systems in [23], and T-S fuzzy systems in [24]. In contrast,

The associate editor coordinating the review of this manuscript and approving it for publication was Bing Li .

$H_\infty$  control is one of the most crucial robust control methods because an external interference can be suppressed; many results have been reported. For instance, [25] and [26] are devoted to the robust  $H_\infty$  control problem for uncertain stochastic nonlinear systems with time-varying delays and stochastic nonlinear uncertain T-S fuzzy systems with time-delay. Robust  $H_\infty$  filtering and control for a class of linear systems with fractional stochastic noise were studied in [27]. Other nice results are to be found in [28]–[31]. Taking advantage of finite-time stability and  $H_\infty$  control, a new control method called the finite-time  $H_\infty$  control was proposed, and various results have been reported. For example, the finite-time  $H_\infty$  control problem for a singular Markovian jump system with an actuator fault was investigated in [32] with a sliding mode control approach being applied. The reference [33] addressed the problem of robust finite-time  $H_\infty$  control of singular stochastic systems and [34] studied the robust and resilient finite-time  $H_\infty$  control problem for uncertain discrete-time nonlinear systems with Markovian jump parameters.

Although the problem of finite-time  $H_\infty$  control has been investigated, there is a lack of literature on Itô-type stochastic nonlinear time-delay systems. To redress this issue, the finite-time  $H_\infty$  control these systems was investigated in our study. Because of the complexity of the systems considered, designing a  $H_\infty$  controller with state feedback and dynamic-output feedback is difficult. By using the Itô formula, the Gronwall inequality, and the matrix inequality technique, the above difficulties are overcome. The main contributions from this study are: (i) a precise statement of the finite-time  $H_\infty$  control problem for Itô-type stochastic nonlinear systems with time-delay that considers both the transient performance and the capability of attenuating the disturbances in the closed-loop systems in a given finite-time interval; (ii) two new conditions developed from matrix inequalities concerning the sufficiency for the existence of two  $H_\infty$  controllers one providing state feedback and the other dynamic-output feedback; and (iii) the establishment of two algorithms that solve the gain parameter settings of the two controllers and that optimize the  $H_\infty$  performance index and maximum time-delay, simultaneously.

The rest of this paper is organized as follows. Section II gives some preliminaries, definitions, and lemmas. In Section III, we provide the conditions of sufficiency for the existence of the finite time  $H_\infty$  controllers for Itô-type stochastic nonlinear time-delay systems. Section IV provides the two algorithms that solve the theorems. Section V presents a numerical example to demonstrate the effectiveness of the proposed method. In the last section, our conclusions are stated.

Notation:  $A'$  denotes the transpose of matrix  $A$ ;  $tr(A)$  denotes the trace of matrix  $A$ ;  $A > 0 (A \geq 0)$  signifies that  $A$  is a positive definite (positive semi-definite) matrix;  $\lambda_{max}(A) (\lambda_{min}(A))$  denotes the maximum (minimum) eigenvalue of matrix  $A$ ;  $I_{n \times n}$  denotes the  $n$ -dimensional identity matrix;  $\mathcal{R}^n$  denotes an  $n$ -dimensional Euclidean space;  $E$

represents the mathematical expectation of a random process; an asterisk “\*” in a matrix marks elements to be obtained by the symmetry of the matrix.

## II. PRELIMINARIES

Consider an Itô-type stochastic nonlinear system with time delay described by

$$\begin{cases} dx(t) = (A_{11}x(t) + A_{12}x(t - \tau) + B_{11}u(t) \\ \quad + B_{12}v(t) + H_0(x(t)))dt + (A_{21}x(t) \\ \quad + A_{22}x(t - \tau) + H_1(x(t)))dw(t), \\ y(t) = C_1x(t), \\ z(t) = C_2x(t), \\ x(t) = \phi(t) \in L^2(w, \mathcal{F}_0, C([-\tau, 0], \mathcal{R}^n)), \\ t \in [-\tau, 0], \end{cases} \quad (1)$$

where  $x(t)$  denotes the state of the system,  $y(t)$  the measurement output,  $u(t)$  the control input,  $z(t)$  the control output,  $\phi(t)$  the initial state function, and  $w(t)$  a one-dimensional standard Wiener process defined on probability space  $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ .  $\mathcal{F}_t$  stands for the smallest  $\sigma$ -algebra generated by  $w(s)$ , with  $0 \leq s \leq t$ , i.e.,  $\mathcal{F}_t = \sigma\{w(s) | 0 \leq s \leq t\}$ .  $A_{11}, A_{12}, A_{21}, A_{22}, B_{11}, B_{12}, C_1, C_2$  are constant matrices with appropriate dimensions.  $\tau > 0$  denotes the time-delay,  $v(t)$  an external disturbance that satisfies

$$\Psi(v(t)) = \{v(t) | E \int_0^T v'(s)v(s)ds < d^2\}. \quad (2)$$

The nonlinear terms  $H_0(x(t))$  and  $H_1(x(t))$  satisfy

$$\| H_i(x) \| \leq \varepsilon \| x \| \quad i = 0, 1, \quad (3)$$

where  $\varepsilon > 0$ .

Next, a new definition of the mean-square finite-time bounded-ness for the system (1) is given.

*Definition 1:* For given  $0 < c_1 < c_2, R > 0, T > 0$ , system (1) ( $u(t) \equiv 0$ ) is said to be the mean-square finite-time bounded with respect to  $(c_1, c_2, T, R, d^2)$  if

$$\sup_{-\tau \leq t_0 \leq 0} x'(t_0)Rx(t_0) \leq c_1^2 \Rightarrow E [x'(t)Rx(t)] < c_2^2,$$

for all  $t \in [0, T]$  and  $v(t) \in \Psi$ .

*Remark 1:* The mean-square finite-time bounded-ness reflects the transient performance of the system in a fixed time interval. That is, the average energy of the system does not exceed a given upper bound in the prescribed time-interval. The transient performance is also important in many practical systems. For example, a large transient voltage can destroy the normal operation of the power system.

Next, some lemmas for obtaining the main results are introduced.

*Lemma 1 (Gronwall Inequality):* [35] Let  $f(t)$  be a nonnegative function; if it satisfies

$$f(t) \leq a + b \int_0^t f(s)ds \quad 0 \leq t \leq T,$$

for some constants  $a \geq 0$  and  $b \geq 0$ , then the following inequality is established:

$$f(t) \leq a \exp(bt) \quad 0 \leq t \leq T.$$

**Lemma 2 [36]:** For given  $x \in R^n, y \in R^m, N \in R^{n \times m}$  and  $\rho > 0$ , then we have

$$2x'Ny \leq \rho x'x + \frac{1}{\rho}y'N'y.$$

**Lemma 3 [37]:** Let  $V(t, x) \in C^{1,2}(R_+, R^n)$  be a scalar function, and  $V(t, x) > 0$ ; for the following stochastic system

$$dx(t) = a(x)dt + b(x)dw(t),$$

the Itô formula of  $V(t, x)$  is given as follows:

$$dV(t, x) = LV(t, x)dt + \frac{\partial V'(t, x)}{\partial x}b(x)dw(t),$$

where

$$LV(t, x) = \frac{\partial V(t, x)}{\partial t} + \frac{\partial V'(t, x)}{\partial x}a(x) + \frac{1}{2}b'(x)\frac{\partial^2 V(t, x)}{\partial x^2}b(x).$$

### III. MAIN RESULTS

The design of the state feedback finite-time  $H_\infty$  controller and that of the dynamic-output feedback finite-time  $H_\infty$  controller are described next.

#### A. STATE FEEDBACK FINITE-TIME $H_\infty$ CONTROL

Consider the following state feedback controller

$$u(t) = Kx(t), \tag{4}$$

where  $K$  is the controller gain matrix to be solved.

A closed-loop system is obtained by substituting (4) into system (1) giving

$$\begin{cases} dx(t) = (\tilde{A}_{11}x(t) + A_{12}x(t - \tau) + B_{12}v(t) \\ \quad + H_0(x(t)))dt + (A_{21}x(t) + A_{22}x(t - \tau) \\ \quad + H_1(x(t)))dw(t), \\ y(t) = C_1x(t), \\ z(t) = C_2x(t), \\ x(t) = \phi(t) \in L^2(w, \mathcal{F}_0, C([-\tau, 0], R^n)), \\ t \in [-\tau, 0], \end{cases} \tag{5}$$

where  $\tilde{A}_{11} = A_{11} + B_{11}K$ .

Next, the problem concerning the state feedback finite-time  $H_\infty$  controller is described.

**Definition 2:** For given scalars  $0 < c_1 < c_2, T > 0, d > 0, \gamma > 0$ , and a matrix  $R > 0$ , if there exists a state feedback controller (4) such that:

(i) the closed-loop system (5) is mean-square finite-time bounded with respect to  $(c_1, c_2, T, R, d^2)$ ;

(ii) for any non-zero disturbance  $v(t)$ , the control output  $z(t)$  satisfies the following inequality with zero initial condition,

$$E \int_0^t z'(s)z(s)ds < \gamma^2 E \int_0^t v'(s)v(s)ds, \quad \forall t \in [0, T], \tag{6}$$

then (4) is said to be a state feedback finite-time  $H_\infty$  controller for system (1).

**Remark 2:** The definition considers both the attenuation level of the disturbance and the mean-square finite-time boundedness, which is widely applied in practical systems. For example, in the solar power supply system, if the load power is too large or the external interference is strong, the normal operation of the system will deteriorate.

The sufficient conditions for the existence of the state feedback finite-time  $H_\infty$  controller (4) are given below. For this purpose, an important lemma is first stated and proved.

**Lemma 4:** For given scalars  $0 < c_1 < c_2, T > 0, d > 0, \gamma > 0$ , and a matrix  $R > 0$ , if there are scalars  $\alpha \geq 0, \beta_1 > 0, \beta_2 > 0, \beta_3 > 0, \tau > 0, \varepsilon > 0$ , and two symmetric positive definite matrices  $P_1, Q_1$  such that

$$\begin{bmatrix} \Gamma_1 & \tilde{P}_1A_{12} + A'_{21}\tilde{P}_1A_{22} & \tilde{P}_1B_{12} \\ * & -\tilde{Q}_1 + 2A'_{22}\tilde{P}_1A_{22} & 0 \\ * & * & -\gamma^2I \end{bmatrix} < 0, \tag{7}$$

$$\beta_1I < P_1 < \beta_2I, \tag{8}$$

$$0 < Q_1 < \beta_3I, \tag{9}$$

$$d^2\gamma^2 + c_1^2(\beta_2 + \tau\beta_3) < c_2^2\beta_1e^{-\alpha T}, \tag{10}$$

hold, where  $\Gamma_1 = 4\varepsilon^2\beta_2I + \tilde{P}_1 + \tilde{Q}_1 + \tilde{A}'_{11}\tilde{P}_1 + \tilde{P}_1\tilde{A}_{11} + 2A'_{21}\tilde{P}_1A_{21} - \alpha\tilde{P}_1 + C'_2C_2, \tilde{P}_1 = R^{\frac{1}{2}}P_1R^{\frac{1}{2}}, \tilde{Q}_1 = R^{\frac{1}{2}}Q_1R^{\frac{1}{2}}$ , then system (5) is mean-square finite-time bounded with respect to  $(c_1, c_2, T, R, d^2)$  and satisfies (6).

**Proof:** The proof is divided into two parts. First, the closed-loop system (5) is proved to be mean-square finite-time bounded.

Note that

$$\begin{bmatrix} C'_2C_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} C'_2 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} C_2 \\ 0 \\ 0 \end{bmatrix}' \geq 0. \tag{11}$$

Therefore, condition (7) implies

$$\begin{bmatrix} \Gamma_2 & \tilde{P}_1A_{12} + A'_{21}\tilde{P}_1A_{22} & \tilde{P}_1B_{12} \\ * & -\tilde{Q}_1 + 2A'_{22}\tilde{P}_1A_{22} & 0 \\ * & * & -\gamma^2I \end{bmatrix} < 0, \tag{12}$$

where  $\Gamma_2 = 4\varepsilon^2\beta_2I + \tilde{P}_1 + \tilde{Q}_1 + \tilde{A}'_{11}\tilde{P}_1 + \tilde{P}_1\tilde{A}_{11} + 2A'_{21}\tilde{P}_1A_{21} - \alpha\tilde{P}_1$ .

From (3) and (8), two inequalities are obtained,

$$H'_0(x)P_1H_0(x) \leq \varepsilon^2\beta_2 \|x\|^2, \tag{13}$$

$$H'_1(x)P_1H_1(x) \leq \varepsilon^2\beta_2 \|x\|^2. \tag{14}$$

Introducing quadratic function  $V(x(t)) = x'(t)\tilde{P}_1x(t) + \int_{t-\tau}^t x'(s)\tilde{Q}_1x(s)ds$  and the differential generation operator of the system (5)  $L_1V(x(t))$ , then, according to the Itô formula, we obtain

$$\begin{aligned} L_1V(x(t)) &= [\tilde{A}_{11}x(t) + A_{12}x(t - \tau) + B_{12}v(t) + H_0(x(t))] \\ &\quad \times \frac{\partial V(x(t))}{\partial x} + \frac{1}{2}[(A_{21}x(t) + A_{22}x(t - \tau) + H_1(x(t)))' \\ &\quad \times \frac{\partial^2 V(x(t))}{\partial x^2} [A_{21}x(t) + A_{22}x(t - \tau) + H_1(x(t))]]; \end{aligned} \tag{15}$$

that is,

$$\begin{aligned}
 L_1 V(x(t)) &= x'(t)[\tilde{Q}_1 + A'_{21}\tilde{P}_1 A_{21} + \tilde{P}_1 \tilde{A}_{11} + \tilde{A}'_{11}\tilde{P}_1]x(t) \\
 &\quad + x'(t-\tau)[-\tilde{Q}_1 + A'_{22}\tilde{P}_1 A_{22}]x(t-\tau) \\
 &\quad + x'(t-\tau)[A'_{12}\tilde{P}_1 + A'_{22}\tilde{P}_1 A_{21}]x(t) \\
 &\quad + x'(t)[\tilde{P}_1 A_{12} + A'_{21}\tilde{P}_1 A_{22}]x(t-\tau) \\
 &\quad + x'(t)\tilde{P}_1 B_{12}v(t) + 2H'_1(x)\tilde{P}_1 A_{22}x(t-\tau) \\
 &\quad + 2H'_0(x)\tilde{P}_1 x(t) + 2H'_1(x)\tilde{P}_1 A_{21}x(t) \\
 &\quad + H'_1(x)\tilde{P}_1 H_1(x) + v'(t)B'_{12}\tilde{P}_1 x(t). \tag{16}
 \end{aligned}$$

From lemma 2, we have

$$\begin{aligned}
 &2H'_0(x)\tilde{P}_1 x(t) + 2H'_1(x)\tilde{P}_1 A_{21}x(t) \\
 &\quad + 2H'_1(x)\tilde{P}_1 A_{22}x(t-\tau) \\
 &\leq 2H'_1(x)\tilde{P}_1 H_1(x) + H'_0(x)\tilde{P}_1 H_0(x) \\
 &\quad + x'(t-\tau)A'_{22}\tilde{P}_1 A_{22}x(t-\tau) \\
 &\quad + x'(t)(\tilde{P}_1 + A'_{21}\tilde{P}_1 A_{21})x(t). \tag{17}
 \end{aligned}$$

Based on (13), (14), (16), and (17), we see that

$$\begin{aligned}
 L_1 V(x(t)) &\leq \begin{bmatrix} x(t) \\ x(t-\tau) \\ v(t) \end{bmatrix} \\
 &\quad \times \begin{bmatrix} \Gamma_3 & \tilde{P}_1 A_{12} + A'_{21}\tilde{P}_1 A_{22} & \tilde{P}_1 B_{12} \\ * & -\tilde{Q}_1 + 2A'_{22}\tilde{P}_1 A_{22} & 0 \\ * & * & 0 \end{bmatrix} \\
 &\quad \times \begin{bmatrix} x(t) \\ x(t-\tau) \\ v(t) \end{bmatrix} \tag{18}
 \end{aligned}$$

where  $\Gamma_3 = 4\varepsilon^2 \beta_2 I + \tilde{P}_1 + \tilde{Q}_1 + \tilde{A}'_{11}\tilde{P}_1 + \tilde{P}_1 \tilde{A}_{11} + 2A'_{21}\tilde{P}_1 A_{21}$ .

In light of (12) and (18), we have

$$L_1 V(x(t)) < \alpha V(x(t)) + \gamma^2 v'(t)v(t). \tag{19}$$

Integrating from 0 to  $t$  and taking the mathematical expectation on both sides of (19), we have

$$\begin{aligned}
 EV(x(t)) &< V(x(0)) + \alpha \int_0^t EV(x(s))ds \\
 &\quad + \gamma^2 E \int_0^t v'(s)v(s)ds. \tag{20}
 \end{aligned}$$

From lemma 1, we see that

$$EV(x(t)) < V(x(0))e^{\alpha t} + \gamma^2 e^{\alpha t} E \int_0^t v'(s)v(s)ds, \tag{21}$$

and because

$$\begin{aligned}
 EV(x(t)) &= E[x'(t)\tilde{P}_1 x(t) + \int_{t-\tau}^t x'(s)\tilde{Q}_1 x(s)ds] \\
 &= E[x'(t)R^{\frac{1}{2}}P_1 R^{\frac{1}{2}}x(t) \\
 &\quad + \int_{t-\tau}^t x'(s)R^{\frac{1}{2}}Q_1 R^{\frac{1}{2}}x(s)ds] \\
 &\geq \lambda_{\min}(P_1)E[x'(t)Rx(t)], \tag{22}
 \end{aligned}$$

$$\begin{aligned}
 V(x(0))e^{\alpha t} &= x'(0)\tilde{P}_1 x(0)e^{\alpha t} + e^{\alpha t} \int_{-\tau}^0 x'(s)\tilde{Q}_1 x(s)ds \\
 &= x'(0)R^{\frac{1}{2}}P_1 R^{\frac{1}{2}}x(0)e^{\alpha t} \\
 &\quad + e^{\alpha t} \int_{-\tau}^0 x'(s)R^{\frac{1}{2}}Q_1 R^{\frac{1}{2}}x(s)ds \\
 &\leq \lambda_{\max}(P_1)c_1^2 e^{\alpha T} + \tau \lambda_{\max}(Q_1)c_1^2 e^{\alpha T}, \tag{23}
 \end{aligned}$$

$$\gamma^2 e^{\alpha t} E \int_0^t v'(s)v(s)ds < \gamma^2 e^{\alpha T} d^2. \tag{24}$$

From (21), (22), (23), and (24), we have

$$\begin{aligned}
 E[x'(t)Rx(t)] &\leq e^{\alpha T} \left( \frac{c_1^2(\lambda_{\max}(P_1) + \tau \lambda_{\max}(Q_1))}{\lambda_{\min}(P_1)} + \frac{d^2 \gamma^2}{\lambda_{\min}(P_1)} \right), \tag{25}
 \end{aligned}$$

and exploiting (8) and (9), (25) becomes

$$E[x'(t)Rx(t)] \leq e^{\alpha T} \left( \frac{c_1^2(\beta_2 + \tau \beta_3)}{\beta_1} + \frac{d^2 \gamma^2}{\beta_1} \right).$$

With condition (10), we obtain  $E[x'(t)Rx(t)] < c_2^2$  for  $t \in [0, T]$ ; that is, the closed-loop system (5) is mean-square finite-time bounded.

Next, we prove that the control output  $z(t)$  satisfies (6) for any non-zero disturbance  $v(t)$  imposing to the zero initial condition. From (7) and (18), we have

$$L_1 V(x(t)) < \alpha V(x(t)) + \gamma^2 v'(t)v(t) - z'(t)z(t). \tag{26}$$

Multiplying both sides of (26) by  $e^{-\alpha t}$ , we have

$$\begin{aligned}
 e^{-\alpha t} L_1 V(x(t)) &< \alpha e^{-\alpha t} V(x(t)) + e^{-\alpha t} [\gamma^2 v'(t)v(t) - z'(t)z(t)]. \tag{27}
 \end{aligned}$$

From lemma 3, we have

$$L_1 [e^{-\alpha t} V(x(t))] = -\alpha e^{-\alpha t} V(x(t)) + e^{-\alpha t} L_1 V(x(t)). \tag{28}$$

Combining (27) and (28), we find

$$L_1 [e^{-\alpha t} V(x(t))] < e^{-\alpha t} [\gamma^2 v'(t)v(t) - z'(t)z(t)]. \tag{29}$$

Integrating from 0 to  $t$ , taking the mathematical expectation on both sides of (29), and imposing the zero initial condition, we have

$$\begin{aligned}
 e^{-\alpha t} EV(x(t)) &< [E \int_0^t e^{-\alpha s} (\gamma^2 v'(s)v(s) - z'(s)z(s))ds] \\
 &< \gamma^2 E \int_0^t e^{-\alpha s} v'(s)v(s)ds \\
 &\quad - E \int_0^t e^{-\alpha s} z'(s)z(s)ds. \tag{30}
 \end{aligned}$$

Because

$$e^{-\alpha t} EV(x(t)) > 0, \tag{31}$$

then (30) implies that

$$E \int_0^t z'(s)z(s)ds < \gamma^2 E \int_0^t v'(s)v(s)ds, \forall t \in [0, T].$$

and the proof is complete.  $\square$

Based on lemma 4, we next derive the following Theorem 1.

*Theorem 1:* For given scalars  $0 < c_1 < c_2, T > 0, d > 0, \gamma > 0$  and a matrix  $R > 0$ , if there are scalars  $\alpha \geq 0, \sigma > 0, \mu > 0, \tau > 0, \varepsilon > 0$ , two symmetric positive definite matrices  $U, W$ , and a matrix  $M$  satisfying

$$\begin{bmatrix} \Psi_{11} & * & * & * & * \\ \Psi_{21} & \Psi_{22} & * & * & * \\ UA'_{12} & 0 & -W & * & * \\ 0 & 0 & \sqrt{3}UA'_{22} & -U & * \\ UB'_{12} & 0 & 0 & 0 & -\gamma^2 I \end{bmatrix} < 0, \quad (32)$$

$$\sigma R^{-1} < U < R^{-1}, \quad (33)$$

$$0 < W < \mu R^{-1}, \quad (34)$$

$$\begin{bmatrix} d^2\gamma^2 - c_2^2 e^{-\alpha T} + c_1^2 \tau \mu & c_1 \\ c_1 & -\sigma \end{bmatrix} < 0, \quad (35)$$

where  $\Psi_{11} = (1 - \alpha)U + W + UA'_{11} + A_{11}U + M'B'_{11} + B_{11}M$ ,  $\Psi_{21} = [\sqrt{3}A_{21}U, C_2U, A_{21}U, 2\varepsilon\sqrt{\beta_2}U]'$ , and  $\Psi_{22} = \text{diag}\{-U, -I, -U, -I\}$ , then (4) is a state feedback finite-time  $H_\infty$  controller. In this case,  $K = MU^{-1}$ .

*Proof:* Noting that  $\tilde{A}_{11} = A_{11} + B_{11}K$ , then according to condition (7), we have

$$\begin{bmatrix} \Gamma_5 & \tilde{P}_1 A_{12} + A'_{21} \tilde{P}_1 A_{22} & \tilde{P}_1 B_{12} \\ * & -\tilde{Q}_1 + 2A'_{22} \tilde{P}_1 A_{22} & 0 \\ * & * & -\gamma^2 I \end{bmatrix} < 0 \quad (36)$$

where  $\Gamma_5 = 4\varepsilon^2 \beta_2 I + \tilde{P}_1 + \tilde{Q}_1 + A'_{11} \tilde{P}_1 + K'B'_{11} \tilde{P}_1 + \tilde{P}_1 A_{11} + \tilde{P}_1 B_{11} K + 2A'_{21} \tilde{P}_1 A_{21} - \alpha \tilde{P}_1 + C'_2 C_2$ .

Obviously, if the following inequality

$$\begin{bmatrix} \Gamma_5 + A'_{21} \tilde{P}_1 A_{21} & \tilde{P}_1 A_{12} & \tilde{P}_1 B_{12} \\ * & -\tilde{Q}_1 + 3A'_{22} \tilde{P}_1 A_{22} & 0 \\ * & * & -\gamma^2 I \end{bmatrix} < 0 \quad (37)$$

holds, then (36) holds.

Using  $\text{diag}\{\tilde{P}_1^{-1}, \tilde{P}_1^{-1}, I\}$  to pre-multiply and post-multiply the inequality (37), we have

$$\begin{bmatrix} \Lambda_{11} & A_{12} \tilde{P}_1^{-1} & B_{12} \tilde{P}_1^{-1} \\ * & \Lambda_{22} & 0 \\ * & * & -\gamma^2 I \end{bmatrix} < 0 \quad (38)$$

where  $\Lambda_{11} = \tilde{P}_1^{-1} \Gamma_5 \tilde{P}_1^{-1} + \tilde{P}_1^{-1} A'_{21} \tilde{P}_1 A_{21} \tilde{P}_1^{-1}$ , and  $\Lambda_{22} = -\tilde{P}_1^{-1} \tilde{Q}_1 \tilde{P}_1^{-1} + 3\tilde{P}_1^{-1} A'_{22} \tilde{P}_1 A_{22} \tilde{P}_1^{-1}$ . Let  $U = \tilde{P}_1^{-1}, M = KU$ , and  $W = \tilde{P}_1^{-1} \tilde{Q}_1 \tilde{P}_1^{-1}$ ; then, using the Schur complement, we obtain (32) from (38).

From the inequality (8) and  $U = \tilde{P}_1^{-1}$ , we obtain

$$\beta_2^{-1} R^{-1} < U < \beta_1^{-1} R^{-1}. \quad (39)$$

Let  $\beta_1^{-1} = 1$  and  $\sigma = \beta_2^{-1}$ ; then, (39) implies inequality (33). According to (39), (9), and  $W = \tilde{P}_1^{-1} \tilde{Q}_1 \tilde{P}_1^{-1}$ , we have

$$0 < W < \beta_3 R^{-1}. \quad (40)$$

Setting  $\mu = \beta_3$ , then (34) is obtained from (40). Considering (33) and (34), (10) is guaranteed with

$$d^2\gamma^2 - c_2^2 e^{-\alpha T} + c_1^2 (\sigma^{-1} + \tau \mu) < 0. \quad (41)$$

Using the Schur complement, (41) is equivalent to (35). Summarizing the process, the proof is complete.  $\square$

*Remark 3:* Note that inequalities given by (32) in Theorem 1 are not linear. However, once the  $\alpha$  and  $\varepsilon$  are fixed, these inequalities can be treated as linear.

### B. DYNAMIC OUTPUT FEEDBACK FINITE-TIME $H_\infty$ CONTROL

The previous subsection assumes that the state variables are available, which does not always hold in practice. In this case, one should estimate  $x(t)$  from the measurement output  $y(t)$ . As usual, consider the following dynamic-output feedback controller

$$\begin{cases} d\hat{x}(t) = (A_f \hat{x}(t) + B_f y(t))dt, \\ u(t) = C_f \hat{x}(t). \end{cases} \quad (42)$$

Substituting (42) into (1) and setting  $\eta(t) = [x'(t) \hat{x}'(t)]'$ , the following augmented closed-loop system is obtained

$$\begin{cases} d\eta(t) = (\bar{A}_{11}\eta(t) + \bar{A}_{12}\eta(t - \tau) + \bar{B}_{12}\bar{v}(t) \\ \quad + \bar{H}_0(x(t)))dt + (\bar{A}_{21}\eta(t) \\ \quad + \bar{A}_{22}\eta(t - \tau) + \bar{H}_1(x(t)))dw(t), \\ y(t) = \bar{C}_1\eta(t), \\ z(t) = \bar{C}_2\eta(t), \end{cases} \quad (43)$$

where  $\bar{v}(t) = [v'(t) \ 0]'$ ,  $\bar{H}_0(x) = [H'_0(x) \ 0]'$ ,  $\bar{H}_1(x) = [H'_1(x) \ 0]'$ ,  $\bar{C}_1 = [C_1 \ 0]$ ,  $\bar{C}_2 = [C_2 \ 0]$ ,

$$\bar{A}_{11} = \begin{bmatrix} A_{11} & B_{11}C_f \\ B_f C_1 & A_f \end{bmatrix}, \quad \bar{A}_{12} = \begin{bmatrix} A_{12} & 0 \\ 0 & 0 \end{bmatrix}, \\ \bar{A}_{21} = \begin{bmatrix} A_{21} & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{A}_{22} = \begin{bmatrix} A_{22} & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{B}_{12} = \begin{bmatrix} B_{12} & 0 \\ 0 & 0 \end{bmatrix}.$$

The problem concerning the dynamic-output feedback finite-time  $H_\infty$  controller is stated next.

*Definition 3:* For given scalars  $0 < c_1 < c_2, T > 0, d > 0, \gamma > 0$ , and a positive definite matrix  $\tilde{R}$ , if there exists a dynamic-output feedback controller (42) such that

- (i) the closed-loop system (43) is mean-square finite-time bounded with respect to  $(c_1, c_2, T, R, d^2)$ , that is,

$$\sup_{-\tau \leq t_0 \leq 0} \eta'(t_0) \tilde{R} \eta(t_0) \leq c_1^2 \Rightarrow E [\eta'(t) \tilde{R} \eta(t)] < c_2^2,$$

where  $\tilde{R} = \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}$ ,  $\forall t \in [0, T], \forall v(t) \in \Psi$ , and

- (ii) for any non-zero disturbance  $v(t)$ , the control output  $z(t)$  satisfies (6) with zero initial condition,

then (42) is said to be a dynamic-output feedback finite-time  $H_\infty$  controller for system (1).

Next, a sufficient condition for the existence of the dynamic-output feedback finite-time  $H_\infty$  controller are given below. First, an important lemma is proved.

*Lemma 5:* For given scalars  $0 < c_1 < c_2, T > 0, d > 0, \gamma > 0$ , and a matrix  $R > 0$ , if there are scalars  $\alpha \geq 0, \gamma > 0$ ,

$\delta_1 > 0, \delta_2 > 0, \delta_3 > 0, \tau > 0, \varepsilon > 0$  and two symmetric positive definite matrices  $P_2, Q_2$  such that

$$\begin{bmatrix} \Gamma_6 & \tilde{P}_2\tilde{A}_{12} + \tilde{A}'_{21}\tilde{P}_2\tilde{A}_{22} & \tilde{P}_2\tilde{B}_{12} \\ * & -\tilde{Q}_2 + 2\tilde{A}'_{22}\tilde{P}_2\tilde{A}_{22} & 0 \\ * & * & -\gamma^2 I \end{bmatrix} < 0, \quad (44)$$

$$\delta_1 I < P_2 < \delta_2 I, \quad (45)$$

$$0 < Q_2 < \delta_3 I, \quad (46)$$

$$d^2\gamma^2 + c_1^2(\delta_2 + \tau\delta_3) < c_2^2\delta_1 e^{-\alpha T}, \quad (47)$$

hold, where  $\Gamma_6 = 4\varepsilon^2\delta_2 I + \tilde{P}_2 + \tilde{Q}_2 + \tilde{A}'_{11}\tilde{P}_2 + \tilde{P}_2\tilde{A}_{11} + 2\tilde{A}'_{21}\tilde{P}_2\tilde{A}_{21} - \alpha\tilde{P}_2 + \tilde{C}'_2\tilde{C}_2, \tilde{P}_2 = \tilde{R}^{\frac{1}{2}}P_2\tilde{R}^{\frac{1}{2}}, \tilde{Q}_2 = \tilde{R}^{\frac{1}{2}}Q_2\tilde{R}^{\frac{1}{2}}$ , then system (43) is mean-square finite-time bounded with respect to  $(c_1, c_2, T, R, d^2)$  and satisfies (6).

*Proof:* The proof is divided into two parts. First, the closed-loop system (43) is proved to be mean-square finite-time bounded.

Note that

$$\begin{bmatrix} \tilde{C}'_2\tilde{C}_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \tilde{C}'_2 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \tilde{C}_2 \\ 0 \\ 0 \end{bmatrix}' \geq 0. \quad (48)$$

Therefore, condition (44) implies

$$\begin{bmatrix} \Gamma_7 & \tilde{P}_2\tilde{A}_{12} + \tilde{A}'_{21}\tilde{P}_2\tilde{A}_{22} & \tilde{P}_2\tilde{B}_{12} \\ * & -\tilde{Q}_2 + 2\tilde{A}'_{22}\tilde{P}_2\tilde{A}_{22} & 0 \\ * & * & -\gamma^2 I \end{bmatrix} < 0, \quad (49)$$

where  $\Gamma_7 = 4\varepsilon^2\delta_2 I + \tilde{P}_2 + \tilde{Q}_2 + \tilde{A}'_{11}\tilde{P}_2 + \tilde{P}_2\tilde{A}_{11} + 2\tilde{A}'_{21}\tilde{P}_2\tilde{A}_{21} - \alpha\tilde{P}_2$ .

From (3) and (45), we have

$$\tilde{H}'_0(x)P_2\tilde{H}_0(x) \leq \varepsilon^2\delta_2 \|x\|^2 \leq \varepsilon^2\delta_2 \|\eta\|^2, \quad (50)$$

$$\tilde{H}'_1(x)P_2\tilde{H}_1(x) \leq \varepsilon^2\delta_2 \|x\|^2 \leq \varepsilon^2\delta_2 \|\eta\|^2. \quad (51)$$

Let  $V(\eta(t)) = \eta'(t)\tilde{P}_2\eta(t) + \int_{t-\tau}^t \eta'(s)\tilde{Q}_2\eta(s)ds$ , and  $L_2V(\eta(t))$  denote the differential generation operator of system (43). According to the Itô formula, we have

$$\begin{aligned} L_2V(\eta(t)) &= [\tilde{A}_{11}\eta(t) + \tilde{A}_{12}\eta(t-\tau) + \tilde{B}_{12}\tilde{v}(t) \\ &\quad + \tilde{H}_0(x(t))]' \frac{\partial V}{\partial \eta} + \frac{1}{2}[(\tilde{A}_{21}\eta(t) \\ &\quad + \tilde{A}_{22}\eta(t-\tau) + \tilde{H}_1(x(t))]' \frac{\partial^2 V}{\partial \eta^2} [\tilde{A}_{21}\eta(t) \\ &\quad + \tilde{A}_{22}\eta(t-\tau) + \tilde{H}_1(x(t))], \end{aligned} \quad (52)$$

that is,

$$\begin{aligned} L_2V(\eta(t)) &= \eta'(t)[\tilde{Q}_2 + \tilde{A}'_{21}\tilde{P}_2\tilde{A}_{21} + \tilde{P}_2\tilde{A}_{11} \\ &\quad + \tilde{A}'_{11}\tilde{P}_2]\eta(t) + \eta'(t-\tau)[- \tilde{Q}_2 \\ &\quad + \tilde{A}'_{22}\tilde{P}_2\tilde{A}_{22}]\eta(t-\tau) \\ &\quad + \eta'(t-\tau)[\tilde{A}'_{12}\tilde{P}_2 + \tilde{A}'_{22}\tilde{P}_2\tilde{A}_{21}]\eta(t) \\ &\quad + \eta'(t)[\tilde{P}_2\tilde{A}_{12} + \tilde{A}'_{21}\tilde{P}_2\tilde{A}_{22}]\eta(t-\tau) \\ &\quad + \eta'(t)\tilde{P}_2\tilde{B}_{12}\tilde{v}(t) + 2\tilde{H}'_1(x)\tilde{P}_2\tilde{A}_{22}\eta(t-\tau) \end{aligned}$$

$$\begin{aligned} &+ 2\tilde{H}'_0(x)\tilde{P}_2\eta(t) + 2\tilde{H}'_1(x)\tilde{P}_2\tilde{A}_{21}\eta(t) \\ &+ \tilde{H}'_1(x)\tilde{P}_2\tilde{H}_1(x) + \tilde{v}'(t)\tilde{B}'_{12}\tilde{P}_2\eta(t). \end{aligned} \quad (53)$$

From lemma 2, we obtain

$$\begin{aligned} &2\tilde{H}'_0(x)\tilde{P}_2\eta(t) + 2\tilde{H}'_1(x)\tilde{P}_2\tilde{A}_{21}\eta(t) \\ &\quad + 2\tilde{H}'_1(x)\tilde{P}_2\tilde{A}_{22}\eta(t-\tau) \\ &\leq 2\tilde{H}'_1(x)\tilde{P}_2\tilde{H}_1(x) + \tilde{H}'_0(x)\tilde{P}_2\tilde{H}_0(x) \\ &\quad + \eta'(t-\tau)\tilde{A}'_{22}\tilde{P}_2\tilde{A}_{22}\eta(t-\tau) \\ &\quad + \eta'(t)(\tilde{P}_2 + \tilde{A}'_{21}\tilde{P}_2\tilde{A}_{21})\eta(t). \end{aligned} \quad (54)$$

Combining (50), (51), (53), and (54), we find

$$\begin{aligned} L_2V(\eta(t)) &\leq \begin{bmatrix} \eta(t) \\ \eta(t-\tau) \\ \tilde{v}(t) \end{bmatrix}' \begin{bmatrix} \Gamma_8 & \tilde{P}_2\tilde{A}_{12} + \tilde{A}'_{21}\tilde{P}_2\tilde{A}_{22} & \tilde{P}_2\tilde{B}_{12} \\ * & -\tilde{Q}_2 + 2\tilde{A}'_{22}\tilde{P}_2\tilde{A}_{22} & 0 \\ * & * & 0 \end{bmatrix} \\ &\quad \times \begin{bmatrix} \eta(t) \\ \eta(t-\tau) \\ \tilde{v}(t) \end{bmatrix} \end{aligned} \quad (55)$$

where  $\Gamma_8 = 4\varepsilon^2\delta_2 I + \tilde{P}_2 + \tilde{Q}_2 + \tilde{A}'_{11}\tilde{P}_2 + \tilde{P}_2\tilde{A}_{11} + 2\tilde{A}'_{21}\tilde{P}_2\tilde{A}_{21}$ . From (49) and (55), we have

$$L_2V(\eta(t)) < \alpha V(\eta(t)) + \gamma^2\tilde{v}'(t)\tilde{v}(t). \quad (56)$$

Integrating from 0 to  $t$  and taking the mathematical expectation on both sides of (56), we have

$$\begin{aligned} EV(\eta(t)) &< V(\eta(0)) + \alpha \int_0^t EV(\eta(s))ds \\ &\quad + \gamma^2 E \int_0^t \tilde{v}'(s)\tilde{v}(s)ds. \end{aligned} \quad (57)$$

From lemma 1, we obtain inequality

$$EV(\eta(t)) < \gamma^2 e^{\alpha t} E \int_0^t \tilde{v}'(s)\tilde{v}(s)ds + V(\eta(0))e^{\alpha t}, \quad (58)$$

and because

$$\begin{aligned} EV(\eta(t)) &= E[\eta'(t)\tilde{P}_2\eta(t) + \int_{t-\tau}^t \eta'(s)\tilde{Q}_2\eta(s)ds] \\ &= E[\eta'(t)R^{\frac{1}{2}}P_2R^{\frac{1}{2}}\eta(t) \\ &\quad + \int_{t-\tau}^t \eta'(s)R^{\frac{1}{2}}Q_2R^{\frac{1}{2}}\eta(s)ds] \\ &\geq \lambda_{\min}(P_2)E[\eta'(t)R\eta(t)], \end{aligned} \quad (59)$$

$$\begin{aligned} V(\eta(0))e^{\alpha t} &= \eta'(0)\tilde{P}_2\eta(0)e^{\alpha t} \\ &\quad + e^{\alpha t} \int_{-\tau}^0 \eta'(s)\tilde{Q}_2\eta(s)ds \\ &= \eta'(0)\tilde{R}^{\frac{1}{2}}P_2\tilde{R}^{\frac{1}{2}}\eta(0)e^{\alpha t} \\ &\quad + e^{\alpha t} \int_{-\tau}^0 \eta'(s)\tilde{R}^{\frac{1}{2}}Q_2\tilde{R}^{\frac{1}{2}}\eta(s)ds \\ &\leq \lambda_{\max}(P_2)c_1^2 e^{\alpha T} + \tau\lambda_{\max}(Q_2)c_1^2 e^{\alpha T}, \end{aligned} \quad (60)$$

$$\gamma^2 e^{\alpha t} E \int_0^t \tilde{v}'(s)\tilde{v}(s)ds < \gamma^2 e^{\alpha T} d^2. \quad (61)$$

Equations (58), (59), (60), and (61), lead straightforwardly to

$$E[\eta'(t)\tilde{R}\eta(t)] \leq e^{\alpha T} \left( \frac{c_1^2(\lambda_{\max}(P_2) + \tau\lambda_{\max}(Q_2))}{\lambda_{\min}(P_2)} + \frac{d^2\gamma^2}{\lambda_{\min}(P_2)} \right). \quad (62)$$

On the basis of (45), (46), and (62), we have

$$E[\eta'(t)\tilde{R}\eta(t)] \leq e^{\alpha T} \left( \frac{c_1^2(\delta_2 + \tau\delta_3)}{\delta_1} + \frac{d^2\gamma^2}{\delta_1} \right).$$

From (47),  $E[\eta'(t)\tilde{R}\eta(t)] < c_2^2$  for  $t \in [0, T]$  obtains; that is, the closed-loop system (43) is mean-square finite-time bounded.

In the second part of the proof, the control output  $z(t)$  is proved to satisfy (6) for any non-zero disturbance  $v(t)$  under zero initial condition. From (44) and (55), we have

$$L_2V(\eta(t)) < \alpha V(\eta(t)) + \gamma^2\bar{v}'(t)\bar{v}(t) - z'(t)z(t). \quad (63)$$

Multiplying both sides of (63) by  $e^{-\alpha t}$ , we obtain

$$e^{-\alpha t}L_2V(\eta(t)) < \alpha e^{-\alpha t}V(\eta(t)) + e^{-\alpha t}[\gamma^2\bar{v}'(t)\bar{v}(t) - z'(t)z(t)]. \quad (64)$$

Using lemma 3, we have

$$L_2[e^{-\alpha t}V(\eta(t))] = -\alpha e^{-\alpha t}V(\eta(t)) + e^{-\alpha t}L_2V(\eta(t)). \quad (65)$$

From (64) and (65), one sees that

$$L_2[e^{-\alpha t}V(\eta(t))] < e^{-\alpha t}[\gamma^2\bar{v}'(t)\bar{v}(t) - z'(t)z(t)]. \quad (66)$$

Integrating from 0 to  $t$  and taking the mathematical expectation on both sides of (66) applying the zero initial condition, we have

$$\begin{aligned} & e^{-\alpha t}EV(\eta(t)) \\ & < E\left[\int_0^t e^{-\alpha s}(\gamma^2\bar{v}'(s)\bar{v}(s) - z'(s)z(s))ds\right] \\ & < \gamma^2E\int_0^t e^{-\alpha s}\bar{v}'(s)\bar{v}(s)ds \\ & \quad - E\int_0^t e^{-\alpha s}z'(s)z(s)ds. \end{aligned} \quad (67)$$

Because

$$e^{-\alpha t}EV(\eta(t)) > 0, \quad (68)$$

then (67) implies

$$E\int_0^t z'(s)z(s)ds < \gamma^2E\int_0^t \bar{v}'(s)\bar{v}(s)ds, \quad \forall t \in [0, T].$$

The proof is complete.  $\square$

On the basis of the above analysis, we prove the following Theorem 2.

**Theorem 2:** For given scalars  $0 < c_1 < c_2, T > 0, d > 0, \gamma > 0$ , and a matrix  $R > 0$ , if there are scalars  $\alpha \geq 0, \gamma > 0, \delta_1 > 0, \delta_2 > 0, \delta_3 > 0, \tau > 0, \varepsilon > 0$ , three symmetric positive definite matrices  $P_{22}, Q_{11}, Q_{22}$ , and two matrices  $X, Y$  satisfying (45), (46), (47), and

where  $\Gamma_9 = 4\varepsilon^2\delta_2I + I + \tilde{Q}_{11} + A'_{11} + A_{11} + 2A'_{21}A_{21} - \alpha I + C'_2C_2, \Gamma_{10} = 4\varepsilon^2\delta_2I + \tilde{P}_{22} + \tilde{Q}_{22} + Y + Y' - \alpha\tilde{P}_{22}$ , then (42) is the dynamic-output feedback finite-time  $H_\infty$  controller. In this case,  $A_f = \tilde{P}_{22}^{-1}Y'$  and  $B_f = \tilde{P}_{22}^{-1}X'$ .

*Proof:* Let

$$\tilde{P}_2 = \begin{bmatrix} I & 0 \\ 0 & \tilde{P}_{22} \end{bmatrix}, \quad \tilde{Q}_2 = \begin{bmatrix} \tilde{Q}_{11} & 0 \\ 0 & \tilde{Q}_{22} \end{bmatrix}.$$

According to (44), we have

$$\begin{bmatrix} \Gamma_9 & \Omega_1 & A_{12} + A'_{21}A_{22} & 0 & B_{12} & 0 \\ * & \Gamma_{11} & 0 & 0 & 0 & 0 \\ * & * & -\tilde{Q}_{11} + 2A'_{22}A_{22} & 0 & 0 & 0 \\ * & * & * & -\tilde{Q}_{22} & 0 & 0 \\ * & * & * & * & -\gamma^2I & 0 \\ * & * & * & * & * & -\gamma^2I \end{bmatrix} < 0, \quad (70)$$

where  $\Omega_1 = C'_1B'_f\tilde{P}_{22} + B_{11}C_f, \Gamma_{11} = 4\varepsilon^2\delta_2I + \tilde{P}_{22} + \tilde{Q}_{22} + A'_f\tilde{P}_{22} + \tilde{P}_{22}A_f - \alpha\tilde{P}_{22}$ .

Let  $X = B'_f\tilde{P}_{22}, Y = A'_f\tilde{P}_{22}$ ; we obtain (69), shown at the bottom of the next page.

This completes the proof.  $\square$

#### IV. NUMERICAL ALGORITHMS

In this section, two algorithms are presented that produced the results obtained above. One finds the minimum value of  $\gamma$ ; the other finds the maximum value of  $\tau$ .

By analyzing (32)–(35) in Theorem 1, we find that if these equations have no feasible solutions when  $\alpha = 0$ , then they will have no feasible solutions for all  $\alpha > 0$ . The specifics of the algorithm are as follows.

##### Algorithm 1

*Step 1:* Set the values of  $c_1, c_2, T, R, d$ , and  $\tau$ .

*Step 2:* Using the linear search algorithm, if a series of  $\alpha_i (i = 1, \dots, n)$  can be found that ensure inequalities (32)–(35) have feasible solutions, then move to *Step 3*; otherwise, move to *Step 7*.

*Step 3:* Let  $i = 1$ , then we take  $\alpha_i$ .

*Step 4:* Solve the following minimization problem:

$$\min_{s.t.(32)-(35)} \gamma.$$

*Step 5:* Let  $i = i + 1$ , if  $i + 1 > n$ , then move to *Step 6*; otherwise, let  $\alpha_i = \alpha_{i+1}$ , and return to *Step 4*.

*Step 6:* There are solutions to this problem; print data and then stop.

*Step 7:* There is no solution to this problem; stop.

By analyzing (32)–(35) in Theorem 1, we find that if these equations have no feasible solutions when  $\alpha = 0$  and  $\tau = 0$ , then they will have no feasible solutions for all  $\alpha > 0$  and  $\tau > 0$ . The specific algorithm is as follows.

**Algorithm 2**

- Step 1: Set the values of  $c_1, c_2, T, R, d$  and  $\gamma$ .
- Step 2: Take a series of  $\alpha_i (i = 1, \dots, n)$  and a series of  $\tau_j (j = 1, \dots, n)$ .
- Step 3: Let  $i = 1$  and set  $\alpha_1 = 0$ .
- Step 4: Let  $j = 1$  and set  $\tau_1 = 0$ .
- Step 5: If  $(\alpha_i, \tau_j)$  ensure (32)–(35) have feasible solutions, then store  $(\alpha_i, \tau_j)$  into  $(\Delta(i), \Delta(j))$ . Let  $\tau_j = \tau_{j+1}$  and move to Step 5; otherwise, move to Step 6.
- Step 6: If  $i + 1 < n$ , set  $\alpha_i = \alpha_{i+1}$ , and with take  $\tau_j$ , return to Step 5; otherwise, skip to Step 7.
- Step 7: Stop.

**V. NUMERICAL EXAMPLES**

The matrix of coefficients of the system (1) is given as follows.

$$\begin{aligned}
 A_{11} &= \begin{bmatrix} -30 & 25 \\ -10.38 & -30.16 \end{bmatrix}, & A_{12} &= \begin{bmatrix} 0.2 & 0.1 \\ 0 & 0.3 \end{bmatrix}, \\
 A_{21} &= \begin{bmatrix} -2.7 & 0.8 \\ 0.9 & -1.6 \end{bmatrix}, & A_{22} &= \begin{bmatrix} -0.2 & -0.5 \\ -0.3 & -1.4 \end{bmatrix}, \\
 B_{11} &= \begin{bmatrix} -4 \\ 3 \end{bmatrix}, & B_{12} &= \begin{bmatrix} 0.1 & -0.1 \\ 0.5 & 0.7 \end{bmatrix}, \\
 C_1 &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, & C_2 &= \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}.
 \end{aligned}$$

Let  $c_1 = 1, c_2 = 4, T = 1, d = 1, \tau = 0.5, \varepsilon = 1.2, R = I$ .

**A. STATE FEEDBACK FINITE-TIME  $H_\infty$  CONTROL**

To find the minimum value of  $\gamma$ , the relationship between  $\gamma$  and  $\alpha$  is obtained using Algorithm 1 (see Fig. 1).

From Fig. 1,  $\gamma$  increases with increasing  $\alpha$ , and the minimum value of  $\gamma = 0.0315$  is obtained when  $\alpha = 0$ .

Setting  $\alpha = 0$ , according to Theorem 1, we obtain

$$\begin{aligned}
 U &= \begin{bmatrix} 0.5254 & 0.0079 \\ 0.0079 & 0.4524 \end{bmatrix}, & M &= [0.7103 \quad 0.1159], \\
 W &= \begin{bmatrix} 5.4891 & -0.1883 \\ -0.0929 & 6.1716 \end{bmatrix}, & K &= [1.3484 \quad 0.2326].
 \end{aligned}$$

Therefore, the state feedback controller is as follows:

$$u(t) = [1.3484 \quad 0.2326]x(t).$$

Setting  $\gamma = 0.1$ , then by Algorithm 2, the relationship between  $\tau$  and  $\alpha$  so as to find the maximum value of  $\tau$  was obtained (see Fig. 2).

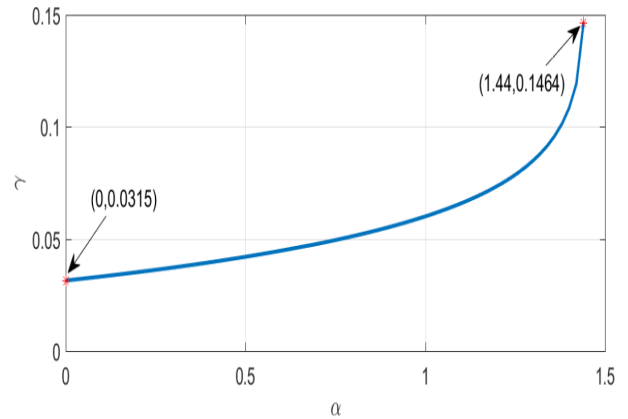


FIGURE 1. Plot of the minimum upper bound of  $\gamma$  when  $\alpha \in [0, 1.5]$ .

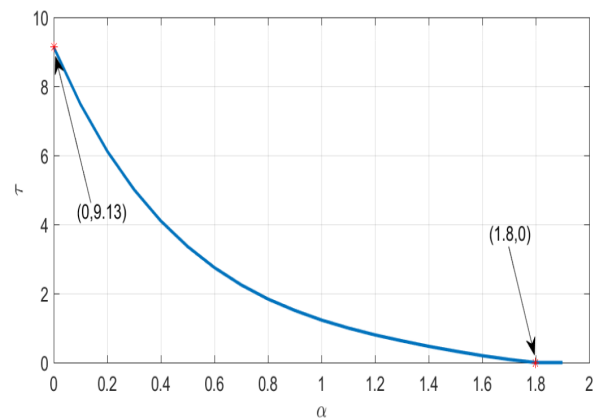


FIGURE 2. When  $\alpha \in [0, 2]$ , the value of  $\tau$ .

From Fig. 2,  $\tau$  decreases with increasing  $\alpha$ , and  $\tau$  has a maximum value in the range of  $\alpha$ ; that is,  $\tau = 9.13$  when  $\alpha = 0$ .

Considering the external disturbance  $v(t) = \sin(t)$  and setting  $x(0) = [0.4, 0.2]^T$ , we obtained the curves for  $x_1(t), x_2(t)$ , and  $E[x'(t)Rx(t)]$  (Fig. 3) from which we find that  $E[x'(0)Rx(0)] = E[x_1^2(0) + x_2^2(0)] = 0.2 \leq c_1^2 = 1$  and  $E[x'(t)Rx(t)] < c_2^2 = 16$ . Therefore, the closed-loop system (5) is mean-square finite-time bounded with respect to  $(1, 4, 1, I, 1)$ .

**B. DYNAMIC OUTPUT FEEDBACK FINITE-TIME  $H_\infty$  CONTROL**

To find the minimum value of  $\gamma$ , the relationship between  $\gamma$  and  $\alpha$  is determined using Algorithm 1 (see Fig. 4).

$$\begin{bmatrix} \Gamma_9 & C_1'X + B_{11}C_f & A_{12} + A_{21}'A_{22} & 0 & B_{12} & 0 \\ * & \Gamma_{10} & 0 & 0 & 0 & 0 \\ * & * & -\tilde{Q}_{11} + 2A_{22}'A_{22} & 0 & 0 & 0 \\ * & * & * & -\tilde{Q}_{22} & 0 & 0 \\ * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & -\gamma^2 I \end{bmatrix} < 0, \tag{69}$$



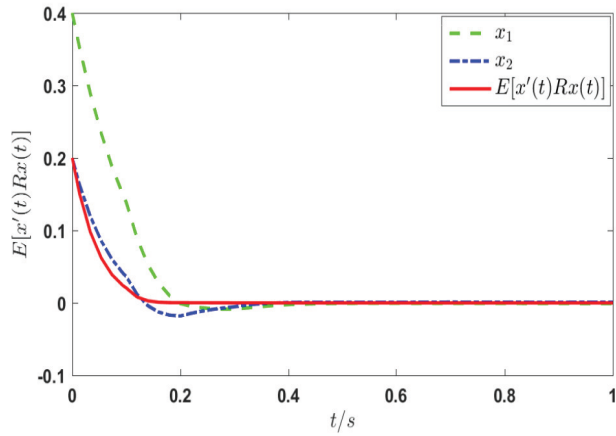


FIGURE 3. Response for  $E[x'(t)Rx(t)]$ .

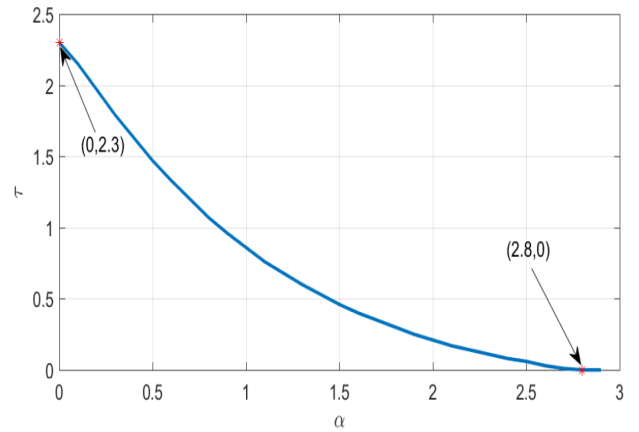


FIGURE 5. When  $\alpha \in [0, 3]$ , the value of  $\tau$ .

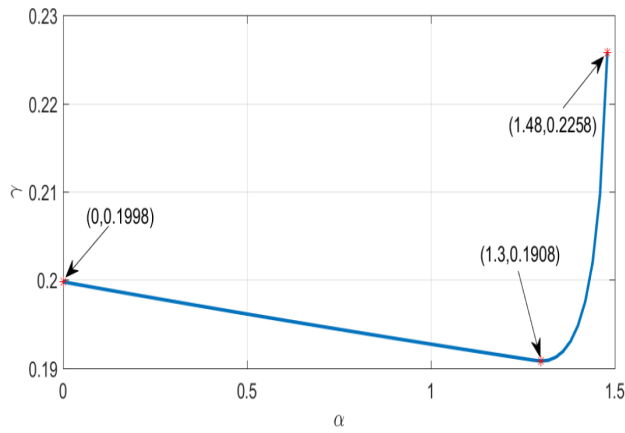


FIGURE 4. Plot of the minimum upper bound of  $\gamma$  when  $\alpha \in [0, 1.5]$ .

From Fig. 4,  $\gamma$  decreases first as  $\alpha$  increases, and then increases as  $\alpha$  increases. The optimal solution  $\gamma = 0.1908$  is obtained when  $\alpha = 1.3$ .

Setting  $\alpha = 1.3$ , then, according to Theorem 2, we derive a set of solutions

$$\begin{aligned} \tilde{Q}_{11} &= \begin{bmatrix} 3.8636 & 0.2661 \\ 0.2661 & 4.9417 \end{bmatrix}, & \tilde{Q}_{22} &= \begin{bmatrix} 3.8779 & 0 \\ 0 & 3.8779 \end{bmatrix}, \\ \tilde{P}_{22} &= \begin{bmatrix} 1.1144 & 0 \\ 0 & 1.1144 \end{bmatrix}, & A_f &= \begin{bmatrix} -208.9 & 0 \\ 0 & -208.9 \end{bmatrix}, \\ B_f &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, & C_f &= [0 \quad 0], \quad \gamma = 0.1908. \end{aligned}$$

Note, for  $B_f = 0, C_f = 0$ , the above solutions do not satisfy the requirements for the coefficients of the dynamic-output feedback controller. Therefore, setting  $X > 0$ , the following solutions are obtained,

$$\begin{aligned} \tilde{Q}_{11} &= \begin{bmatrix} 3.9163 & 0.2632 \\ 0.2632 & 4.9930 \end{bmatrix}, & \tilde{Q}_{22} &= \begin{bmatrix} 3.9376 & 0 \\ 0 & 3.9376 \end{bmatrix}, \\ \tilde{P}_{22} &= \begin{bmatrix} 1.1087 & 0 \\ 0 & 1.1087 \end{bmatrix}, & A_f &= \begin{bmatrix} -1245.3 & -55.5 \\ -55.5 & -836.2 \end{bmatrix}, \\ B_f &= \begin{bmatrix} 124.6 & -203.1 \\ -203.1 & 1620.7 \end{bmatrix}, & C_f &= [30.9107 \quad -56.7711]. \end{aligned}$$

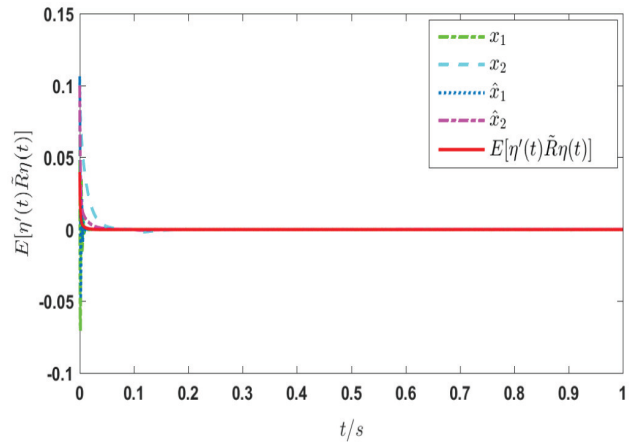


FIGURE 6. Response for  $E[\eta'(t)\tilde{R}\eta(t)]$ .

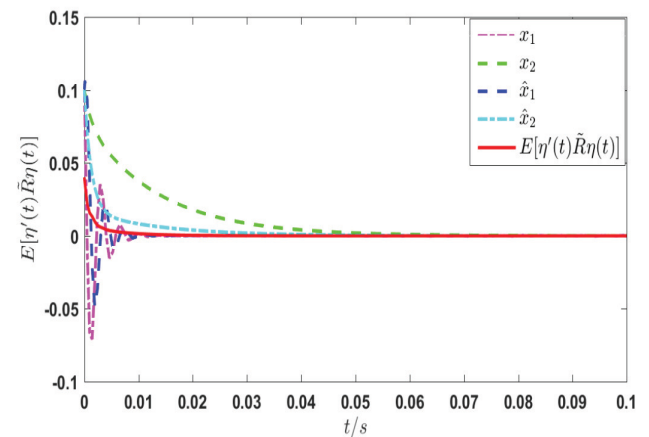


FIGURE 7. Response for  $E[\eta'(t)\tilde{R}\eta(t)]$  when  $t \in [0, 0.1]$ .

Setting  $\gamma = 0.2$ , the relationship between  $\tau$  and  $\alpha$  (Fig. 5) shows that  $\tau$  decreases with increasing  $\alpha$ . Its maximum value is  $\tau = 2.3$  when  $\alpha = 0$ .

Considering the external disturbance  $v(t) = \sin(t)$  and letting  $\eta(0) = [0.1, 0.1, 0.1, 0.1]'$ , we obtain the curves for  $x_1(t), x_2(t), \hat{x}_1(t), \hat{x}_2(t)$ , and  $E[\eta'(t)\tilde{R}\eta(t)]$  (Fig. 6). Magnifying the plot in the interval  $t \in [0, 0.1]$ , (Fig. 7), we see that  $E[\eta'(0)R\eta(0)] = 0.04 \leq c_1^2 = 1$  and

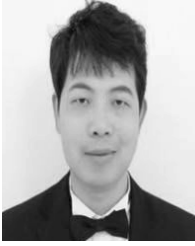
$E[\eta'(t)R\eta(t)] < c_2^2 = 16$ . Therefore, the closed-loop system (43) is mean-square finite-time bounded with respect to  $(1, 4, 1, I, 1)$ .

## VI. CONCLUSION

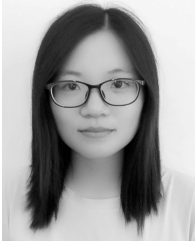
We investigated the finite-time  $H_\infty$  control problem for the Itô-type stochastic nonlinear time-delay systems. Two kinds of controllers, state feedback and dynamic-output feedback finite-time  $H_\infty$  controllers, were designed. Moreover, their corresponding algorithms used in solving the state feedback controller and dynamic-output feedback controller were presented. They also simultaneously optimize the  $H_\infty$  performance index and determine the maximum time-delay  $\tau$ .

## REFERENCES

- [1] Z. Song, P. Li, J. Zhai, Z. Wang, and X. Huang, "Global fixed-time stabilization for switched stochastic nonlinear systems under rational switching powers," *Appl. Math. Comput.*, to be published, doi: 10.1016/j.amc.2019.124856.
- [2] H. Min, S. Xu, B. Zhang, and Q. Ma, "Globally adaptive control for stochastic nonlinear time-delay systems with perturbations and its application," *Automatica*, vol. 102, pp. 105–110, Apr. 2019.
- [3] S. Sui, Y. Li, and S. Tong, "Adaptive fuzzy control design and applications of uncertain stochastic nonlinear systems with input saturation," *Neurocomputing*, vol. 156, pp. 42–51, May 2015.
- [4] H. Hua, J. Cao, G. Yang, and G. Ren, "Voltage control for uncertain stochastic nonlinear system with application to energy Internet: Non-fragile robust  $H_\infty$  approach," *J. Math. Anal. Appl.*, vol. 463, no. 1, pp. 93–110, Jul. 2018.
- [5] T. Bian and Z.-P. Jiang, "New results in global stabilization for stochastic nonlinear systems," *Control Theory Technol.*, vol. 14, no. 1, pp. 57–67, Feb. 2016.
- [6] X. Yu, J. Yin, and S. Khoo, "Generalized Lyapunov criteria on finite-time stability of stochastic nonlinear systems," *Automatica*, vol. 107, pp. 183–189, Sep. 2019.
- [7] F. Wang, B. Chen, Y. Sun, and C. Lin, "Finite time control of switched stochastic nonlinear systems," *Fuzzy Sets Syst.*, vol. 365, pp. 140–152, Jun. 2019.
- [8] B. Li, Z. Wang, Q.-L. Han, and H. Liu, "Input-to-State stabilization in probability for nonlinear stochastic systems under quantization effects and communication protocols," *IEEE Trans. Cybern.*, vol. 49, no. 9, pp. 3242–3254, Sep. 2019.
- [9] B. Li, Z. Wang, L. Ma, and H. Liu, "Observer-based event-triggered control for nonlinear systems with mixed delays and disturbances: The input-to-state stability," *IEEE Trans. Cybern.*, vol. 49, no. 7, pp. 2806–2819, Jul. 2019.
- [10] W. Qi, M. Gao, C. K. Ahn, J. Cao, J. Cheng, and L. Zhang, "Quantized fuzzy finite-time control for nonlinear semi-Markov switching systems," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, early access, Dec. 25, 2019, doi: 10.1109/TCSII.2019.2962250.
- [11] Z. Hao and Z. Lu, "The improved stability criterion of uncertain stochastic system with time-delay," *Optik*, vol. 130, pp. 854–863, Feb. 2017.
- [12] Y. Yang, Z. Yu, S. Li, and J. Sun, "Adaptive neural output feedback control for stochastic nonlinear time-delay systems with input and output quantization," *Neurocomputing*, vol. 282, pp. 146–156, Mar. 2018.
- [13] H. Wang, B. Chen, and C. Lin, "Adaptive neural control for strict-feedback stochastic nonlinear systems with time-delay," *Neurocomputing*, vol. 77, no. 1, pp. 267–274, Feb. 2012.
- [14] B. Li, Z. Wang, and Q.-L. Han, "Input-to-State stabilization of delayed differential systems with exogenous disturbances: The event-triggered case," *IEEE Trans. Syst., Man, Cybern. Syst.*, vol. 49, no. 6, pp. 1099–1109, Jun. 2019.
- [15] M. Gao, L. Sheng, and W. Zhang, "Stochastic  $H_2/H_\infty$  control of nonlinear systems with time-delay and state-dependent noise," *Appl. Math. Comput.*, vol. 266, pp. 429–440, Sep. 2015.
- [16] H. Xue and M. Popov, "Analysis of switching transient overvoltages in the power system of floating production storage and offloading vessel," *Electric Power Syst. Res.*, vol. 115, pp. 3–10, Oct. 2014.
- [17] T. Zhang, F. Deng, and W. Zhang, "Finite-time stability and stabilization of linear discrete time-varying stochastic systems," *J. Franklin Inst.*, vol. 356, no. 3, pp. 1247–1267, Feb. 2019.
- [18] Z. Yan, Y. Song, and J. H. Park, "Finite-time stability and stabilization for stochastic Markov jump systems with mode-dependent time delays," *ISA Trans.*, vol. 68, pp. 141–149, May 2017.
- [19] S. He, Q. Ai, C. Ren, J. Dong, and F. Liu, "Finite-time resilient controller design of a class of uncertain nonlinear systems with time-delays under asynchronous switching," *IEEE Trans. Syst., Man, Cybern. Syst.*, vol. 49, no. 2, pp. 281–286, Feb. 2019.
- [20] J. Cheng, J. H. Park, X. Zhao, J. Cao, and W. Qi, "Static output feedback control of switched systems with quantization: A nonhomogeneous sojourn probability approach," *Int. J. Robust Nonlinear Control*, vol. 29, no. 17, pp. 5992–6005, Nov. 2019.
- [21] W. Qi, G. Zong, and H. R. Karimi, "Finite-time observer-based sliding mode control for quantized semi-Markov switching systems with application," *IEEE Trans. Ind. Informat.*, vol. 16, no. 2, pp. 1259–1271, Feb. 2020.
- [22] Z. Yan, Y. Song, and X. Liu, "Finite-time stability and stabilization for Itô-type stochastic Markovian jump systems with generally uncertain transition rates," *Appl. Math. Comput.*, vol. 321, pp. 512–525, Mar. 2018.
- [23] G. Wang, L. Liu, Q. Zhang, and C. Yang, "Finite-time stability and stabilization of stochastic delayed jump systems via general controllers," *J. Franklin Inst.*, vol. 354, no. 2, pp. 938–966, Jan. 2017.
- [24] W. Qi, X. Yang, C. K. Ahn, J. Cao, and J. Cheng, "Input-output finite-time sliding-mode control for T-S fuzzy systems with application," *IEEE Trans. Syst., Man, Cybern. Syst.*, early access, Dec. 5, 2019, doi: 10.1109/TSMC.2019.2954854.
- [25] C. Wang and Y. Shen, "Robust  $H_\infty$  control for stochastic systems with nonlinearity, uncertainty and time-varying delay," *Comput. Math. Appl.*, vol. 63, no. 5, pp. 985–998, 2012.
- [26] T. Senthilkumar and P. Balasubramaniam, "Robust  $H_\infty$  control for nonlinear uncertain stochastic T-S fuzzy systems with time delays," *Appl. Math. Lett.*, vol. 24, no. 12, pp. 1986–1994, 2011.
- [27] S. Lu and W. Zhang, "Robust  $H_\infty$  filtering and control for a class of linear systems with fractional stochastic noise," *Phys. A, Stat. Mech. Appl.*, vol. 526, Jul. 2019, Art. no. 120958.
- [28] Y. Ren, W. Wang, W. Zhou, and M. Shen, "Stochastic incremental  $H_\infty$  control for discrete-time switched systems with disturbance dependent noise," *Inf. Sci.*, vol. 513, pp. 519–535, Mar. 2020.
- [29] H. Liu, X. Li, X. Liu, and H. Wang, "Backstepping-based decentralized bounded- $H_\infty$  adaptive neural control for a class of large-scale stochastic nonlinear systems," *J. Franklin Inst.*, vol. 356, no. 15, pp. 8049–8079, 2019.
- [30] D. Zhang, J. Cheng, D. Zhang, and K. Shi, "Nonfragile  $H_\infty$  control for periodic stochastic systems with probabilistic measurement," *ISA Trans.*, vol. 86, pp. 39–47, Mar. 2019.
- [31] X. Xing, Y. Liu, and B. Niu, " $H_\infty$  control for a class of stochastic switched nonlinear systems: An average dwell time method," *Nonlinear Anal., Hybrid Syst.*, vol. 19, pp. 198–208, Feb. 2016.
- [32] Y. Ma and Y. Liu, "Finite-time  $H_\infty$  sliding mode control for uncertain singular stochastic system with actuator faults and bounded transition probabilities," *Nonlinear Anal., Hybrid Syst.*, vol. 33, pp. 52–75, Aug. 2019.
- [33] Y. Zhang, C. Liu, and X. Mu, "Robust finite-time  $H_\infty$  control of singular stochastic systems via static output feedback," *Appl. Math. Comput.*, vol. 218, no. 9, pp. 5629–5640, Jan. 2012.
- [34] Y. Zhang, Y. Shi, and P. Shi, "Resilient and robust finite-time  $H_\infty$  control for uncertain discrete-time jump nonlinear systems," *Appl. Math. Model.*, vol. 49, pp. 612–629, Sep. 2017.
- [35] Z. Yan, G. Zhang, and W. Zhang, "Finite-time stability and stabilization of linear Itô stochastic systems with state and control-dependent noise," *Asian J. Control*, vol. 15, no. 1, pp. 270–281, Jan. 2013.
- [36] Z. Yan, G. Zhang, and J. Wang, "Non-fragile robust finite-time  $H_\infty$  control for nonlinear stochastic Itô systems using neural network," *Int. J. Control, Autom. Syst.*, vol. 10, no. 5, pp. 873–882, Oct. 2012.
- [37] Z. Yan, W. Zhang, and G. Zhang, "Finite-time stability and stabilization of Itô stochastic systems with Markovian switching: Mode-dependent parameter approach," *IEEE Trans. Autom. Control*, vol. 60, no. 9, pp. 2428–2433, Sep. 2015.



**ZHIGUO YAN** (Member, IEEE) received the B.E. degree in electrical engineering and automation from Northeast Normal University, in 2005, and the M.E. degree in control theory and control engineering from Shandong Polytechnic University, in 2008, and the Ph.D. degree in control theory and control engineering from Tianjin University, in 2011. He is currently a Professor with the School of Electrical Engineering and Automation, Qilu University of Technology. He has authored or coauthored over 60 journals and conference papers. His research interests include robust control, stochastic control, and robot control.



**MIN ZHANG** was born in Zibo, Shandong, China, in 1995. She received the B.S. degree in communications engineering from the Qilu University of Technology (Shandong Academy of Sciences), in 2018, where she is currently pursuing the M.S. degree. Her current research direction is stability analysis of stochastic systems.



**YUNXIA SONG** received the B.S. and M.S. degrees from the Department of Electrical Engineering and Automations, Qilu University of Technology (Shandong Academy of Sciences), Jinan, China, in 2014 and 2018, respectively. She is currently pursuing the Ph.D. degree with the Department of Control Science and Engineering, Harbin Institute of Technology, Harbin, China. Her research interests include time-delay systems and stochastic systems.



**SHIYU ZHONG** was born in Yantai, Shandong, China, in 1993. She received the B.S. degree in automation and M.S. degree in control science and engineering from the Qilu University of Technology (Shandong Academy of Sciences), in 2016 and 2019, respectively. She is currently pursuing the Ph.D. degree in control theory and control engineering with the Shandong University of Science and Technology.

...