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Monitoring Reliability for Three-Parameter Frechet Distribution Using Control Charts

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ABSTRACT Control Charts for attributes have been widely adopted for examining the fraction non-conforming or non-conformities in a process. However, Shewhart control charts may face some practical problems when the process fraction of non-conformities is very low. While dealing with high quality parameters (low defects), a precise solution is to use time between events (TBE) charts. In the present study, control charts for time between failures have been developed considering that the inter-failure time follows Frechet distribution. Maximum likelihood estimation method (MLE) and Probability weighted moment method (PWMM) are taken into account for estimation purposes. We have also used cumulative sums for inter-failure times to monitor the reliability of three-parameter Frechet distribution. The distribution of sum of Frechet random variates has been obtained with the help of moment approximation. Control limits of cumulative chart for different values of shape parameter have been obtained. Two real data sets are analyzed for illustrative purposes.

INDEX TERMS Control charts, moment approximations, average run length, Frechet distribution.

I. INTRODUCTION

A control chart is a statistical tool used to distinguish between variation in a process resulting from common causes and variation resulting from special causes. It presents a graphical display of process stability/instability over time. One of the assumptions of control charts is that the underlying distribution of the quality characteristic is normal, but there are situations where we have to deal with skewed data. While Dealing with such non-normal circumstances, Shewhart control charts may give misleading results about the process. It often leads to an increase in Type-I risk with the increase in skewness. In case of moderate to large departure from normality, there are two adaptations. First, we can use transformations to make our data approximately normal and then use typical Shewhart control limits. The other choice is to study the behavior of process carefully to find the actual underlying distribution of the quality characteristic.

Currently, many skewed distributions have been used to model the lifetime of products. As in life testing of high quality products, the data turns out to be highly skewed, so extreme value distributions perform very well in the field of reliability engineering and electronics as compared to normal distribution. There are many statistical distributions used for reliability analysis, e.g. exponential, Weibull, Log-normal, Gamma, Frechet etc. In present study, we are using Frechet distribution (FD) to model the failure times of a system or component(s), which was first introduced by a French mathematician Maurice Frechet [9]. We are exploiting FD by means of time between failure control charts.

A lot of study has been conducted based on control charts using different distributions and also on the parameter estimation of FD. For example, Abbas and Tang [3] considered maximum likelihood estimators (MLEs) and least square estimators of FD with two parameters based on Type II censored sample. Abid [5] estimated the MLEs, moment estimators, regression estimators, percentile estimators, least square estimators and L-moments estimators of FD. Xie *et al.* [23] suggested to use T chart for Poisson and exponentially distributed processes. Rao and Sricharani [1] worked on Time control charts through non homogenous Poisson process based on Dagum distribution. Rosaiah *et al.* [4] developed Shewhart control charts based on percentiles for Gumbel distribution which is also a positively skewed distribution. A method to

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monitor reliability for a three parameter Weibull distribution was proposed by Surucu and Sazak [21]. The parameters of the distribution were estimated and the study was extended for cumulative time elapsed between r failures. The lower and upper control limits of T_r control chart were obtained by using percentage points of the corresponding distribution. To obtain the cumulative chart, the distribution of sum of independent Weibull random variates was approximated using two moment Normal approximation and three moment Chi-square approximation. This work can be extended using a different estimation technique and for different distributions. Therefore, in our study we have adopted the same procedure for constructing control limits for another life testing distribution named FD. The problem of estimating parameters of the distribution was set using ML and PWM methods.

II. TIME BETWEEN FAILURE CHARTS

Usually, Shewhart 'c'and 'u' charts can be used to monitor the process of non-conformities or defects, but for precise results we deal with high quality in our production processes leading to zero or low defect levels. These types of rare events cannot be better explained by 'c' and 'u' chart. In this case, Calvin [8] developed a new chart named time between events (TBE) chart. This chart is used to monitor inter-arrival TBEs. There are many versions of TBE charts including cumulative count of conforming chart, cumulative quantity control chart, time between first failure (T) and time between r^{th} failure (T_r)-charts, cumulative sum (CUSUM) and exponentially distributed moving average (EWMA) T-charts and so on. In the present study, T-chart and T_r-chart will be considered as our model is based on time between failures of a production process.

A. FRECHET DISTRIBUTION

Many generalizations of Exponential distribution are useful to model failure times, like Gamma, Weibull, and Inverse Weibull distribution etc. In this article, three-parameter Frechet distribution (FD) will be studied, to model time between failures and construct control charts for various estimates of model parameters. FD is a special case of generalized extreme value distributions (EVDs) with cumulative distribution function (CDF) given as

$$F(x) = exp\left[-\left(\frac{\beta}{x-\gamma}\right)^{\alpha}\right], \quad x > 0, \tag{1}$$

and the probability density function (PDF) given as

$$f(x,\alpha,\beta,\gamma) = \frac{\alpha}{\beta}\left(\frac{\beta}{x-\gamma}\right)^{\alpha+1} exp\left[-\left(\frac{\beta}{x-\gamma}\right)^{\alpha}\right], \quad x > 0, \alpha, \beta > 0, -\infty < \gamma < \infty. \tag{2}$$

where α, β, γ, are shape, scale and location parameters respectively. To monitor the failure time of systems or components, one can utilize LCL and UCL. If the time between failures plotted on the chart is below the LCL, it is an indication of increasing failure rate or system deterioration. If the plotted time is above the UCL, it shows improvement in the process. We will estimate the location, shape and scale parameters of FD using maximum likelihood estimation (MLE) and Probability weighted moments (PWM) method.

TABLE 1. Average ML estimates for $\alpha = 3$, $\beta = 1$ and different values of γ along with their CLs and MSE (within parenthesis) for n = 30.

γ (LCL, UCL)	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\widehat{LCL}$	$\widehat{UCL}$
0.5 (0.6105,2.3765)	2.9321 (1.4861)	0.9852 (0.2616)	0.5279 (0.2362)	0.6808	2.4284
1 (1.1105,2.8765)	2.5713 (0.5051)	0.7873 (0.0687)	1.1911 (0.0557)	1.2587	2.8664
2 (2.1105,3.8765)	2.5547 (0.9117)	0.8539 (0.1250)	2.1275 (0.1148)	2.2166	3.9567
3 (3.6105,5.37565)	2.4607 (0.7684)	0.7430 (0.1544)	3.2379 (0.1341)	3.3056	4.8491
3.5 (3.6105,5.3765)	3.3000 (1.8904)	1.0217 (0.2673)	3.4561 (0.2652)	3.6482	5.2965
4 4.1105,5.8765)	2.7757 (1.1531)	0.9584 (0.3387)	4.0445 (0.2652)	4.1831	5.8935

The results given in Table I and 2 are the average ML estimates obtained by solving these normal equations for 30 and 60 samples.

III. CONTROL CHARTS FOR FIRST FAILURE

The control limits for first failure can easily be obtained by following the suggestion of Xie *et al.* [23]. He suggested that the approximate LCL and UCL reduce to lower and upper percentage points of the assumed distribution for first failure. The percentage points of the assumed distribution will be obtained using its quantile function which is

$$F^{-1(x)} = \hat{\gamma} + \frac{\hat{\beta}}{(-\log p)^{\frac{1}{\hat{\alpha}}}} \tag{3}$$

By substituting $p = 1 - \frac{\lambda}{2}$ and $p = \frac{\lambda}{2}$ in (3), we get LCL and UCL for T-chart as given in (4) and (5) respectively where λ is the probability of rejecting true null hypothesis.

$$LCL = \hat{\gamma} + \frac{\hat{\beta}}{\left(-\log\left(1-\frac{\lambda}{2}\right)\right)^{\frac{1}{\hat{\alpha}}}} \tag{4}$$

$$UCL = \hat{\gamma} + \frac{\hat{\beta}}{\left(-\log\left(\frac{\lambda}{2}\right)\right)^{\frac{1}{\hat{\alpha}}}} \tag{5}$$

where $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$ are estimated using ML and PWM methods.

A. MAXIMUM LIKELIHOOD ESTIMATION

Let X_1, X_2, X_3..., X_n be a random sample of size 'n' from three parameter FD, then the likelihood function of (2) is

$$L(x,\alpha,\beta,\gamma) = \prod_{i=1}^{n}\left(\frac{\alpha}{\beta}\left(\frac{\beta}{x_i-\gamma}\right)^{\alpha+1} \times exp\left[-\left(\frac{\beta}{x_i-\gamma}\right)^{\alpha}\right]\right), \tag{6}$$

TABLE 2. Average ML estimates for $\alpha = 3$, $\beta = 1$ and different values of location parameter γ along with their control limits and MSE (within parenthesis) for n = 60.

Υ (LCL, UCL)	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\widehat{LCL}$	$\widehat{UCL}$
0.5 0.6105,2.3765	3.4614 (1.4563)	1.1456 (0.1769)	0.3484 (0.1809)	0.5467	2.3559
1 1.1105,2.8765	3.1639 (0.8890)	0.9948 (0.0750)	0.9970 (0.0728)	1.1405	2.8452
2 2.1105,3.8765	3.3102 (1.5709)	1.1016 (0.2633)	1.9165 (0.2247)	2.1078	3.8781
3 3.1105,4.8765	3.3374 (1.4941)	1.0297 (0.1253)	2.9473 (0.1203)	3.1180	4.8116
3.5 3.6105,5.3765	2.9304 (0.3354)	0.9047 (0.0373)	3.6220 (0.0487)	3.7248	5.3632
4 4.1105,5.8765	2.8530 (0.9327)	1.0062 (0.0792)	4.0189 (0.0735)	4.1410	6.0712

It is more convenient to work with log likelihood function. The log likelihood function of (6) is

$$l = \log L(x, \alpha, \beta, \gamma) = \sum_{i=1}^{n} \log\left(\frac{\alpha}{\beta}\right) + (\alpha+1)\sum_{i=1}^{n}\log\left(\frac{\beta}{x_i-\gamma}\right) - \sum_{i=1}^{n}\log\left(\frac{\beta}{x_i-\gamma}\right)^{\alpha}, \quad (7)$$

Partial derivatives of (7) with respect to α, β, γ and gives us (8), (9) and (10) respectively

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} + n\log\beta - \sum_{i=1}^{n}(x_i-\gamma) + \sum_{i=1}^{n}\left(\left(\frac{x_i-\gamma}{\beta}\right)^{-\alpha}\log\left(\frac{x_i-\gamma}{\beta}\right)\right), = 0, \quad (8)$$

$$\frac{\partial l}{\partial \beta} = \frac{n\alpha}{\beta} - \frac{\alpha}{\beta}\sum_{i=1}^{n}\left(\frac{x_i-\gamma}{\beta}\right)^{-\alpha} = 0, \quad (9)$$

$$\frac{\partial l}{\partial \gamma} = (\alpha+1)\sum_{i=1}^{n}\frac{1}{x_i-\gamma} - \frac{\alpha}{\beta}\sum_{i=1}^{n}\left(\frac{x_i-\gamma}{\beta}\right)^{-\alpha-1} = 0, \quad (10)$$

The solution to the above system of normal equations is not possible explicitly. Here we used the Laplace approximation in the LearnBayes package of the R software version (i386 3.6.1) to get the point estimates.

The results given in Table 1 and 2 are the average ML estimates obtained by solving these normal equations for 30 and 60 samples.

Figure 1, 2 and 3 illustrates T-charts for first failure using ML estimates of the parameters. In the T-chart presented in Figure 2, a small amount of shift is created in the location of the process and hence T chart is unable to detect this shift. But in Figure 3, the quantity of shift created is somehow large and therefore T chart has detected the shift in the process location.

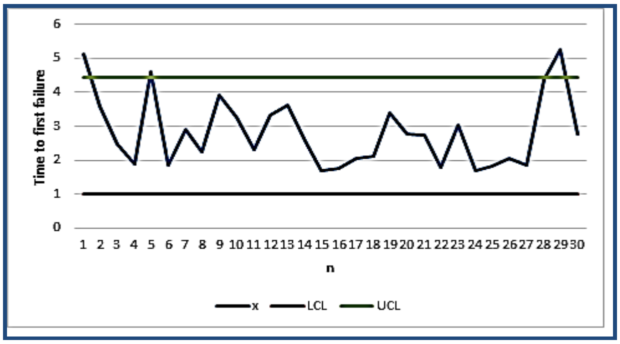

FIGURE 1. T-chart for first failure when $\alpha = 3$, $\beta = 1$, $\gamma = 2$ for n = 60 failure numbers using ML Estimates.

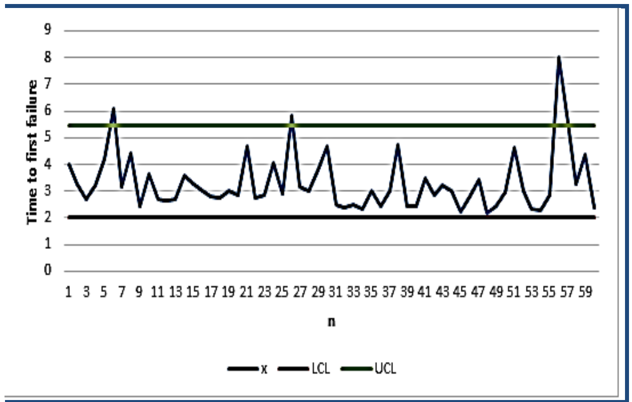

FIGURE 2. T-chart with $\alpha = 3$, $\beta = 1$, $\gamma = 4$ for first 60failures and $\alpha = 3$, $\beta = 1$, $\gamma = 3.5$ for next 60 failures using ML estimates.

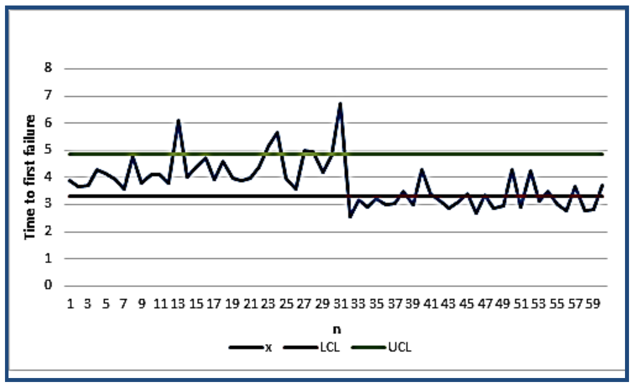

FIGURE 3. T-chart for first failure with $\alpha = 3$, $\beta = 1$, $\gamma = 3$ for first 30 failures and $\alpha = 3$, $\beta = 1$, $\gamma = 2$ for next 30 failures using ML estimates.

It can be clearly noted that the charts constructed here in Figure 1, 2 and 3 are in statistical control since no sample point falls below the LCL. However, when the process possesses a shift in the location, T-charts can detect this shift only when the quantity of the shift is large. Otherwise, T-chart does not detect small amount of shift.

B. PROBABILITY WEIGHTED MOMENT METHOD

Probability weighted moment (PWM) method was proposed by Greenwood *et al.* [12]. It has the following form:

$$M_{q:r,s} = E\left(x^q (F(x))^r (1-F(x))^s\right), \quad (11)$$

TABLE 3. Average PWM estimates for $\alpha = 1$, $\beta = 0.5$ and different values of γ along with their CLs and MSE (within parenthesis) for n = 30.

ϒ LCL, UCL	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\widehat{LCL}$	$\widehat{UCL}$
1 1.0007,4.3038	1.6629 (0.6629)	1.0841 (0.5841)	0.4900 (-0.5100)	0.5104	3.8644
1.5 1.5007,4.8038	1.6398 (0.6398)	1.0858 (0.5858)	0.9958 (-0.5042)	1.0151	4.4301
2 2.0007,5.3038	1.6459 (0.6459)	1.1097 (0.6097)	1.4734 (-0.5266)	1.4934	4.9685
2.5 2.5007,8038	1.6151 (0.6151)	1.0783 (0.5783)	2.0032 (-0.4968)	2.0212	5.4744

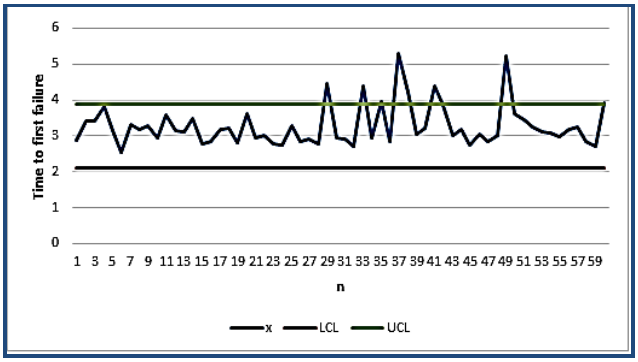

FIGURE 4. T-chart for first failure when $\alpha = 1$, $\beta = 0.5$, $\gamma = 1$ for n = 30 failure numbers using PWM estimates.

Hosking *et al.* [14] presented the following form

$$B_r = E\left[XF(x)^r\right], \quad r = 0, 1, 2, 3 \ldots, \tag{12}$$

The PWM equation is

$$B_r = \frac{1}{r+1}\left(\gamma + \left(\beta^\alpha (r+1)\right)^{\frac{1}{\alpha}} \Gamma\left(1 - \frac{1}{\alpha}\right)\right), \quad \alpha > 1, \tag{13}$$

The corresponding unbiased sample PWM's proposed by Landwehr *et al.* [17] are

$$\hat{B}_0 = \frac{1}{n}\sum\nolimits_{i=1}^{n} x_i, \tag{14}$$

$$\hat{B}_r = \frac{\sum_{i=1}^{n}\left(\frac{x-i}{r}\right)x_i}{n\left(\frac{x-i}{r}\right)}, \tag{15}$$

By equating these population and sample moments we will get our required estimators. Table 3 presents PWM estimates for the underlying parameters of the distribution.

T-charts based on PWM estimates has been presented in Figure 4 and 5 where Figure 5 represents control chart with a shift after thirtieth sample. From these figures, it can be observed that the charts depict in control process behavior. But from Figure 5, it has been clearly evident that when shift occurs in the location of the process, T-chart based on

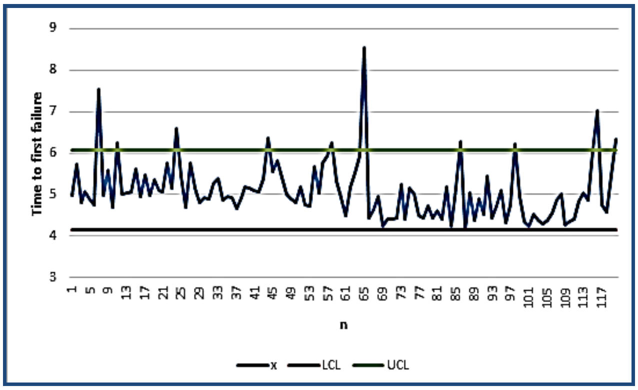

FIGURE 5. T-chart with $\alpha = 1$, $\beta = 0{:}5$, $\gamma = 2.5$ for first 30 failures and $\alpha = 1$, $\beta = 0.5$, $\gamma = 2$ for next 30 failures using PWM estimates.

PWM estimates is unable to detect this shift. It is to be noted that the quantity of the shift observed here is small therefore T chart is unable to detect small shifts but it may happen that the same charts detects shift in the process when the quantity of shift is large. To overcome this flaw, the behavior of the process is examined using cumulative chart which is called time between r^{th} failure charts (T_r-charts) in the terminology of TBE charts which is discussed in the later section.

IV. CONTROL CHARTS FOR MORE THAN ONE FAILURE

It has been observed that T-chart is unable to detect small shift in the location of the process. Therefore, we have chosen another TBE charts with better performance which is called T_r-chart. T_r-chart is sometimes called CUSUM T-chart as it uses sum of failure times observed. For the construction of T_r-charts, the distribution of cumulative time elapsed between a fixed number of failures is required. But in the literature of FD, we do not know the exact distribution of sum of Frechet random variables. Therefore, to identify the underlying distribution, two moment approximations have been used. Surucu and Sazak [21] introduced a new variable for cumulative time elapsed between 'r' failures as

$$Y_i = \sum\nolimits_{j=r(i-1)+1}^{ir} X_j, \quad (i = 1, 2, 3, \ldots, m). \tag{16}$$

where X_j ($j = 1, 2, 3, \ldots, N$) are independently and identically distributed (IID) Frechet random variates. Since we have been dealing with the condition of constructing control limits based on percentage points of the distribution, therefore, the limits cannot be obtained without having the distribution of Y_i. Here the random variable Y_i represents the sum of Frechet random variates. In order to obtain the distribution of Y_i, numerous approximations are available to approximate one distribution from another distribution. Some of those are based on equating moments of both distributions. Here, two moment Normal approximation and three moment Chi-square approximation will be considered, for details see [10] and [22].

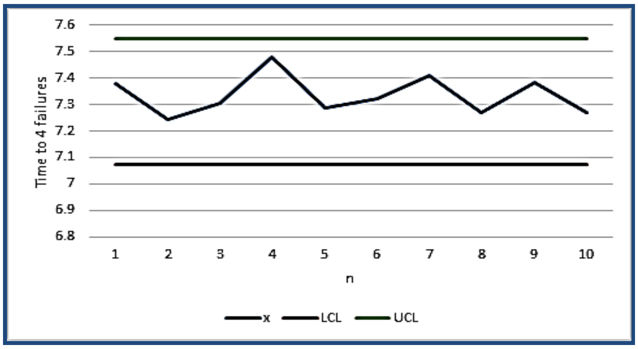

FIGURE 6. T_4 -chart with 40 cumulative observations for $\alpha = 12$, $\beta = 0.5$, $\gamma = 1.3$ using Normal approximation.

A. TWO MOMENT NORMAL APPROXIMATION

This approximation will give us better results for percentage points of Y if the following conditions are roughly satisfied

$$\left|\beta_1^*\right| < 0.5 \quad \text{and } 2.8 < \left|\beta_2^*\right| < 3.2 \tag{17}$$

Here simulated skewness and kurtosis values have been used to check the above conditions. Suppose

$$w_1 = (y - c)/h \sim N(0, 1) \tag{18}$$

where 'c' and 'h'are constants and can be obtained by equating moments on both sides of equation (18) and are equal to $c = \acute{\mu}_1 = E(Y)$ and $h = \sqrt{\mu_2} \equiv \sqrt{V(Y)}$; see [10]. Since $-z_{\frac{\lambda}{2}}$ and $z_{\frac{\lambda}{2}}$ are the lower and upper $\left(\lambda/2\right)$ *th* percentage points of W_1 respectively, therefore the LCL and UCL of Y which are the percentage points of the distribution of Y are

$$\hat{L}_r = c - hz\frac{\lambda}{2} \tag{19}$$

$$\hat{U}_r = c + hz\frac{\lambda}{2} \tag{20}$$

While approximating the distribution of sum of Frechet random variables, the results for different values of shape parameters are presented in Table 4. Since these approximations are based on the moments of the distribution, therefore we cannot take the value of shape parameter less than four. The reason for this is the limitation of FD that its value of moments does not exist for the value of the shape less than four. The LCL and UCL of T_r chart based on Normal approximations are presented in Table 5. Using two moment Normal approximation, T_r-charts for r = 4 are given in Figure 6 and 7. Small shift is created in Figure 7 which the chart has detected clearly and rapidly.

Recall that when the quantity of shift is small, T-chart is unable to detect that shift. In contrast, T_r-chart based on Normal approximation gives fast detection of shift even if the quantity of shift is small.

B. THREE MOMENT CHI-SQUARE APPROXIMATION

If the following conditions are satisfied

$$\sqrt{\beta_1^*} > 0 \quad \text{and } |\beta_2^* - (3 + 1.5\beta_1^*)| \leq 0.5$$

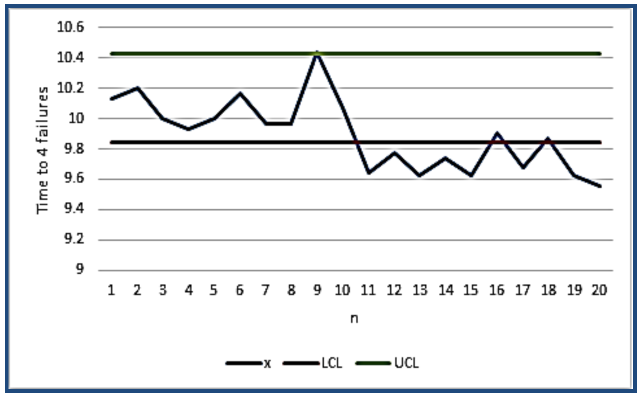

FIGURE 7. T_4 -chart with first 40 cumulative observations for α =10, $\beta = 0.5$, $\gamma = 2$ and next 40 cumulative observations for $\alpha = 10$, $\beta = 0.5$, $\gamma = 1.9$ using normal approximation.

TABLE 4. Moment approximations for the distribution of sum of Frechet random variable for r = 4.

⇓	Shape	$\|\beta_1^*\|$	$\|\beta_2^*\|$	$\sqrt{\beta_1}^*$	$\|\beta_2^*-(3+1.5\beta_1^*)\|$
3-M χ^2	5.8	2.1199	6.7887	1.4560	0.6088
	5.9	2.0407	6.4615	1.4285	0.4005
	6.0	1.9678	6.1695	1.4028	0.2178
	6.1	1.9005	5.9074	1.3786	0.0566
	6.2	1.8383	5.6710	1.3558	0.0865
	6.3	1.7806	5.4567	1.3344	0.2141
	6.4	1.7269	5.2617	1.3141	0.3286
2-M Norm	9.5	0.9652	2.8856	0.9824	1.5622
	10	0.9123	2.7446	0.9552	1.6239
	10.5	0.8674	2.6268	0.9313	1.6742
	11	0.8286	2.5270	0.9103	1.7160
	11.5	0.7950	2.4413	0.8916	1.7511
	12	0.7655	2.3671	0.8749	1.7812
	13	0.7161	2.2449	0.8462	1.8293
	13.5	0.6953	2.1940	0.8339	1.8490
	14	0.6766	2.1485	0.8225	1.8664

TABLE 5. Control limits for Tr-chart using normal approximation when β = 0.5.

α	γ	$\widehat{LCL}$	$\widehat{UCL}$	α	γ	$\widehat{LCL}$	$\widehat{UCL}$
	2.0	9.8241	10.4869		2	9.874201	10.3480
	1.8	9.0241	9.6869		1.8	9.074201	9.5480
	1.7	8.6241	9.2869		1.6	10.1484	11.0960
9	1.5	7.8241	8.4869	12	1.5	7.874201	8.3480
	1.4	7.4241	8.0869		1.3	7.074201	7.5480
	1.2	6.6241	7.2869		1	5.874201	6.3480
	1.0	5.8241	6.4869				
	2	9.8448	10.4297		2	9.8799	10.3322
	1.9	9.4448	10.0297		1.8	9.0799	9.5322
	1.7	10.4896	11.6594		1.5	7.8799	8.3322
10	1.5	7.8448	8.4297	12.5	1.3	7.0799	7.5322
	1.3	7.0448	7.6297		1	5.8799	6.3322
	1.0	7.6896	8.8594		0.9	5.4799	5.9322
	0.8	6.8896	8.0594		0.5	3.8799	4.3322

Then $w_2 = {(y-c)}/{h} \sim \chi^2_{(v)}$ gives better approximation for the distribution of Y, where $V = \frac{8}{\beta_1^*}$, $h = \frac{\sqrt{\mu_2}}{2V}$ and $c = \mu_1' - hV$; see [20]. The control limits are similarly obtained as in (19) and (20).

$$\hat{L}_r = c + h\chi^2_{\left(\lambda/2, V\right)'} \tag{21}$$

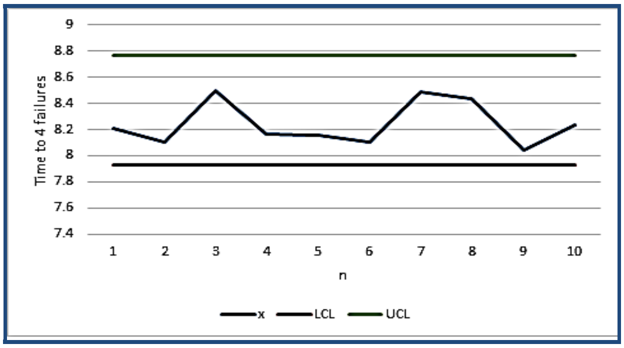

FIGURE 8. T_4 -chart with 40 cumulative observations for $\alpha = 6.2$, $\beta = 0.5$, $\gamma = 1.5$ using chi-square approximation.

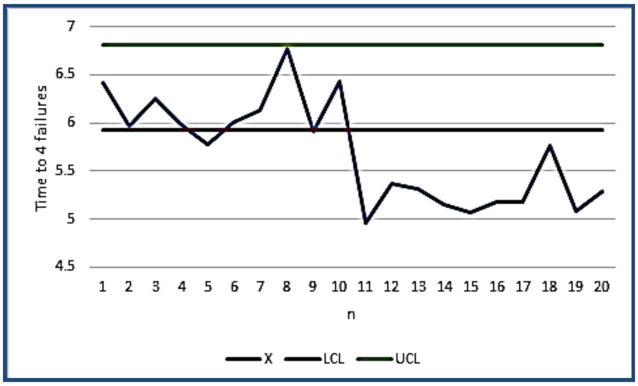

FIGURE 9. T-chart with first 40 cumulative observations for $\alpha = 6$, $\beta = 0.5$, $\gamma = 1$ and next 40 cumulative observations for $\alpha = 6$, $\beta = 0.5$, $\gamma = 0.8$ using chi-square approximation.

$$\hat{U}_r = c + h\chi^2_{\left(1-\lambda/2, V\right)}. \tag{22}$$

Control limits based on Chi-square approximation for different values of shape parameter are presented in Table 6. Figure 8 and 9 give us control charts for cumulative time elapsed between four failures in a production process. These charts are based on Chi-square approximation. These charts also show in control behavior and there is no false alarm in the chart. Also, we are using T_r-charts because individual T charts are not able to detect small shifts in the process behavior as described earlier. It is also clearly shown in Figure 9 that if small shift appears in the location of the process, T_r-chart rapidly detects this shift. But T chart in the previous sections was not able to detect small quantity of shift. Those charts were only detecting shift of approximately 1σ or more. Therefore, these types of TBE charts are preferable than the previous ones.

V. AVERAGE RUN LENGTH

The probability of not detecting shift in a control chart is the probability that all the points plot in control when process is out of control and it is denoted by p. The power of the chart i.e., $1 - \pi$ is the probability of detecting shift in the process. The average number of points that plot before the chart shows

TABLE 6. Control limits for T_r-chart using χ^2 approximation with $\beta = 0.5$.

α	ϒ	$\widehat{LCL}$	$\widehat{UCL}$	α	ϒ	$\widehat{LCL}$	$\widehat{UCL}$
5.8	1.5	7.9295	8.8454	6.2	1.5	7.9263	8.7697
	1.4	7.5295	8.4454		1.0	5.9263	6.7697
	1.2	6.7295	6.6454		0.8	5.1263	5.9697
	1.0	5.9295	6.8454		0.5	3.9263	4.7697
	0.8	5.1295	6.0454		0.3	3.1263	3.9697
	0.5	3.9295	4.8454				
5.9	1.5	7.9284	8.8251	6.3	1.5	7.9259	8.7528
	1.3	7.1284	8.0251		1.0	5.9259	6.7528
	1.0	5.9284	6.8251		0.5	3.9259	4.7528
	0.8	5.1284	6.0251		0.3	3.1259	3.9528
	0.5	3.9284	4.8251				
6.0	1.5	7.9275	8.8058	6.4	1.5	7.9256	8.7367
	1.3	7.1275	8.0058		1.0	5.9256	6.7367
	1.0	5.9275	6.8058		0.5	3.9256	4.7367
	0.8	5.1275	6.0058	6.5	1.5	7.9254	8.7213
	0.5	3.9275	4.8058		1.0	5.9254	6.7213
	0.3	3.1275	4.0058		0.5	3.9254	4.7213
6.1	1.5	7.9268	8.7873	6.6	1.5	7.9253	8.7064
	1.0	5.9268	6.7873		1.0	5.9253	6.7064
	0.5	3.9268	4.7873		0.5	3.9254	4.7064

an out of control signal is defined as ARL. Generally ARL can be expressed as

$$ARL_0 = \frac{1}{\lambda}, \tag{23}$$

when the process is in control. Here λ is the probability of false alarm in the process so that the chart shows out of control signal while the process is in control. When the process is out of control, ARL is

$$ARL_1 = \frac{1}{1-\pi}, \tag{24}$$

But these formulae does not work always when the charts are not Shewhart type therefore for T_r control chart Xie *et al.* [23] gave the following formula for the ARL of Erlang distribution

$$\hat{ARL}_r = \frac{E\left(Y_r\right)}{1-\pi}. \tag{25}$$

To calculate π, we need exact distribution of the corresponding variable. So here we cannot make use of the above rule as the distribution of sum of Frechet random variables is not known exactly. Therefore, we will use the definition of π. Since π is the probability that failure time falls within control limits while having the process shifted to a new point, therefore;

$$\pi = P\left[LCL < T_r < UCL|H_1\right], \tag{26}$$

After standardization of (26), we have

$$\pi = F\left[\frac{UCL_r - c}{h}\right] - F\left[\frac{LCL_r - c}{h}\right], \tag{27}$$

TABLE 7. ARL for chi-square approximation when $\alpha = 5.9$, $\beta = 1$ and $\gamma = 1$.

Shift(δ)	π	1- π	ARL	ANOS
0.1	0.0771	0.9229	4.2477	16.9908
0.2	0.0587	0.9413	4.1649	16.6595
0.3	0.0445	0.9555	4.1029	16.4116
0.4	0.0336	0.9664	4.0565	16.2060
0.5	0.0253	0.9748	4.0218	16.0871
0.6	0.0189	0.9811	3.9958	15.9831
0.7	0.0141	0.9859	3.9764	15.9054
0.8	0.0105	0.9895	3.9619	15.8476

TABLE 8. ARL for normal approximation when $\beta = 0.5$.

Shift(δ)	π	1-π	ARL
When α =9	ϒ=1	ARL_0=123.110	
0.2	0.3424	0.6576	9.3604
0.4	0.0028	0.9972	6.1727
0.6	1.3298e-07	0.9999	6.1555
>0.8	0	1	6.1555
When α=10	ϒ=1	ARL_0=122.7251	
0.2	0.2355	0.7645	6.9812
0.4	0.0003	0.9997	5.3390
>0.6	≅ 0	≅ 1	5.3337
When α=11	ϒ=0.8	ARL_0=122.4561	
0.2	0.1502	0.8498	6.2633
0.4	2.7649e05	0.99997	5.3229
>0.6	≅ 0	≅ 1	5.3228
When α=12	ϒ=1	ARL_0=122.2219	
0.2	0.0886	0.9114	6.7051
0.4	1.589e-06	0.9999	6.1111
>0.6	≅ 0	≅ 1	6.1111
When α=12.5	ϒ=1	ARL_0=122.1206	
0.2	0.0660	0.9340	6.5374
0.4	3.2997e-07	0.9999	6.1060
>0.6	≅ 0	≅ 1	6.1060

therefore equation (25) becomes

$$\hat{ARL}_r = \frac{E(Y_r)}{1 - \left[F\left(\frac{UCL_r - c}{h}\right) - F\left(\frac{LCL_r - c}{h}\right)\right]}. \quad (28)$$

If the probability of the detection of shift is estimated, we can easily estimate ARL for the distribution of sum of Frechet random variables using equation (26). In the present study, we have used the same formula to find ARL for T_r-charts. The ARL for Chi-square approximation is given in Table 7 and that of Normal approximation is given in Table 8. ARL values depict that when a small shift in the process occurs, CUSUM chart clearly detects this shift. The ARL is small for small shift in the process and the probability of detecting shift becomes very close to one when shift is about or greater than 0.6 sigma. Therefore, it is clear that T_r-chart has rapid detection of shift than individual T-charts. Hence we prefer T_r-charts in a production process to reduce deterioration.

VI. DATA ANALYSIS

To illustrate, we have used real data sets taken from life testing experiments. While fitting any distribution to a real

TABLE 9. Estimates for example 1.

⇓	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\widehat{LCL}$	$\widehat{UCL}$
MLE	3.0190	823.0821	-526.7316	-434.4757	1011.689
PWM	4.1390	1269.311	-963.8083	-706.5792	1039.26

The value of KS-test for example 1 along with their corresponding P-values are given below

DML = 0.0696, P-value = 0.9957

DPWM = 0.0667 P-value = 0.9976

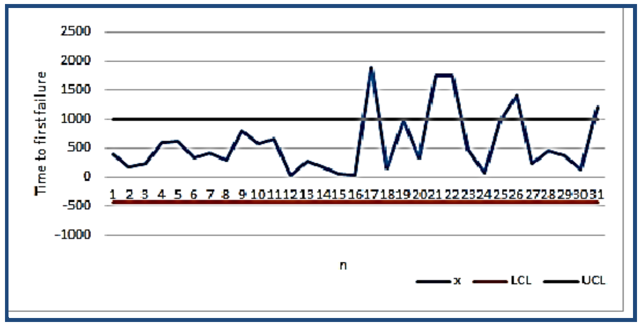

FIGURE 10. T-chart for example 1 using ML estimates.

data set, first we have to check that whether the underlying model is suitable for data set. For this purpose, we have been using a graphical method and Kolmogorov-Smirnov (KS) test: see [16], [20] in our study.

Example 1: This data set exhibits 31 observations of lifetime of lamps used in projectors. The projection hours were recorded in hours when each lamp burned out; see [11]. Data for failure of 31 lamps are presented here:

387, 182, 244, 600, 627, 332, 418, 300, 798, 584, 660, 39, 274, 174, 50, 34, 1895, 158, 974, 345, 1755, 1752, 473, 81, 954, 1407, 230, 464, 380, 131, 1205.

Parameter estimates for example 1 are given in Table 9. The estimates are found using ML and PWM method. The control limits are also computed using percentiles of the distribution as described in section 3.

The value of KS-test for example 1 along with their corresponding P-values are given below

$$DML = 0.0696, \quad P - value = 0.9957$$
$$DPWM = 0.0667 \quad P - value = 0.9976$$

T-chart for real data set 1 using ML estimates has been presented in Figure 10 and using PWM estimates is presented in Figure 11. Same as the charts based on simulation results, these charts also depict that the process is in statistical control since all the values of failure times plot are above LCL. The LCL of these control charts become negative but if it is not convenient for researchers to take negative LCL, they can use zero as their LCL because in life testing experiments there is no need to work with negative points. Therefore, the technique works for real life testing processes and T-chart is more appropriate for this data.

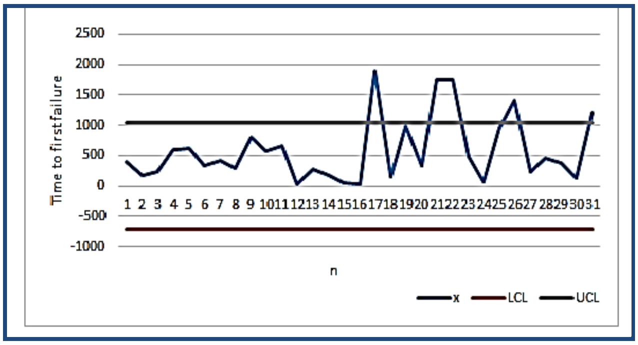

FIGURE 11. T-chart for example 1 using PWM estimates.

TABLE 10. Estimates for example 2.

⇓	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\widehat{LCL}$	$\widehat{UCL}$
MLE	15.5404	420.6071	-365.888	-90.94817	109.0596
PWM	13.6445	381.6477	-327.739	-92.57664	110.5532

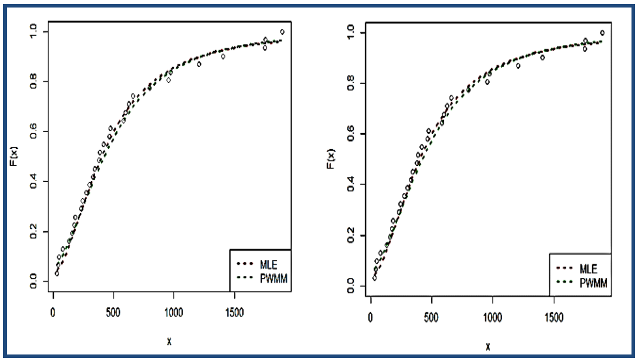

FIGURE 12. Goodness of fit plots for real data set 1 and 2.

Example 2: Another data set is taken from Lawless [18] about time to failure of 23 Ball Bearings and is presented here:

17.88, 28.92, 33.00, 173.40, 42.12, 45.60, 48.80, 51.84, 51.96, 54.12, 55.56, 67.80, 68.44, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 41.52.

The value of D-statistic along with their p-values for example 2 based on ML and PWM methods have been given as DML = 0.1021, P-value = 0.9503

$$DPWM = 0.0987, \quad P - \text{value} = 0.9622$$

Parameter estimates for example 2 along with their estimated CLs have been presented in Table 10. T-charts can also be constructed for data set 2 as for data set 1.

It is clear from Figure 12 that FD has suitable fit for both the data sets. Further, KS-test is also used to test the two tailed hypothesis of goodness of fit. The significance level used to test the hypothesis is $\lambda = 0{:}05$. Since the KS distance is small and the p-value of KS-test for both real data sets is greater than 0.05. Therefore, the null hypothesis cannot be rejected and it is concluded that FD is suitable to model both the datasets.

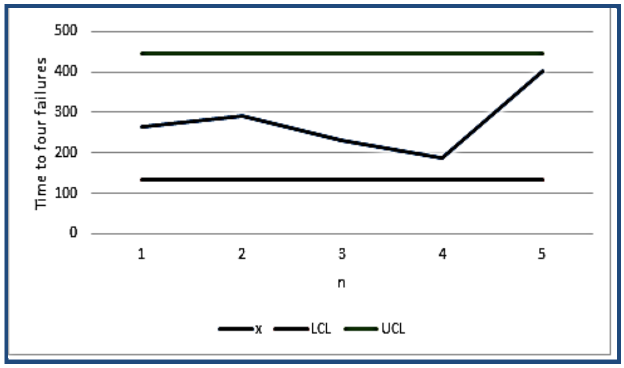

FIGURE 13. T_4 -chart for example 2 using control limits based on Normal approximation.

As mentioned earlier, T-charts have the shortcoming of not detecting the shift when the quantity of shift is small. Therefore, for the data given in example 2, T_r-chart is being constructed. For example 1, we see that the value of shape parameter estimated by both ML and PWM method is less than 5.7, therefore the cumulative observations of this data set do not follow Chi-square or Normal distribution approximately. For example 2, Normal approximation can be used while making T_4 chart. The data have been plotted on T_4 chart for time to four failures shown in Figure 13. PWM estimates have been used for the approximation of the distribution of sum and control limits of the chart have been estimated using percentage points of Normal distribution.

VII. CONCLUSION

In this paper, an attempt have been made to develop a model for time between failures of a process with three-parameter FD. To monitor the reliability of high quality processes, we make use of T chart and Tr chart whose limits are based on ML and PWM estimates. To obtain T_{r-} chart, the distribution of sum of Frechet random variable is required. For this purpose, two moment approximations are utilized that provide promising results. T-charts show in control process behavior but Tr charts are proven more effective due to fast detection of shift when the process deteriorates. One can extend this study by considering different estimation technique for the model parameters, constructing different types of TBE charts like CCC charts, CQC charts, synthetic T charts etc. approximating the distribution of sum of Frechet random variables by other moment approximations like four moment F- approximations or any other approximation technique or taking some other life testing distribution to model failure times of a system or component.

DATA AVAILABILITY

The data are given in the paper.

CONFLICT OF INTEREST

The authors declare that there is no conflict of interest regarding the publication of this paper.

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