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The Conditions for a Linear Vibration System to Have Only Pure Rotation Modes

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ABSTRACT This paper proposes a geometric approach to the conditions for mode decoupling of a vibration system of an elastically supported single rigid body and presents the conditions that the system has only pure rotation modes of vibration. A small oscillation of a rigid body is indeed a repetitive screw motion and thus vibration modes are expressed by screws in general, which results in the difficulty involved in solving a vibration problem. The complexity of a vibration system can be alleviated for both analysis and synthesis if the system has only rotation modes. In order to acquire the decoupling techniques, this paper begins by investigating a stiffness matrix which can be separated into the sum of two rank 3 stiffness matrices, which are realizable by using co-reciprocal line vectors. From the co-reciprocity, the separable stiffness matrix can be regarded as a linear transformation between two 3-systems of screws containing only line vectors. Using the properties of the linear transformation and the screw systems, the conditions for mode decoupling, or the conditions for only pure rotation modes are derived and described by geometric relations between inertia and stiffness, and three cases of vibration systems with simple geometric nature are identified.

INDEX TERMS Screw theory, linear vibration, mode decoupling, planes of symmetry, pure rotation mode.

NOMENCLATURE

A. NOTATIONS

AAA	Induced wrench space spanned by $\widehat{\boldsymbol{\alpha}}_1$, $\widehat{\boldsymbol{\alpha}}_2$ and $\widehat{\boldsymbol{\alpha}}_3$
AAB	Induced wrench space spanned by $\widehat{\boldsymbol{\alpha}}_1$, $\widehat{\boldsymbol{\alpha}}_2$ and $\widehat{\boldsymbol{\beta}}_3$
ABA	Induced wrench space spanned by $\widehat{\boldsymbol{\alpha}}_1$, $\widehat{\boldsymbol{\beta}}_2$ and $\widehat{\boldsymbol{\alpha}}_3$
ABB	Induced wrench space spanned by $\widehat{\boldsymbol{\alpha}}_1$, $\widehat{\boldsymbol{\beta}}_2$ and $\widehat{\boldsymbol{\beta}}_3$
BAA	Induced wrench space spanned by $\widehat{\boldsymbol{\beta}}_1$, $\widehat{\boldsymbol{\alpha}}_2$ and $\widehat{\boldsymbol{\alpha}}_3$
BAB	Induced wrench space spanned by $\widehat{\boldsymbol{\beta}}_1$, $\widehat{\boldsymbol{\alpha}}_2$ and $\widehat{\boldsymbol{\beta}}_3$
BBA	Induced wrench space spanned by $\widehat{\boldsymbol{\beta}}_1$, $\widehat{\boldsymbol{\beta}}_2$ and $\widehat{\boldsymbol{\alpha}}_3$
BBB	Induced wrench space spanned by $\widehat{\boldsymbol{\beta}}_1$, $\widehat{\boldsymbol{\beta}}_2$ and $\widehat{\boldsymbol{\beta}}_3$
$\widehat{\boldsymbol{\alpha}}_i$, $\widehat{\boldsymbol{\beta}}_i$	Line vectors uniquely determined from stiffness matrix
$\boldsymbol{\Delta}$	Matrix interchanging the ray and axis co-ordinates of a screw
h_i, g_i	Pitches of the principal screws $\widehat{\boldsymbol{p}}_i$, $\widehat{\boldsymbol{q}}_i$
$\boldsymbol{j}$	$6 \times n$ matrix consisting of line vectors of springs
$\boldsymbol{k}$	Diagonal matrix whose diagonal elements are k_i's
$\boldsymbol{K}$	Stiffness matrix
$\boldsymbol{K}_\alpha$, $\boldsymbol{K}_\beta$	Separated stiffness matrices with rank 3
k_i	Spring constant of $\widehat{\boldsymbol{s}}_i$
$\hat{\boldsymbol{l}}_i$	Free vector determined from mass matrix
λ_i, μ_i	Eigenvalues associated with $\widehat{\boldsymbol{p}}_i$, $\widehat{\boldsymbol{q}}_i$
$\boldsymbol{M}$	Mass matrix
$\widehat{\boldsymbol{n}}_i$	Line vector determined from mass matrix
$\widehat{\boldsymbol{p}}_i$, $\widehat{\boldsymbol{q}}_i$	Principal screws of $\boldsymbol{K}$ with eigenvalues $\boldsymbol{\lambda}_i$, $\boldsymbol{\mu}_i$
$\boldsymbol{r}_i$	Position vector of a line vector
$\boldsymbol{s}_i$	Unit direction vector of a line vector
$\widehat{\boldsymbol{s}}_i$	Line vector indicating axis of spring with constant k_i
$\widehat{\boldsymbol{X}}_i$	Vibration mode represented in axis co-ordinates
$\widehat{\boldsymbol{x}}_\alpha$	Line vector belonging to **AAA** (expressed in ray co-ordinates)
$\widehat{\boldsymbol{X}}_\alpha$	Line vector belonging to **AAA** (expressed in axis co-ordinates)
$\widehat{\boldsymbol{x}}_\beta$	Line vector belonging to **BBB** (expressed in ray co-ordinates)
$\widehat{\boldsymbol{X}}_\beta$	Line vector belonging to **BBB** (expressed in axis co-ordinates)

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I. INTRODUCTION

Equations of motion of a resiliently suspended rigid body in three-dimensional space are given by six linear coupled 2nd-order differential equations. The complexity involved in solving the equations can be significantly reduced by decoupling the equations. So far, there has been little research on this issue of decoupling. Derby [1] presented the decoupling of the equations of motion by making 3×3 off-diagonal submatrices of mass and stiffness matrices zero matrices simultaneously. Harris and Piersol [2] investigated the methods to decouple the equations of motion with respect to a plane(s) of symmetry by observing the elements of the stiffness matrix when the mass matrix is diagonalized. Considering that the mass matrix can always be diagonalized by a co-ordinate transformation, it can be said that the complexity of solving the equations of motion is mainly associated with the stiffness matrix.

It was Ball [3] who first described oscillation of a rigid body as a small screw motion. From the viewpoint of geometry, a screw associated with vibration mode means that it is in general repetitive screw motion about the axis of screw. The complex motions give rise to complicated geometrical nature of spatial vibration systems. In contrast, if the decoupling techniques presented in [1], [2] are applied, all the vibration modes are pure rotations and(or) translations. Especially, if a vibrating system has a plane of symmetry, the modes are decoupled into in-plane and out-of-plane modes. The in- and out-of-plane modes, respectively, refer to pure rotational vibrations of a rigid body about vibration axes perpendicular to the plane of symmetry and lying on the plane. Jang et el [4] showed that there exist relatively simple geometrical relationships between vibration axes of in- and out-of-plane modes. Recently, some researchers [5]–[7] presented design methods of planar vibration system by use of geometric properties of in-plane modes. Park and Choi [8] utilized geometric properties of out-of-plane modes to design energy harvester with desired resonant frequencies. The transparent geometrical nature of vibrating systems may provide useful tools for design of a vibration system. Thus, the identification of vibration systems with simple geometric properties is the motivation for this work.

Several approaches have been studied to find conditions for plane(s) of symmetry. Dan and Choi [9], [10] derived conditions for plane(s) of symmetry by observing stiffness matrices diagonalizable by a co-ordinate transformation. Hong and Choi [11] proposed the geometric approach to conditions for a vibrating system with diagonalizable stiffness matrices to have a plane(s) of symmetry. However, it has been known that a stiffness matrix is not always diagonalizable by a co-ordinate transformation [12]. Although Jang and Choi [13] presented general conditions for a plane(s) of symmetry regardless of diagonalization of stiffness matrix, geometric relations between inertia and stiffness were not described.

Patterson and Lipkin [14], [15] demonstrated several propositions of the principal screws of the potential defined by Ball [3]. Huang [16] represented a spatial stiffness matrix as the sum of outer products of principal screws with unit magnitude (eigenscrew decomposition). Griffis and Duffy [17] showed that the correlation of stiffness mapping can be expressed in terms of the stiffness constants of springs and Jacobian matrix consisting of line vectors of springs. On the other hand, not all of the stiffness matrices are realizable by means of parallel connections of springs. Lončarić [12] investigated the condition for realizability of a stiffness matrix. Many researchers [18]–[23] have made much efforts to develop the methods of the realization of a given stiffness using parallel connections of springs.

This paper proposes a geometric approach to the conditions for having only pure rotation modes. The general conditions for a plane of symmetry are developed as well as the other two conditions that can be applied to simplify a vibration system. This paper is organized as follows: Section II introduces theoretical preliminaries and describes the necessary condition for having only pure rotation modes that is derived from the orthogonality of vibration modes with respect to stiffness matrix. In Section III, we define a special term 'separable,' which is used to mean that the stiffness matrix of rank 6 can be separated into the sum of two rank 3 symmetric matrices that are realizable by using co-reciprocal line vectors. From the co-reciprocity, the separable stiffness matrix is viewed as a linear transformation between two 3-systems of screws. In Section IV, the mass matrix is also considered as a linear transformation and the conditions for a vibration system to have only pure rotation modes is derived using the linear transformations corresponding to mass and stiffness matrices. In what follows, the conditions are described as the geometrical relationships between the inertia and separable stiffness. In Section V, a mounting system with separable stiffness matrix is used as a numerical example to demonstrate the general condition for a plane of symmetry. Finally, Section VI presents our conclusions.

II. PRELIMINARLIES

A. GEOMETRICAL EXPRESSION OF VIBRATION MODES

For a single rigid body supported by a number of line (and/or torsion) springs in three-dimensional space, a symmetric and positive definite rank 6 stiffness matrix $\boldsymbol{K}(\in R^{6\times 6})$ can be expressed as follows [17]–[19]

$$\boldsymbol{K} = \boldsymbol{jkj}^T = \left[\widehat{\boldsymbol{s}}_1 \cdots \widehat{\boldsymbol{s}}_n\right] \begin{bmatrix} k_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & k_n \end{bmatrix} \left[\widehat{\boldsymbol{s}}_1 \cdots \widehat{\boldsymbol{s}}_n\right]^T$$
$$= \sum_{i=1}^{n} k_i \widehat{\boldsymbol{s}}_i \widehat{\boldsymbol{s}}_i^T, \tag{1}$$

where $\boldsymbol{j}$ is a $6 \times n$ matrix whose column vector $\widehat{\boldsymbol{s}}_i$ represents the axis of a line (or torsion) spring with the stiffness constant k_i. The axis of a line spring can be expressed by a line vector of zero pitch as $\widehat{\boldsymbol{s}}_i = \left[\boldsymbol{s}_i^T; (\boldsymbol{r}_i \times \boldsymbol{s}_i)^T\right]^T$ and that of a torsion spring can be written as $\widehat{\boldsymbol{s}}_i = \left[\mathbf{0}_{3\times 1}^T; \boldsymbol{s}_i^T\right]^T$ which is a free vector, where $\boldsymbol{s}_i \left(\in R^{3\times 1}\right)$ denotes a unit direction vector and $\boldsymbol{r}_i \left(\in R^{3\times 1}\right)$ is the position vector to $\widehat{\boldsymbol{s}}_i$. The line and free vectors satisfy the following relation:

$$\widehat{s}_i^T \Delta \widehat{s}_i = 0, \tag{2}$$

where $\Delta = \begin{bmatrix} 0_{3\times 3} & I_{3\times 3} \\ I_{3\times 3} & 0_{3\times 3} \end{bmatrix}$ and $I_{3\times 3}$ is the 3×3 identity matrix. The line vector $\widehat{s}_i$ is said to be self-reciprocal when it satisfies (2). From (1), it is clear that the trace of $K\Delta$ becomes zero:

$$\mathrm{tr}\,(K\Delta) = \sum\nolimits_{i=1}^{n} k_i \widehat{s}_i^T \Delta \widehat{s}_i = 0. \tag{3}$$

It means that the stiffness matrix K can be realized by using parallel connections of line springs when it satisfies (3) [12].

Now, the equation of motion for undamped free vibration can be given by

$$M\ddot{X} + KX = 0, \tag{4}$$

where $M(\in R^{6\times 6})$ is the mass matrix. A small harmonic displacement can be written in terms of a screw as

$$X = \widehat{X} e^{j\omega t}, \tag{5}$$

where the amplitude $\widehat{X}(\in R^{6\times 1})$ is expressed in Plücker's axis co-ordinates. Substituting (5) into (4) gives:

$$\left(K - \omega^2 M\right)\widehat{X} = 0. \tag{6}$$

From (6), six eigenvectors $\widehat{X}_i (i = 1, \ldots, 6)$ corresponding to vibration modes are obtained as general screws. If $\widehat{X}_i$ is a line vector (of zero pitch), then the vibration mode is a pure rotation about the line (Fig. 1(a)). If the axis of vibration goes to infinity (Fig. 1(b)), $\widehat{X}_i$ becomes a free vector and the vibration mode becomes pure translation. It is noted that the line and free vectors are depicted respectively by a single line and double lines in the figures, throughout this paper. Since a translation can be thought of as a rotation about an infinitely distant axis, the mode $\widehat{X}_i$ will be called a pure rotation mode if it is self-reciprocal as follows:

$$\widehat{X}_i^T \Delta \widehat{X}_i = 0. \tag{7}$$

B. NECESSARY CONDITION FOR ONLY PURE ROTATION MODES

A necessary condition for a vibration system to have only pure rotation modes can be derived from the orthogonality of vibration modes with respect to the stiffness matrix. Orthogonal character of vibration modes is defined by

$$S^T K S = \begin{bmatrix} \tilde{k}_1 & & 0 \\ & \ddots & \\ 0 & & \tilde{k}_6 \end{bmatrix}, \tag{8}$$

where $S = \left[\widehat{X}_1 \cdots \widehat{X}_6\right]$ and $\tilde{k}_i \equiv \widehat{X}_i^T K \widehat{X}_i$ for $i = 1, \ldots, 6$. From (8), K^{-1} can be obtained as

$$K^{-1} = \sum\nolimits_{i=1}^{6} \frac{1}{\tilde{k}_i} \widehat{X}_i \widehat{X}_i^T. \tag{9}$$

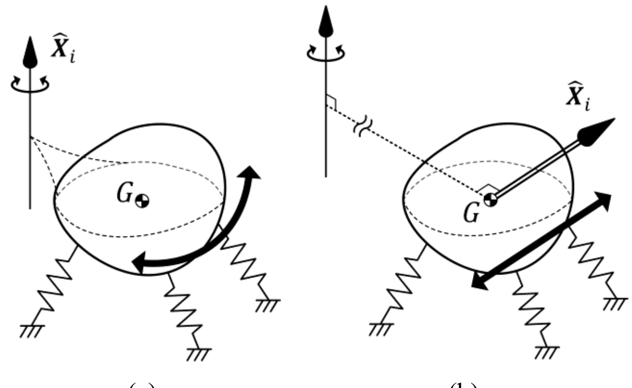

FIGURE 1. Pure rotation modes: (a) finitely distant axis and (b) infinitely distant axis.

Thus, the trace of ΔK^{-1} becomes

$$\mathrm{tr}\left(\Delta K^{-1}\right) = \sum\nolimits_{i=1}^{6} \frac{1}{\tilde{k}_i} \widehat{X}_i^T \Delta \widehat{X}_i. \tag{10}$$

Equation (10) implies that $\mathrm{tr}\left(\Delta K^{-1}\right) = 0$ if all the vibration modes are pure rotations, i.e., $\widehat{X}_i^T \Delta \widehat{X}_i = 0$ for all i. In other words, necessary condition for a vibration system to have only pure rotation modes can be described as:

$$\mathrm{tr}\left(\Delta K^{-1}\right) = 0. \tag{11}$$

C. EIGENSTRUCTURE OF STIFFNESS MATRIX

We suppose that an elastically supported rigid body is displaced near equilibrium and a wrench is generated along a screw. When the displacement and the induced wrench are on the same screw, the screw is called the principal screw of the potential [3] for the given stiffness matrix K. From the statics relation between the small displacement and the wrench, the following eigenequation can be written as:

$$\lambda_i \widehat{p}_i = K\Delta \widehat{p}_i \quad (\text{for } i = 1, \ldots, 6), \tag{12}$$

where $\widehat{p}_i \left(\equiv \left[p_i^T; p_{oi}^T\right]^T\right)$ denotes the principal screw of the potential expressed in Plücker's ray co-ordinates and λ_i is the corresponding eigenvalue. The matrix Δ interchanges Plücker's ray and axis co-ordinates of a screw. That is, the axis co-ordinates of $\widehat{p}_i$ can be obtained as $\Delta \widehat{p}_i = \left[p_{oi}^T; p_i^T\right]^T$. For a symmetric and positive definite K, there are six linearly independent principal screws of the potential with non-zero finite pitches and eigenvalues [3], [14]. The reciprocal product of two screws is given by

$$\widehat{p}_i^T \Delta \widehat{p}_j = \left(h_i + h_j\right)\cos\theta_{ij} - d_{ij}\sin\theta_{ij}, \tag{13}$$

where h_i is the pitch of $\widehat{p}_i$, and θ_{ij} and d_{ij} are respectively the angle and the shortest distance between the axes of $\widehat{p}_i$ and $\widehat{p}_j$. The principal screws of the potential are co-reciprocal:

$$\widehat{p}_i^T \Delta \widehat{p}_i = 2h_i \text{ and } \widehat{p}_i^T \Delta \widehat{p}_j = 0 \quad (i \neq j). \tag{14}$$

Using (14), $\boldsymbol{K}$ can be expressed in terms of the principal screws of the potential as [11], [16]

$$\boldsymbol{K} = \sum_{i=1}^{6} \frac{\lambda_i}{2h_i} \widehat{\boldsymbol{p}}_i \widehat{\boldsymbol{p}}_i^T. \tag{15}$$

From (15), the trace of $\boldsymbol{K\Delta}$ is equal to the sum of the eigenvalues:

$$\mathrm{tr}\,(\boldsymbol{K\Delta}) = \sum_{i=1}^{6} \lambda_i. \tag{16}$$

Similarly, the trace of $\boldsymbol{\Delta K}^{-1}$ is given by

$$\mathrm{tr}\left(\boldsymbol{\Delta K}^{-1}\right) = \sum_{i=1}^{6} \frac{1}{\lambda_i}. \tag{17}$$

Therefore, it becomes clear that in order for a vibration system with realizable stiffness matrix to have only pure rotation modes, the sum of the eigenvalues of $\boldsymbol{K\Delta}$ and the sum of their reciprocals must vanish simultaneously.

III. SEPARATION OF RANK 6 STIFFNESS MATRICES INTO SUM OF TWO MATRICES OF RANK 3

A. CONDITION FOR SEPARABLE STIFFNESS MATRIX OF RANK 6

We may begin by defining a special term, '*separable*.' In this paper, the term '*separable*' is used to mean that the stiffness matrix of rank 6 can be separated into (the sum of) two rank 3 symmetric matrices that are realizable by using co-reciprocal line vectors. The following proposition about separable stiffness matrices is introduced with proof.

Proposition: The stiffness matrix $\boldsymbol{K}$ of rank 6 is separable if and only if the matrix $\boldsymbol{K\Delta}$ has three pairs of eigenvalues and the sum of each of three pairs becomes zero.

To prove this proposition, for the 6×6 symmetric and positive definite stiffness matrix $\boldsymbol{K}$, suppose that three pairs of eigenvalues of $\boldsymbol{K\Delta}$ satisfy the following conditions:

$$\lambda_1 + \mu_1 = 0, \lambda_2 + \mu_2 = 0, \text{ and } \lambda_3 + \mu_3 = 0, \tag{18}$$

where λ_i and μ_i are the eigenvalues of principal screws of the potential $\widehat{\boldsymbol{p}}_i$ and $\widehat{\boldsymbol{q}}_i$, respectively. Equation (15) can be rewritten as the sum of three rank 2 matrices $\boldsymbol{K}_{2i}$ $(i = 1, 2, 3)$:

$$\boldsymbol{K} = \sum_{i=1}^{3} \left(\frac{\lambda_i}{2h_i} \widehat{\boldsymbol{p}}_i \widehat{\boldsymbol{p}}_i^T + \frac{\mu_i}{2g_i} \widehat{\boldsymbol{q}}_i \widehat{\boldsymbol{q}}_i^T \right) = \sum_{i=1}^{3} \boldsymbol{K}_{2i},$$

where h_i and g_i are the pitches of the principal screws of the potential, $\widehat{\boldsymbol{p}}_i$ and $\widehat{\boldsymbol{q}}_i$, respectively. Clearly, $\mathrm{tr}\,(\boldsymbol{K}_{2i}\boldsymbol{\Delta}) = 0$ and $\mathrm{rank}\,(\boldsymbol{K}_{2i}) = 2$. From (1), $\boldsymbol{K}_{2i}$ can be expressed in terms of two linearly independent line vectors $\widehat{\boldsymbol{\alpha}}_i$ and $\widehat{\boldsymbol{\beta}}_i$ belonging to $\mathrm{span}\,(\widehat{\boldsymbol{p}}_i, \widehat{\boldsymbol{q}}_i)$ as follows [22]:

$$\boldsymbol{K}_{2i} = k_{\alpha i} \widehat{\boldsymbol{\alpha}}_i \widehat{\boldsymbol{\alpha}}_i^T + k_{\beta i} \widehat{\boldsymbol{\beta}}_i \widehat{\boldsymbol{\beta}}_i^T, \tag{19}$$

where $k_{\alpha,i}$ and $k_{\beta,i}$ are the spring constants corresponding to $\widehat{\boldsymbol{\alpha}}_i$ and $\widehat{\boldsymbol{\beta}}_i$, respectively. Since $\widehat{\boldsymbol{\alpha}}_i \in \mathrm{span}(\widehat{\boldsymbol{p}}_i, \widehat{\boldsymbol{q}}_i)$, its self-reciprocity gives the following relation:

$$\begin{aligned} \widehat{\boldsymbol{\alpha}}_i^T \boldsymbol{\Delta} \widehat{\boldsymbol{\alpha}}_i &= \left(P_i \widehat{\boldsymbol{p}}_i + Q_i \widehat{\boldsymbol{q}}_i\right)^T \boldsymbol{\Delta} \left(P_i \widehat{\boldsymbol{p}}_i + Q_i \widehat{\boldsymbol{q}}_i\right) \\ &= 2h_i P_i^2 + 2g_i Q_i^2 = 0 \end{aligned} \tag{20}$$

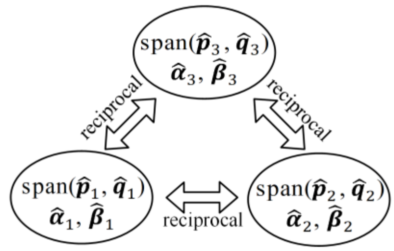

FIGURE 2. Reciprocal relations between three pairs of line vectors, $(\widehat{\alpha}_1, \widehat{\beta}_1)$, $(\widehat{\alpha}_2, \widehat{\beta}_2)$, and $(\widehat{\alpha}_3, \widehat{\beta}_3)$.

where P_i and Q_i are scalar multiples. From (20), we obtain

$$P_i = \pm\sqrt{-\frac{g_i}{h_i}} Q_i. \tag{21}$$

In (15), h_i and λ_i have the same sign since $\boldsymbol{K}$ is positive definite. Accordingly, $-\frac{g_i}{h_i}$ must be positive since $\lambda_i = -\mu_i$, and thereby (21) ensures that there exist only 2 unit line vectors belonging to $\mathrm{span}(\widehat{\boldsymbol{p}}_i, \widehat{\boldsymbol{q}}_i)$.That is, $\widehat{\boldsymbol{\alpha}}_i$ and $\widehat{\boldsymbol{\beta}}_i$ are uniquely determined. Now, $\boldsymbol{K}$ can be expressed in terms of $\widehat{\boldsymbol{\alpha}}_i$'s and $\widehat{\boldsymbol{\beta}}_i$'s:

$$\boldsymbol{K} = \sum_{i=1}^{3} \left(k_{\alpha i} \widehat{\boldsymbol{\alpha}}_i \widehat{\boldsymbol{\alpha}}_i^T + k_{\beta i} \widehat{\boldsymbol{\beta}}_i \widehat{\boldsymbol{\beta}}_i^T \right). \tag{22}$$

Then, $\boldsymbol{K}$ can be separated as $\boldsymbol{K} = \boldsymbol{K}_\alpha + \boldsymbol{K}_\beta$, where

$$\boldsymbol{K}_\alpha = \sum_{i=1}^{3} \left(k_{\alpha i} \widehat{\boldsymbol{\alpha}}_i \widehat{\boldsymbol{\alpha}}_i^T \right), \tag{23a}$$

and

$$\boldsymbol{K}_\beta = \sum_{i=1}^{3} \left(k_{\beta i} \widehat{\boldsymbol{\beta}}_i \widehat{\boldsymbol{\beta}}_i^T \right). \tag{23b}$$

Recalling that the principal screws of the potential are co-reciprocal, it becomes obvious that $\mathrm{span}(\widehat{\boldsymbol{p}}_1, \widehat{\boldsymbol{q}}_1)$, $\mathrm{span}(\widehat{\boldsymbol{p}}_2, \widehat{\boldsymbol{q}}_2)$, and $\mathrm{span}(\widehat{\boldsymbol{p}}_3, \widehat{\boldsymbol{q}}_3)$ are reciprocal to each other (Fig. 2). Accordingly, the line vectors $\widehat{\boldsymbol{\alpha}}_i$'s and $\widehat{\boldsymbol{\beta}}_i$'s satisfy the relations

$$\widehat{\boldsymbol{\alpha}}_i^T \boldsymbol{\Delta} \widehat{\boldsymbol{\alpha}}_j = \widehat{\boldsymbol{\beta}}_i^T \boldsymbol{\Delta} \widehat{\boldsymbol{\beta}}_j = 0 \quad (\text{for } i, j = 1, 2, 3), \tag{24a}$$

and

$$\widehat{\boldsymbol{\alpha}}_i^T \boldsymbol{\Delta} \widehat{\boldsymbol{\beta}}_j = 0 \quad (\text{for } i \neq j). \tag{24b}$$

Equation (24a) means that $(\widehat{\boldsymbol{\alpha}}_1, \widehat{\boldsymbol{\alpha}}_2, \widehat{\boldsymbol{\alpha}}_3)$ and $(\widehat{\boldsymbol{\beta}}_1, \widehat{\boldsymbol{\beta}}_2, \widehat{\boldsymbol{\beta}}_3)$ are respectively the co-reciprocal line vectors. Consequently, $\boldsymbol{K}_\alpha$ and $\boldsymbol{K}_\beta$ are realizable by co-reciprocal line vectors, and therefore $\boldsymbol{K}$ is separable.

Conversely, suppose that the rank 6 stiffness matrix $\boldsymbol{K}$ is separable. By definition of the term '*separable*,' $\boldsymbol{K}$ given by (1) can be considered to be

$$\boldsymbol{K} = \boldsymbol{jkj}^T = \sum_{i=1}^{3} \left(k_i \widehat{\boldsymbol{s}}_i \widehat{\boldsymbol{s}}_i^T \right) + \sum_{i=4}^{6} \left(k_i \widehat{\boldsymbol{s}}_i \widehat{\boldsymbol{s}}_i^T \right), \tag{25}$$

where

$$\widehat{\boldsymbol{s}}_i^T \boldsymbol{\Delta} \widehat{\boldsymbol{s}}_j = \widehat{\boldsymbol{s}}_{i+3}^T \boldsymbol{\Delta} \widehat{\boldsymbol{s}}_{j+3} = 0 \quad (\text{for } i, j = 1, 2, 3). \tag{26}$$

Since $\boldsymbol{K}$ has rank 6, $\widehat{\boldsymbol{s}}_i$'s are linearly independent. Thus, the principal screw of the potential $\widehat{\boldsymbol{p}}$ can be expressed by a linear combination of $\widehat{\boldsymbol{s}}_i$'s

$$\widehat{\boldsymbol{p}} = \boldsymbol{jc}, \tag{27}$$

where $\boldsymbol{c} \in R^{6\times 1}$. Substituting (25) and (27) into (12) and premultiplying both sides of (12) by $\boldsymbol{j}^{-1}$ yields

$$\lambda \boldsymbol{c} = \boldsymbol{kj}^T \boldsymbol{\Delta j c}. \tag{28}$$

Equation (28) means that λ is the eigenvalue of $\boldsymbol{kj}^T\boldsymbol{\Delta j}$. $\boldsymbol{K}$ is positive definite, and thus the diagonal elements of $\boldsymbol{k}$ are positive. Since the two matrices $\boldsymbol{kj}^T\boldsymbol{\Delta j}$ and $\sqrt{\boldsymbol{k}}\boldsymbol{j}^T\boldsymbol{\Delta j}\sqrt{\boldsymbol{k}}$ are similar, they share the eigenvalue λ. From (26), $\sqrt{\boldsymbol{k}}\boldsymbol{j}^T\boldsymbol{\Delta j}\sqrt{\boldsymbol{k}}$ can be expressed as

$$\sqrt{\boldsymbol{k}}\boldsymbol{j}^T\boldsymbol{\Delta j}\sqrt{\boldsymbol{k}} = \begin{bmatrix} \boldsymbol{0}_{3\times 3} & \boldsymbol{V} \\ \boldsymbol{V}^T & \boldsymbol{0}_{3\times 3} \end{bmatrix}, \tag{29}$$

where $V_{ij} = \sqrt{k_i k_{j+3}}\widehat{\boldsymbol{s}}_i^T\boldsymbol{\Delta}\widehat{\boldsymbol{s}}_{j+3}$ and V_{ij} denotes the (i, j) element of $\boldsymbol{V}$. The characteristic equation of (29) is

$$\det\left(\lambda \boldsymbol{I}_{6\times 6} - \begin{bmatrix} \boldsymbol{0}_{3\times 3} & \boldsymbol{V} \\ \boldsymbol{V}^T & \boldsymbol{0}_{3\times 3} \end{bmatrix}\right) = 0, \tag{30}$$

or,

$$\det\left(\lambda^2 \boldsymbol{I}_{3\times 3} - \boldsymbol{V}^T\boldsymbol{V}\right) = 0. \tag{31}$$

Clearly, λ^2 is the eigenvalue of the positive definite matrix $\boldsymbol{V}^T\boldsymbol{V}$, and therefore λ is real and given by $\lambda = \pm\lambda_i (i = 1, 2, 3)$. This concludes that there are three pairs of eigenvalues, and two eigenvalues of each pair add up to zero. The proof of the proposition is complete.

The necessary and sufficient conditions for separation of the stiffness matrix implies that if $\boldsymbol{K}$ is separable, then $\mathrm{tr}(\boldsymbol{K\Delta}) = 0$ and $\mathrm{tr}\left(\boldsymbol{\Delta K}^{-1}\right) = 0$ since $\lambda_i + \mu_i = 0$ and $\frac{1}{\lambda_i} + \frac{1}{\mu_i} = 0$. Therefore, the separable stiffness matrices are realizable and qualify for only pure rotation modes.

B. THREE CASES OF SEPARABLE STIFFNESS MATRICES OF RANK 6

For the given rank 3 stiffness matrix $\boldsymbol{K}_\alpha$ that can be realized by co-reciprocal line vectors, the wrench $\widehat{\boldsymbol{w}}$ induced by a small displacement $\widehat{\boldsymbol{D}}$ due to $\boldsymbol{K}_\alpha$ can be obtained as the linear combination of line vectors $\widehat{\boldsymbol{s}}_i$'s:

$$\widehat{\boldsymbol{w}} = \boldsymbol{K}_\alpha\widehat{\boldsymbol{D}} = \sum_{i=1}^{3} k_i\left(\widehat{\boldsymbol{s}}_i^T\widehat{\boldsymbol{D}}\right)\widehat{\boldsymbol{s}}_i. \tag{32}$$

Clearly, $\widehat{\boldsymbol{w}}^T\boldsymbol{\Delta}\widehat{\boldsymbol{w}} = 0$ for any $\widehat{\boldsymbol{D}}$. It means that span($\widehat{\boldsymbol{s}}_1$, $\widehat{\boldsymbol{s}}_2$, $\widehat{\boldsymbol{s}}_3$) is a 3-system of screws containing only lines and(or) free vectors. As listed in Table 1, $\boldsymbol{K}_\alpha$ can be classified into four cases according to Hunt's special 3-systems containing only line and free vectors [24]. h'_1, h'_2, and h'_3 are the pitches of principal screws of 3-systems. This classification gives four geometric constraints of springs (Fig. 3). These constraints can also be obtained from (13). If both $\widehat{\boldsymbol{s}}_i$ and $\widehat{\boldsymbol{s}}_j$ are line vectors (of zero pitch), their reciprocity implies that they meet or are parallel. If $\widehat{\boldsymbol{s}}_i$ and $\widehat{\boldsymbol{s}}_j$ are a line and a free vector, respectively, they are orthogonal to each other since $\widehat{\boldsymbol{s}}_i^T\boldsymbol{\Delta}\widehat{\boldsymbol{s}}_j = \boldsymbol{s}_i^T\boldsymbol{s}_j = 0$.

TABLE 1. Classification of 3-systems of Screws containing only Line Vectors [24].

Case (α, β)	Hunt's classification of special 3-systems	Possible line vectors
$\boldsymbol{K}_1$	The second ($h'_1 = h'_2 = h'_3 = 0$)	A bundle of lines
$\boldsymbol{K}_2$	The fourth ($h'_1 = h'_2 = 0, h'_3 = \infty$)	Lines on a plane and free vectors normal to the plane
$\boldsymbol{K}_3$	The fifth ($h'_1 = 0, h'_2 = h'_3 = \infty$)	Lines normal to a plane and free vectors parallel to the plane
$\boldsymbol{K}_4$	The sixth ($h'_1 = h'_2 = h'_3 = \infty$)	Only free vectors

For two free vectors, they are always reciprocal to each other. Figure 3 illustrates the spring systems satisfying the geometric constraints obtained from the reciprocal relations between line vectors. It should be noted here that the line and the free vectors correspond to respectively the line and torsion springs in Fig. 3.

Now, we may obtain $10(= {}_5C_2)$ cases of separable stiffness matrices because we should select two matrices from four cases in Table 1 with repetition. However, for rank 6 stiffness matrices, there are only three cases:

1) $\boldsymbol{K}_1 + \boldsymbol{K}_2$,
2) $\boldsymbol{K}_2 + \boldsymbol{K}_3$,
3) $\boldsymbol{K}_1 + \boldsymbol{K}_4$.

This can be shown briefly as follows: If the given rank 6 stiffness matrix is separated in one of seven cases other than the above three cases, then there always exists at least one line vector $\widehat{\boldsymbol{u}}$ belonging to both span($\widehat{\boldsymbol{s}}_1, \widehat{\boldsymbol{s}}_2, \widehat{\boldsymbol{s}}_3$) and span($\widehat{\boldsymbol{s}}_4, \widehat{\boldsymbol{s}}_5, \widehat{\boldsymbol{s}}_6$) as illustrated in Fig. 4, and thus $\boldsymbol{K\Delta}\widehat{\boldsymbol{u}}$ becomes null vector

$$\left(\boldsymbol{K}_\alpha + \boldsymbol{K}_\beta\right)\boldsymbol{\Delta}\widehat{\boldsymbol{u}} = \left(\sum_{i=1}^{3}\left(k_i\widehat{\boldsymbol{s}}_i\widehat{\boldsymbol{s}}_i^T\right) + \sum_{i=4}^{6}\left(k_i\widehat{\boldsymbol{s}}_i\widehat{\boldsymbol{s}}_i^T\right)\right)\boldsymbol{\Delta}\widehat{\boldsymbol{u}} = \boldsymbol{0}.$$

It means that the sum of two matrices becomes singular. This contradiction concludes that there are only three cases of separations of rank 6 stiffness matrices: 1) $\boldsymbol{K}_1 + \boldsymbol{K}_2$, 2) $\boldsymbol{K}_2 + \boldsymbol{K}_3$, and 3) $\boldsymbol{K}_1 + \boldsymbol{K}_4$.

C. SEPARATION OF STIFFNESS MATRICES OF RANK 6

If a rank 6 stiffness matrix $\boldsymbol{K}$ is separable, the line vectors $\widehat{\boldsymbol{\alpha}}_i$'s and $\widehat{\boldsymbol{\beta}}_i$'s are uniquely determined from (19), (20), and (21). The geometrical interpretation of reciprocity and linear independency between $\widehat{\boldsymbol{\alpha}}_i$'s and $\widehat{\boldsymbol{\beta}}_i$'s of (24) can be geometrically interpreted to obtain 7 possible cases of combinations of separations using $\widehat{\boldsymbol{\alpha}}_i$'s and $\widehat{\boldsymbol{\beta}}_i$'s (Fig. 5). In addition, it is noted that Case7 (Fig. 5(g)) is identical to the condition for diagonalization of the stiffness matrix by a co-ordinate transformation [11].

Now, $\boldsymbol{K}$ can be separated using $\widehat{\boldsymbol{\alpha}}_i$'s and $\widehat{\boldsymbol{\beta}}_i$'s for each case. For Case1 (Fig. 5(a)), the stiffness matrix can be separated into

$$\boldsymbol{K}_1 = \sum_{i=1}^{3}\left(k_{\alpha i}\widehat{\boldsymbol{\alpha}}_i\widehat{\boldsymbol{\alpha}}_i^T\right) \text{ and } \boldsymbol{K}_2 = \sum_{i=1}^{3}\left(k_{\beta i}\widehat{\boldsymbol{\beta}}_i\widehat{\boldsymbol{\beta}}_i^T\right).$$

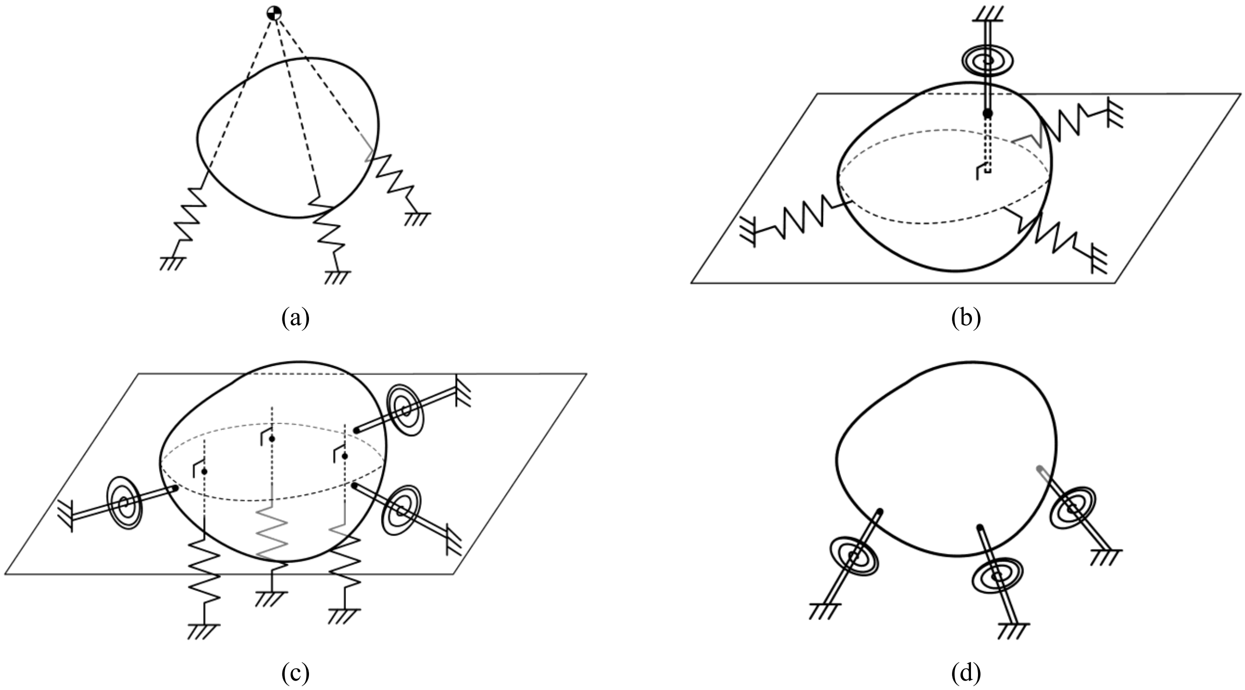

FIGURE 3. Geometric constraints of springs of (a) K_1, (b) K_2, (c) K_3, and (d) K_4.

It can be said that the induced wrench space of $\boldsymbol{K}$, span($\widehat{\boldsymbol{\alpha}}_1$, $\widehat{\boldsymbol{\alpha}}_2$, $\widehat{\boldsymbol{\alpha}}_3$, $\widehat{\boldsymbol{\beta}}_1$, $\widehat{\boldsymbol{\beta}}_2$,$\widehat{\boldsymbol{\beta}}_3$), is separated as span($\widehat{\boldsymbol{\alpha}}_1$, $\widehat{\boldsymbol{\alpha}}_2$, $\widehat{\boldsymbol{\alpha}}_3$) $\oplus$ span($\widehat{\boldsymbol{\beta}}_1$, $\widehat{\boldsymbol{\beta}}_2$,$\widehat{\boldsymbol{\beta}}_3$). Separation for the other cases can also be represented by direct sum of induced wrench spaces accordingly. All the possible separation cases are summarized in Table 2, where the following notation is used:

$$\begin{aligned}
\mathbf{AAA} &\equiv \text{span}\left(\widehat{\boldsymbol{\alpha}}_1, \widehat{\boldsymbol{\alpha}}_2, \widehat{\boldsymbol{\alpha}}_3\right), \\
\mathbf{BBB} &\equiv \text{span}\left(\widehat{\boldsymbol{\beta}}_1, \widehat{\boldsymbol{\beta}}_2, \widehat{\boldsymbol{\beta}}_3\right), \\
\mathbf{ABB} &\equiv \text{span}\left(\widehat{\boldsymbol{\alpha}}_1, \widehat{\boldsymbol{\beta}}_2, \widehat{\boldsymbol{\beta}}_3\right), \\
\mathbf{BAA} &\equiv \text{span}\left(\widehat{\boldsymbol{\beta}}_1, \widehat{\boldsymbol{\alpha}}_2, \widehat{\boldsymbol{\alpha}}_3\right), \\
\mathbf{ABA} &\equiv \text{span}\left(\widehat{\boldsymbol{\alpha}}_1, \widehat{\boldsymbol{\beta}}_2, \widehat{\boldsymbol{\alpha}}_3\right), \\
\mathbf{BAB} &\equiv \text{span}\left(\widehat{\boldsymbol{\beta}}_1, \widehat{\boldsymbol{\alpha}}_2, \widehat{\boldsymbol{\beta}}_3\right), \\
\mathbf{AAB} &\equiv \text{span}\left(\widehat{\boldsymbol{\alpha}}_1, \widehat{\boldsymbol{\alpha}}_2, \widehat{\boldsymbol{\beta}}_3\right), \\
\mathbf{BBA} &\equiv \text{span}\left(\widehat{\boldsymbol{\beta}}_1, \widehat{\boldsymbol{\beta}}_2, \widehat{\boldsymbol{\alpha}}_3\right).
\end{aligned}$$

D. SEPARABLE STIFFNESS MATRIX AS LINEAR TRANSFORMATION

For the given separable stiffness matrix $\boldsymbol{K} = \sum_{i=1}^{3}\left(k_{\alpha i}\widehat{\boldsymbol{\alpha}}_i\widehat{\boldsymbol{\alpha}}_i^T + k_{\beta i}\widehat{\boldsymbol{\beta}}_i\widehat{\boldsymbol{\beta}}_i^T\right)$, let $\widehat{\boldsymbol{X}}_\alpha$ and $\widehat{\boldsymbol{X}}_\beta$ be small displacements belonging to **AAA** and **BBB**, respectively. Since $\widehat{\boldsymbol{X}}_\alpha$ and $\widehat{\boldsymbol{X}}_\beta$ are expressed in axis co-ordinates, $\boldsymbol{\Delta}\widehat{\boldsymbol{X}}_\alpha \in$ **AAA** and $\boldsymbol{\Delta}\widehat{\boldsymbol{X}}_\beta \in$ **BBB**. Obviously, **AAA** and **BBB** are self-reciprocal, and thereby the wrenches induced by $\widehat{\boldsymbol{X}}_\alpha$ and $\widehat{\boldsymbol{X}}_\beta$ can be obtained as

$$\boldsymbol{K}\widehat{\boldsymbol{X}}_\alpha = \sum_{i=1}^{3} k_{\beta i}\left(\widehat{\boldsymbol{\beta}}_i^T\widehat{\boldsymbol{X}}_\alpha\right)\widehat{\boldsymbol{\beta}}_i,$$

and

$$\boldsymbol{K}\widehat{\boldsymbol{X}}_\beta = \sum_{i=1}^{3} k_{\alpha i}\left(\widehat{\boldsymbol{\alpha}}_i^T\widehat{\boldsymbol{X}}_\beta\right)\widehat{\boldsymbol{\alpha}}_i.$$

From these two equations, $\boldsymbol{K}$ can be considered as a linear operator that transforms a line vector belonging to **AAA** into the one belonging to **BBB** and vice versa

$$\boldsymbol{K}\widehat{\boldsymbol{X}}_\alpha \in \mathbf{BBB} \text{ and } \boldsymbol{K}\widehat{\boldsymbol{X}}_\beta \in \mathbf{AAA}. \tag{33}$$

In the similar manner, it can also be shown that $\boldsymbol{K}$ is a linear transformation between each pair of 3-systems of screws: (**BAA** and **ABB**), (**ABA** and **BAB**), and (**AAB** and **BBA**).

IV. CONDITIONS FOR HAVING ONLY PURE ROTATION MODES

The mass matrix $\boldsymbol{M}$ is diagonalized at the co-ordinate frame coincident with the principal axes of inertia. It means that $\boldsymbol{M}$ is separable and the line vectors are uniquely determined from (19), (20), and (21) as three orthogonal lines $\widehat{\boldsymbol{n}}_i (i = 1, 2, 3)$ and free vectors $\hat{\boldsymbol{l}}_i (i = 1, 2, 3)$. The three lines $\widehat{\boldsymbol{n}}_i$'s are aligned with the principal axes of inertia, respectively (Fig. 6) [11]. Using $\widehat{\boldsymbol{n}}_i$'s and $\hat{\boldsymbol{l}}_i$'s, $\boldsymbol{M}$ can be expressed as

$$\boldsymbol{M} = \sum_{i=1}^{3}\left(m\widehat{\boldsymbol{n}}_i\widehat{\boldsymbol{n}}_i^T + I_{ii}\hat{\boldsymbol{l}}_i\hat{\boldsymbol{l}}_i^T\right),$$

where m is the mass and I_{ii} denotes the moment of inertia with respect to the ith axis. The separation of $\boldsymbol{M}$ can be described as

$$\begin{aligned}
&\text{span}\left(\widehat{\boldsymbol{n}}_1, \widehat{\boldsymbol{n}}_2, \widehat{\boldsymbol{n}}_3\right) \oplus \text{span}\left(\hat{\boldsymbol{l}}_1, \hat{\boldsymbol{l}}_2, \hat{\boldsymbol{l}}_3\right), \\
&\text{span}\left(\widehat{\boldsymbol{n}}_1, \hat{\boldsymbol{l}}_2, \hat{\boldsymbol{l}}_3\right) \oplus \text{span}\left(\hat{\boldsymbol{l}}_1, \widehat{\boldsymbol{n}}_2, \widehat{\boldsymbol{n}}_3\right),
\end{aligned}$$

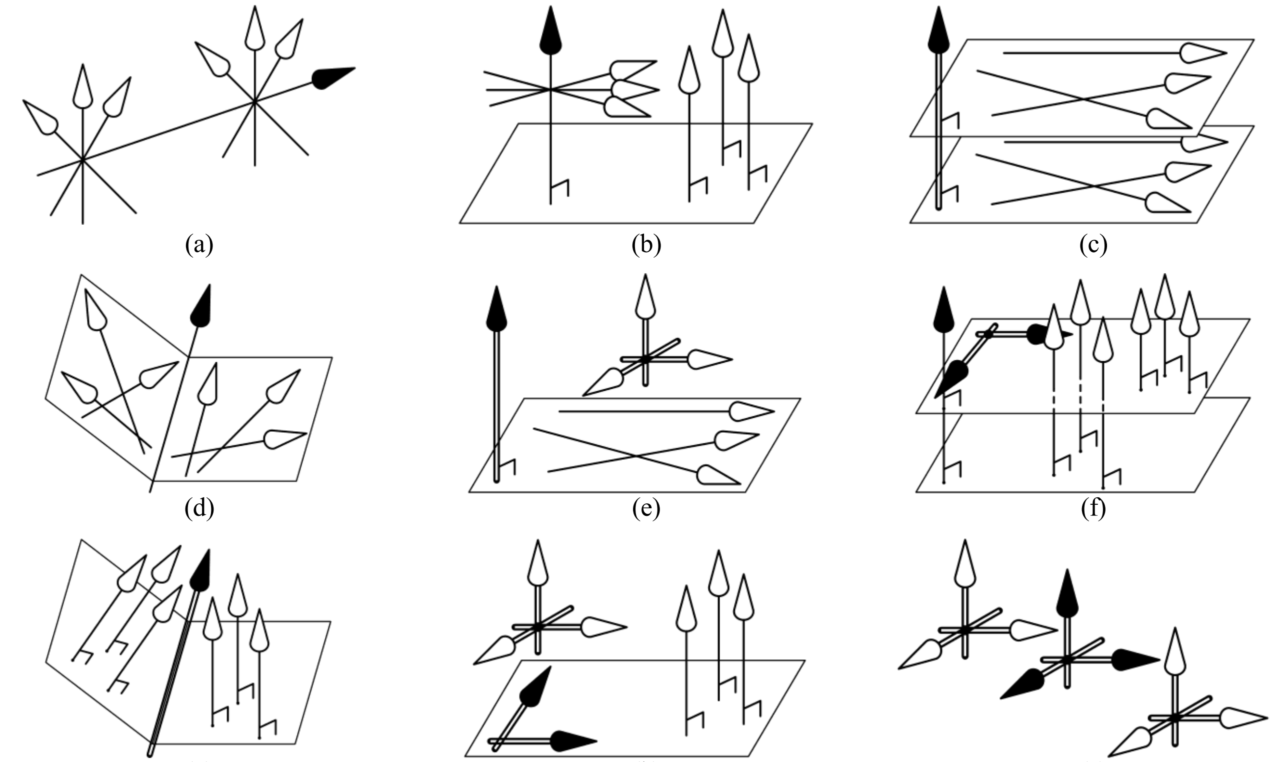

FIGURE 4. Geometrical representation of null space $\Delta\hat{u}$ (black arrowhead) of sum of two rank 3 stiffness matrices: (a) K_1+K_1, (b) K_1+K_3, (c) K_2+K_2 if the planes are parallel, (d) K_2+K_2 if the planes are not parallel, (e) K_2+K_4, (f) K_3+K_3 if the lines are parallel, (g) K_3+K_3 if the lines are not parallel, (h) K_3+K_4, and (i)K_4+K_4.

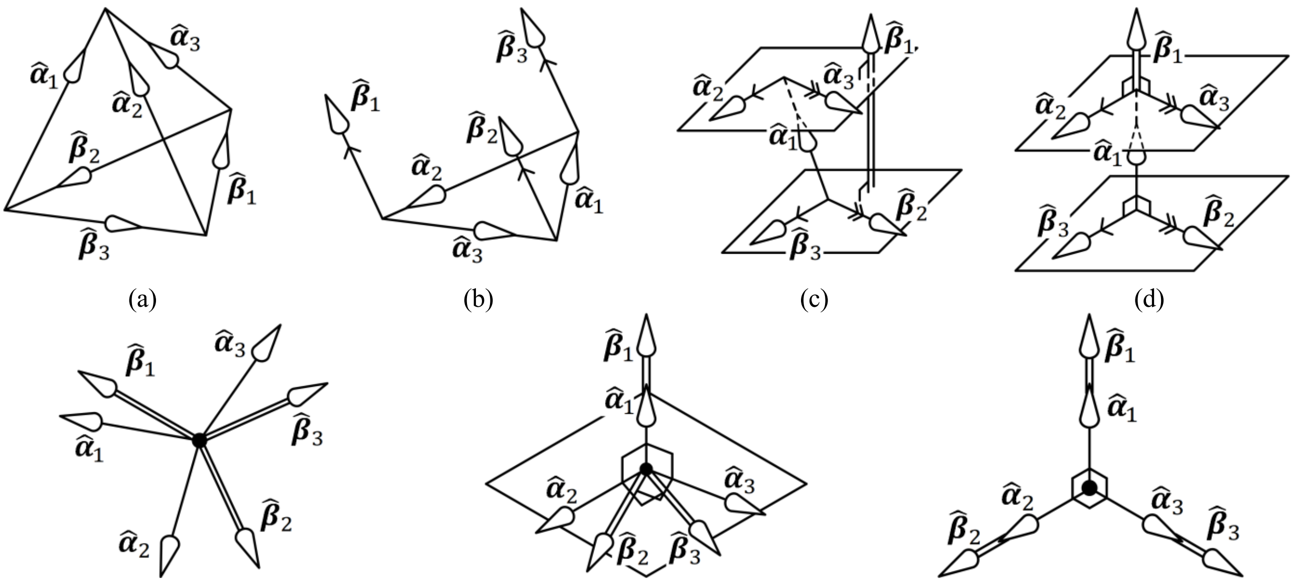

FIGURE 5. Possible cases of line vectors of rank 6 separable stiffness matrices: (a) Case1, (b) Case2, (c) Case3, (d) Case4, (e) Case5, (f) Case6, and (g) Case7.

$$\mathrm{span}\left(\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{l}}_2, \hat{\boldsymbol{n}}_3\right) \oplus \mathrm{span}\left(\hat{\boldsymbol{l}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{l}}_3\right),$$
$$\mathrm{span}\left(\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{l}}_3\right) \oplus \mathrm{span}\left(\hat{\boldsymbol{l}}_1, \hat{\boldsymbol{l}}_2, \hat{\boldsymbol{n}}_3\right).$$

The above separations of the inertia matrix $\boldsymbol{M}$ are equivalent to Case7 for the separable stiffness matrix $\boldsymbol{K}$. The separations of $\boldsymbol{M}$ implies that $\boldsymbol{M}$ can also be regarded as a linear

TABLE 2. Exhaustive list of separations of rank 6 stiffness matrices.

Case	Methods to separate	Case	Methods to separate
Case1	AAA ⊕ BBB, ABB ⊕ BAA, BAB ⊕ ABA, BBA ⊕ AAB $\rightarrow \boldsymbol{K}=\boldsymbol{K}_1+\boldsymbol{K}_2$	Case3 (4)	AAA ⊕ BBB, ABB ⊕ BAA $\rightarrow \boldsymbol{K}=\boldsymbol{K}_1+\boldsymbol{K}_2$ ABA ⊕ BAB, AAB ⊕ BBA $\rightarrow \boldsymbol{K}=\boldsymbol{K}_2+\boldsymbol{K}_3$
Case2	AAA ⊕ BBB $\rightarrow \boldsymbol{K}=\boldsymbol{K}_2+\boldsymbol{K}_3$ BAA ⊕ ABB, ABA ⊕ BAB, AAB ⊕ BBA $\rightarrow \boldsymbol{K}=\boldsymbol{K}_1+\boldsymbol{K}_2$	Case5 (6, 7)	AAA ⊕ BBB $\rightarrow \boldsymbol{K}=\boldsymbol{K}_1+\boldsymbol{K}_4$ BAA ⊕ ABB, ABA ⊕ BAB, AAB ⊕ BBA $\rightarrow \boldsymbol{K}=\boldsymbol{K}_2+\boldsymbol{K}_3$

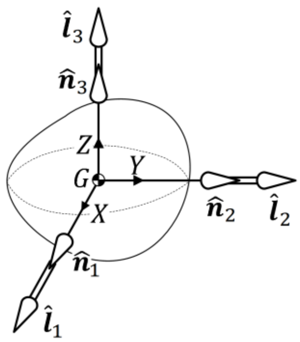

FIGURE 6. Three orthogonal lines and free vectors uniquely determined from mass matrix.

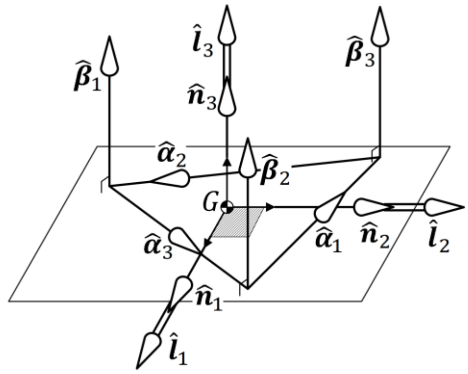

FIGURE 7. Geometric conditions for a plane of symmetry.

transformation between the 3-systems of screws, as $\boldsymbol{K}$ is a linear transformation between 3-systems of screws. In this section, we derive the conditions for only pure rotation modes using the linear transformations between screw systems. In what follows, it is shown that the existence of only pure rotation modes depends on geometrical relationship between $\boldsymbol{M}$ and $\boldsymbol{K}$.

A. CONDITIONS FOR ONLY PURE ROTATION MODES

When the separable stiffness matrix is given by (22), that is, $\boldsymbol{K} = \sum_{i=1}^{3}\left(k_{\alpha i}\widehat{\boldsymbol{\alpha}}_i\widehat{\boldsymbol{\alpha}}_i^T + k_{\beta i}\widehat{\boldsymbol{\beta}}_i\widehat{\boldsymbol{\beta}}_i^T\right)$, the conditions for only pure rotation modes can be obtained by considering the mass matrix $\boldsymbol{M}$ as a linear transformation. In the same manner as $\boldsymbol{K}$ in (33), if $\boldsymbol{M}$ is assumed to be a linear transformation between **AAA** and **BBB** such that

$$\boldsymbol{M}\widehat{\boldsymbol{X}}_\alpha \in \mathbf{BBB} \text{ and } \boldsymbol{M}\widehat{\boldsymbol{X}}_\beta \in \mathbf{AAA}, \tag{34}$$

then the vibration mode $\widehat{\boldsymbol{X}}$ is a pure rotation mode since it belongs to **AAA** or **BBB**. To prove this, let $\widehat{\boldsymbol{x}}_{\alpha i}(i = 1, 2, 3)$ and $\widehat{\boldsymbol{x}}_{\beta j}(j = 1, 2, 3)$ be linearly independent line vectors selected from **AAA** and **BBB**, respectively. Since $\widehat{\boldsymbol{X}}$ is expressed in axis co-ordinates, it can be written as

$$\widehat{\boldsymbol{X}} = \sum_{i=1}^{3}\left(\alpha_i\boldsymbol{\Delta}\widehat{\boldsymbol{x}}_{\alpha i}\right) + \sum_{j=1}^{3}\left(\beta_j\boldsymbol{\Delta}\widehat{\boldsymbol{x}}_{\beta j}\right), \tag{35}$$

where α_i and β_j are constants. Substituting (35) into (6) gives

$$\sum_{i=1}^{3}\alpha_i\left(-\omega^2\boldsymbol{M}+\boldsymbol{K}\right)\boldsymbol{\Delta}\widehat{\boldsymbol{x}}_{\alpha i} = -\sum_{j=1}^{3}\beta_j\left(-\omega^2\boldsymbol{M}+\boldsymbol{K}\right)\boldsymbol{\Delta}\widehat{\boldsymbol{x}}_{\beta j}. \tag{36}$$

On the assumption of (34), the left and right sides of (36) belong to **BBB** and **AAA**, respectively. Since **BBB** and **AAA** are linearly independent, (36) holds only when both sides of (36) become zero simultaneously, and consequently, we obtain

$$\sum_{i=1}^{3}\alpha_i\left(-\omega^2\boldsymbol{M}+\boldsymbol{K}\right)\boldsymbol{\Delta}\widehat{\boldsymbol{x}}_{\alpha i}=\mathbf{0}, \tag{37a}$$

$$\sum_{j=1}^{3}\beta_j\left(-\omega^2\boldsymbol{M}+\boldsymbol{K}\right)\boldsymbol{\Delta}\widehat{\boldsymbol{x}}_{\beta j}=0. \tag{37b}$$

These equations imply that $\widehat{\boldsymbol{x}}$ belongs to either **BBB** or **AAA** and therefore $\widehat{\boldsymbol{x}}^T\boldsymbol{\Delta}\widehat{\boldsymbol{x}} = 0$, i.e., $\widehat{\boldsymbol{X}}^T\boldsymbol{\Delta}\widehat{\boldsymbol{X}} = 0$ since $\widehat{\boldsymbol{x}} = \boldsymbol{\Delta}\widehat{\boldsymbol{X}}$. Consequently, $\widehat{\boldsymbol{X}}$ is a pure rotation mode, which completes the proof.

Additionally, $\boldsymbol{K}$ is a linear transformation between two 3-systems belonging to one of the following four pairs: (**AAA** and **BBB**), (**BAA** and **ABB**), (**ABA** and **BAB**), and (**AAB** and **BBA**). Therefore, the condition for only pure rotation modes is such that $\boldsymbol{M}$ and $\boldsymbol{K}$ must be the linear transformations between two 3-systems of screws belonging to one of the above four pairs.

B. GEOMETRIC DESCRIPTION OF CONDITIONS FOR ONLY PURE ROTATION MODES

In this subsection, for each separation of $\boldsymbol{K}$, we present geometric conditions in order for $\boldsymbol{M}$ to be the linear transformation between two 3-systems belonging to one of the following four pairs: (**AAA** and **BBB**), (**BAA** and **ABB**), (**ABA** and **BAB**), and (**AAB** and **BBA**).

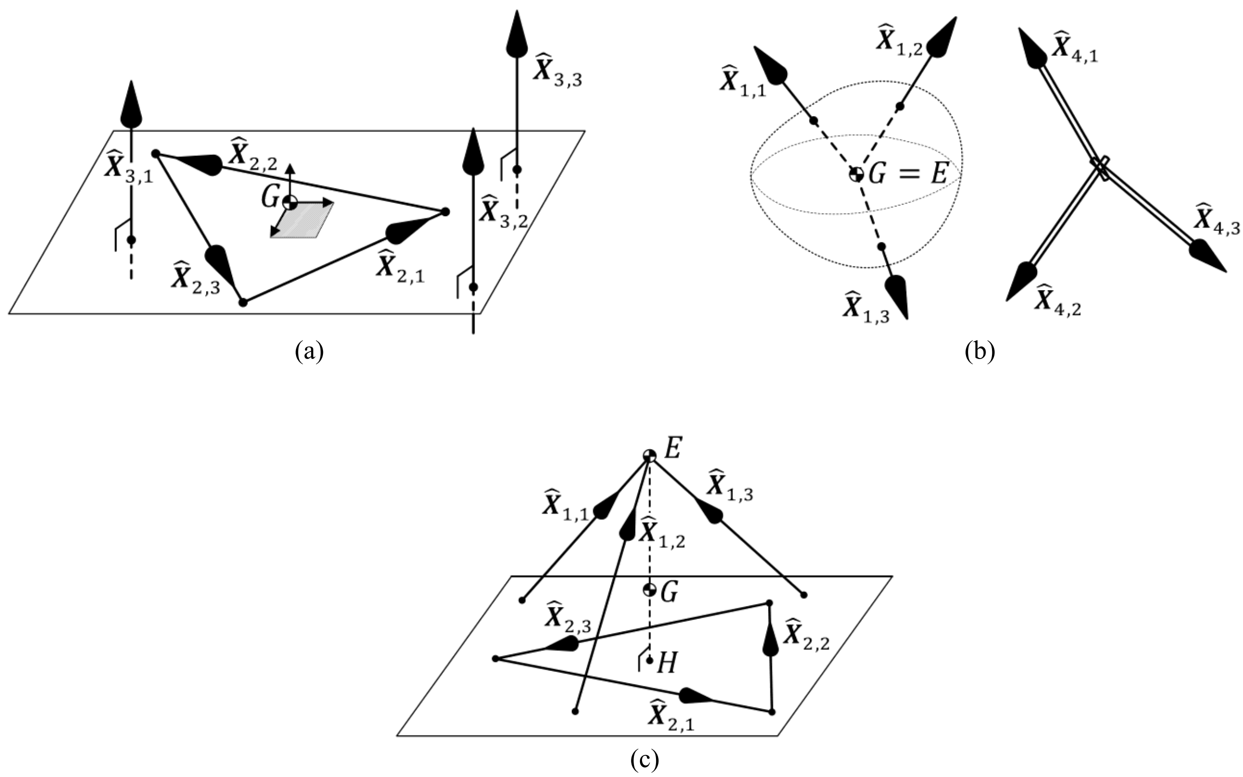

FIGURE 8. Vibration axes of pure rotation modes: (a) decoupled with respect to plane of symmetry, (b) determined as three lines and free vectors, and (c) with tetrahedral configuration.

TABLE 3. Screw system conditions for plane of symmetry.

Case	Conditions
Case1	-
Case2	$\mathbf{AAA} = \text{span}(\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{l}}_3)$ and $\mathbf{BBB} = \text{span}(\hat{\boldsymbol{l}}_1, \hat{\boldsymbol{l}}_2, \hat{\boldsymbol{n}}_3)$
Case3 (4)	$\mathbf{AAB} = \text{span}(\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{l}}_3)$ and $\mathbf{BBA} = \text{span}(\hat{\boldsymbol{l}}_1, \hat{\boldsymbol{l}}_2, \hat{\boldsymbol{n}}_3)$ or $\mathbf{ABA} = \text{span}(\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{l}}_3)$ and $\mathbf{BAB} = \text{span}(\hat{\boldsymbol{l}}_1, \hat{\boldsymbol{l}}_2, \hat{\boldsymbol{n}}_3)$
Case5 (6, 7)	$\mathbf{AAB} = \text{span}(\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{l}}_3)$ and $\mathbf{BBA} = \text{span}(\hat{\boldsymbol{l}}_1, \hat{\boldsymbol{l}}_2, \hat{\boldsymbol{n}}_3)$ or $\mathbf{ABA} = \text{span}(\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{l}}_3)$ and $\mathbf{BAB} = \text{span}(\hat{\boldsymbol{l}}_1, \hat{\boldsymbol{l}}_2, \hat{\boldsymbol{n}}_3)$ or $\mathbf{BAA} = \text{span}(\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{l}}_3)$ and $\mathbf{ABB} = \text{span}(\hat{\boldsymbol{l}}_1, \hat{\boldsymbol{l}}_2, \hat{\boldsymbol{n}}_3)$

For the stiffness matrix $\boldsymbol{K}$ corresponding to Case2 (Table 2), if it is assumed that all the lines belonging to **AAA** are contained in one of the principal planes of inertia and all the lines belonging to **BBB** are perpendicular to the principal plane (Fig. 7), the following relations between 3-systems of screws can be obtained

$$\mathbf{AAA} = \text{span}\left(\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{l}}_3\right) \text{ and }$$
$$\mathbf{BBB} = \text{span}\left(\hat{\boldsymbol{l}}_1, \hat{\boldsymbol{l}}_2, \hat{\boldsymbol{n}}_3\right). \quad (38)$$

TABLE 4. System parameters.

Mass	24 kg
Principal moments of inertia	$I_{XX} = 0.5800$, $I_{YY} = 0.2848$, and $I_{ZZ} = 0.7048$ kg·m^2
Stiffness constants of each mounting element	$k_x = k_y = 1.5 \times 10^5$ and $k_z = 0.98 \times 10^5$ N/m

For any line vectors $\hat{\boldsymbol{x}}_2 (\in \mathbf{AAA})$ and $\hat{\boldsymbol{x}}_3 (\in \mathbf{BBB})$, $\boldsymbol{M}\Delta\hat{\boldsymbol{x}}_2$ and $\boldsymbol{M}\Delta\hat{\boldsymbol{x}}_3$ can be computed as

$$\boldsymbol{M}\Delta\hat{\boldsymbol{x}}_2 = m\left(\hat{\boldsymbol{n}}_3^T \Delta\hat{\boldsymbol{x}}_2\right)\hat{\boldsymbol{n}}_3 + I_{11}\left(\hat{\boldsymbol{l}}_1^T \Delta\hat{\boldsymbol{x}}_2\right)\hat{\boldsymbol{l}}_1 + I_{22}\left(\hat{\boldsymbol{l}}_2^T \Delta\hat{\boldsymbol{x}}_2\right)\hat{\boldsymbol{l}}_2,$$

and

$$\boldsymbol{M}\Delta\hat{\boldsymbol{x}}_3 = m\left(\hat{\boldsymbol{n}}_1^T \Delta\hat{\boldsymbol{x}}_3\right)\hat{\boldsymbol{n}}_1 + m\left(\hat{\boldsymbol{n}}_2^T \Delta\hat{\boldsymbol{x}}_3\right)\hat{\boldsymbol{n}}_2 + I_{33}\left(\hat{\boldsymbol{l}}_3^T \Delta\hat{\boldsymbol{x}}_3\right)\hat{\boldsymbol{l}}_3.$$

Accordingly, $\boldsymbol{M}\Delta\hat{\boldsymbol{x}}_2 \in \mathbf{BBB}$ and $\boldsymbol{M}\Delta\hat{\boldsymbol{x}}_3 \in \mathbf{AAA}$. It concludes that the vibration systems with geometric relations illustrated in Fig. 7 have only pure rotation modes. Furthermore, since $\hat{\boldsymbol{x}}$ must belong to either **AAA** or **BBB**, the vibration axes are grouped into those normal to the principal plane and the others lying on the plane (Fig. 8(a)). In this case, the principal plane becomes a plane of symmetry [11]. It means that a vibration system with the stiffness matrix that can be expressed as $\boldsymbol{K} = \boldsymbol{K}_2 + \boldsymbol{K}_3$ can have a plane of symmetry. In Table 3, the conditions for a plane of symmetry

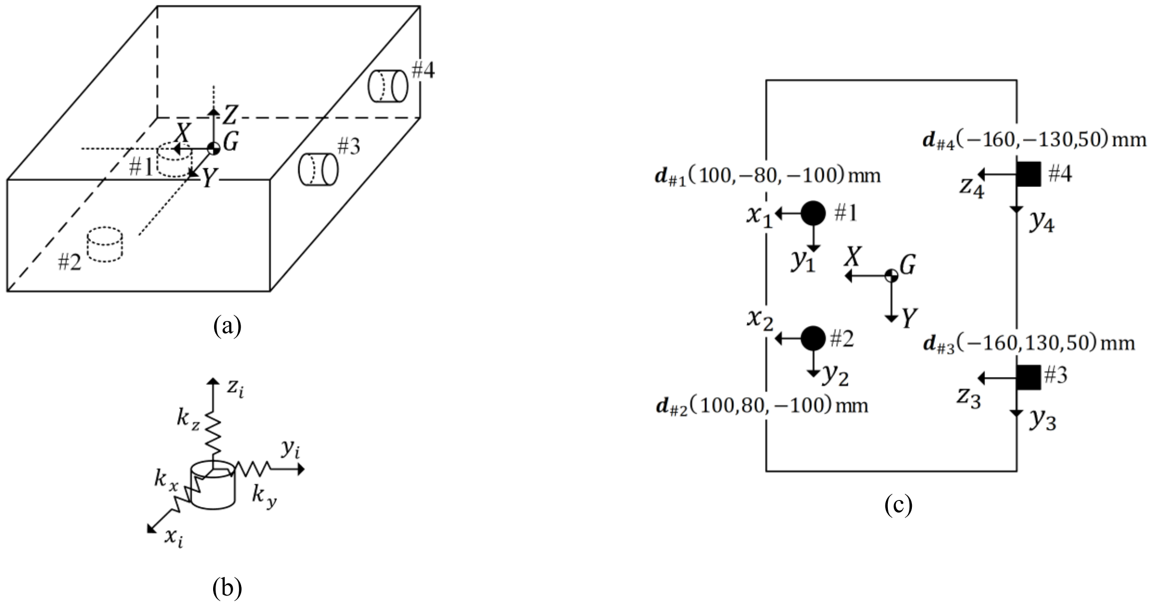

FIGURE 9. Four-point mounting system: (a) overview of the system, (b) modelling mounting element as three orthogonal line springs, and (c) locations of local co-ordinate frames.

are summarized. It is noted that span($\widehat{\boldsymbol{n}}_1, \widehat{\boldsymbol{n}}_2, \hat{\boldsymbol{l}}_3$) is selected as one of the principal planes.

For a stiffness matrix $\boldsymbol{K}(= \boldsymbol{K}_1 + \boldsymbol{K}_4)$ corresponding to one of Case5, 6, and 7, when the intersecting point E of the bundle of lines contained in **AAA** coincides with the center of mass G, we get

$$\mathbf{AAA} = \text{span}\left(\widehat{\boldsymbol{n}}_1, \widehat{\boldsymbol{n}}_2, \widehat{\boldsymbol{n}}_3\right) \text{ and}$$
$$\mathbf{BBB} = \text{span}\left(\hat{\boldsymbol{l}}_1, \hat{\boldsymbol{l}}_2, \hat{\boldsymbol{l}}_3\right). \tag{39}$$

It becomes clear from (39) that $\boldsymbol{M}\boldsymbol{\Delta}\widehat{\boldsymbol{x}}_1 \in \mathbf{BBB}$ and $\boldsymbol{M}\boldsymbol{\Delta}\widehat{\boldsymbol{x}}_4 \in \mathbf{AAA}$ for any line vectors $\widehat{\boldsymbol{x}}_1 (\in AAA)$ and $\widehat{\boldsymbol{x}}_4 (\in \mathbf{BBB})$. Therefore, all the modes are pure rotation modes. As illustrated in Fig. 8(b), the vibration modes are obtained as three lines passing through the center of mass and three free vectors. Consequently, for $\boldsymbol{K}(= \boldsymbol{K}_1 + \boldsymbol{K}_4)$, the condition to have only pure rotation modes is that E is coincident with G.

It becomes clear from (39) that $\boldsymbol{M}\boldsymbol{\Delta}\widehat{\boldsymbol{x}}_1 \in \mathbf{BBB}$ and $\boldsymbol{M}\boldsymbol{\Delta}\widehat{\boldsymbol{x}}_4 \in \mathbf{AAA}$ for any line vectors $\widehat{\boldsymbol{x}}_1$ ($\in$**AAA**) and $\widehat{\boldsymbol{x}}_4$ ($\in$**BBB**). Therefore, all the modes are pure rotation modes. As illustrated in Fig. 8(b), the vibration modes are obtained as three lines passing through the center of mass and three free vectors. Consequently, for $\boldsymbol{K}(= \boldsymbol{K}_1 + \boldsymbol{K}_4)$, the condition to have only pure rotation modes is that E is coincident with G.

Lastly, the conditions of $\boldsymbol{K}(= \boldsymbol{K}_1 + \boldsymbol{K}_2)$ to have only pure rotation modes are presented. Without loss of generality, if **AAA** and **BBB** are respectively the induced wrench spaces of $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$, then we can determine the intersecting point E of **AAA** and the foot of perpendicular H drawn from E to the plane of **BBB** (Fig. 8(c)). Now, we suppose that the inertia properties of a rigid body of a vibrating system satisfies that

1) $I_{11} = I_{22} = I_{33} = I$,

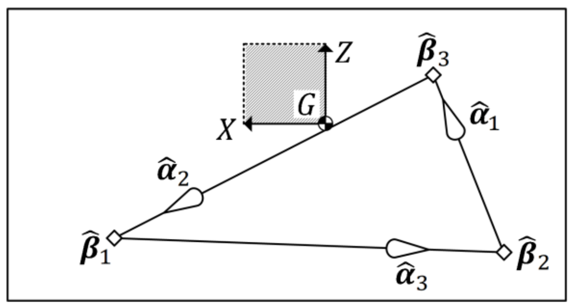

FIGURE 10. Uniquely determined line vectors lying on the principal plane and ones perpendicular to the plane (◇).

2) G is internal division of $\overline{EH}$ and $\overline{EG} \cdot \overline{GH} = \frac{I}{m}$. Then, the mass matrix $\boldsymbol{M}$ is the linear operator with following characteristic (See Appendix):

$$\boldsymbol{M}\boldsymbol{\Delta}\widehat{x}_1 \in \mathbf{BBB} \text{ and } \boldsymbol{M}\boldsymbol{\Delta}\widehat{x}_2 \in \mathbf{AAA}. \tag{40}$$

for any line vectors $\widehat{\boldsymbol{x}}_1$($\in$**AAA**) and $\widehat{\boldsymbol{x}}_2$($\in$**BBB**). Consequently, there are only pure rotation modes and they can be grouped into two sets of vibration axes: 1) a set of three lines passing through E and 2) the other set of three lines lying on the plane as illustrated in Fig. 8(c).

So far, three conditions for only pure rotation modes are derived. Referring to Fig. 8, it can be said that there exist three configurations of two groups of pure rotation modes. These findings are valuable since the simple geometry may be used to design a spatial vibration system systematically.

V. NUMERICAL EXAMPLE

In this section, we use the four-point mounting system as a numerical example similar to the one presented in [11] to

TABLE 5. Stiffness matrix and separation.

Stiffness matrix	$\mathbf{K}=\begin{bmatrix} 4.960 & 0 & 0 & 0 & -0.2020 & 0 \\ 0 & 6.0 & 0 & 0.1500 & 0 & -0.1800 \\ 0 & 0 & 4.960 & 0 & 0.2840 & 0 \\ 0 & 0.1500 & 0 & 0.1007 & 0 & 0.0540 \\ -0.2020 & 0 & 0.2840 & 0 & 0.1313 & 0 \\ 0 & -0.1800 & 0 & 0.0540 & 0 & 0.1591 \end{bmatrix}\times 10^5$
Eigenvalues	$\begin{matrix}\lambda_1=-9.8270\times 10^4 \\ \mu_1=9.8270\times 10^4\end{matrix}, \begin{matrix}\lambda_2=-5.3028\times 10^4 \\ \mu_2=5.3028\times 10^4\end{matrix}$, and $\begin{matrix}\lambda_3=-8.1671\times 10^4 \\ \mu_3=8.1671\times 10^4\end{matrix}$
Uniquely determined line vectors	$\widehat{\boldsymbol{\alpha}}_1=[0.4786 \quad 0 \quad 0.8780 \quad 0 \quad 0.0632 \quad 0]^T$ $\widehat{\boldsymbol{\alpha}}_2=[0.8396 \quad 0 \quad -0.5431 \quad 0 \quad -0.0154 \quad 0]^T$ $\widehat{\boldsymbol{\alpha}}_3=[-0.9986 \quad 0 \quad -0.0536 \quad 0 \quad 0.3868 \quad 0]^T$ $\widehat{\boldsymbol{\beta}}_1=[0 \quad 1 \quad 0 \quad 0.3591 \quad 0 \quad 0.5269]^T$ $\widehat{\boldsymbol{\beta}}_2=[0 \quad 1 \quad 0 \quad 0.4030 \quad 0 \quad -0.2916]^T$ $\widehat{\boldsymbol{\beta}}_3=[0 \quad 1 \quad 0 \quad -0.0208 \quad 0 \quad -0.0606]^T$
Separation of stiffness	$\mathbf{K}_2=\sum_{i=1}^{3}(k_{\alpha i}\widehat{\boldsymbol{\alpha}}_i\widehat{\boldsymbol{\alpha}}_i^T)=\begin{bmatrix} 4.960 & 0 & 0 & 0 & -0.2020 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4.960 & 0 & 0.2840 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.2020 & 0 & 0.2840 & 0 & 0.1313 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\times 10^5$ $\mathbf{K}_3=\sum_{i=1}^{3}(k_{\beta i}\widehat{\boldsymbol{\beta}}_i\widehat{\boldsymbol{\beta}}_i^T)=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6.0 & 0 & 0.1800 & 0 & -0.1800 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1500 & 0 & 0.1007 & 0 & 0.0540 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.1800 & 0 & 0.0540 & 0 & 0.1591 \end{bmatrix}\times 10^5$

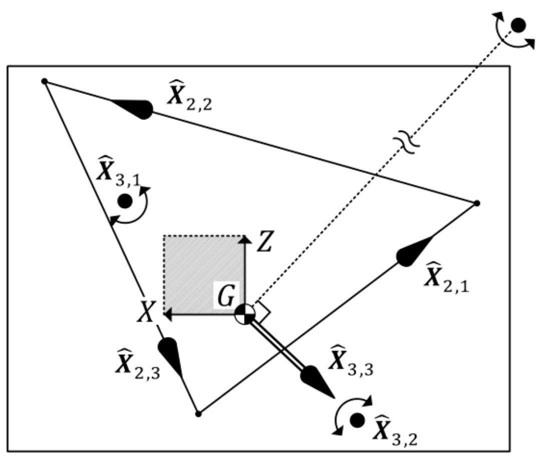

FIGURE 11. Vibration axes lying on the plane of symmetry and ones perpendicular to the plane (•).

demonstrate the general condition for a plane of symmetry. It will be shown that the stiffness matrix of the mounting system illustrated in Fig. 9(a) is separable but not diagonalizable.

Referring to Fig. 9(a), a rigid body is supported by four mounts. Each mount is modelled as three orthogonal line springs intersecting at a point (Fig. 9(b)). The reference frame $G-XYZ$ coincident with the principal axes of inertia and the local co-ordinate frames are aligned with orthogonal springs (Fig. 9(c)). The system parameters are given in Table 4. The stiffness of a mounting element at the corresponding local co-ordinate frame can be computed as

$$\boldsymbol{K}_e=\boldsymbol{j}_e\left[k_e\right]\boldsymbol{j}_e^T=\begin{bmatrix} 1.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.98 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\times 10^5,$$

where $\boldsymbol{j}_e=\left[\widehat{\boldsymbol{s}}_x\ \widehat{\boldsymbol{s}}_y\ \widehat{\boldsymbol{s}}_z\right]\in R^{6\times 3}$ and $[k_e]=\mathrm{diag}\left(k_x,k_y,k_z\right)=\mathrm{diag}\left(1.5,1.5,0.98\right)\times 10^5$ N/m. Here, k_x and k_y are the principal shear stiffnesses, and k_z is the principal compressive stiffness. Also, the co-ordinates of orthogonal three lines can be given by (at the local frame)

$$\widehat{\boldsymbol{s}}_x=\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0\end{bmatrix}^T,$$
$$\widehat{\boldsymbol{s}}_y=\begin{bmatrix}0 & 1 & 0 & 0 & 0 & 0\end{bmatrix}^T,$$
$$\widehat{\boldsymbol{s}}_z=\begin{bmatrix}0 & 0 & 1 & 0 & 0 & 0\end{bmatrix}^T.$$

The stiffness of the ith mount at $G-XYZ$ can be calculated by using the co-ordinate transformation

$$\boldsymbol{K}_{\#i}=\boldsymbol{E}_{\#i}^T\boldsymbol{K}_e\boldsymbol{E}_{\#i}, \tag{41}$$

where $\boldsymbol{E}_{\#i}$ is given by

$$\boldsymbol{E}_{\#i}=\begin{bmatrix}\boldsymbol{R}_{\#i} & \boldsymbol{R}_{\#i}\boldsymbol{D}_{\#i} \\ 0_{3\times 3} & \boldsymbol{R}_{\#i}\end{bmatrix}\in R^{6\times 6}.$$

$\boldsymbol{R}_{\#i}\left(\in R^{3\times 3}\right)$ is the rotation matrix consisting of directional cosines of the reference frame with respect to the ith local

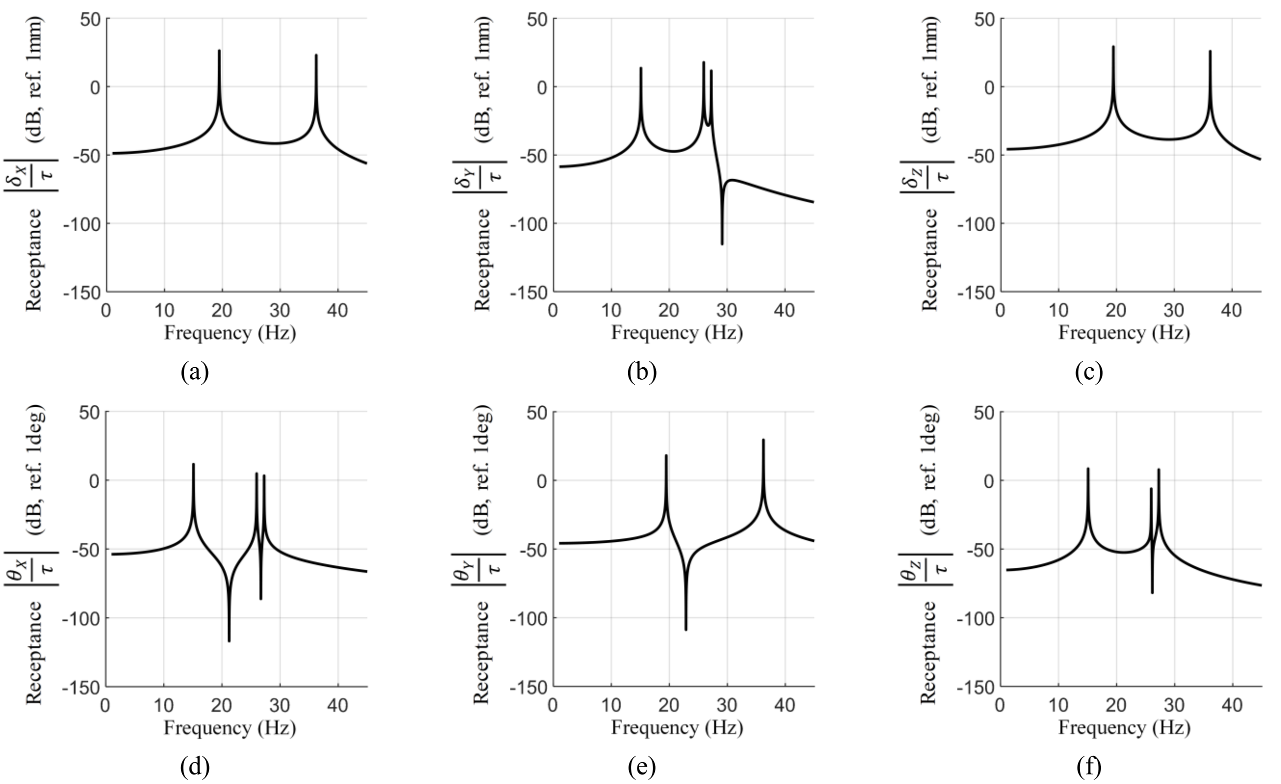

FIGURE 12. Receptance spectra.

frame and $\boldsymbol{D}_{\#i}$ $(\in R^{3\times 3})$ is the skew symmetric matrix of the translational vector $\boldsymbol{d}_{\#i} = \begin{bmatrix} x_{\#i} \ y_{\#i} \ z_{\#i} \end{bmatrix}^T$ to origin of the ith local frame (Fig. 9(c))

$$\boldsymbol{D}_{\#i} = \begin{bmatrix} 0 & z_{\#i} & -y_{\#i} \\ -z_{\#i} & 0 & x_{\#i} \\ y_{\#i} & -x_{\#i} & 0 \end{bmatrix}.$$

Using (41), the overall stiffness matrix $\boldsymbol{K} = \boldsymbol{K}_{\#1} + \boldsymbol{K}_{\#2} + \boldsymbol{K}_{\#3} + \boldsymbol{K}_{\#4}$ is calculated and given in Table 5. The observation of the eigenvalues of $\boldsymbol{K\Delta}$ shows that $\boldsymbol{K}$ satisfies the condition for separation (Table 5). From (19), (20), and (21), we can obtain the unit line vectors $\widehat{\boldsymbol{\alpha}}_i$'s and $\widehat{\boldsymbol{\beta}}_i$'s. Referring to Fig. 10, the geometrical relationship between $\widehat{\boldsymbol{\alpha}}_i$'s and $\widehat{\boldsymbol{\beta}}_i$'s implies that $\boldsymbol{K}$ corresponds to Case2. Thus, $\boldsymbol{K}$ cannot be diagonalized by co-ordinate transformation but it can be separated as $\boldsymbol{K} = \boldsymbol{K}_2 + \boldsymbol{K}_3$.

As shown in Fig. 10, since **AAA** is contained in the ZX-plane and **BBB** is perpendicular to the plane, the vibration system satisfies the conditions for a plane of symmetry. To verify the conditions, the modes of vibration are obtained as

$$\widehat{\boldsymbol{X}}_{2,1} = \begin{bmatrix} 0 & 0.0589 & 0 & -0.8194 & 0 & 0.5732 \end{bmatrix}^T,$$
$$\widehat{\boldsymbol{X}}_{2,2} = \begin{bmatrix} 0 & 0.2453 & 0 & 0.9618 & 0 & 0.2736 \end{bmatrix}^T,$$
$$\widehat{\boldsymbol{X}}_{2,3} = \begin{bmatrix} 0 & 0.0762 & 0 & -0.5063 & 0 & -0.8623 \end{bmatrix}^T,$$
$$\widehat{\boldsymbol{X}}_{3,1} = \begin{bmatrix} -0.0270 & 0 & 0.0380 & 0 & 1 & 0 \end{bmatrix}^T,$$
$$\widehat{\boldsymbol{X}}_{3,2} = \begin{bmatrix} 0.1475 & 0 & -0.2074 & 0 & 1 & 0 \end{bmatrix}^T,$$
$$\widehat{\boldsymbol{X}}_{3,3} = \begin{bmatrix} -0.8149 & 0 & -0.5796 & 0 & 0 & 0 \end{bmatrix}^T.$$

Since all the modes are self-reciprocal, they are pure rotation modes. Referring to Fig. 11, the two groups $(\widehat{\boldsymbol{X}}_{2,1}, \widehat{\boldsymbol{X}}_{2,2}, \widehat{\boldsymbol{X}}_{2,3})$ and $(\widehat{\boldsymbol{X}}_{3,1}, \widehat{\boldsymbol{X}}_{3,2}, \widehat{\boldsymbol{X}}_{3,3})$ are respectively vibration axes of out-of-plane and in-plane modes. It can be said that the ZX-plane becomes the plane of symmetry of the vibration system. The result can also be verified by the frequency response of the rigid body $\widehat{\boldsymbol{X}} \equiv \begin{bmatrix} \delta_X \ \delta_Y \ \delta_Z \ \theta_X \ \theta_Y \ \theta_Z \end{bmatrix}^T$. For the torque with unit magnitude $\boldsymbol{\tau} = \begin{bmatrix} 0.2888 \ 0.9554 \ 0.0620 \end{bmatrix}^T$ applied to the rigid body, the receptance spectrum of each component of $\widehat{\boldsymbol{X}}$ is plotted in Fig. 12. The receptance spectra show that two distinct sets $(\delta_X, \delta_Z, \theta_Y)$, $(\delta_Y, \theta_X, \theta_Z)$ are decoupled. This implies that the modes are decoupled with respect to ZX-plane.

VI. CONCLUSION

In this paper, a geometric approach to the conditions for mode decoupling is proposed, which can be used to significantly simplify the vibration analysis and the synthesis of vibration modes. The major contribution of this research lies in the study of vibration systems of which six vibration modes are determined as line vectors, or pure rotation modes. An important property of the pure rotation mode is self-reciprocity. To derive self-reciprocity, a stiffness matrix that can be separated into sum of two rank 3 stiffness matrices that

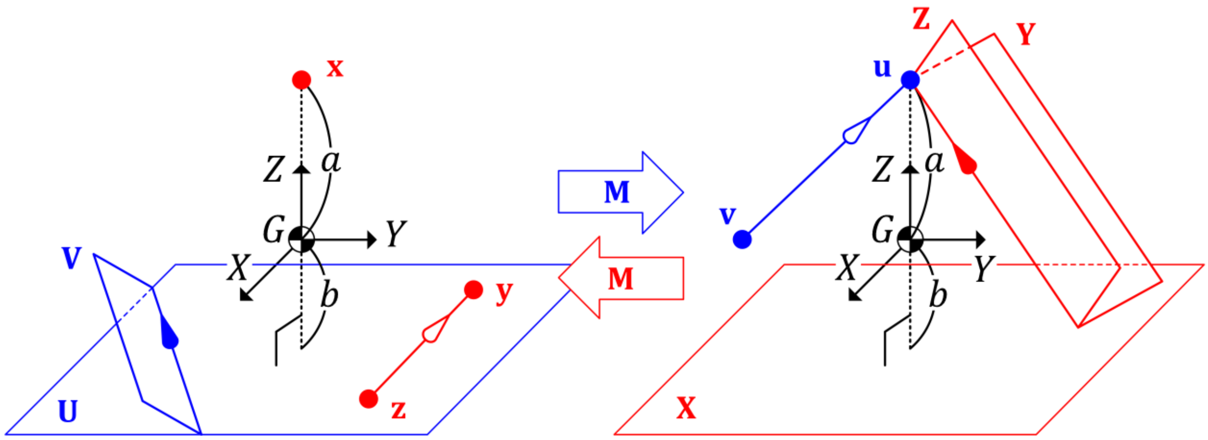

FIGURE 13. Geometrical representation of correlation.

are realizable by co-reciprocal line vectors is investigated. All the possible combinations of a separable stiffness matrix are classified based on the special 3-systems of screws containing only line vectors. The separable stiffness matrix can be viewed as a linear transformation between two self-reciprocal 3-systems of screws. It is shown that the existence of only pure rotation modes depends on the geometrical relationships between the inertia and stiffness matrices, and the three cases of spatial vibration systems with simple geometry are identified. This theory of mode decoupling will be used for the development of a design method of a vibration system in a subsequent study.

APPENDIX

Equation (40) implies that the mass matrix $\boldsymbol{M}$ transforms the lines passing through the point E of **AAA** into the lines lying on the plane of **BBB** and vice versa. In addition, $\boldsymbol{M}$ is a correlation that maps a line expressed in axis co-ordinates into one in ray co-ordinates [25]. The axis co-ordinates of a line are determined by the intersection of two planes. If their homogeneous co-ordinates are given by

$$\mathbf{U} = \begin{bmatrix} U_0 & U_1 & U_2 & U_3 \end{bmatrix}^T,$$
$$\mathbf{V} = \begin{bmatrix} V_0 & V_1 & V_2 & V_3 \end{bmatrix}^T,$$

the axis co-ordinates can be obtained as

$$\widehat{\boldsymbol{S}}_1 = \begin{bmatrix} S_{01} & S_{02} & S_{03} & S_{23} & S_{31} & S_{12} \end{bmatrix}^T,$$

where $S_{ij} = U_i V_j - V_i U_j$. Similarly, for the given homogeneous co-ordinates of two points where

$$\mathbf{x} = \begin{bmatrix} x_0 & x_1 & x_2 & x_3 \end{bmatrix}^T,$$
$$\mathbf{y} = \begin{bmatrix} y_0 & y_1 & y_2 & y_3 \end{bmatrix}^T,$$

the join of two points determines the ray co-ordinates of a line

$$\widehat{\boldsymbol{s}}_1 = \begin{bmatrix} s_{01} & s_{02} & s_{03} & s_{23} & s_{31} & s_{12} \end{bmatrix}^T,$$

where $s_{ij} = x_i y_j - y_i x_j$. From the geometrical meanings of ray and axis co-ordinates, (40) means that the planes containing a line are mapped to the points lying on a line. Suppose that there are two correlations between points (**x**, **y**, **u**, and **v**) and planes (**X**, **Y**, **U**, and **V**) as follows:

$$\mathbf{X} = [\boldsymbol{\pi}]\,\mathbf{x},$$
$$\mathbf{Y} = [\boldsymbol{\pi}]\,\mathbf{y},$$
$$\mathbf{u} = [\boldsymbol{\Pi}]\,\mathbf{U},$$
$$\mathbf{v} = [\boldsymbol{\Pi}]\,\mathbf{V},$$

where $[\boldsymbol{\pi}] \equiv \mathrm{diag}\left(\frac{\sqrt{I}}{m}, \frac{1}{\sqrt{I}}, \frac{1}{\sqrt{I}}, \frac{1}{\sqrt{I}}\right)$ and $[\boldsymbol{\Pi}] \equiv \mathrm{diag}\left(\frac{m}{\sqrt{I}}, \sqrt{I}, \sqrt{I}, \sqrt{I}\right)$. If we let $\widehat{\boldsymbol{S}}_2$ and $\widehat{\boldsymbol{s}}_2$ denote respectively the intersection of **X** and **Y** and the join of **u** and**v**, then they can be written as

$$\widehat{\boldsymbol{S}}_2 = \boldsymbol{M}^{-1}\widehat{\boldsymbol{s}}_1 \text{ and } \widehat{\boldsymbol{s}}_2 = \boldsymbol{M}\widehat{\boldsymbol{S}}_1, \tag{A1}$$

where $\boldsymbol{M}$ =diag(m, m, m, I, I, I). Equation (13) implies that $[\boldsymbol{\pi}]$ maps the points lying on $\widehat{\boldsymbol{s}}_1$ to the planes containing $\widehat{\boldsymbol{S}}_2$ and $[\boldsymbol{\Pi}]$ maps the planes containing $\widehat{\boldsymbol{S}}_1$ to the points on $\widehat{\boldsymbol{s}}_2$.

Now, we prove (40) using $[\boldsymbol{\pi}]$ and $[\boldsymbol{\Pi}]$. When $\boldsymbol{x}$ and $\boldsymbol{U}$ arecoincident with the intersecting point of **AAA** and the plane of **BBB**, without loss of generality, their homogeneous co-ordinates can be expressed as

$$\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 & a \end{bmatrix}^T,$$
$$\mathbf{U} = \begin{bmatrix} b & 0 & 0 & 1 \end{bmatrix}^T,$$

where $ab = \frac{I}{m}$. Thus, we obtain

$$\mathbf{X} = [\boldsymbol{\pi}]\,\mathbf{x} = \left(\frac{a}{\sqrt{I}}\right)\begin{bmatrix} b\ 0\ 0\ 1 \end{bmatrix}^T, \tag{A2a}$$
$$\mathbf{u} = [\boldsymbol{\Pi}]\,\mathbf{U} = \left(b\sqrt{I}\right)\begin{bmatrix} 1 & 0 & 0 & a \end{bmatrix}^T. \tag{A2b}$$

Equation (A2) means that **x** and **U** are coincident with **u** and **X**, respectively, since they are given in terms of homogenous co-ordinates. From (A2b), it can be said that $\boldsymbol{M}$ maps all the lines lying on **U** (expressed in axis co-ordinates) to the lines passing through **u** (expressed in ray co-ordinates) (blue arrows in Fig. 13). This conclusion proves that $\boldsymbol{M}\boldsymbol{\Delta}\widehat{\boldsymbol{x}}_2 \in \mathbf{AAA}$.

In what follows, if $\mathbf{y}$ and $\mathbf{z}$ are arbitrarily selected as the points on $\mathbf{U}$, the incidence of two points $\mathbf{y}$, $\mathbf{z}$ and plane $\mathbf{U}$ yields

$$\mathbf{y}^T\mathbf{U} = 0 \text{ and } \mathbf{z}^T\mathbf{U} = 0. \tag{A3}$$

Since $[\boldsymbol{\pi}][\boldsymbol{\Pi}] = \mathbf{I}_{4\times4}$, (A3) can be rewritten as

$$\mathbf{u}^T\mathbf{Y} = 0 \text{ and } \mathbf{u}^T\mathbf{Z} = 0. \tag{A4}$$

where $\mathbf{Y} = [\boldsymbol{\pi}]\mathbf{y}$ and$\mathbf{Z} = [\boldsymbol{\pi}]\mathbf{z}$. Equation (A4) means that the join of two points on $\mathbf{U}$ is mapped to the intersection of two planes including $\mathbf{u}$ by $\boldsymbol{M}^{-1}$. Since the transformation is one-to-one, it can be said that $\boldsymbol{M}$ maps all the lines passing through $\mathbf{u}$ (expressed in axis co-ordinates) to the lines lying on $\mathbf{U}$ (expressed in ray co-ordinates) (red arrows in Fig. 13). This result proves that $\boldsymbol{M}\boldsymbol{\Delta}\widehat{\boldsymbol{x}}_1 \in \mathbf{BBB}$.

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