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Fixed-Time Bipartite Containment of Multi-Agent Systems Subject to Disturbance

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ABSTRACT This paper studies the bipartite containment of multi-agent systems (MASs) subject to the bounded disturbance. Different from the subsistent related works on this topic, the settling time irrespective of initial value can be calculated in advance. Two distributed control algorithms are provided correspondingly for the first-order systems and second-order systems. Based on algebraic theory, properties of the Laplacian matrix and fixed-time stability theory, it shows that fixed-time bipartite containment can be achieved via the presented control law. Simulations are eventually employed to prove the correctness and effectiveness of theoretical results.

INDEX TERMS Fixed-time bipartite containment, multi-agent systems, disturbance.

I. INTRODUCTION

In recent years, the research of control systems has obtained tremendous concern due to its wide application, especially for network control systems [1]–[6] and multi-agent systems [7]–[11]. Under circumstance of MASs, the agents could decrease the effect of probable agent faults, reduce the energy expenditure of the entire system and so on. Among various study of MASs, consensus is a fundamental and significant problem, which means agents can reach an agreement via information exchange. Moreover, sufficient criteria are often given to reach consensus by stability theories and other mathematical theories [12]–[14]. According to the number of leaders, consensus can be roughly divided into three categories, namely, leaderless consensus [15], leader-follower consensus [16] and containment control [17].

Note that there might need multiple leaders to complete complicated tasks in reality. For example, some robots must enter into safety area when others are equipped with sensors to detect the hazardous obstacle in practical applications. Naturally, the containment control comes into existence, where followers will converge into the dynamic convex hull formed by leaders. Up to now, various control methods have been developed to solve containment problem, such as adaptive control [18], feedback control [19], observer-based approach [20] and so on. Yuan and Zeng [19] dealt with output containment control, where sufficient criterions were presented by linear algebraic equations and matrix inequalities. Han *et al.* [20] used an observer-based control approach to handle the containment control for MASs with exogenous disturbances.

Noticeably, the interaction topology between agents quoted above is just collaborative network. Actually, competition is as common and important as cooperation. Thus, the relation of agents must be directly denoted as signed digraphs with positive/negative weights signifying trust/distrust. By taking advantage of signed graph [21], Meng [22] extended containment control to bipartite containment control. It was shown that followers can enter the convex hull formed by each leader's pathway and its symmetric one. Indeed, there have been some connected outcomes on the bipartite containment lately. Based on the feedback control, Zuo *et al.* [23] solved bipartite containment for heterogeneous multi-agent systems on signed digraphs. He indicated that the output-feedback control was more practical than the state feedback case if the full state information of agents was hard to obtain in applications. Meng and Gao [24] investigated the high-order bipartite containment tracking in time-varying cooperation-competition topology. Zhou *et al.* [25] proposed an observer-based event-triggered controller to solve bipartite

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containment of MASs with input quantization, where a nonlinear decomposition approach was applied to build a relation between the actual control signal and the quantized one.

However, the bipartite containment is asymptotically reached in the previous results. Remarkably, convergence rate is a vital index for evaluating the performance of the control algorithms. Thus, the finite-time control algorithm [26], [27] is developed and it has the merits of faster convergence and better disturbance rejection. Regrettably, the finite-time control is strongly correlative with the initial value, that is, the setting time changes dramatically with the initial value. Therefore, it is of enormous necessity to research fixed-time control due to the mentioned superiorities above. Up to now, there have been abundant literatures about fixed-time control, such as fixed-time synchronization [28], fixed-time formation control [29], fixed-time consensus [30], and fixed-time flocking [31].

Inspired by the foregoing researches, we make an endeavor to work out the fixed-time bipartite containment by fixed-time stability theory [32], as far as we know, no one has studied yet. The main contributions of the paper can be summarized as follows: (i) In the paper, the proposed fixed-time control law guarantee all agents reach the bipartite containment whatever there's a disturbance or not. (ii) Compared with asymptotic bipartite containment [22]–[25], the controller is designed based on fixed-time stability theory such that the convergence time of the system is independent of starting value in the paper. Therefore, the settle time can be speculated accurately, in turn, it can be set at any arbitrary value in advance by adjusting the control parameters. (iii) In fact, external disturbances are inevitably generated and their influence can not be ignored. In comparison to first-order bipartite containment control [22], we further take into account the disturbances in the first-order dynamics of agents and extend it to second-order systems with disturbances. It is more practical and meaningful in theory and application.

The remainder of this article is made up of several parts. In Section II, necessary preliminaries as well as problem formulations are introduced briefly. In Section III, we drive the protocol to solve fixed-time bipartite containment for first-order systems with disturbance. In addition, another fixed-time protocol was proposed so as to deal with the bipartite containment problem of second-order multiagent systems. The simulations and summings-up are provided in Section IV and Section V, respectively.

II. PRELIMINARIES AND PROBLEM FORMULATION

A. NOTATIONS

Throughout this article, let R, R^l and $R^{h\times l}$ be the real number, the $l-$dimensional Euclidean space and the set of all $h \times l$ real matrices, respectively. $\mathbf{1}$ represents the vector with all the entries being one, meanwhile, I represents the identity matrix with the appropriate dimensions. $A \otimes B$ denotes the Kronecker product of matrices A and B. Moreover, A^T, $\lambda_{\min}(A)$, $\lambda_{\max}(A)$ correspondingly denote the transpose, minimum eigenvalue and maximum eigenvalue for a given matrix A. $diag(\eta) = diag(\eta_1, \eta_2, \ldots, \eta_l)$ is a diagonal matrix with diagonal elements $\eta_1, \eta_2, \ldots, \eta_l$. In addition, for $p > 0$ and for a vector $x = [x_1, x_2, \ldots, x_l]^T$, we define $\|x\|_p = (\sum_{i=1}^{l} |x_i|^p)^{1/p}$ and $sig^p(x) = [sign(x_1)|x_1|^p, sign(x_2)|x_2|^p, \ldots, sign(x_l)|x_l|^p]^T$, where $sign(\cdot)$ is the standard sign function.

B. GRAPH THEORY AND SOME LEMMAS

Consider h leaders and l followers with cooperative-antagonistic connections G here. Regard $F = \{1, 2, \ldots, l\}$, $L = \{l+1, l+2, \ldots, l+h\}$ as the sets of followers and leaders. $G_f(v, \varepsilon)$ contains the whole relations between followers, where $v = \{1, 2, \ldots, l\}$ and $\varepsilon \subset v \times v$ stand for the finite, nonempty vertex set and link set, accordingly. We assume that $a_{ij} > 0$ indicates the cooperative interaction from follower j to follower i, and $a_{ij} < 0$ means the competitive interaction, otherwise $a_{ij} = 0$. Undirected graphs G_f is taken into account, and evidently $A^T = A$. $D = diag(d_1, d_2, \ldots, d_l)$ is the in-degree matrix with $d_i = \sum_{j=1}^{l} |a_{ij}|, i \in F$, meanwhile, the Laplacian matrix L_s of graph G_f is defined as

$$l_{ij} = \begin{cases} \sum_{k=1}^{l} |a_{ik}|, & j = i, \\ -a_{ij}, & j \neq i. \end{cases}$$

Additionally, $G_r = diag(g_i^r) \in R^{l\times l}$, $i \in F$, $r \in L$ signifies the information transmission from leader r to follower i. $g_i^r \neq 0$ if the ith follower can obtain information from the rth leader, and $g_i^r = 0$, otherwise. Hence, $\bar{G}_r = diag(|g_i^r|)$, $\varphi_r = \frac{1}{h}L^s + \bar{G}_r$.

Assumption 1: For every follower $i \in F$ in the signed digraph G, there is at least one leader $r \in L$ which has a directed path from this leader to the follower. And the graph G is structural balance.

Lemma 1 [22]: Under Assumptions 1, define

$$\bar{\psi}_k = \frac{1}{m}\bar{L} + \bar{G}_k, \quad \psi_k^s = \frac{1}{m}L^s + \bar{G}_k,$$

then, ψ_k^s and $\sum_{k=n+1}^{n+m} \psi_k^s$ are positive-definite and non-singular M-matrices. That is,

(i) The eigenvalues of ψ_k^s and $\sum_{k=n+1}^{n+m} \psi_k^s$ have positive real parts;

(ii)$(\psi_k^s)^{-1}$ and $(\sum_{k=n+1}^{n+m} \psi_k^s)^{-1}$ exist and both are nonnegative.

Remark 1: Assumption 1 in this study is the condition of Lemma 1, therefore it is listed in assumptions.

Lemma 2 [33]: Let $x \in R^N$ and $q > p > 0$, then $\|\cdot\|_q \leq \|\cdot\|_p \leq N^{\frac{1}{p}-\frac{1}{q}}\|\cdot\|_q$.

Lemma 3 [32]: If there exists a continuous radially unbounded and positive definite function $V(x)$ such that $\dot{V}(x) \leq -aV^{p}(x) - \beta V^{q}(x)$ for some $a, \beta > 0, p > 1$, $0 < q < 1$, then the system is globally fixed-time stable and the settling time function T can be estimated by

$$T \leq T_{\max} = \frac{1}{a(p-1)} + \frac{1}{\beta(1-q)}.$$

Furthermore, if $p = 1 + \frac{1}{u}$ and $q = 1 - \frac{1}{u}$ with $u > 1$ are selected, the settling time function T can be estimated by a less conservative bound

$$T_{\max} = \frac{\pi u}{2\sqrt{a\beta}}.$$

Lemma 4: For structurally balanced graph G, a Laplacian candidate function and state error is accordingly expressed as

$$V(t) = \frac{1}{2}\xi^{T}(t)\left(\sum_{v=l+1}^{l+h}(\varphi_v \otimes I)\right)\xi(t),$$
$$e(t) = \left(\sum_{v=l+1}^{l+h}(\varphi_v \otimes I)\right)\xi(t),$$

where $\sum_{v=l+1}^{l+h}(\varphi_v \otimes I)$ is a positive matrix and $\xi = [\xi_1, \xi_2, \ldots, \xi_l]^T$, afterwards there satisfies

$$\frac{2V(t)}{\lambda_{\max}((\sum_{v=l+1}^{l+h}(\varphi_v \otimes I))^{-1})} \leq \|e(t)\|_2^2 \leq \frac{2V(t)}{\lambda_{\min}((\sum_{v=l+1}^{l+h}(\varphi_v \otimes I))^{-1})}.$$

Proof: See the Appendix.

C. PROBLEM FORMULATION

In this section, the first-order systems and second-order systems with bounded disturbance will be discussed.

1) BIPARTITE CONTAINMENT FOR FIRST-ORDER SYSTEMS WITH DISTURBANCE

Consider the systems with $l + h$ agents, in which there are l followers tagged as $i = 1, 2, \ldots, l$ and h leaders tagged as $r = l+1, l+2, \ldots, l+h$. The dynamics of the leader-follower agents is given by

Followers:

$$\dot{x}_i(t) = u_i(t) + w_i(t), \tag{1}$$

Leaders:

$$\dot{x}_r(t) = w_r(t), \tag{2}$$

where x_i, u_i, w_i represent respectively state, control input and disturbance of the follower i, and x_r, w_r represent respectively state and disturbance of the leader r. Besides, $|w_i| \leq c$ and $|w_r| \leq c$ are all permitted.

Assumption 2: For the external bounded disturbance, it satisfies $\|w_i\|_{i \in F \cup L} \leq c$, in which c is a positive constant.

Define the consensus error as the following.

$$e_{ix}(t) = \sum_{j=1}^{l} a_{ij}\left(sign\left(a_{ij}\right)x_i(t) - x_j(t)\right) + \sum_{r=l+1}^{l+h} g_i^r(sign\left(g_i^r\right)x_i(t) - x_r(t)).$$

Definition 1: For multi-agent dynamics (1) and (2), the fixed-time bipartite containment control is addressed if there is a controller u_i and a fixed-time $T > 0$ regardless of the initial value, such that the states of followers enter into the convex hulls $co(X_h)$, where

$$co(X_h) = \{\sum_{i=l+1}^{l+h}(\alpha_i x_i - \beta_i x_i)\,|\alpha_i \geq 0, \beta_i \geq 0, \sum_{i=l+1}^{l+h}(\alpha_i + \beta_i) = 1\}.$$

Lemma 5: Under Assumptions 1 and 2, for first-order dynamics (1) and (2), the bipartite containment control introduced in definition 1 is figured out if $\lim_{t \to t_1} e_{ix}(t) = 0$ and $e_{ix}(t) = 0$ for $\forall t \geq t_1$.

Proof: The process of proof is similar to lemma 3 in [22] and therefore it is omitted here.

2) BIPARTITE CONTAINMENT OF SECOND-ORDER SYSTEMS WITH DISTURBANCE

A group of $l + h$ agents is considered, in which there are l followers tagged as $i = 1, 2, \ldots, l$ and h leaders tagged as $r = l+1, l+2, \ldots, l+h$. The dynamics of the leader-follower agents is given by

Followers:

$$\dot{x}_i(t) = v_i(t),$$
$$\dot{v}_i(t) = u_i(t) + w_i(t), \tag{3}$$

Leaders:

$$\dot{x}_r(t) = v_r(t),$$
$$\dot{v}_r(t) = w_r(t), \tag{4}$$

where x_i, v_i, u_i, w_i represent respectively position, velocity, control input and disturbance of the agent i, and x_r, v_r, w_r represent respectively position, velocity, and disturbance of the leader r. Besides, $|w_i| \leq c$ and $|w_r| \leq c$ are all permitted.

Assumption 3: For the external bounded disturbance, it satisfies $\|w_i\|_{i\in F\cup L} \leq c$, in which c is a positive constant.

Define the consensus errors as the following

$$e_{ix}(t) = \sum_{j=1}^{l} a_{ij}\left(sign\left(a_{ij}\right)x_i(t) - x_j(t)\right) + \sum_{r=l+1}^{l+h} g_i^r(sign\left(g_i^r\right)x_i(t) - x_r(t)),$$

$$e_{iv}(t) = \sum_{j=1}^{l} a_{ij}\left(sign\left(a_{ij}\right)v_i(t) - v_j(t)\right) + \sum_{r=l+1}^{l+h} g_i^r(sign\left(g_i^r\right)v_i(t) - v_r(t)).$$

Definition 2: For multi-agent dynamics (3) and (4), the fixed-time bipartite containment control is solved if there is a controller u_i and a fixed-time $T > 0$ independent of the initial value, such that the positions and velocities of followers respectively enter into the convex hulls $co(X_h)$ and $co(V_h)$, where

$$co(X_h) = \{\sum_{i=l+1}^{l+h}(\alpha_i x_i - \beta_i x_i)\,|\alpha_i \geq 0, \beta_i \geq 0, \sum_{i=l+1}^{l+h}(\alpha_i + \beta_i) = 1\},$$

$$co(V_h) = \{\sum_{i=l+1}^{l+h}(\alpha_i v_i - \beta_i v_i)\,|\alpha_i \geq 0, \beta_i \geq 0, \sum_{i=l+1}^{l+h}(\alpha_i + \beta_i) = 1\}.$$

Lemma 6: Under Assumptions 1 and 3, considering second-order dynamics (3) and (4), the bipartite containment problem introduced in definition 2 is solved if $\lim_{t\to t_2} e_{ix}(t) = 0$, $\lim_{t\to t_2} e_{iv}(t) = 0$ and $e_{ix}(t) = e_{iv}(t) = 0$ for $\forall t \geq t_2$.

Proof: The proof process is similar to lemma 3 in [22] and therefore it is omitted here.

III. MAIN RESULTS

In this part, two control inputs will be respectively raised in order to address the first-order/second-order fixed-time bipartite containment with bounded disturbance.

A. BIPARTITE CONTAINMENT FOR FIRST-ORDER SYSTEMS WITH DISTURBANCE

For systems (1) and (2), we proposed the protocol based only on relative states as follows

$$u_i(t) = -sig(e_{ix}(t))^{2-\frac{1}{d}} - gsig(e_{ix}(t))^{\frac{1}{d}} - f\,sign(e_{ix}(t)), \quad (5)$$

where $g > 0, d > 1$, d is a positive odd integer and f is designed later.

Theorem 1: Assume that Assumption 1 and 2 hold. For the multi-agent systems (1) and (2) with the control input (5), in which $f \geq c + \left\|(\sum_{k=l+1}^{l+h}(\varphi_k \otimes I))^{-1}\sum_{r=l+1}^{l+h}(G_r \otimes I)\right\|_1 c$, the bipartite containment can be addressed in fixed time

$$T_1 \leq \frac{d\pi l^{\frac{d-1}{4d}}}{2\sqrt{g}(d-1)}\lambda_{\max}((\sum_{r=l+1}^{l+h}(\varphi_r \otimes I))^{-1}).$$

Proof: The tracking error can be expressed in a compact form

$$\begin{aligned} e_x(t) &= \left(\left(\bar{D} - A\right) \otimes I\right)x(t) + \sum_{r=l+1}^{l+h}\left(\bar{G}_r \otimes I\right)x(t) - \sum_{r=l+1}^{l+h}(G_r \otimes I)(1 \otimes x_r(t)) \\ &= \sum_{r=l+1}^{l+h}(\varphi_r \otimes I)x(t) - \sum_{r=l+1}^{l+h}(G_r \otimes I)\bar{x}_r(t) \\ &= \sum_{v=l+1}^{l+h}(\varphi_v \otimes I)(x(t) - \left(\sum_{k=l+1}^{l+h}(\varphi_k \otimes I)\right)^{-1} \times \sum_{r=l+1}^{l+h}(G_r \otimes I)\bar{x}_r(t)), \end{aligned}$$

where $x = [x_1, x_2, \ldots, x_l]^T$ and $\bar{x}_r = 1 \otimes x_r$. Remark $\delta = x - \left(\sum_{k=l+1}^{l+h}(\varphi_k \otimes I)\right)^{-1}\sum_{r=l+1}^{l+h}(G_r \otimes I)\bar{x}_r$. So the tracking error equations can be equivalently described by

$$e_x(t) = \sum_{v=l+1}^{l+h}(\varphi_v \otimes I)\delta(t).$$

Differentiate $\delta(t)$ as follows

$$\dot{\delta}(t) = -sig(e_x(t))^{2-\frac{1}{d}} - gsig(e_x(t))^{\frac{1}{d}} - f\,sign(e_x(t)) + W_1(t) - (\sum_{k=l+1}^{l+h}(\varphi_k \otimes I))^{-1}\sum_{r=l+1}^{l+h}(G_r \otimes I)\bar{W}_r(t),$$

where $W_1 = [w_1, w_2, \ldots, w_l]^T$, $\bar{W}_r = 1 \otimes w_r$. From Lemma 4, one derives

$$\begin{aligned} \dot{V}(t) &= \delta^T(t)\sum_{v=l+1}^{l+h}(\varphi_v \otimes I)\dot{\delta}(t) \\ &= e_x{}^T(t)(-sig(e_x(t))^{2-\frac{1}{d}} - gsig(e_x(t))^{\frac{1}{d}} - f\,sign(e_x(t)) \\ &\quad + W_1(t) - (\sum_{k=l+1}^{l+h}(\varphi_k \otimes I))^{-1}\sum_{r=l+1}^{l+h}((G_r \otimes I)\bar{W}_r(t))) \\ &\leq -\|e_x(t)\|_{3-\frac{1}{d}}^{3-\frac{1}{d}} - g\|e_x(t)\|_{1+\frac{1}{d}}^{1+\frac{1}{d}} - \|e_x(t)\|(f - c \\ &\quad - \left\|(\sum_{k=l+1}^{l+h}(\varphi_k \otimes I))^{-1}\sum_{r=l+1}^{l+h}(G_r \otimes I)\right\|_1 c) \end{aligned}$$

$$\leq -\frac{l^{\frac{1-d}{2d}}2^{\frac{3d-1}{2d}}}{\lambda_{\max}^{\frac{3d-1}{2d}}((\sum_{k=l+1}^{l+h}(\varphi_k\otimes I))^{-1})}(V(t))^{\frac{3d-1}{2d}} -\frac{2^{\frac{d+1}{2d}}g}{\lambda_{\max}^{\frac{d+1}{2d}}((\sum_{k=l+1}^{l+h}(\varphi_k\otimes I))^{-1})}(V(t))^{\frac{d+1}{2d}}. \quad (6)$$

Here, the above inequality is realized by Lemma 2. Based on (6) and Lemma 3, it comes to a conclusion that $V(t)$ achieves zero in fixed time, which means the sliding mode $e_x = 0$ will be kept in fixed time. Obviously, the fixed-time bipartite containment is realised according to Lemma 5. The rigorous proof is completed.

Corollary 1: If assumption 1 holds, considering the special case $\underset{i\in F\cup L}{w_i} = 0$ for the systems (1) and (2), the bipartite containment will be also achieved under the controller (5).

Proof: To avoid being tedious, the proof can refer to Theorem 1, thus is omitted.

B. BIPARTITE CONTAINMENT FOR SECOND-ORDER SYSTEMS WITH DISTURBANCE

Next, we focus on bipartite containment for second-order systems (3) and (4). The process is composed of two parts. First, in view of state error and velocity error, a terminal sliding mode vector is developed. Second, we propose a distributed control protocol based on fixed-time stability theory.

The tracking error can be described by a matrix

$$e_x(t) = \sum_{v=l+1}^{l+h}(\varphi_v\otimes I)(x(t) - \left(\sum_{k=l+1}^{l+h}(\varphi_k\otimes I)\right)^{-1} \times \sum_{r=l+1}^{l+h}(G_r\otimes I)\bar{x}_r(t)),$$
$$e_v(t) = \sum_{v=l+1}^{l+h}(\varphi_v\otimes I)(v(t) - \left(\sum_{k=l+1}^{l+h}(\varphi_k\otimes I)\right)^{-1} \times \sum_{r=l+1}^{l+h}(G_r\otimes I)\bar{v}_r(t)),$$

where $x = [x_1, x_2, \ldots, x_l]^T$, $v = [v_1, v_2, \ldots, , v_l]^T$, $\bar{x}_r = 1\otimes x_r$ and $\bar{v}_r = 1\otimes v_r$.

Note

$$\xi(t) = x(t) - \left(\sum_{k=l+1}^{l+h}(\varphi_k\otimes I)\right)^{-1}\sum_{r=l+1}^{l+h}(G_r\otimes I)\bar{x}_r(t),$$
$$\eta(t) = v(t) - \left(\sum_{k=l+1}^{l+h}(\varphi_k\otimes I)\right)^{-1}\sum_{r=l+1}^{l+h}(G_r\otimes I)\bar{v}_r(t),$$

for convenience. So the tracking error equations can be equivalently expressed by

$$e_x(t) = \left(\sum_{v=l+1}^{l+h}(\varphi_v\otimes I)\right)\xi(t),$$
$$e_v(t) = \left(\sum_{v=l+1}^{l+h}(\varphi_v\otimes I)\right)\eta(t). \quad (7)$$

Based on above discussion, with the help of fixed-time stability theory, we construct a sliding mode for each follower agent as

$$s(t) = \eta(t) + sig^e(e_x(t)), \quad (8)$$

where $0 < e < 1$. To reach the aforementioned bipartite containment problem of second-order multi-agents, the following control input is represented as

$$u(t) = -e|e_x(t)|^{e-1} - sig^{2-\frac{1}{g}}(s(t)) - vsig^{\frac{1}{g}}(s(t)) - bsign(s(t)), \quad (9)$$

where $v > 0, g > 1, g$ is positive odd integer and b is designed later.

Theorem 2: Assume that Assumption 1 and 3 hold. For the multi-agent systems (3) and (4) with the control input (9), in which $b \geq c + \left\|\left(\sum_{k=l+1}^{l+h}(\varphi_k\otimes I)\right)^{-1}\sum_{r=l+1}^{l+h}(G_r\otimes I)\right\|_1 c$, the bipartite containment can be addressed in fixed time

$$T_3 \leq T_2 + 2\frac{(0.5\xi^T(T_2)(\sum_{v=l+1}^{l+h}(\varphi_v\otimes I))\xi(T_2))^{\frac{1-e}{2}}}{(1-e)\left(\sqrt{2/\lambda_{\max}((\sum_{r=l+1}^{l+h}(\varphi_r\otimes I))^{-1})}\right)^{e+1}},$$

where $T_2 = \frac{g\pi l^{\frac{g-1}{4g}}}{2\sqrt{v(g-1)}}\lambda_{\max}((\sum_{r=l+1}^{l+h}(\varphi_r\otimes I))^{-1})$.

Proof: Step 1: Substituting dynamics (3) and (4) into (9), one generates

$$\dot{\eta}(t) = -e|e_x(t)|^{e-1} - sig^{2-\frac{1}{g}}(s(t)) - vsig^{\frac{1}{g}}(s(t)) - bsign(s(t)) + W_1(t) - (\sum_{k=l+1}^{l+h}(\varphi_k\otimes I))^{-1} \times \sum_{r=l+1}^{l+h}(G_r\otimes I)\bar{W}_r(t),$$

where $W_1 = [w_1, w_2, \ldots, w_l]^T$, $\bar{W}_r = 1\otimes w_r$.

Set up the Lyapunov function $V_1(t) = \frac{1}{2}s^T(t)s(t)$. From Lemma 5, it deduces

$$\dot{V}_1(t) = s^T(t)\dot{s}(t) = s^T(t)(-sig^{2-\frac{1}{g}}(s(t)) - vsig^{\frac{1}{g}}(s(t)) - bsign(s(t))$$

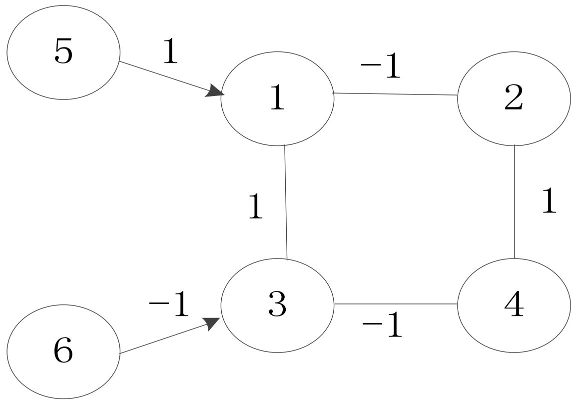

FIGURE 1. Diagraph with six agents.

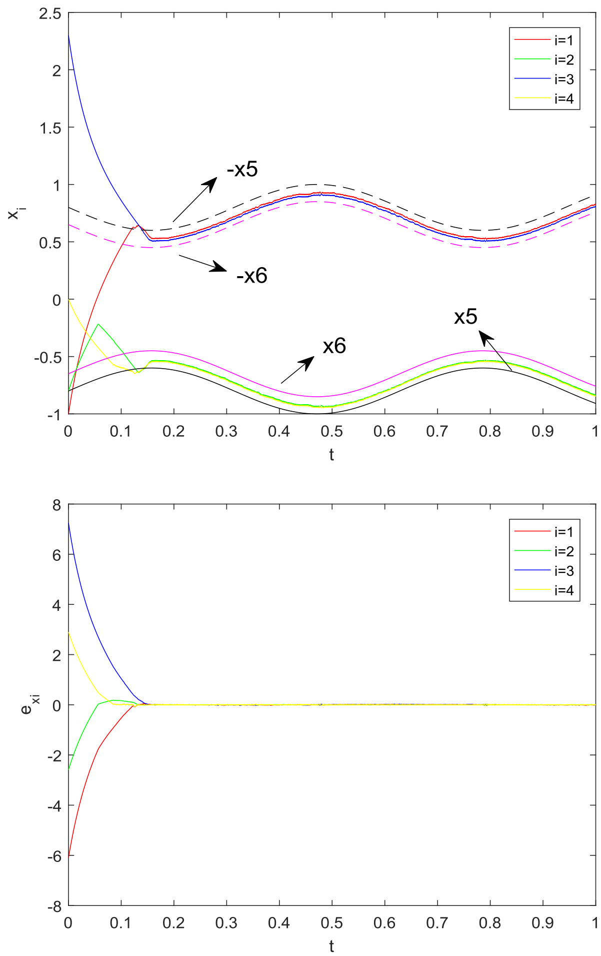

FIGURE 2. States and state errors of first-order MASs with disturbance.

$$+W_1(t) - \left(\sum_{k=l+1}^{l+h}(\varphi_k \otimes I)\right)^{-1}\sum_{r=l+1}^{l+h}(G_r \otimes I)\bar{W}_r(t))$$
$$\leq -\|s(t)\|_{3-\frac{1}{g}}^{3-\frac{1}{g}} - v\|s(t)\|_{1+\frac{1}{g}}^{1+\frac{1}{g}} - \|s(t)\|(b-c$$
$$-\left\|\left(\sum_{k=l+1}^{l+h}(\varphi_k \otimes I)\right)^{-1}\sum_{r=l+1}^{l+h}(G_r \otimes I)\right\|_1 c)$$

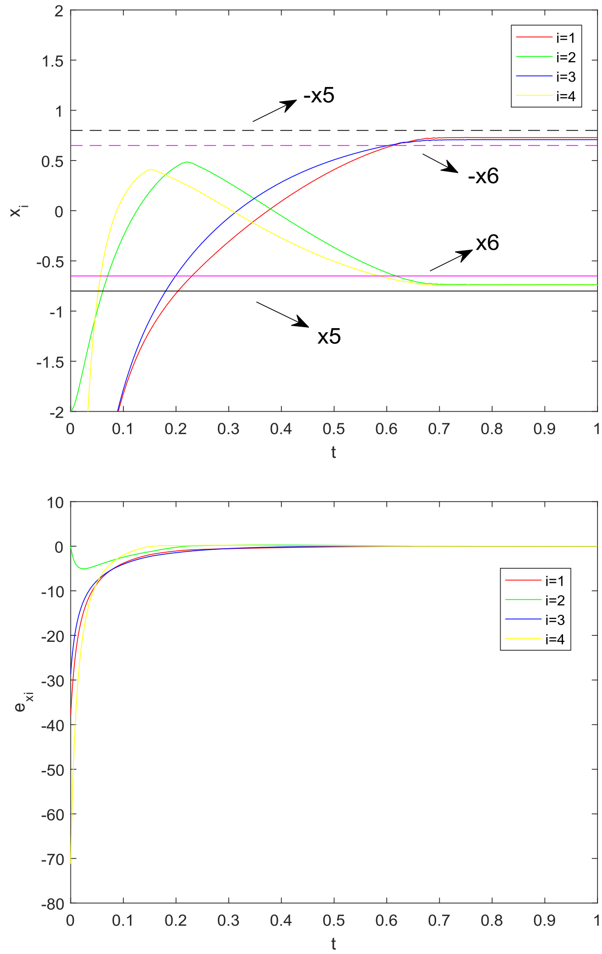

FIGURE 3. States and state errors of first-order MASs without disturbance.

$$\leq -l^{\frac{1-g}{2g}}\|s(t)\|_2^{\frac{3g-1}{g}} - v\|s(t)\|_2^{1+\frac{1}{g}}$$
$$\leq -\frac{l^{\frac{1-g}{2g}}2^{\frac{3g-1}{2g}}}{\lambda_{\max}^{\frac{3g-1}{2g}}(\left(\sum_{k=l+1}^{l+h}(\varphi_k \otimes I)\right)^{-1})}V_1(t)^{\frac{3g-1}{2g}}$$
$$-\frac{2^{\frac{g+1}{2g}}v}{\lambda_{\max}^{\frac{g+1}{2g}}(\left(\sum_{k=l+1}^{l+h}(\varphi_k \otimes I)\right)^{-1})}V_1(t)^{\frac{g+1}{2g}}. \quad (10)$$

Here, the above inequality is realized by Lemma 2. Evidently, it follows from (10) and Lemma 3 that $V_1(t)$ achieves zero in fixed time, which means that the sliding mode $s(t) = 0$ will be kept in fixed time. Its upper bound is

$$T_2 = \frac{g\pi l^{\frac{g-1}{4g}}}{2\sqrt{v}(g-1)}\lambda_{\max}((\sum_{r=l+1}^{l+h}(\varphi_r \otimes I))^{-1}).$$

Step 2: On the basis of above detailed analysis, if the sliding mode surface $s(t) = 0$ meets, it is easy to reach that $\eta(t) = -sig^e(e_x(t))$ as $i = 1, 2, \ldots, l$. For the purpose of making $(e_x(t), e_v(t))$ converge to $(0, 0)$, Lyapunov function

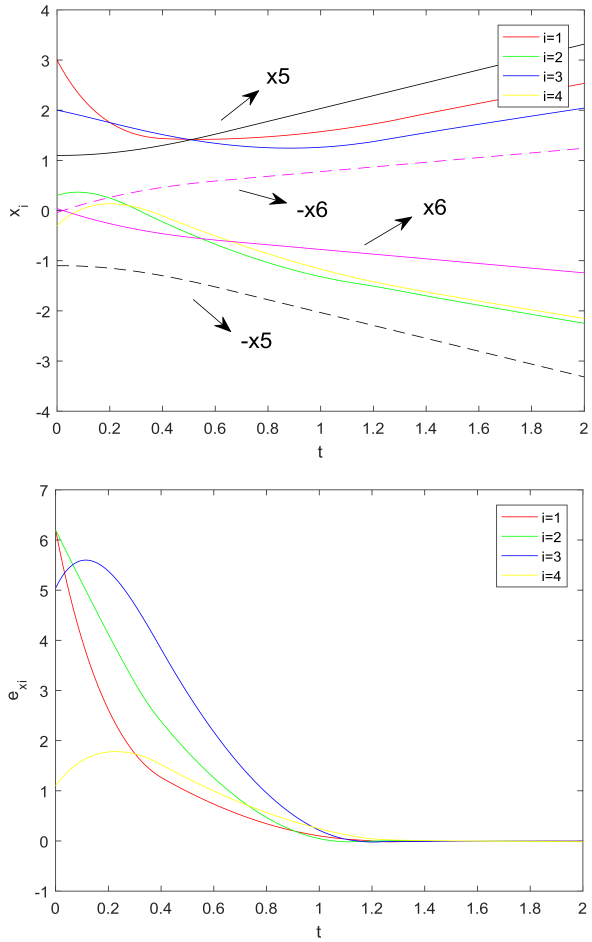

FIGURE 4. Positions and position errors of second-order MASs with disturbance.

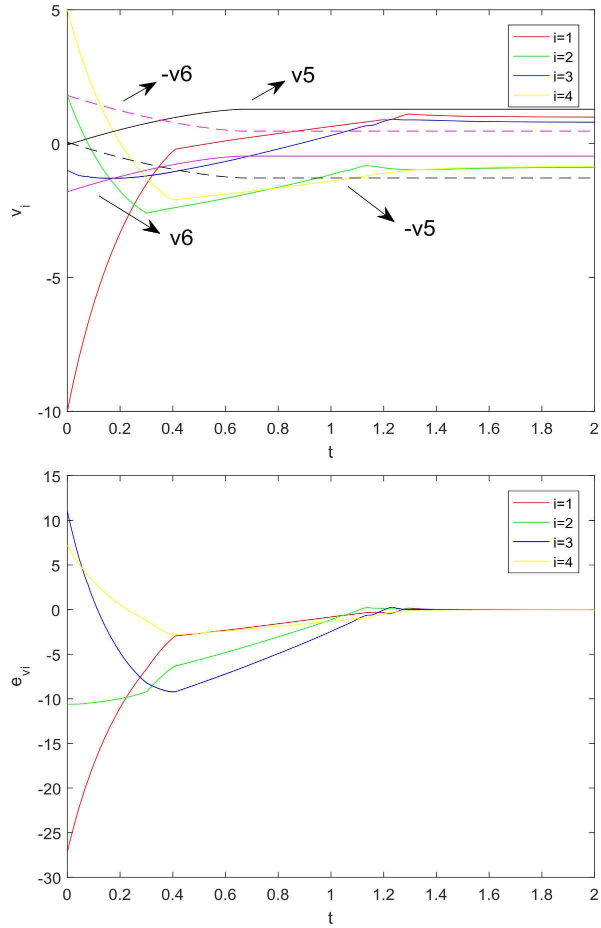

FIGURE 5. Velocities and velocity errors of second-order MASs with disturbance.

candidate is elected as

$$V_2(t) = \frac{1}{2}\xi^T(t)\left(\sum_{v=l+1}^{l+h}(\varphi_v \otimes I)\right)\xi(t).$$

One yields the time derivative of $V_2(t)$

$$\begin{aligned}\dot{V}_2(t) &= \xi^T(t)\left(\sum_{v=l+1}^{l+h}(\varphi_v \otimes I)\right)\eta(t)\\ &= -e_x{}^T(t)sig^e(e_x(t))\\ &\leq -\|e_x(t)\|_2^{e+1}\\ &\leq -\frac{2^{\frac{e+1}{2}}}{\lambda_{\max}^{\frac{e+1}{2}}(\left(\sum_{k=l+1}^{l+h}(\varphi_k \otimes I)\right)^{-1})}V_2(t)^{\frac{e+1}{2}}. \quad (11)\end{aligned}$$

Together with lemma 3, it indicates that $\xi(t)$ can come up to zero. Due to (7) and (11), it draws a conclusion that the equilibrium point $(e_{ix}(t), e_{iv}(t))$ can achieve $(0, 0)$ in finite time. The upper bound of settlement time satisfies

$$T_3 \leq T_2 + 2\frac{(0.5\xi^T(T_2)(\sum_{v=l+1}^{l+h}(\varphi_v \otimes I))\xi(T_2))^{\frac{1-e}{2}}}{(1-e)\left(\sqrt{2/\lambda_{\max}((\sum_{r=l+1}^{l+h}(\varphi_r \otimes I))^{-1})}\right)^{e+1}}.$$

Obviously, the fixed-time bipartite containment is realised according to Lemma 6. The rigorous proof is completed.

Remark 2: Since the initial time T_2 of $\xi(T_2)$ in the afore-mentioned formula is independent of the initial condition, that is to mean, the settling time of the entire multiagent systems is also irrespective of the starting condition.

Corollary 2: If assumption 1 holds, considering the special case $\underset{i\in F\cup L}{w_i} = 0$ for the system (3) and (4), the bipartite containment will be achieved under the control protocol (9).

Proof: To avoid being tedious, the proof can refer to Theorem 2, thus is omitted.

IV. ILLUSTRATIVE EXAMPLE

We study a multiagent system including two leaders labelled as 5-6 and four followers labelled as 1-4 with the

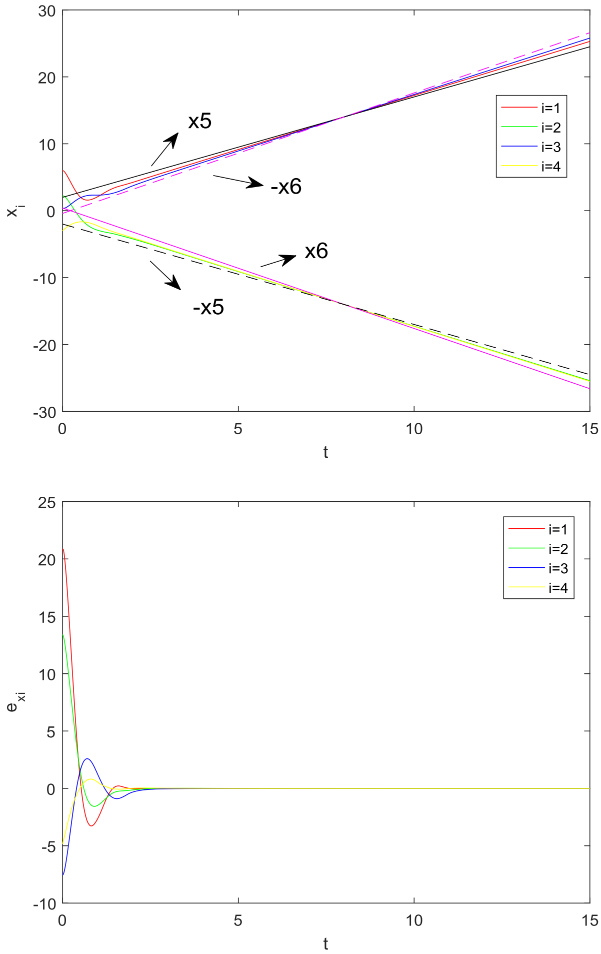

FIGURE 6. Positions and position errors of second-order MASs without disturbance.

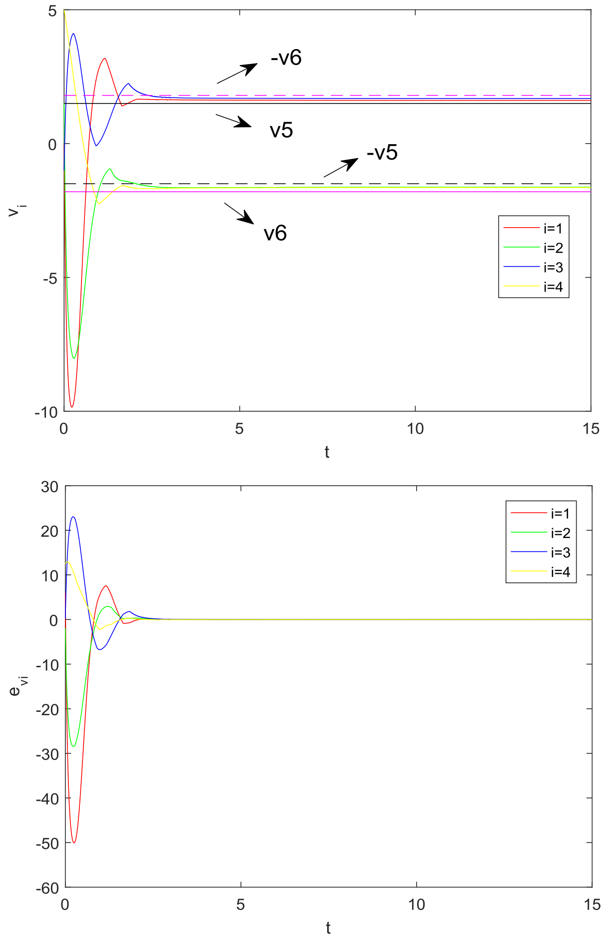

FIGURE 7. Velocities and velocity errors of second-order MASs without disturbance.

communication topology FIGURE.1. Four simulation examples will be introduced to prove the correctness and the validity of the theoretical results.

Example 1 (First-Order Systems): In this example, the control law (5) is utilized to explain fixed-time bipartite containment for the system (1) and (2). The original value is designed as $x(0) = [-1, -0.8, 2.3, 0, -0.8, -0.65]^T$.

Case 1 (Without Disturbance): $w_i = 0$, $i = 1, 2, \ldots, 6$, $g = 1, d = 3, f = 6$.

Case 2 (With Disturbance): $w_i = 2\cos(10t)$, $i = 1, 2, \ldots 6$, $g = 5, d = 3, f = 0$.

The state trajectory and state error is shown in FIGURE.2 with disturbance while the state trajectory and state error is shown in FIGURE.3 without disturbance.

Example 2 (Second-Order Systems): In this example, considering the system (3) and (4) under the protocol (9), the simulation results are presented in FIGURE.4-7. The starting information and parameters are designed as

Case 1 (Without Disturbance):

$$x(0) = [6, 2.2, 0.35, -3, 2, 0.4]^T,$$
$$v(0) = [0, 1.52, -1, 5, 1.5, -1.8]^T,$$
$$g = 6, \ e = 0.95, \ v = 3, \ b = 0.$$

Case 2 (With Disturbance):

$$x(0) = [3, 0.3, 2, -0.3, 1.1, 0.04]^T,$$
$$v(0) = [-10, 1.8, -1, 5, -0.05, -1.8]^T, g = 2, e = 0.483,$$
$$v = 6, \ b = 6.$$

Additionally, $w_i = 2arccos(1.5t), i = 1, 2, \ldots, 6$ in case 2. The simulations are shown in FIGURE.4-5 with disturbance while the simulations are shown in FIGURE.6-7 without disturbance.

V. CONCLUSION

In this study, two bipartite containment protocols are respectively put forward for the first-order/second-order systems on the basis of Lyapunov function and fixed-time stability theory. We can find that the settling time is not associated with starting state but only to rely on parameters and network connectivity. The availability of correlative theory has been demonstrated via simulations. Besides, some parameter estimation approaches [34]–[38] can be combined with the method in this article for network time-delay systems [39]–[43] with unknown parameters.

Future works will be on the bipartite containment in the hybrid cases such as with switching topology or with impulse [44], [45].

APPENDIX

Proof of Lemma 4: According to the definition of a Laplacian candidate function and state error above, it deduces

$$\begin{aligned} V(t) &= \frac{1}{2}e^T(t)(\sum_{v=l+1}^{l+h}(\varphi_v\otimes I))^{-1}e(t) \\ &\geq \frac{1}{2}\lambda_{\min}((\sum_{v=l+1}^{l+h}(\varphi_v\otimes I))^{-1})e^T(t)e(t) \\ &= \frac{1}{2}\lambda_{\min}((\sum_{v=l+1}^{l+h}(\varphi_v\otimes I))^{-1})\,\|e(t)\|_2^2. \end{aligned}$$

Therefore,

$$\|e(t)\|_2^2 \leq \frac{2V(t)}{\lambda_{\min}((\sum_{v=l+1}^{l+h}(\varphi_v\otimes I))^{-1})}.$$

Similarly, $\|e(t)\|_2^2 \geq \frac{2V(t)}{\lambda_{\max}((\sum_{v=l+1}^{l+h}(\varphi_v\otimes I))^{-1})}$ is obtained.

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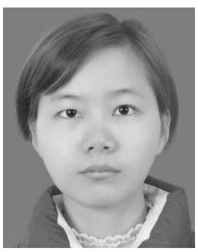

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