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Hamiltonian Cycle in Folded Hypercubes With Highly Conditional Edge Faults

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ABSTRACT As an extension of the n-dimensional hypercube Q_n, the n-dimensional folded hypercube denoted as FQ_n, which can be structured from Q_n adding an edge to every pair of vertices with complementary addresses.FQ_n possesses many properties superior to those of Q_n, such as diameter, fault diameter, connectivity, and so on. In this paper, let FF_e denote the set of faulty edges in FQ_n and assume that each vertex is incident to at least three fault-free edges in $FQ_n - FF_e$. Then, we show that $FQ_n - FF_e$ contains a fault-free Hamiltonian cycle of length 2^n, where $n \geq 3$ and $|FF_e| \leq 3n - 7$.

INDEX TERMS Interconnection networks, hypercubes, folded hypercubes, Hamiltonian cycle.

I. INTRODUCTION

Since processors in multiprocessor systems are communicated by an interconnection network (network for short), the demand performance of designing networks are important and indispensable. Therefore, a large number of network topologies are widely designed by researchers. Among the previously proposed well-known network topologies, the hypercube [1] has several excellent properties such as recursive structure, symmetry, small diameter, regularity, relatively short mean internode distance, low degree, and much small edge complexity, which are very important for designing massively parallel or distributed systems [2]. Since the hypercube structure has many excellent properties, many variant or extended structures by the hypercube have been proposed [3]–[5]. Among them, one of the excellent topology is *folded hypercubes* [3], which is an extension of the hypercube, constructed by adding an edge to every pair of vertices that are the farthest apart. The folded hypercube has been shown that which can improve the performance of the hypercube in many measurements [3], [6].

The embedding of the *guest network G* into the *host network H* is a one-to-one mapping from the vertex set of G into the vertex set H [2]. Cycle (Linear array), which is an important network for parallel and distributed computing. Cycle is suitable for designing simple algorithms with low communication costs, and which is also can be used as control/data flow structures for the distributed computing in arbitrary networks [2], [7]. Therefore, these excellent applications motivate us to embed the cycle properties in networks.

Since edges and/or vertices may fail when a network is put into use, it is meaningful to consider faulty networks. Usually, two models are used to consider the fault-tolerant embedding properties in faulty networks. One is the *standard fault model*, which means that the distribution of faulty edges and/or faulty vertices are not restricted; the other one is the Latifi's *conditional fault model* [8], which means that each vertex in networks is incident to at least g fault-free neighbors, where $g \geq 2$. In order to describe conditional fault model with different condition g conveniently, we denote *g-conditional* to represent that each vertex is incident to at least g fault-free neighbors in a faulty network. In this paper, our research topic is focusing on the edge fault-tolerant cycle embedding property in the n-dimensional folded hypercube network FQ_n. Previously, properties of the fault- tolerant cycle embedding in FQ_n has been studied in [6], [9]–[21]. Let FF_v and FF_e denote the sets of faulty vertices and faulty edges in FQ_n, respectively. We briefly summarize of previously reported properties by researchers and our main property of this paper for fault-tolerant cycle embedding in FQ_n as Table 1.

The rest of this paper is organized as follows: In Section 2, some necessary definitions and notations are presented.

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TABLE 1. A briefly summarize of the fault-tolerant cycle embedding properties in FQ_n.

Range of faults	Fault model	Property of cycle embedding in FQ_n	Original
$\|FF_e\| \le n-1$	Standard	A cycle of length 2^n	[6]
$\|FF_e\| \le n-1$	Standard	Every fault-free edge lies on a cycle of every even length from 4 to 2^n and every fault-free edge lies on a cycle of every odd length from $n+1$ to 2^n-1 if n being even.	[22]
$\|FF_e\| \le 2n-3$	2-Conditional	A cycle of length 2^n if $n \ge 3$	[23]
$\|FF_e\| \le n-1$; $\|FF_v\| + \|FF_e\| \le 2n-4$	Standard	A cycle of length at least $2^n - 2\|FF_v\|$ if $n \ge 3$	[24]
$\|FF_v\| + \|FF_e\| \le 2n-4$	2-Conditional	A cycle of length at least $2^n - 2\|FF_v\|$ if $n \ge 3$	[9]
$\|FF_e\| \le 2n-3$	2-Conditional	A cycle of every even length from 4 to 2^n if $n \ge 2$ and a cycle of every odd length from $n+1$ to 2^n-1 if $n \ge 2$ being even.	[10]
$\|FF_e\| \le 2n-5$	2-Conditional	Every fault-free edge lies on a cycle of every odd length from $n+1$ to 2^n-1 if $n \ge 4$ being even.	[12]
$\|FF_e\| \le 2n-4$	2-Conditional	Every fault-free edge lies on a cycle of every even length from 6 to 2^n if $n \ge 5$ being odd.	[13]
$\|FF_v\| \le n-2$	Standard	Every fault-free edge lies on a cycle of every even length from 4 to $2^n - 2\|FF_v\|$ if $n \ge 3$; and every fault-free edge lies on a cycle of every odd length from $n+1$ to $2^n - 2\|FF_v\| - 1$ if $n \ge 2$ being even.	[11]
$\|FF_v\| + \|FF_e\| \le n-2$	Standard	Every fault-free edge lies on a cycle of every even length from 4 to $2^n - 2\|FF_v\|$ if $n \ge 3$; and every fault-free edge lies on a cycle of every odd length from $n+1$ to $2^n - 2\|FF_v\| - 1$ if $n \ge 2$ being even.	[16]
$\|FF_v\| + \|FF_e\| \le n-1$	Standard	A cycle of every even length from 4 to $2^n - 2\|FF_v\|$ if $n \ge 3$; and a cycle of every odd length from $n+1$ to $2^n - 2\|FF_v\| - 1$ if $n \ge 4$ being even.	[14]
$\|FF_v\| \le n-7$	4-conditional	A cycle of every even length from 4 to $2^n - 2\|FF_v\|$ if $n \ge 4$; and a cycle of every odd length from $n+1$ to $2^n - 2\|FF_v\| - 1$ if $n \ge 4$ being even.	[15]
$\|FF_v\| \le 2n-5$	4-conditional	A cycle of every even length from 4 to $2^n - 2\|FF_v\|$ if $n \ge 3$; and a cycle of every odd length from $n+1$ to $2^n - 2\|FF_v\| - 1$ if $n \ge 4$ being even.	[17]
$\|FF_e\| \le 3n-7$	3-conditional	A Hamiltonian cycle of length 2^n if $n \ge 3$.	this paper

Section 3 provides the main property of the Hamiltonian cycle embedding in folded hyperucbes. Conclusions are given in Section 4.

II. PRELIMINARIES

The topology of an interconnection network usually can be conveniently represented as a graph G. Let $G = (V, E)$ be a graph in which V is a finite set and E is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. We can say that V is the vertex set and E is the edge set. The sets of $V(G)$ and $E(G)$ can be denoted as the vertex set and the edge set of G, respectively. The number of vertices and edges in G can be denoted as $|V(G)|$ and $|E(G)|$, respectively. Vertices u and v are adjacent if the edge $(u, v) \in E(G)$. Vertices u and v are called the *end-vertices* of an edge $e = (u, v)$. We call u is adjacent to v, and vice versa. Furthermore, we also call u (or v) is incident to $e = (u, v)$. Let F_e and F_v denote the sets of faulty edges and faulty vertices in G, where $F_e \subseteq E(G)$ and $F_v \subseteq V(G)$, respectively. We use $G - F_v - F_e$ to denote the subgraph which obtained by deleting F_v and F_e from G. A graph $G = (V_0 \cup V_1, E)$ is bipartite if $V_0 \cap V_1 = \phi$ and every edge in $E(G)$ joins V_0 with V_1. We also call V_0 and V_1 with different partite sets. A path $P[v_0, v_m] = \langle v_0, v_1, \ldots, v_m\rangle$ is a sequence of adjacent vertices in which all the vertices $v_0, v_1, \ldots, v_m$ are distinct. We call v_0 and v_m are the *end-vertices* of the path. Further- more, a path may contain a *sub-path*, denoted as $\langle v_0, v_1, \ldots, vi, P[v_i, v_j], v_j, v_{j+1}, \ldots, v_m\rangle$, where $P[v_i, v_j] = \langle v_i, v_{i+1}, \ldots, vj-1, v_j\rangle$. The length of a path $P[v_0, v_m]$ can be denoted as $l(P[v_0, v_m])$ which means the number of edges in $P[v_0, v_m]$. A cycle in G is a sequence of vertices $\langle v_0, v_1, \ldots, v_m, v_0\rangle$, $m \ge 2$, where all the vertices $v_0, v_1, \ldots, v_m$ are distinct and any two consecutive vertices are adjacent. A *Hamiltonian* cycle is a cycle with length $|V(G)|$ and a Hamiltonian path is a path with length $|V(G)| - 1$. An isomorphism from G to H is a one-to-one and onto function $\pi : |V(G)| \to |V(H)|$ such that $(u, v) \in |E(G)|$ if and only if $(\pi(u), \pi(v)) \in |E(H)|$. Therefore, we can denote that $G \cong H$, if there is an isomorphism from G to H. An *automorphism* of the graph G is an isomorphism from G to G. An edge is *fault-free* if the two end-vertices and the edge between them are not faulty. A path is *fault-free* if it contains no faulty edges. A graph G is *Hamiltonian-connected* if there is a Hamiltonian path joining any two vertices of G [5]. A bipartite graph G is *Hamiltonian-laceable* if there exists a Hamiltonian path between any two vertices from different partite sets. A Hamiltonian-laceable graph G=$(V_0 \cup V_1, E)$ is *strong* [25] if there is a path of length $|V_0| + |V_1| - 2$ between any two vertices of the same partite set. A Hamiltonian-laceable graph $G = (V_0 \cup V_1, E)$ is *hyper-Hamiltonian laceable* [26] if for any vertex $v \in V_i$, $i \in \{0, 1\}$, there is a path in $G - v$ between any two vertices of V_{1-i} with length $|V_0| + |V_1| - 2$. For graph-theoretic terminologies and notations not mentioned here, see [27].

Let n be a positive interger. The graph of the n-dimensional hypercube (Q_n for short) contains 2^n vertices. Every vertex of Q_n is labeled by an n-bit binary strings

$V(Q_n) = \{x_n x_{n-1} \ldots x_i \ldots x_1 | x_i \in \{0, 1\}, 1 \le i \le n$ from $\underbrace{00\ldots0}_{n}$ to $\underbrace{11\ldots1}_{n}$. Every edge $e = (u, v) \in E(Q_n)$ connects two vertices u and v if and only if u and v with different labels in exactly one bit, i.e., $u = b_n b_{n-1} \ldots b_k \ldots b_1$ and $v = b_n b_{n-1} \ldots \overline{b_k} \ldots b_1$, where b_k is the 1's complement of b_k. We call that e is an edge of dimension k. Clearly, In Q_n, each vertex has degree n, each dimension possesses 2^{n-1} edges, and $|E(Q_n)| = n \times 2^{n-1}$.

Let $u = u_n u_{n-1} \ldots u_i \ldots u_1$ and $v = v_n v_{n-1} \ldots v \ldots v_1$ are two n-bit binary strings; and let $v = u^{(k)}$, where $1 \le k \le n$, if $v_k = 1 - u_k$ and $v_i = u_i$ for all $i \ne k$ and $1 \le i \le n$. Furthermore, let $v = \bar{u}$ if $v_i = 1 - u_i$ for all $1 \le i \le n$. The *Hamming distance* $d_H(u, v)$ between two vertices u and v is the number of different bits in the corresponding strings of the vertices. The *Hamming weight* $hw(u)$ of the vertex $u = u_n u_{n-1} \ldots u_i \ldots u_1$ is the number of i's such that $u_i = 1$, where $1 \le k \le n$. Note that Q_n is a bipartite graph with two partite sets $\{u \mid hw(u) \text{isodd}\}$ e and $u|hw(u)$ iseven$\}$. Let $d_{Q_n}(u, v)$ be the distance of the shortest path between two vertices u and v in Q_n. Then, $d_{Q_n}(u, v) = d_H(u, v)$.

An n-dimensional folded hypercube (FQ_n for short) is a regular Q_n augmented by adding an edge (also called *complementary edge)* to every pair of vertices that are the farthest apart, i.e., A vertex whose address is $u = u_n u_{n-1} \ldots u_1$, it now has one more edge to the vertex $\bar{u} = \bar{u}_n \bar{u}_{n-1} \ldots \bar{u}_1$, in addition to its original n edges in Q_n. Therefore, FQ_n has 2^{n-1} more edges than Q_n. We use E_c to denote the set of these complementary edges, and use E_r to denote the set of regular edges in FQ_n. Formally, $E_r = E_1 \cup E_2 \cup \ldots \cup E_n$, where E_i denote the i-dimensional edge set in FQ_n, $1 \le i \le n$. Therefore, $E(FQ_n) = E_r \cup E_c = \{e = (u, v) | d_H(u, v) = 1 \in E_r$ and $d_H(u, v) = n \in E_c$. Examples of FQ_2 and FQ_3 are shown in Figure 1. It has been shown that FQ_n is $(n+1)$-regular, $(n+1)$-connected, vertex-transitive, and edge-transitive [22]. Moreover, FQ_n has been shown that for every odd $n \ge 3$ is bipartite [26].

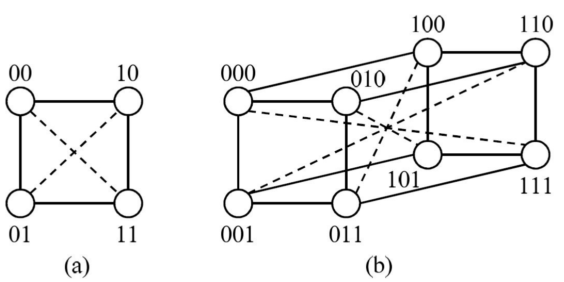

FIGURE 1. Graphs (a) FQ_2 and (b) FQ_3, in which complementary edges are represented as dashed lines.

A regular hypercube Q_n can be partitioned into two subcubes Q_{n-1} along dimension i, where $1 \le i \le n$. We denote the subcubes as $Q^{0i}_{n-1} = *^{n-i}0*^{i-1}$ and $Q^{1i}_{n-1} = *^{n-i}1*^{i-1}$, in which the values of the ith bit of the vertices are 0 and 1, respectively. Formally, Q^{0i}_{n-1} (respectively, Q^{1i}_{n-1}) is a subgraph of FQ_n induced by $\{x_n \ldots x_i \ldots x_1 \in V(Q_n) | x_i = 0$ (respectively, $\{x_n \ldots x_i \ldots x_1 \in V(Q_n) | x_i = 1\}$). In brief, we denote the subcubes Q^{0i}_{n-1} (respectively, Q^{1i}_{n-1}) as Q^{0}_{n-1} (respectively, Q^{1}_{n-1}) if the dimension i is not ambiguity.

Definition 1: [28] An i-partition on FQ_n, where $1 \le i \le n$, partitions along dimension i into two subcubes Q^{0i}_{n-1} (briefly, Q^{0}_{n-1}) and Q^{1i}_{n-1} (briefly, Q^{1}_{n-1}). Moreover, every complementary edge in E_c joins Q^{0}_{n-1} with Q^{1}_{n-1}.

In this paper, let F_e (respectively, FF_e) denote the set of faulty edges in Q_n (respectively, FQ_n). In FQ_n, let F_c denote the set of faulty complementary edges in E_c; and for $1 \le i \le n$, let F_i denote the set of i-dimensional faulty edges in E_r. Then, $|FF_e| = \sum_{i=1}^{n} |F_i| + |F_c|$. By Definition 1, if we execute an i-partition on FQ_n to form two $(n-1)$-dimensional hypercubes Q^{0}_{n-1} and Q^{1}_{n-1}, than $F^0_e = FF_e \cap E(Q^{0}_{n-1})$ and $F^1_e = FF_e \cap E(Q^{1}_{n-1})$. Hence, $FF_e = F^0_e \cup F_i \cup F_c \cup F^1_e$, where $1 \le i \le n$.

In the remainder of this section, we consider some previously reported properties in Q_n or FQ_n, which are useful to our main proof.

Lemma 1: [6] *$FQ_n - FF_e$ contains a Hamiltonian cycle of length 2^n, where $|FF_e| \le n - 1$ and $n \ge 2$.*

Lemma 2: [23] *$FQ_n - FF_e$ for $n \ge 3$ contains a Hamiltonian cycle of length 2^n if $|FF_e| \le 2n - 3$ and each vertex in $FQ_n - FF_e$ is incident to at least two fault-free edges.*

Lemma 3: [29] *$Q_n - F_e$ for $n \ge 5$ and $|F_e| \le 3n - 8$ contains a Hamiltonian cycle of length 2^n if the following two constraints are satisfied:*

1. *each vertex in $Q_n - F_e$ is incident to at least two fault-free edges;*
2. *there do not exist a pair of non-adjacent vertices in a 4-cycle whose degree are both two.*

Lemma 4: [22] *In FQ_n, there is an automorphism π of FQ_n such that $\pi(E_i) = E_j$ and $FQ_n - E_j \cong Q_n$ for i, $j \in \{1, 2, \ldots, n, c$.*

Lemma 5: [22] *Let u and v are any two different vertices in Q_n with $|F_e| \le n - 2$, where $n \ge 2$. Then, $Q_n - F_e$ contains a fault-free path $P[u, v]$ of length l with $d_{Q_n}(u, v) + 2 \le l \le 2^n - 1$ and $(l - d_{Q_n}(u, v))$ is a multiple of 2.*

According to the above Lemma 5, the following corollary can be directly derived.

Corollary 1: *Let u and u are any two different vertices in Q_n with $|F_e| \le n-2$, where $n \ge 2$. Then, $Q_n - F_e$ contains a fault-free path $P[u, v]$ of length $2^n - 1$ or $2^n - 2$ when $d_{Q_n}(u, v)$ is odd or even, respectively.*

Lemma 6: [26] $n \ge 3$ be an integer. The n-dimensional hypercube Q_n is hyper- hamiltonian-laceable.

III. EMBEDDING HAMILTONIAN CYCLE IN FAULTY FOLDED HY- PERCUBES

In this section, assume that $|FF_e| \le 3n - 7$ nd each vertex is incident to at least three fault-free edges in $FQ_n - FF_e$. We show that $FQ_n - FF_e$ contains a fault-free Hamiltonian cycle of length 2^n, where $n \ge 3$.

Lemma 7: *For $n \ge 3$, $FQ_n - FF_e$ contains a fault-free Hamiltonian cycle of length 2^n if $|FF_e| \le 3n - 7$ and each* *vertex in $FQ_n - FF_e$ is incident to at least three fault-free edges.*

Proof: We consider the cases for $n = 3$, $n = 4$, and $n \geq 5$.

Case 1: For $n = 3$. Since $|FF_e| \leq 3 \times 3 - 7 = 3 - 1 = 2$, by Lemma 1, there exists a fault-free Hamiltonian cycle of length 2^n in $FQ_3 - FF_e$.

Case 2: For $n = 4$. Since $|FF_e| \leq 3 \times 4 - 7 = 2 \times 4 - 3 = 5$ and each vertex in $FQ_n - FF_e$ is incident to at least three fault-free edges, by Lemma 2, there exists a fault-free Hamiltonian cycle of length 2^n in $FQ_4 - FF_e$.

Case 3: For $n \geq 5$. In this case, by Lemma 4, we can assume that the distribution of faulty edges in FQ_n as that $|F_c| \geq |F_n| \geq \ldots \geq |F_1|$. Since $\sum_{i=1}^{n} |F_i| \geq |F_c| = |FF_e| = 3n - 7 \geq n + 3$ for $n \geq 5$ and $|F_c| \geq |F_n| \geq \ldots \geq |F_1|$, we have that $|F_c| \geq 2$, the number of faulty edges in $FQ_n - E_c \cong Q_n$ is at most $3n - 9$ and each vertex in $FQ_n - E_c \cong Q_n$ is incident to at least two fault-free edges. Furthermore, in $FQ_n - E_c \cong Q_n$, there is at most one pair of non-adjacent vertices in a 4-cycle whose degree are both two. If there are two pair of non-adjacent vertices in a 4-cycle whose degree are both two in $FQ_n - E_c \cong Q_n$, then $|FF_e| \geq 4(n-2) = 4n - 8 > 3n - 7$, for $n \geq 5$, which contradicts the assumption that $|FF_e| \leq 3n - 7$. (see Figure 2)

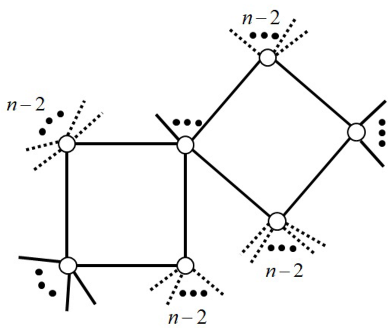

FIGURE 2. Illustration of Case 3 in the proof of Lemma 7.

Therefore, we have the following scenarios.

Case 3.1: In $FQ_n - E_c \cong Q_n$, there is no pair of non-adjacent vertices in a 4-cycle whose degree are both two. Since the number of faulty edges in $FQ_n - E_c \cong Q_n$ is at most $3n - 9 < 3n - 8$ and each vertex in $FQ_n - E_c \cong Q_n$ is incident to at least two fault-free edges, by Lemma 3, there exists a fault-free Hamiltonian cycle of length 2^n in $FQ_n - FF_e$.

Case 3.2: In $FQ_n - E_c \cong Q_n$, there exists one pair of non-adjacent vertices in a 4-cycle whose degree are both two. Without loss of generality, we can assume that these two non-adjacent vertices in a 4-cycle are labeled as u and v, and their degree are exactly both two in $FQ_n - E_c \cong Q_n$. Moreover, vertices u and v are both incident to the fault-free i-dimensional edge and j-dimensional edge to form the fault-free 4-cycle, where $i \neq j$ and $i, j \in 1, 2, \ldots, n$. Then, the 4- cycle can be represented as $\langle u, u^{(i)(j)} = v, v^{(j)} = u^{(j)}, u \rangle$ in $FQ_n - E_c \cong Q_n$. (see Figure 3) Note that

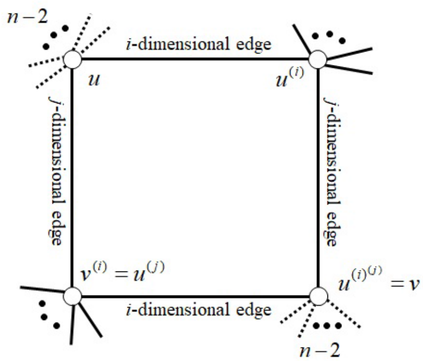

FIGURE 3. Illustration of Case 3.2 in the proof of Lemma 7.

$|F_n| \geq |F_{n-1}| \geq \ldots \geq |F_1|$ in $FQ_n - E_c \cong Q_n$, without loss of generality, we can assume that $i = 1$ and $j = 2$. Since each vertex in FQ_n is incident to at least three fault-free edges, we know that $(u, \bar{u})$ and $(v, \bar{v})$ are both fault-free in $FQ_n - FF_e$. Then, we consider the following subcases.

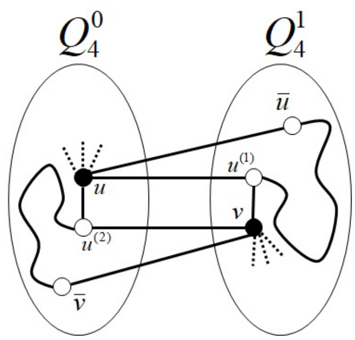

FIGURE 4. Illustration of Case 3.2.1 in the proof of Lemma 7.

Case 3.2.1: For $n = 5$. In this case, $|FF_e| \leq 3 \times 5 - 7 = 8$. Since $|F_c| \geq |F_5| \geq \ldots \geq |F_1|$ and there exists one pair of non-adjacent vertices in a 4- cycle whose degree are both two in $FQ_5 - E_c \cong Q_5$, we have that $|F_c| = |F_5| = |F_4| = |F_3| = 2$ and $|F_2| = |F_1| = 0$ in FF_e. By Definition 1, FQ_n can be partitioned along dimension 1 to form two subcubes Q_4^0 and Q_4^1 such that the vertex sets $u, u^{(2)}$ and $u^{(1)}, v$ are in different subcubes. Without loss of generality, we can assume that $u, u^{(2)} \subseteq V(Q_4^0)$ and $u^{(1)}, v \subseteq Q_4^1$. Note that$FQ_5$ is a bipartite graph. The vertex sets u, v and $u^{(1)}, u^{(2)}, \bar{u}, \bar{v}$ are in different partite sets. Since every edge in $\left(u, u^{(1)}\right), (u, \bar{u}), \left(u, u^{(2)}\right), (v, \bar{v}), \left(u, u^{(2)}\right), (u^{(1)}, v)\}$ is fault-free and $|F_c| = 2$, we know that the distribution of the remaining six faulty edges are three of them all incident to u in Q_4^0, and the other three of them all are incident to v in Q_4^1. Moreover, if we eliminate the vertices u and v in Q_4^0 and Q_4^1, respectively. Then, $Q_4^0 - u$ and $Q_4^1 - v$ are both fault- free. Therefore, by Lemma 6, there exists fault-free paths $P_0[u^{(2)}, \bar{v}]$ and $P_1[u^{(1)}, \bar{u}]$ of length $2^4 - 2$ in $Q_4^0 - u$ and $Q_4^1 - v$, respectively. Then, $\langle u, u^{(2)}, P_0[u^{(2)}, \bar{v}], \bar{v}, v, u^{(1)}, P_1[u^{(1)}, \bar{u}], \bar{u} \rangle$ forms a Hamiltonian cycle of length $2 \times (2^4 - 2) + 4 = 2^5$. (see Figure 4).

Case 3.2.2: For $n \geq 6$. In this case, by Definition 1, FQ_n can be partitioned along some dimension $k \in \{3, 4, \ldots, n$ to form two subcubes Q^0_{n-1} and Q^1_{n-1} such that $(u, u^{(k)})$ is a faulty edge. Obviously, $(v, v^{(k)})$ is also a faulty edge because of there are only three fault-free edges $(v, v^{(1)})$, $(v, v^{(2)})$, and $(v, \bar{v})$ incident to v. Since $k \notin 1, 2\}$, we know that the 4-cycle $\langle u, u^{(1)}, v, u^{(2)}, u\rangle$ will be in the same subcube. Without loss of generality, we can assume that $k = n$ and the 4-cycle $\langle u, u^{(1)}, v, u^{(2)}, u\rangle$ is in Q^0_{n-1}. Note that $|F_c| \geq |F_n| \geq \ldots \geq |F_1|$ and $\langle u, u^{(1)}, v, u^{(2)}, u\rangle$ is in Q^0_{n-1}, we have that $|F_c| \geq 2$, $|F_c| + |F_n| \geq 4$, $2(n-3) \leq F^0_e \leq 3n - 11$, and $F^1_e \leq (3n-7) - 2(n-3) - 4 = n - 5$, for $n \geq 6$. Therefore, $|F_c| + |F_n| + F^1_e \leq n-1$. Furthermore, we can choose a faulty edge $(u, u^{(i)}) \in F^0_e$ or $(v, v^{(j)}) \in F^0_e$ such that $\left(u^{(i)}, \overline{u^{(i)}}\right)$ or $\left(v^{(i)}, \overline{u^{(i)}}\right)$ is fault-free, where $i, j \in \{3, 4, \ldots, n-1\}$. (Note that $u^{(i)} \neq v^{(j)}$) in Q^0_{n-1} because of $d_H\left(u^{(i)}, v^{(j)}\right) = 2$ or $4 \neq 0$, where $, j \in \{3, 4, \ldots, n-1\}$. If such a faulty edge does not exist, then $|F_c| \geq 2(n-3)$, which contradicts the assumption that $|F_c| + |F_n| + |F^1_e| \leq n-1$, for $n \geq 6$. Without loss of generality, let $\left(u^{(4)}, \overline{u^{(4)}}\right)$ is fault-free. We assume that the faulty edge $(u, u^{(4)})$ is temporarily fault-free, then there is no pair of non- adjacent vertices in a 4-cycle whose degree are both two in Q^0_{n-1}. Note that $3n-7-2(n-3)-4 \leq n-5$, we have that each vertex except u and v is incident to at least three fault-free edges in Q^0_{n-1}. Since $(u, u^{(4)})$ is temporarily fault-free, $F^0_e \leq (3n-11) - 1 < 3(n-1) - 8$. By Lemma 3, there exist a fault-free Hamiltonian cycle C_0 of length 2^{n-1} in Q^0_{n-1}. Then, we have the following subcases.

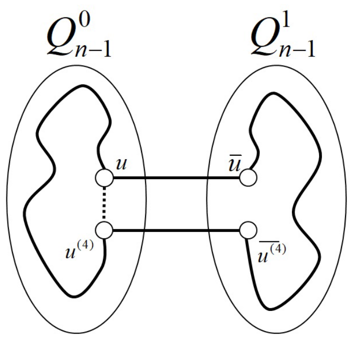

FIGURE 5. Illustration of Case 3.2.2.1 in the proof of Lemma 7.

Case 3.2.2.1: $(u, u^{(4)}) \in C_0$. In this case, C_0 can be represented as $\langle u, P_0[u, u^{(4)}], u^{(4)}, u\rangle$ in Q^0_{n-1}. Note that $\left(u^{(4)}, \overline{u^{(4)}}\right)$ and $(u, \bar{u})$ are both fault-free. Since $F^1_e \leq n-5 < (n-1) - 2 = n-3$ for $n \geq 6$ and $d_H\left(u^{(4)}, \bar{u}\right)$ is odd, by Corollary 1, there exists a fault-free path $P[\overline{u^{(4)}}, \bar{u}]$ of length $2^{n-1} - 1$ in Q^1_{n-1}. Therefore, $\langle u, P_0[u, u^{(4)}], \overline{u^{(4)}}, P_1[\overline{u^{(4)}}, \bar{u}], \bar{u}, u\rangle$ forms a Hamiltonian cycle of length $(2^{n-1}-1)+2+(2^{n-1}-1) = 2^n$. (see Figure 5)

Case 3.2.2.2: $(u, u^{(4)}) \notin C_0$. We can choose an edge $(a, b) \in C_0$ such that $(a, \bar{a})$ and $(b, \bar{b})$ both are fault-free. (If no such an edge exists, then $|F_c| \geq \left\lceil \frac{2^{n-1}}{2} \right\rceil = 2^{n-2} > n-1$ for $n \geq 6$, which contradicts the assumption that $|F_c| + |F_n| + |F^1_e| \leq n-1$) Therefore, C_0 can be represented as $\langle a, P_0[a, b], b, a\rangle$ in Q^0_{n-1}. Then, the proof of this case is similar to that in Case 3.2.2.1.

By combining above cases, we complete the proof. **Q.E.D.**

Theorem 1 Let FF_e denote the set of faulty edges in FQ_n and assume that each vertex is incident to at least three fault-free edges in $FQ_n - FF_e$. Then, $FQ_n - FF_e$ contains a fault-free Hamiltonian cycle of length 2^n, where $n \geq 3$ and $|FF_e| \leq 3n - 7$.

IV. CONCLUDING REMARKS

In recent years, with the development of Very Large Scale Integration (VLSI) and Wafer- Scale Integration (WSI), it is necessary to design more processors on the wafer. Therefore, the multiprocessor system becomes more and more important and prevalent. As the number of processors increasing in a multiprocessor system, the reliability of the parallel computing system becomes a significant issue. Therefore, the reliability analysis problems of the fault-tolerant embedding research field has played an important role and a meaningful research topics. Many researchers have focused on the vertex fault-tolerant, edge fault-tolerant, or both vertex and edge fault-tolerant embedding properties of some specific multiprocessor network topologies. In this paper, we consider the highly edge fault-tolerant Hamiltonian cycle embedding in the extended n-dimensional hypercube structure, which is called the n-dimensional folded hypercube FQ_n. We let FF_e denote the set of faulty edges in FQ_n and assume that each vertex is incident to at least three fault-free edges in $FQ_n - FF_e$. Then, we show that a fault-free Hamiltonian cycle of length 2^n can be embedded in $FQ_n - FF_e$, where $n \geq 3$ and $|FF_e| \leq 3n - 7$.

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