IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received February 11, 2020, accepted March 19, 2020, date of publication April 22, 2020, date of current version May 28, 2020.
Digital Object Identifier 10.1109/ACCESS.2020.2989558

Second-Order Consensus for Multi-Agent Systems With Time-Varying Delays Based on Delay-Partitioning

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This work was supported in part by the Natural Science Foundation of Tianjin under Grant 19JCTPJC48600, Grant 18JCYBJC88100, and Grant 17JCYBJC18500.

ABSTRACT A consensus problem for second-order agents network with time-varying delays under the directed fixed topology is investigated in this paper. For the convenience of analysing, the original system is converted into an equivalent system associated with disagreement terms. By applying a variable delay-partitioning method, increased computational complexity is solved and obtain less conservatism. A suitable Lyapunov-Krasovskii functional (LKF) is constructed, combining the reciprocally convex combination lemmas with Wirtinger-based inequality to deal with integral items for the derivative of the LKF and further reduce the conservatism. Following the linear matrix inequality theory, a sufficient condition is presented to make all agents asymptotically reach consensus. Finally, simulations are given to illustrate the effectiveness of the proposed results.

INDEX TERMS Second-order agents, consensus, fixed topology, time-varying delays, delay-partitioning approach.

I. INTRODUCTION

In the past few decades, distributed cooperative control of multi-agent systems have attracted more and more research interest and widely applied to various fields such as sensor networks [1], [2], formation control [3], [4], consensus problem [5]–[8], group synchronization [9], spacecraft attitude tracking control [10], [11], distributed optimization calculation and control engineering. As one of the key problems of distributed cooperative control, the consensus problem means that a group of agents reaches consensus on a certain physical quantity based on local interaction rules. Many scholars have studied the consensus problem from various perspectives. For instance, event-triggered security consensus control [12], sampled-data consensus [13]–[15], adaptive control [16], [17] and so on.

In practical applications, time delays always inevitably occur in the process of information exchange between multiple agents, the delay causes undesirable agent dynamic behaviour, deteriorates the performance of agents to reach consensus and becomes a principal factor of instability in a real multi-agent system. Therefore, the research on the consensus of multi-agent systems with time delays has attracted extensive attention. Among previous studies, the time delays are mainly divided into two categories which include time-invariant delays and time-varying delays. Initially, [18] and [19] proved that first-order agents and second-order agents could reach average-consensus in the presence of time-invariant delays, respectively. Consider that most of complicated multi-agent systems generally exhibit time-varying behaviour, in reality, different consensus problems of multi-agent systems with time-varying delays were solved [20]–[24]. All the above papers on consensus problem show that the time delays problem is typical and significant.

Delay-partitioning is regarded as a practical approach to reduce the conservatism of time-delayed systems, which is broadly used in various time-delay systems. Unfortunately, few of the mentioned references apply the delay-partitioning approach to multi-agent systems with time-varying delays.

The associate editor coordinating the review of this manuscript and approving it for publication was Bing Li.

Qin *et al.* in [25] employed a delay-partitioning approach to process the time-invariant delays of second-order agents system and proved that the conservatism of the system could be reduced with the number of delay interval divisions increasing. However, time-varying delays were not considered in the paper. Furthermore, the delay-partitioning method of [25] is based on the idea of evenly dividing the delay interval and the computational complexity increases with the number of delay interval divisions.

On the other hand, it has been point out that the leading indicator of measuring the conservatism is the maximum allowable delay. Mainly, integral inequalities play a crucial role in the derivation of delay-dependent stability criteria for time-delayed systems. Jensen inequality was utilized to deal with the derivative of Lyapunov Krasovskii functional (LKF) containing double integral terms in [25]. Although this inequality is frequently used to evaluate the derivative of LKF for time-delayed systems, stability criteria based on the inequality is relatively conservative, which limits the further improvement of existing results.

Inspired by the above-mentioned work, we study consensus for second-order agents networks with time-varying delays under the directed fixed topology. For the convenience of analysing, the original system is converted into an equivalent system associated with disagreement terms. Different from reference [25], here, we consider time-varying delays. By adopting the same delay-partitioning method as [26], based on the idea of variable interval delay dividing, segments with two variable sizes can yield less conservatism than those with n equal sizes. Meanwhile, the computational complexity is also reduced. With the Lyapunov stability theory and linear matrix inequality theory, we construct a suitable Lyapunov Krasovskii functional (LKF) and perform the stability analysis of the system for two cases which respectively correspond to two variable subintervals. To further reduce the conservatism, the reciprocally convex combination lemmas, combined with Wirtinger-based inequality, following the same procedure as used in [27], based on the form of linear matrix inequalities (LMIs), a sufficient condition is proposed eventually to make all agents reach consensus asymptotically. In brief, the innovation of this paper is first reflected in the fact that a novel delay-partitioning method to deal with time-varying delays, which was rarely used in multi-agent systems in previous literature, and time-varying delays are more general in practice. Furthermore, searching the optimal division point is the key to this method. Compared with the traditional delay-partitioning idea, the advantage of this method is that it can reduce both computational complexity and system conservatism. Another contribution is that using the reciprocally convex combination lemmas and Wirtinger-based inequality in the LKF, which is used to analyze the stability of multi-agent systems.

Main Structure: Formulating the problem formulation and several concepts related graph theory in Section 2 and providing the stability analysis for the second-order agents network with time-varying delays in Section 3. In Section 4, simulation results are presented, and some conclusions are given in Section 5.

Notations: Apply the following notations throughout this paper. $\mathbb{R}^n$ indicates the n-dimensional Euclidean space. The notations $\mathbb{R}^{n\times m}$ and $\mathbb{R}^{n\times n}$ are the set of $n \times m$ real matrices and n-order real square matrices, respectively. $\mathbb{Z}_+$ stands for a set of positive integers, and any scalar that belongs to this set is a positive integer. I_n is the $n \times n$-dimensional identity matrix, and I is the identity matrix of compatible dimensions. The symmetric block of a symmetric matrix will be denoted by $*$. The notation A^T represents the transpose of matrix A. For any matrices A, B of appropriate dimension, the matrix $diag\{A, B\}$ means the diagonal matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$.

II. PRELIMINARIES AND PROBLEM FORMULATION

A. PRELIMINARIES

Define $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ as a weighted directed graph, which consists of n nodes. $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ are regarded as a set of these nodes and a set of edges, respectively. The weighted adjacency matrix of G is described by $\mathcal{A} = \left[a_{ij}\right]$ whose all adjacency elements a_{ij} are non-negative. $\Gamma = \{1, 2, \ldots, n\}$ represents a finite set of the node indexes and $N_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}\}$ denotes the set of neighbors of node v_i. Take adjacency elements related the edges of G as positive values if there exists an edge between node v_i and v_j, i.e., $e_{ij} = (v_i, v_j) \in \mathcal{E} \iff a_{ij} > 0$. Furthermore, we suppose $a_{ii} = 0$ for all $i \in \Gamma$.

A node of G is balanced if and only if the equality $deg_{in}(v_i) = deg_{out}(v_i)$ holds, where $deg_{in}(v_i) = \sum_{j=1}^{n} a_{ji}$ and $deg_{out}(v_i) = \sum_{j=1}^{n} a_{ij}$ represent the in-degree and out-degree of the node v_i, respectively. G is a balanced graph if and only if all of its nodes are balanced.

Let L be the Laplacian matrix of the weighted digraph G, which is defined as $L = \Delta - \mathcal{A}$, where $\Delta = [\Delta_{ii}]$ is a diagonal matrix with $\Delta_{ii} = deg_{out}(v_i)$. If G is balanced, then $\mathbf{1}_n^T L = 0$, where $\mathbf{1}_n$ is a column vector with its all entries are one.

We say that square matrix $W \in \mathbb{R}^{n\times n}$ is balanced if and only if $\mathbf{1}_n^T W = \mathbf{0}$ and $W\mathbf{1}_n = \mathbf{0}$.

If there has a path between any two different nodes $v_i, v_j \in \mathcal{E}$ in G, then G is called a strongly connected graph.

Replace all directed edges of a directed graph with undirected edges, if the obtained graph is connected, then the directed graph is a weakly connected graph.

Lemma 1 (Wirtinger-Based Inequality [28]): Given a symmetric positive definite matrix Ω, the following inequality holds for any continuously differential function $\delta : [a, b] \to \mathbb{R}^n$:

$$\int_a^b \dot{\delta}(u)^T \Omega \dot{\delta}(u) du \geq \frac{1}{b-a} \begin{bmatrix} \varsigma_1 \\ \varsigma_2 \end{bmatrix}^T \widetilde{\Omega} \begin{bmatrix} \varsigma_1 \\ \varsigma_2 \end{bmatrix}, \tag{1}$$

where

$$\varsigma_1 = \delta(b) - \delta(a),\ \varsigma_2 = \delta(b) + \delta(a) - \frac{2}{b-a}\int_a^b \delta(u)du,$$
$$\widetilde{\Omega} = diag\{\Omega, 3\Omega\}.$$

Lemma 2 (Reciprocally Convex Combination Lemmas [29]): For a given symmetric positive matrix E, assume that there exists a matrix $F \in \mathbb{R}^{n\times n}$ such that

$$\begin{bmatrix} E & F \\ * & E \end{bmatrix} \succeq 0, \tag{2}$$

Then for all scalar $\beta \in (0, 1)$, one has

$$\begin{bmatrix} \frac{1}{\beta}E & 0 \\ * & \frac{1}{1-\beta}E \end{bmatrix} = \Lambda(\beta) + \begin{bmatrix} E & F \\ * & E \end{bmatrix}, \tag{3}$$

where

$$\begin{aligned} \Lambda(\beta) &= \begin{bmatrix} \frac{1}{\beta}E & 0 \\ * & \frac{1}{1-\beta}E \end{bmatrix} - \begin{bmatrix} E & F \\ * & E \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{\frac{1-\beta}{\beta}}I & 0 \\ * & \sqrt{-\frac{\beta}{1-\beta}}I \end{bmatrix} \begin{bmatrix} E & F \\ * & E \end{bmatrix} \\ &\times \begin{bmatrix} \sqrt{\frac{1-\beta}{\beta}}I & 0 \\ * & \sqrt{-\frac{\beta}{1-\beta}}I \end{bmatrix} \succeq 0. \end{aligned}$$

Lemma 3 [30]: For a given matrix

$$\Psi = \begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{bmatrix} \in \mathbb{R}^{n\times n},$$

The following statements hold.

(1) 0 with multiplicity 1 and n with multiplicity $n-1$ are the eigenvalues of Ψ.

(2) The right eigenvector related the zero eigenvalue of Ψ is the column vector $\mathbf{1}_n$, while the left eigenvector related the zero eigenvalue of Ψ is the row vector $\mathbf{1}_n^T$.

(3) $\Theta_n \in \mathbb{R}^{n\times n}$ denotes an orthogonal matrix which satisfies $\Theta_n^T \Psi \Theta_n = \begin{bmatrix} nI_{n-1} & 0 \\ 0 & 0 \end{bmatrix}$. $\frac{\mathbf{1}_n}{\sqrt{n}}$ is the last column of Θ_n. $\Xi \in \mathbb{R}^{n\times n}$ is regarded as the Laplacian matrix of any balanced graph, then $\Theta_n^T \Xi \Theta_n = \begin{bmatrix} \vartheta_1 & 0 \\ 0 & 0 \end{bmatrix}$, $\vartheta_1 \in \mathbb{R}^{(n-1)\times(n-1)}$.

Lemma 4 [25]: Given any matrix $\Sigma \in \mathbb{R}^{2mn}$ and nonzero $\omega \in \mathbb{R}^{2mn}$ satisfying $(\epsilon_i \otimes \mathbf{1}_{2n})^T \omega = 0$, where $i = 1, 2, \ldots, m$, $m \in \mathbb{Z}_+$, the inequality $\omega^T \Sigma \omega < 0$ holds if and only if the associated matrix $\bar{\Sigma} \in \mathbb{R}^{m(2n-1)}$ of Σ is negative definite, where $\epsilon_i \in \mathbb{R}^m$ represents the column vector with one on the ith entry and zeros elsewhere.

B. PROBLEM FORMULATION

A second-order network system consists of n agents is considered. Each node in the directed graph G corresponds to an agent of the system. We assume that the following double integrator dynamics are used to model ith agent:

$$\dot{x}_i = \upsilon_i \quad \dot{\upsilon}_i = u_i, \tag{4}$$

where the position state and speed state of each agent in the network are represented by x_i and υ_i, respectively. u_i denotes the control input. We say all agents of (4) asymptotically reach consensus if for any initial conditions,

$$\lim_{t\to\infty}(x_i(t) - x_j(t)) = 0, \quad \lim_{t\to\infty}\upsilon_i(t) = 0, \quad \forall i, j \in \mathcal{V}. \tag{5}$$

We determine to utilize the following consensus protocol so as to solve the second-order consensus problem of (4) with time-varying delays:

$$u_i(t) = -2\kappa\upsilon_i + \sum_{j\in N_i} a_{ij}(t)(x_j(t-\tau(t)) - x_i(t-\tau(t))), \tag{6}$$

where the velocity damping gain κ is positive, i.e. $\kappa > 0$, the time-varying delays functional $\tau(t)$ satisfying

$$\tau_1 \le \tau(t) \le \tau_2, \tag{7}$$
$$\dot{\tau}(t) \le \nu, \tag{8}$$

Let

$$\varphi = [x_1, y_1, \ldots, x_n, y_n]^T, \; y_i = \frac{1}{\kappa}\upsilon_i + x_i,$$
$$A = \begin{bmatrix} -\kappa & \kappa \\ \kappa & -\kappa \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1/\kappa & 0 \end{bmatrix}.$$

By (6), we can rewrite the network dynamics (4) into the following matrix form

$$\dot{\varphi}(t) = (I_n \otimes A)\varphi(t) - (L \otimes B)\varphi(t-\tau(t)), \tag{9}$$

where L signifies the Laplacian matrix of the weakly connected and balanced directed graph G, the symbol '$\otimes$' indicates the Kronecker product. The Kronecker product of two positive definite matrices is also positive definite.

We introduce an disagreement function $\delta(t)$ for convergence analysis of the protocol (6), which has the same definition as the disagreement function mentioned in [25]. The similar disagreement dynamics induced by (9) shows the form as follows:

$$\dot{\delta}(t) = (I_n \otimes A)\delta(t) - (L \otimes B)\delta(t-\tau(t)). \tag{10}$$

where $\delta(t) \in \mathbb{R}^{2n\times 2n}$ denotes the disagreement vector satisfying $\sum_{i=1}^{2n} \delta_i(t) = 0$.

Remark 1: Actually, the strongly connected and balanced digraph and weakly connected and balanced digraph both describe the same kind of graph and a weakly connected and balanced digraph must be strongly connected. However, for a given directed graph, the weakly connected and balanced digraph is easier to verify. Hence, it is more appropriate to consider weakly connected and balanced digraph in this paper rather than strongly connected and balanced digraph.

III. STABILITY ANALYSIS

In this section a stability analysis of system (9) is performed, a variable delay-partitioning method proposed in [26] is applied to process the time-varying delay $\tau(t)$, where the delay interval $[\tau_1, \tau_2]$ was partitioned into two variable segments: $[\tau_1, \gamma]$ and $[\gamma, \tau_2]$, where γ is a positive scalar satisfying $\tau_1 < \gamma < \tau_2$.

For notation simplification, let

$$\epsilon_s = \Big[\underbrace{0,\cdots,0}_{s-1}, I, \underbrace{0,\cdots,0}_{10-s}\Big]^T, \quad s = 1,2,\ldots,10.$$

$$\xi_1(t) = [\varrho_1^T(t), \eta_1^T(t), \dot{\delta}^T(t)]^T, \tag{11}$$

$$\xi_2(t) = [\varrho_2^T(t), \eta_2^T(t), \dot{\delta}^T(t)]^T, \tag{12}$$

where

$$\begin{aligned}
&\varrho_1(t)\\
&= [\delta^T(t), \delta^T(t-\tau_1), \delta^T(t-\tau(t)), \delta^T(t-\gamma),\\
&\quad \delta^T(t-\tau_2)]^T,\\
&\varrho_2(t)\\
&= [\delta^T(t), \delta^T(t-\tau_1), \delta^T(t-\gamma), \delta^T(t-\tau(t)),\\
&\quad \delta^T(t-\tau_2)]^T,\\
&\eta_1(t)\\
&= [\frac{1}{\tau_1}\int_{t-\tau_1}^{t}\delta^T(s)ds, \frac{1}{\tau(t)-\tau_1}\int_{t-\tau(t)}^{t-\tau_1}\delta^T(s)ds,\\
&\quad \times\frac{1}{\gamma-\tau(t)}\int_{t-\gamma}^{t-\tau(t)}\delta^T(s)ds, \frac{1}{\tau_2-\gamma}\int_{t-\tau_2}^{t-\gamma}\delta^T(s)ds]^T,\\
&\eta_2(t)\\
&= [\frac{1}{\tau_1}\int_{t-\tau_1}^{t}\delta^T(s)ds, \frac{1}{\gamma-\tau_1}\int_{t-\gamma}^{t-\tau_1}\delta^T(s)ds,\\
&\quad \times\frac{1}{\tau(t)-\gamma}\int_{t-\tau(t)}^{t-\gamma}\delta^T(s)ds, \frac{1}{\tau_2-\tau(t)}\int_{t-\tau_2}^{t-\tau(t)}\delta^T(s)ds]^T.
\end{aligned}$$

Remark 2: Based on Algorithm 1 in literature [26], we set $\gamma = \tau_1 + \lambda\tau_{1,2}$ can search the optimal division point γ of the variable delay-partitioning approach, where $\tau_{1,2} = \tau_2 - \tau_1$, λ is a scalar which is positive and satisfies $0 < \lambda < 1$.

On the basis of the previous preparations, we intend to investigate the consensus of second-order agents networks with time-varying delays. Here we construct an appropriate Lyapunov Krasovskii functional $V(\delta_t)$ and introduce the above mentioned variable delay-partitioning approach, Lemma 2.4 can be used to determine the negative definiteness of $\dot{V}(\delta_t)$. The main result in this section is stated by the following theorem.

Theorem 1: By protocol (6), second-order consensus problem for directed network of agents with fixed topology and time-varying delay $\tau(t)$ can be globally asymptotically solved if there exists scalars γ, ν, τ_1, τ_2 and symmetric positive matrices $\bar{P}, \bar{Q}, \bar{R}, \bar{S}_i, \bar{X}_i, \bar{Y}_j, \bar{Z}_j \in \mathbb{R}^{(2n-1)\times(2n-1)}$, (i = 1, 2; j = 1, 2, 3) and any matrix $\bar{M}_l$, $(l = 1, 2, 3, 4)$ with proper dimensions, such that the following LMIs hold.

$$\Phi_1 = 2\Upsilon_1 + \Upsilon_2 + \Upsilon_3 + \Upsilon_4 + 2\Upsilon_5 < 0 \tag{13}$$

$$\Phi_2 = 2\Pi_1 + \Pi_2 + \Pi_3 + \Pi_4 + 2\Pi_5 < 0 \tag{14}$$

where

$$\begin{aligned}
\Upsilon_1 &= [\epsilon_1,\ \tau_1\epsilon_6 + (\tau(t)-\tau_1)\epsilon_7 + (\gamma-\tau(t))\epsilon_8,\ (\tau_2-\gamma)\epsilon_9]\\
&\quad\times\begin{bmatrix}\bar{P} & \bar{Y}_1 & \bar{Y}_2\\ * & \bar{Q} & \bar{Y}_3\\ * & * & \bar{R}\end{bmatrix}\begin{bmatrix}\epsilon_{10}^T\\ \epsilon_1^T-\epsilon_4^T\\ \epsilon_4^T-\epsilon_5^T\end{bmatrix},\\
\Upsilon_2 &= (\epsilon_1\bar{S}_1\epsilon_1^T + \epsilon_4(\bar{S}_2-\bar{S}_1)\epsilon_4^T - \epsilon_5 S_2\epsilon_5^T),\\
\Upsilon_3 &= (\epsilon_1\bar{X}_1\epsilon_1^T + \epsilon_2(\bar{X}_2-\bar{X}_1)\epsilon_2^T - (1-\nu)\epsilon_3\bar{X}_2\epsilon_3^T),\\
\Upsilon_4 &= \epsilon_{10}(\tau_1^2\bar{Z}_1 + (\gamma-\tau_1)^2\bar{Z}_2 + (\tau_2-\gamma)^2\bar{Z}_3)\epsilon_{10}^T\\
&\quad - [\epsilon_1-\epsilon_2,\quad \epsilon_1+\epsilon_2-2\epsilon_6]\,\tilde{\bar{Z}}_1\begin{bmatrix}\epsilon_1^T-\epsilon_2^T\\ \epsilon_1^T+\epsilon_2^T-2\epsilon_6^T\end{bmatrix}\\
&\quad - [\epsilon_2-\epsilon_3,\ \epsilon_2+\epsilon_3-2\epsilon_7,\ \epsilon_3-\epsilon_4,\ \epsilon_3+\epsilon_4-2\epsilon_8]\\
&\quad\times\begin{bmatrix}\beta_1\tilde{\bar{Z}}_2 & 0\\ 0 & \frac{1}{1-\beta_1}\tilde{\bar{Z}}_2\end{bmatrix}\begin{bmatrix}\epsilon_2^T-\epsilon_3^T\\ \epsilon_2^T+\epsilon_3^T-2\epsilon_7^T\\ \epsilon_3^T-\epsilon_4^T\\ \epsilon_3^T+\epsilon_4^T-2\epsilon_8^T\end{bmatrix}\\
&\quad - [\epsilon_4-\epsilon_5,\quad \epsilon_4+\epsilon_5-2\epsilon_9]\,\tilde{\bar{Z}}_3\begin{bmatrix}\epsilon_4^T-\epsilon_5^T\\ \epsilon_4^T+\epsilon_5^T-2\epsilon_9^T\end{bmatrix},\\
\Upsilon_5 &= [\epsilon_1\bar{M}_1 + \epsilon_{10}\bar{M}_2][-\epsilon_{10} + \bar{\Xi}_1\epsilon_1 - \bar{\Xi}_2\epsilon_3],\\
\Pi_1 &= [\epsilon_1,\ \tau_1\epsilon_6 + (\gamma-\tau_1)\epsilon_7 + (\tau(t)-\gamma)\epsilon_8,\ (\tau_2-\tau(t))\epsilon_9]\\
&\quad\times\begin{bmatrix}\bar{P} & \bar{Y}_1 & \bar{Y}_2\\ * & \bar{Q} & \bar{Y}_3\\ * & * & \bar{R}\end{bmatrix}\begin{bmatrix}\epsilon_{10}^T\\ \epsilon_1^T-\epsilon_3^T\\ \epsilon_3^T-\epsilon_5^T\end{bmatrix},\\
\Pi_2 &= (\epsilon_1\bar{S}_1\epsilon_1^T + \epsilon_3(\bar{S}_2-\bar{S}_1)\epsilon_3^T - \epsilon_5\bar{S}_2\epsilon_5^T),\\
\Pi_3 &= (\epsilon_1\bar{X}_1\epsilon_1^T + \epsilon_2(\bar{X}_2-\bar{X}_1)\epsilon_2^T - (1-\nu)\epsilon_4\bar{X}_2\epsilon_4^T),\\
\Pi_4 &= \epsilon_{10}(\tau_1^2\bar{Z}_1 + (\gamma-\tau_1)^2\bar{Z}_2 + (\tau_2-\gamma)^2\bar{Z}_3)\epsilon_{10}^T\\
&\quad - [\epsilon_1-\epsilon_2,\quad \epsilon_1+\epsilon_2-2\epsilon_6]\,\tilde{\bar{Z}}_1\begin{bmatrix}\epsilon_1^T-\epsilon_2^T\\ \epsilon_1^T+\epsilon_2^T-2\epsilon_6^T\end{bmatrix}\\
&\quad - [\epsilon_2-\epsilon_3,\ \epsilon_2+\epsilon_3-2\epsilon_7]\,\tilde{\bar{Z}}_2\begin{bmatrix}\epsilon_2^T-\epsilon_3^T\\ \epsilon_2^T+\epsilon_3^T-2\epsilon_7^T\end{bmatrix}\\
&\quad - [\epsilon_3-\epsilon_4,\ \epsilon_3+\epsilon_4-2\epsilon_8,\ \epsilon_4-\epsilon_5,\ \epsilon_4+\epsilon_5-2\epsilon_9]\\
&\quad\times\begin{bmatrix}\beta_2\tilde{\bar{Z}}_3 & 0\\ 0 & \frac{1}{1-\beta_2}\tilde{\bar{Z}}_3\end{bmatrix}\begin{bmatrix}\epsilon_3^T-\epsilon_4^T\\ \epsilon_3^T+\epsilon_4^T-2\epsilon_8^T\\ \epsilon_4^T-\epsilon_5^T\\ \epsilon_4^T+\epsilon_5^T-2\epsilon_9^T\end{bmatrix},\\
\Pi_5 &= [\epsilon_1\bar{M}_3 + \epsilon_{10}\bar{M}_4][-\epsilon_{10} + \bar{\Xi}_1\epsilon_1 - \bar{\Xi}_2\epsilon_4],\\
\bar{\Xi}_1 &= \Theta_1^T(I_n\otimes A)\Theta_1,\ \bar{\Xi}_2 = \Theta_1^T(L\otimes B)\Theta_1,
\end{aligned}$$

and Θ_1 denotes the first $2n-1$ columns of Θ_{2n}, related definitions are given in Lemma 2.3.

Proof: Let $I_n \otimes A = \Xi_1$ and $L \otimes B = \Xi_2$. Then we can take the network dynamics (9) as

$$\dot{\varphi}(t) = \Xi_1\varphi(t) - \Xi_2\varphi(t-\tau(t)), \tag{15}$$

and the disagreement dynamics (10) is rewritten as

$$\dot{\delta}(t) = \Xi_1\delta(t) - \Xi_2\delta(t-\tau(t)), \tag{16}$$

Given the Lyapunov-Krasovskii functional $V(\delta_t)$ as follows:

$$V(\delta_t) = \sum_{i=1}^{4} V_i(\delta_t), \tag{17}$$

where

$$\begin{aligned}
V_1(\delta_t) &= \begin{bmatrix} \delta(t) \\ \int_{t-\gamma}^{t} \delta(s)ds \\ \int_{t-\tau_2}^{t-\gamma} \delta(s)ds \end{bmatrix}^T \begin{bmatrix} P & Y_1 & Y_2 \\ * & Q & Y_3 \\ * & * & R \end{bmatrix} \begin{bmatrix} \delta(t) \\ \int_{t-\gamma}^{t} \delta(s)ds \\ \int_{t-\tau_2}^{t-\gamma} \delta(s)ds \end{bmatrix}, \\
V_2(\delta_t) &= \int_{t-\gamma}^{t} \delta^T(s)S_1\delta(s)ds + \int_{t-\tau_2}^{t-\gamma} \delta^T(s)S_2\delta(s)ds, \\
V_3(\delta_t) &= \int_{t-\tau_1}^{t} \delta^T(s)X_1\delta(s)ds + \int_{t-\tau(t)}^{t-\tau_1} \delta^T(s)X_2\delta(s)ds, \\
V_4(\delta_t) &= \tau_1 \int_{-\tau_1}^{0}\int_{t+\theta}^{t} \dot{\delta}^T(s)Z_1\dot{\delta}(s)dsd\theta \\
&\quad +(\gamma-\tau_1)\int_{-\gamma}^{-\tau_1}\int_{t+\theta}^{t} \dot{\delta}^T(s)Z_2\dot{\delta}(s)dsd\theta \\
&\quad +(\tau_2-\gamma)\int_{-\tau_2}^{-\gamma}\int_{t+\theta}^{t} \dot{\delta}^T(s)Z_3\dot{\delta}(s)dsd\theta,
\end{aligned}$$

Note, matrices $P, Q, R, S_i, X_i, Y_j, Z_j \in \mathbb{R}^{2n\times 2n}(i = 1, 2; j = 1, 2, 3)$ are positive definite. For $\tau(t) \in [\tau_1, \gamma]$. Along the trajectories of the dynamics (16), the time derivative of $V(\delta_t)$ in (17) is calculated as follows

$$\dot{V}(\delta_t) = \sum_{i=1}^{4} \dot{V}_i(\delta_t), \tag{18}$$

where

$$\begin{aligned}
&\dot{V}_1(\delta_t) \\
&= 2\begin{bmatrix} \delta(t) \\ \int_{t-\gamma}^{t} \delta(s)ds \\ \int_{t-\tau_2}^{t-\gamma} \delta(s)ds \end{bmatrix}^T \begin{bmatrix} P & Y_1 & Y_2 \\ * & Q & Y_3 \\ * & * & R \end{bmatrix} \\
&\quad \times \begin{bmatrix} \dot{\delta}(t) \\ \delta(t)-\delta(t-\gamma) \\ \delta(t-\gamma)-\delta(t-\tau_2) \end{bmatrix} \\
&= 2\xi_1^T(t) \\
&\quad \times \left[\epsilon_1,\ \tau_1\epsilon_6+(\tau(t)-\tau_1)\epsilon_7+(\gamma-\tau(t))\epsilon_8,\ (\tau_2-\gamma)\epsilon_9\right] \\
&\quad \times \begin{bmatrix} P & Y_1 & Y_2 \\ * & Q & Y_3 \\ * & * & R \end{bmatrix} \begin{bmatrix} \epsilon_{10}^T \\ \epsilon_1^T-\epsilon_4^T \\ \epsilon_4^T-\epsilon_5^T \end{bmatrix} \xi_1(t) \\
&= 2\xi_1^T(t)\widetilde{\Theta}\widetilde{\Theta}^T \\
&\quad \times \left[\epsilon_1,\ \tau_1\epsilon_6+(\tau(t)-\tau_1)\epsilon_7+(\gamma-\tau(t))\epsilon_8,\ (\tau_2-\gamma)\epsilon_9\right] \\
&\quad \times \begin{bmatrix} P & Y_1 & Y_2 \\ * & Q & Y_3 \\ * & * & R \end{bmatrix} \begin{bmatrix} \epsilon_{10}^T \\ \epsilon_1^T-\epsilon_4^T \\ \epsilon_4^T-\epsilon_5^T \end{bmatrix} \widetilde{\Theta}\widetilde{\Theta}^T\xi_1(t) \\
&= 2\xi_1^T(t)\widetilde{\Theta}\Upsilon_1\widetilde{\Theta}^T\xi_1(t),
\end{aligned} \tag{19}$$

where $\widetilde{\Theta} = diag\{\Theta_{2n}, \Theta_{2n}, \ldots, \Theta_{2n}\}_{20n\times 20n}$,

$$\begin{aligned}
&\dot{V}_2(\delta_t) \\
&= \delta^T(t)S_1\delta(t) + \delta^T(t-\gamma)(S_2-S_1)\delta(t-\gamma) \\
&\quad -\delta^T(t-\tau_2)S_2\delta(t-\tau_2), \\
&= \xi_1^T(t)(\epsilon_1S_1\epsilon_1^T + \epsilon_4(S_2-S_1)\epsilon_4^T - \epsilon_5S_2\epsilon_5^T)\xi_1(t) \\
&= \xi_1^T(t)\widetilde{\Theta}\widetilde{\Theta}^T(\epsilon_1S_1\epsilon_1^T + \epsilon_4(S_2-S_1)\epsilon_4^T - \epsilon_5S_2\epsilon_5^T) \\
&\quad \times\widetilde{\Theta}\widetilde{\Theta}^T\xi_1(t) \\
&= \xi_1^T(t)\widetilde{\Theta}\Upsilon_2\widetilde{\Theta}^T\xi_1(t), \\
&\dot{V}_3(\delta_t) \\
&= \delta^T(t)X_1\delta(t) + \delta^T(t-\tau_1)(X_2-X_1)\delta(t-\tau_1) \\
&\quad -(1-\dot{\tau}(t))\delta^T(t-\tau(t))X_2\delta(t-\tau(t)),
\end{aligned} \tag{20}$$

By $\dot{\tau}(t) \le \nu$, we have

$$\begin{aligned}
&\dot{V}_3(\delta_t) \\
&\le \delta^T(t)X_1\delta(t) + \delta^T(t-\tau_1)(X_2-X_1)\delta(t-\tau_1) \\
&\quad -(1-\nu)\delta^T(t-\tau(t))X_2\delta(t-\tau(t)) \\
&= \xi_1^T(t)(\epsilon_1X_1\epsilon_1^T + \epsilon_2(X_2-X_1)\epsilon_2^T \\
&\quad -(1-\nu)\epsilon_3X_2\epsilon_3^T)\xi_1(t) \\
&= \xi_1^T(t)\widetilde{\Theta}\widetilde{\Theta}^T(\epsilon_1X_1\epsilon_1^T + \epsilon_2(X_2-X_1)\epsilon_2^T \\
&\quad -(1-\nu)\epsilon_3X_2\epsilon_3^T)\widetilde{\Theta}\widetilde{\Theta}^T\xi_1(t) \\
&= \xi_1^T(t)\widetilde{\Theta}\Upsilon_3\widetilde{\Theta}^T\xi_1(t), \\
&\dot{V}_4(\delta_t) \\
&= \tau_1^2\dot{\delta}^T(t)Z_1\dot{\delta}(t) - \tau_1\int_{t-\tau_1}^{t} \dot{\delta}^T(s)Z_1\dot{\delta}(s)ds \\
&\quad \times(\gamma-\tau_1)^2\dot{\delta}^T(t)Z_2\dot{\delta}(t) - (\gamma-\tau_1)\int_{t-\gamma}^{t-\tau_1} \dot{\delta}^T(s)Z_2\dot{\delta}(s)ds \\
&\quad \times(\tau_2-\gamma)^2\dot{\delta}^T(t)Z_3\dot{\delta}(t) - (\tau_2-\gamma)\int_{t-\tau_2}^{t-\gamma} \dot{\delta}^T(s)Z_3\dot{\delta}(s)ds,
\end{aligned} \tag{21}$$

From Lemma 2.1 and Lemma 2.2, we get

$$\begin{aligned}
&\tau_1\int_{t-\tau_1}^{t} \dot{\delta}^T(s)Z_1\dot{\delta}(s)ds \\
&\quad \ge \begin{bmatrix} \zeta_0 \\ \zeta_1 \end{bmatrix}^T \widetilde{Z}_1 \begin{bmatrix} \zeta_0 \\ \zeta_1 \end{bmatrix}, \\
&(\gamma-\tau_1)\int_{t-\gamma}^{t-\tau_1} \dot{\delta}^T(s)Z_2\dot{\delta}(s)ds \\
&\quad \ge \begin{bmatrix} \zeta_2 \\ \zeta_3 \\ \zeta_4 \\ \zeta_5 \end{bmatrix}^T \begin{bmatrix} \beta_1\widetilde{Z}_2 & 0 \\ 0 & \frac{1}{1-\beta_1}\widetilde{Z}_2 \end{bmatrix} \begin{bmatrix} \zeta_2 \\ \zeta_3 \\ \zeta_4 \\ \zeta_5 \end{bmatrix}, \\
&(\tau_2-\gamma)\int_{t-\tau_2}^{t-\gamma} \dot{\delta}^T(s)Z_3\dot{\delta}(s)ds \\
&\quad \ge \begin{bmatrix} \zeta_6 \\ \zeta_7 \end{bmatrix}^T \widetilde{Z}_3 \begin{bmatrix} \zeta_6 \\ \zeta_7 \end{bmatrix},
\end{aligned}$$

where

$$
\begin{aligned}
\widetilde{Z}_i &= diag\{Z_i, 3Z_i\}, \forall i = 1, 2, 3,\\
\beta_1 &= \frac{\tau(t)-\tau_1}{\gamma-\tau_1}, \forall \beta_1 \in [0, 1],\\
\zeta_0 &= \delta(t) - \delta(t-\tau_1),\\
\zeta_1 &= \delta(t) + \delta(t-\tau_1) - \frac{2}{\tau_1}\int_{t-\tau_1}^{t} \delta(s)ds,\\
\zeta_2 &= \delta(t-\tau_1) - \delta(t-\tau(t)),\\
\zeta_3 &= \delta(t-\tau_1) + \delta(t-\tau(t)) - \frac{2}{\tau(t)-\tau_1}\int_{t-\tau(t)}^{t-\tau_1} \delta(s)ds,\\
\zeta_4 &= \delta(t-\tau(t)) - \delta(t-\gamma),\\
\zeta_5 &= \delta(t-\tau(t)) + \delta(t-\gamma) - \frac{2}{\gamma-\tau(t)}\int_{t-\gamma}^{t-\tau(t)} \delta(s)ds,\\
\zeta_6 &= \delta(t-\gamma) - \delta(t-\tau_2),\\
\zeta_7 &= \delta(t-\gamma) + \delta(t-\tau_2) - \frac{2}{\tau_2-\gamma}\int_{t-\tau_2}^{t-\gamma} \delta(s)ds.
\end{aligned}
$$

therefore

$$
\begin{aligned}
&\dot{V}_4(\delta_t)\\
&\le \xi_1^T(t)\widetilde{\Theta}\widetilde{\Theta}^T\epsilon_{10}(\tau_1^2 Z_1 + (\gamma-\tau_1)^2 Z_2\\
&\quad +(\tau_2-\gamma)^2 Z_3)\epsilon_{10}^T\\
&\quad -\left[\epsilon_1-\epsilon_2,\ \epsilon_1+\epsilon_2-2\epsilon_6\right]\widetilde{Z}_1\begin{bmatrix}\epsilon_1^T-\epsilon_2^T\\ \epsilon_1^T+\epsilon_2^T-2\epsilon_6^T\end{bmatrix}\\
&\quad -\left[\epsilon_2-\epsilon_3,\ \epsilon_2+\epsilon_3-2\epsilon_7,\ \epsilon_3-\epsilon_4,\ \epsilon_3+\epsilon_4-2\epsilon_8\right]\\
&\quad \times\begin{bmatrix}\beta_1\widetilde{Z}_2 & 0\\ 0 & \frac{1}{1-\beta_1}\widetilde{Z}_2\end{bmatrix}\begin{bmatrix}\epsilon_2^T-\epsilon_3^T\\ \epsilon_2^T+\epsilon_3^T-2\epsilon_7^T\\ \epsilon_3^T-\epsilon_4^T\\ \epsilon_3^T+\epsilon_4^T-2\epsilon_8^T\end{bmatrix}\\
&\quad -\left[\epsilon_4-\epsilon_5,\ \epsilon_4+\epsilon_5-2\epsilon_9\right]\widetilde{Z}_3\begin{bmatrix}\epsilon_4^T-\epsilon_5^T\\ \epsilon_4^T+\epsilon_5^T-2\epsilon_9^T\end{bmatrix}\\
&\quad \times\widetilde{\Theta}\widetilde{\Theta}^T\xi_1(t)\\
&= \xi_1^T(t)\widetilde{\Theta}\Upsilon_4\widetilde{\Theta}^T\xi_1(t), \qquad (22)
\end{aligned}
$$

We introduce a null identity as follows:

$$
2\left[\delta^T(t)M_1 + \dot{\delta}^T(t)M_2\right] \times\left[-\dot{\delta}^T(t) + \Xi_1\delta(t) - \Xi_2\delta^T(t-\tau(t))\right] = 0,
$$

where M_1 and M_2 are regarded as two slack matrices, one has

$$
\begin{aligned}
&2\xi_1^T(t)\left[\epsilon_1 M_1 + \epsilon_{10}M_2\right]\left[-\epsilon_{10} + \Xi_1\epsilon_1 - \Xi_2\epsilon_3\right]\xi_1(t)\\
&= 2\xi_1^T(t)\widetilde{\Theta}\widetilde{\Theta}^T\left[\epsilon_1 M_1 + \epsilon_{10}M_2\right]\left[-\epsilon_{10} + \Xi_1\epsilon_1 - \Xi_2\epsilon_3\right]\\
&\quad\times\widetilde{\Theta}\widetilde{\Theta}^T\xi_1(t)\\
&= 2\xi_1^T(t)\widetilde{\Theta}\Upsilon_5\widetilde{\Theta}^T\xi_1(t), \qquad (23)
\end{aligned}
$$

where the corresponding notations $\Upsilon_1 - \Upsilon_5$ in equations (19)-(23) are defined in Theorem 1, M_1 and M_2 are arbitrary matrices with appropriate dimensions.

Note that $\Xi_1, \Xi_2, P, Q, R, S_i, X_i, Y_j$ and $Z_j (i = 1, 2; j = 1, 2, 3)$ all belong to balanced matrices, combining (19)-(23) with Lemma 2.3, we can obtain

$$
\dot{V}(\delta_t) \le \xi_1^T(t)\Phi_1\xi_1(t), \tau(t) \in [\tau_1, \gamma], \qquad (24)
$$

where Φ_1 is defined in Theorem 1.

Since $\sum_{i=1}^{2n}\delta_i(t) = 0$, by Lemma 2.4 and (19)-(23), a effective result is obtained,i.e., for any nonzero vector $\xi_1(t)$ which satisfies $(\epsilon_i\otimes\mathbf{1}_{2n})^T\xi_1(t) = 0$ for any $i \in \{1, 2, \ldots, 10\}$, where $\epsilon_i \in \mathbb{R}^{10\times 10}$ is the same notation as that defined in Lemma 2.4 for the case m = 10, the inequality in (13) is a sufficient condition to make $\dot{V}(\delta_t) \le \xi_1^T(t)\Phi_1\xi_1(t) < 0$ hold.

For $\tau(t) \in [\gamma, \tau_2]$, along the trajectories of the dynamics (16), the derivative of $V(\delta_t)$ in (17) with respect to t is taken by

$$
\dot{V}(\delta_t) = \sum_{i=1}^{4}\dot{V}_i(\delta_t), \qquad (25)
$$

According to the same routine as used in the case $\tau(t) \in [\tau_1, \gamma]$,we have

$$
\begin{aligned}
&\dot{V}_1(\delta_t)\\
&= 2\xi_2^T(t)\widetilde{\Theta}\Pi_1\widetilde{\Theta}^T\xi_2(t), \qquad (26)\\
&\dot{V}_2(\delta_t)\\
&= \xi_2^T(t)\widetilde{\Theta}\Pi_2\widetilde{\Theta}^T\xi_2(t), \qquad (27)\\
&\dot{V}_3(\delta_t)\\
&\le \xi_2^T(t)\widetilde{\Theta}\Pi_3\widetilde{\Theta}^T\xi_2(t), \qquad (28)\\
&\dot{V}_4(\delta_t)\\
&\le \xi_2^T(t)\widetilde{\Theta}\widetilde{\Theta}^T\epsilon_{10}(\tau_1^2 Z_1 + (\gamma-\tau_1)^2 Z_2\\
&\quad +(\tau_2-\gamma)^2 Z_3)\epsilon_{10}^T\\
&\quad -\left[\epsilon_1-\epsilon_2,\ \epsilon_1+\epsilon_2-2\epsilon_6\right]\widetilde{Z}_1\begin{bmatrix}\epsilon_1^T-\epsilon_2^T\\ \epsilon_1^T+\epsilon_2^T-2\epsilon_6^T\end{bmatrix}\\
&\quad -\left[\epsilon_2-\epsilon_3,\ \epsilon_2+\epsilon_3-2\epsilon_7\right]\widetilde{Z}_2\begin{bmatrix}\epsilon_2^T-\epsilon_3^T\\ \epsilon_2^T+\epsilon_3^T-2\epsilon_7^T\end{bmatrix}\\
&\quad -\left[\epsilon_3-\epsilon_4,\ \epsilon_3+\epsilon_4-2\epsilon_8,\ \epsilon_4-\epsilon_5,\ \epsilon_4+\epsilon_5-2\epsilon_9\right]\\
&\quad \times\begin{bmatrix}\beta_2\widetilde{Z}_3 & 0\\ 0 & \frac{1}{1-\beta_2}\widetilde{Z}_3\end{bmatrix}\begin{bmatrix}\epsilon_3^T-\epsilon_4^T\\ \epsilon_3^T+\epsilon_4^T-2\epsilon_8^T\\ \epsilon_4^T-\epsilon_5^T\\ \epsilon_4^T+\epsilon_5^T-2\epsilon_9^T\end{bmatrix}\widetilde{\Theta}\widetilde{\Theta}^T\xi_2(t)\\
&= \xi_2^T(t)\widetilde{\Theta}\Pi_4\widetilde{\Theta}^T\xi_2(t),\\
&\beta_2 = \frac{\tau(t)-\gamma}{\tau_2-\gamma}, \quad \forall\beta_2 \in [0, 1]. \qquad (29)
\end{aligned}
$$

Similar to (23), one has:

$$
2\xi_2^T(t)\widetilde{\Theta}\Pi_5\widetilde{\Theta}^T\xi_2(t) = 0, \qquad (30)
$$

where the related notations $\Pi_1 - \Pi_5$ for (26)-(30) are defined in Theorem 1, M_3 and M_4 are arbitrary matrices with appropriate dimensions.

Through (26)-(30), the following inequality is obtained

$$
\dot{V}(\delta_t) \le \xi_2^T(t)\Phi_2\xi_2(t), \tau(t) \in [\gamma, \tau_2]. \qquad (31)
$$

where Φ_2 is defined in Theorem 1.

Similarly, by Lemma 2.4 and (26)-(30), we obtain that for any nonzero vector $\xi_2(t)$ which satisfies the equality $(\epsilon_i \otimes \mathbf{1}_{2n})^T\xi_2(t) = 0$ for any $i \in \{1, 2, \ldots, 10\}$ and a sufficient condition to guarantee $\dot{V}(\delta_t) \le \xi_2^T(t)\Phi_2\xi_2(t) < 0$ is that the inequality in (14).

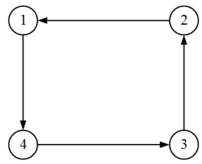

FIGURE 1. Network topology (1) of the multi-agent system.

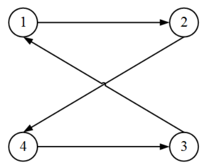

FIGURE 2. Network topology (2) of the multi-agent system.

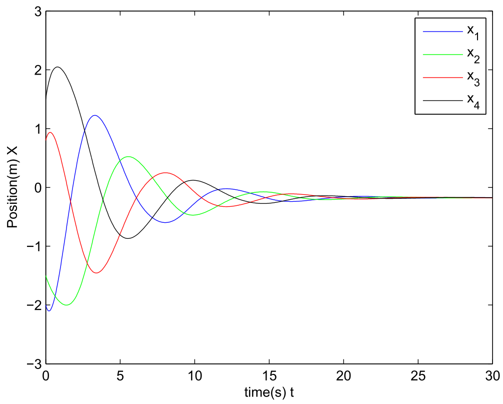

FIGURE 3. Position trajectories of the network (1).

Thus, it is clear that, if (13) and (14) hold, then $\dot{V}(\delta_t) < 0$ holds for $\tau(t) \in [\tau_1, \tau_2]$. This completes the proof.

IV. SIMULATION RESULTS

To demonstrate the validity of the theoretical results obtained in the previous sections, we give numerical simulations in this section.

Example 1: We study the consensus of a second-order multi-agent system consists of 4 agents, whose topology is shown in Fig. 1. It is a weakly connected and balanced digraph with 0–1 adjacency elements. According to Theorem 1, $\tau_1 = 0.1$, $\tau_2 = 1.288$, $\nu = 0.1$. The parameter κ of protocol (6) is taken as $\kappa = 1$. The scalar $\gamma = 0.4559$, which is the optimal division point of the time-varying delays interval $[\tau_1, \tau_2]$. Take account into two cases $\tau(t) \in [\tau_1, \gamma]$ and $\tau(t) \in [\gamma, \tau_2]$, feasible solutions are found by employing the LMI toolbox of MATLAB, respectively. We suppose that $\tau(t) = t - 0.4sin(t)$ and the initial values of position states and speed states are $[-2, -1.5, 0.8, 1.5]$, $[-1, -0.8, 1, 1.8]$, separately. Fig. 3 and Fig. 4 denote the position trajectories and velocity trajectories of system (4) using protocol (6) under the topology of Fig. 1, respectively.

Example 2: Consider the consensus of a second-order multi-agent system, whose topology is shown in Fig. 2. Obviously, it is weakly connected and balanced. According to Theorem 1, we assume τ_1, τ_2, ν, κ and γ are given the same values as those in Example 1 and $\tau(t) = t - 0.4sin(t)$. The initial values of position states and speed states are $[-3, -0.5, 1.8, 2.5]$, $[-2, -1.8, 0.1, 0.8]$, separately. Fig. 5 and Fig. 6 denote the position trajectories and velocity trajectories of system (4) using protocol (6) under the topology of Fig. 2, respectively.

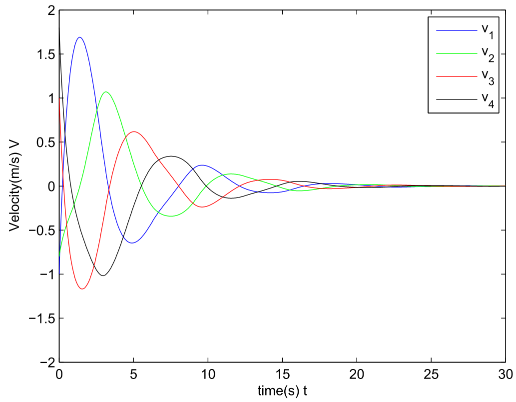

FIGURE 4. Velocity trajectories of the network (1).

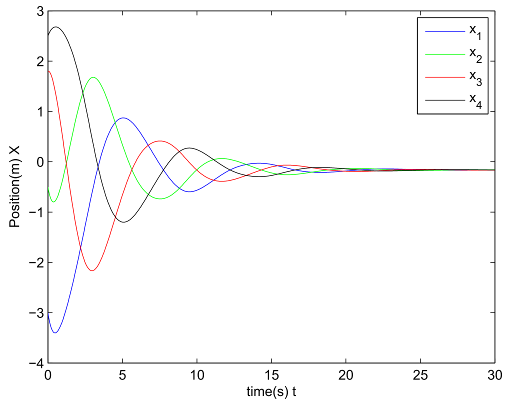

FIGURE 5. Position trajectories of the network (2).

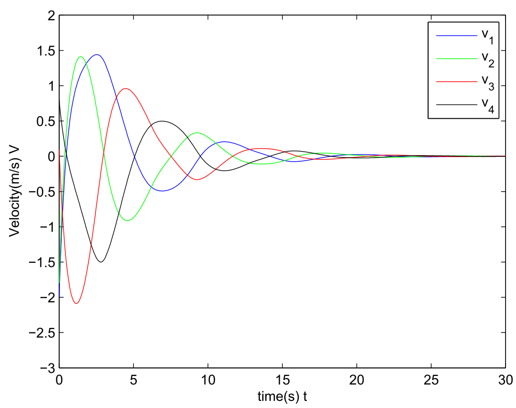

FIGURE 6. Velocity trajectories of the network (2).

Clearly, the disagreement of agents is monotonically decreasing and tends to 0 eventually, i.e. the positions converge to a common value while $t \to \infty$, which is reflected in both Fig. 3 and Fig. 5. Moreover, from Fig. 4 and Fig. 6 one can observe that the velocities converge to 0 while $t \to \infty$.

In summary, the consensus conditions (5) hold, that is to say all agents can asymptotically reach consensus.

V. CONCLUSION

In this paper, a second-order consensus problem of multi-agent systems with time-varying delays is studied, which has broad prospects in practical applications such as UAV formation control, intelligent robots, cooperative control of unmanned vehicles, and attitude control of artificial satellites. In our approach, the delay interval is divided into two variable segments instead of n equal segments, which can reduce the computational complexity. By appropriately constructing a LKF, Wirtinger-based inequality is combined with the reciprocally convex combination lemmas to deal with the derivative of the LKF, the derived results guarantee consensus under a directed fixed topology and effectively tolerate relatively large bounded communication delay. Nevertheless, the results proposed in this paper are still somewhat conservative, which may be improved by the improved Wirtinger-based inequality and it will appear in our future work. Motivated by the work in [31] and [32], the two directions for future research would be to investigate the group consensus of leader-following systems with time delays via pinning control and the finite-time consensus problem of multi-agent systems with time-varying delays, respectively.

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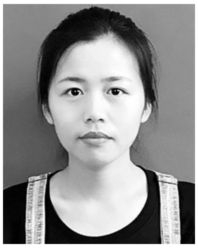

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