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Moving-Window-Based Adaptive Fitting H-Infinity Filter for the Nonlinear System Disturbance

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ABSTRACT The uncertain disturbance in the system signals can lead to biased state estimates and, in turn, can lead to deterioration in the performance of state estimation for a nonlinear dynamic system. In order to address these issues, this paper develops an adaptive fitting H-infinity filter (AFHF) based moving-window by combining the novel noise estimator with fitting H-infinity filtering. Specifically speaking, the novel noise estimator is designed to estimate the process and measurement noise characteristics during a fixed window epoch on the basis of the moving-window technique. Subsequently, the noise characteristics at each window epoch is regarded as the input noise means and covariances of fitting H-infinity filtering at next epoch. Further, the attenuation level is adaptively calculated at each time step to change the structure of AFHF. The Monte-Carlo simulations and INS/GPS integrated navigation experiments are set up for the sake of verifying the superior performance of the proposed filtering with uncertain disturbances.

INDEX TERMS Robust estimation, adaptive fitting H-infinity filter, uncertain system disturbance, the noise estimator.

I. INTRODUCTION

State estimation for the dynamic system is an important research field. Its applications include integrated navigation, fault diagnosis, target tracking, signal processing, information fusion and so on [1]–[3]. Since almost all actual systems inherently involve nonlinearity of one way or another, nonlinear filtering has received considerable attention to estimating a nonlinear system via measurements [4]–[7]. The commonly used nonlinear Kalman-based filters, which are the major research area of state estimation, include the extended Kalman filter (EKF) [8]–[10], unscented Kalman filter (UKF) [9], cubature Kalman filter (CKF) [11]–[13] and particle filter (PF) [14], [15].

EKF is a typical example based on Taylor analytical approximation [8]–[10]. It uses Taylor approximation as the original nonlinear function [8], which is easy to implement.

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However, the use of Taylor approximation causes a model error, making it unsuitable for the cases that the system is not derivable or the linearizable degree of systems is strong [9], [10]. Further, since it is using an approximation at one point to replace the entire random distribution area that ignores the randomness of the state, which affects the filtering performance. In addition, EKF also needs to calculate the complex Jacobian matrix, which is a cumbersome process. Given as above, various extensions of EKF, such as second-order extended Kalman filter [16] and high-gain extended Kalman filter [17], were studied. However, these methods improve the approximate performance of EKF at the expense of increased computational complexity. Both UKF and CKF are based on a numerical approximation to improve EKF with the third-order accuracy for state estimation [18], and also eliminate the cumbersome calculation of Jacobian matrices, leading to a much easier implementation. However, they are unable to deal with non-Gaussian noise.

Based on the Monte-Carlo method, PF approximates the posterior distribution utilizing a cluster of random samples to settle a nonlinear system estimation with non-Gaussian noise. However, its performance depends on the number of randomly selected particles, which is the main barrier to implement PF in practical applications [1]. Further, PF also suffers from the problem of particle degeneracy [14]. Its computational process is much more complicated than that of the above methods [15]. In general, the above filters are based on Bayesian estimation, requiring accurate system models with exact noise statistics.

Robust filtering methods were reported to solve the performance degradation involved in nonlinear Kalman-based filters [18], [19]. The H_∞ filter as a kind of robust filter is based on H_∞ norm as the performance criterion for a variety of system uncertainties [19]–[21]. Different from Kalman-based filters which minimize the variance of estimation errors, the H_∞ filter minimizes the effect of the worst-case disturbances on system estimation [21]. The modeling errors or the system uncertainties are treated as unknown but bounded noises, which is more flexible than the prior condition of noise statistics in Kalman-based filters. However, the H_∞ filter is designed for linear systems, and then unsuitable for nonlinear systems [22].

The extended H_∞ Kalman filter (EHKF) [23] can handle nonlinear systems with unknown-but-bounded noise, however, it suffers from the same disadvantages as EKF. To overcome its problems, unscented transformation, spherical-radial cubature rules and fitting transformation [24] are applied to the structure of EHKF or conventional H_∞ filter, leading to the unscented H_∞ Kalman filter (UHKF) [25], [26], cubature H_∞ Kalman filter (CHKF) [27] and fitting H_∞ filter (FHF) [28].

FHF [28] in the previous paper of authors is a derivative-free nonlinear H_∞ filter based on numerical approximation with a low computational cost. It adopts fitting transformation [24], [28] to approximate the nonlinear system model by minimizing the errors between the nonlinear function and the multivariate fitting function and then applies the structure of the conventional H_∞ filter to estimate the system state. The use of fitting transformation not only avoids complex computation but also provides a numerical coefficient matrix, especially when a nonlinear function lacks an analytical expression [28]. It is also robust to uncertain system disturbance since the use of the upper bound compared with Kalman-based filters. Nevertheless, the estimated bias at each step time can only be partially compensated by the upper bound especially when the mean or covariance of uncertain noise is greatly biased.

In engineering practice, it is common that the system involves errors or uncertainties due to various sources such as sensor manufacturing errors, sensor failures or mechanical disturbances, and various environmental factors such as air resistance, weather conditions and radiation [29]. Further,

the process model is only an approximation to a physical system, and thus it inevitably involves uncertain error [29]. Due to the existence of these uncertain disturbances, the estimated precision of traditional filtering methods is relatively poor or even divergent [30], [31]. For solving the above problems, the adaptive windowing approach is presented to estimate system noise statistics [30]. It applies the innovation or residual item about historical epochs to evaluate the system noise characteristics at the present epoch. It can calculate the noise statistics online and is easy to implement. Wang *et al.* developed a novel H_∞ filter based on the adaptive windowing to handle the unknown prior information of noise statistics for improving the estimation precision [31]. However, the adaptive windowing method combined with the H_∞ filter is only used to improve the estimation accuracy of uncertain linear systems. Meanwhile, the fixed upper bound of H_∞ based filtering that determines the filtering robustness is not always the optimal value. Thus, filtering robustness can be reduced or even lead to filtering interrupt when it is applied to the practical dynamic systems.

In order to solve the above challenges of nonlinear robust estimation, we propose a novel adaptive fitting H_∞ filter (AFHF) based on the moving-window technique. Specifically, the main contributions of this paper are mainly state as follows.

- 1) The novel nonlinear noise estimator is derived under the concept of the moving window technique and FHF. By estimating the process and measurement noise characteristics during a fixed window epoch, it provides the noise means and covariances for state estimation with uncertain system disturbances.
- 2) For a nonlinear system with an uncertain disturbance, AFHF is proposed by combining the novel noise estimator with FHF algorithm, which is to resist the influence due to noise uncertainties involved in a dynamic system. Primarily, the process and measurement noise characteristics at a window epoch provided by the noise estimator are regarded as the input noise means and covariances of FHF at the next time. With the compensation of the noise signals by the noise estimator, the precision of state estimation can be effectively improved for nonlinear dynamic systems with uncertain disturbance. Moreover, the local attenuation level γ_k in AFHF is adaptively calculated at each time step to change the structure of AFHF. This time-varying factor is more flexible than the conventional constant mode. It is used to select the suboptimal upper bound of estimation errors during $[0, k - 1]$. Thus, it is eventually to improve the filtering flexibility and robustness in the estimation process.

Finally, experiments and comparative analyses are conducted to demonstrate the superior precision, robustness and flexibility of the proposed AFHF for nonlinear systems with uncertain disturbances.

II. PRELIMINARIES

A. FITTING H-INFINITY FILTER

Consider the discrete-time nonlinear dynamic state-space model [25] as

$$\begin{cases} \mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k) + \mathbf{w}_k \\ \mathbf{z}_k = \mathbf{h}(\mathbf{x}_k) + \mathbf{v}_k \end{cases} \quad (1)$$

where $\mathbf{x}_k \in \mathbf{R}^n$ is the n-dimensional state; $\mathbf{z}_k \in \mathbf{R}^m$ is the m-dimensional measurement; \mathbf{w}_k is the process noise with non-zero mean and unknown or deterministic covariance \mathbf{Q} as well as bounded energy, i.e., $\sum_{k=0}^{\infty} \mathbf{w}_k^T \mathbf{w}_k < \infty$; \mathbf{v}_k is the measurement noise with non-zero mean and unknown or deterministic covariance \mathbf{R} , as well as bounded energy, i.e., $\sum_{k=0}^{\infty} \mathbf{v}_k^T \mathbf{v}_k < \infty$; \mathbf{w}_k and \mathbf{v}_k are uncorrelated with each other.

Define $\|\mathbf{f}\|_2^2 = \mathbf{f}^T \mathbf{f}$ or $\|\mathbf{f}\|_S^2 = \mathbf{f}^T \mathbf{S} \mathbf{f}$, $\forall \mathbf{f} \in \mathbf{R}^n$, where \mathbf{S} is a symmetric and positive definite matrix. The standard transfer operator (the cost function) of $H\infty$ norm is represented by

$$J_k = \sup_{\mathbf{x}_0, \mathbf{w}, \mathbf{v} \in \ell_2} \frac{\sum_{j=0}^k \|\mathbf{x}_j - \hat{\mathbf{x}}_j\|_2^2}{\|\mathbf{x}_0 - \hat{\mathbf{x}}_0\|_{\mathbf{P}_0}^2 + \sum_{j=0}^k (\|\mathbf{w}_j\|_{\mathbf{Q}}^2 + \|\mathbf{v}_j\|_{\mathbf{R}}^2)} < \gamma_o^2 \quad (2)$$

where $\gamma_o > 0$ is the attenuation level specified by users and \mathbf{P}_0 is the covariance of the state estimation error at the initial time. The operator $\|\cdot\|$ is the Euclidean norm in the n-dimensional space \mathbf{R}^n .

In the numerator of (2), $\mathbf{x}_j - \hat{\mathbf{x}}_j$ is the estimation error of state vector at time j and $\sum_{j=0}^k \|\mathbf{x}_j - \hat{\mathbf{x}}_j\|_2^2$ represents the total energy of estimation errors from the initial time to the time k . In the denominator, $\sum_{j=0}^k (\|\mathbf{w}_j\|_{\mathbf{Q}}^2 + \|\mathbf{v}_j\|_{\mathbf{R}}^2)$ is used to describe the total energy of state and measurement noises from the initial time to the time k and $\|\mathbf{x}_0 - \hat{\mathbf{x}}_0\|_{\mathbf{P}_0}^2$ is the energy of setting error at the initial time, where the vector \mathbf{w}_j and \mathbf{v}_j are disturbances with unknown statistics. At step time k , function cost J_k owns the upper bound γ_o^2 .

The procedure of FHF includes the following steps:

Step1. Update the state and measurement parameters by fitting transformation as shown in Algorithm 1.

$$\begin{cases} \hat{\mathbf{x}}_k = [\hat{\mathbf{x}}_k^T \ 1]^T \\ \bar{\Phi}_k \leftarrow \text{FT}(\mathbf{f}(\cdot), \hat{\mathbf{x}}_k, \mathbf{P}_k) \\ \bar{\mathbf{H}}_k \leftarrow \text{FT}(\mathbf{h}(\cdot), \hat{\mathbf{x}}_k, \mathbf{P}_k) \end{cases} \quad (3)$$

where $\mathbf{f}(\cdot)$ and $\mathbf{h}(\cdot)$ are the state and measurement functions; $\bar{\Phi}_k$ and $\bar{\mathbf{H}}_k$ are the approximate matrices obtained in Algorithm 1.

Step2. The following condition must be satisfied.

$$\mathbf{M}_k = \mathbf{P}_k^{-1} - \gamma_o^{-2} \mathbf{I}_n + \mathbf{H}_k^T \mathbf{R}_k^{-1} \mathbf{H}_k > 0 \quad (4)$$

where γ_o is the attenuation level to adjust the robustness and accuracy of state estimation.

Algorithm 1 Fitting Transformation Algorithm(FT)

Input : $\ell(\cdot)$, $\bar{\mathbf{x}}_0 \in \mathbf{R}^{(n+1) \times 1}$ and $\mathbf{P}_0 \in \mathbf{R}^{n \times n}$

Output : the coefficient matrix $\hat{\bar{\mathbf{A}}}$

1. Generate $2n$ sample points of $\bar{\mathbf{x}}_{00}$ and their weight coefficients

$$[\bar{\mathbf{X}}, \mathbf{W}] = \text{sp}[\bar{\mathbf{x}}_{00}, \mathbf{P}_{00}]$$

where $\bar{\mathbf{X}} = \begin{bmatrix} \mathbf{x}_{00} + \sqrt{\mathbf{P}_{00}} \xi \\ \mathbf{1}_{1 \times 2n} \end{bmatrix}$, $\mathbf{W} = \frac{1}{2n} \mathbf{I}_{2n}$, the unit points are $\xi = [\sqrt{n} \mathbf{I}_n, -\sqrt{n} \mathbf{I}_n]$ and $\mathbf{1}_{1 \times 2n}$ represents a $1 \times 2n$ vector with all elements are 1.

2. The vector \mathbf{Z} is obtained by $\ell(\bar{\mathbf{X}})$ as

$$\mathbf{Z} = [\ell(\bar{\mathbf{x}}^1), \dots, \ell(\bar{\mathbf{x}}^{2n})]$$

where $\bar{\mathbf{X}} = [\bar{\mathbf{x}}^1, \dots, \bar{\mathbf{x}}^{2n}]$ is a $(n+1) \times 2n$ matrix, $\ell(\cdot)$ denotes the analytic function ($\mathbf{f}(\cdot)$ or $\mathbf{h}(\cdot)$) and \mathbf{Z} is a $m \times 2n$ matrix.

3. Estimation of the matrix $\bar{\mathbf{A}}$ by WLS algorithm, i.e.

$$\hat{\bar{\mathbf{A}}} = [(\bar{\mathbf{X}} \mathbf{W} \bar{\mathbf{X}}^T)^{-1} \bar{\mathbf{X}} \mathbf{W} \mathbf{Z}^T]^T$$

Step3. Update the state estimation $\hat{\mathbf{x}}_{k+1}$ and the covariance \mathbf{P}_{k+1} as follow

$$\begin{cases} \mathbf{K}_k = \mathbf{M}_k^{-1} \mathbf{H}_k^T \mathbf{R}_k^{-1} \\ \mathbf{P}_{k+1} = \bar{\Phi}_k \mathbf{M}_k^{-1} \bar{\Phi}_k^T + \mathbf{Q} \\ \hat{\mathbf{x}}_{k+1} = \bar{\Phi}_k \hat{\mathbf{x}}_k + \bar{\Phi}_k \mathbf{K}_k (\mathbf{z}_k - \bar{\mathbf{H}}_k \hat{\mathbf{x}}_k) \end{cases} \quad (5)$$

where Φ_k and \mathbf{H}_k are the numerical Jacobian matrices of $\mathbf{f}(\cdot)$ and $\mathbf{h}(\cdot)$, and they are the matrices $\bar{\Phi}_k$ and $\bar{\mathbf{H}}_k$ without the last column, respectively.

Remark 1: The prerequisite of the FHF filtering process, i.e., $J_k < \gamma_o^2$ described in (2), means that the estimation error $\Delta \mathbf{x}_k$ is bounded. Letting $\gamma_o^2 \rightarrow \infty$, FHF is same as the linear fitting Kalman filter [24]. Some explanations of the sampling sets $\{\bar{\mathbf{X}}, \mathbf{Z}\}$ in FT can be found in [29].

Remark 2: For FHF, $\mathbf{f}(\cdot)$ is known (or unknown) with (without) an analytic expression and $\mathbf{h}(\cdot)$ always known by sensor types. In algorithm 1, $\ell(\cdot)$ denotes the analytic function ($\mathbf{f}(\cdot)$ or $\mathbf{h}(\cdot)$). If $\mathbf{f}(\cdot)$ or $\mathbf{h}(\cdot)$ is known, it can be taken as the input function of FT algorithm, and then the fitting matrix $\bar{\Phi}_k$ or $\bar{\mathbf{H}}_k$ of process or measurement system can be gotten by (3), respectively. If state function $\mathbf{f}(\cdot)$ is unknown, we can obtain its fitting matrix according to Appendix A.

B. PROBLEM DESCRIPTION

It can be seen from the previous section that FHF suffers from the following problems:

a) Assume that the original process noises $\mathbf{w}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{Q})$ ($\mathcal{N}(\mathbf{m}, \mathbf{B})$ represents a normal distribution with the mean \mathbf{m} and the covariance \mathbf{B}), if the process noise statistics are

biased, i.e.

$$\mathbf{w}'_k \sim \mathcal{N}(\mathbf{q}_0, \mathbf{Q}_0) \quad (6)$$

where $\mathbf{Q}_0 = \mathbf{Q} + \Delta\mathbf{Q}$ ($\Delta\mathbf{Q}$ is the positive definite matrix).

Letting $\gamma_o^2 \rightarrow \infty$, the Kalman-based filter is the unbiased method when the exact bias characteristics are known. Substituting (6) into (5) yields

$$\mathbf{P}'_k = \Phi_{k-1}[(\mathbf{P}'_{k-1})^{-1} + \mathbf{H}_{k-1}^T \mathbf{R}^{-1} \mathbf{H}_{k-1}]^{-1} \Phi_{k-1}^T + \mathbf{Q}_0 \quad (7)$$

$$\mathbf{K}'_k = [(\mathbf{P}'_{k-1})^{-1} + \mathbf{H}_{k-1}^T \mathbf{R}^{-1} \mathbf{H}_{k-1}]^{-1} \mathbf{H}_{k-1}^T \mathbf{R}^{-1} \quad (8)$$

$$\hat{\mathbf{x}}'_{k+1} = \bar{\Phi}_k \hat{\mathbf{x}}_k + \mathbf{q}_0 + \Phi_k \mathbf{K}'_k (\mathbf{z}_k - \bar{\mathbf{H}}_k \hat{\mathbf{x}}_k) \quad (9)$$

Comparing (7) and (8) with the first two items in (5), it is evident that \mathbf{K}_k , \mathbf{P}_k , \mathbf{K}'_k and \mathbf{P}'_k are increased by adjusting γ_o and $\Delta\mathbf{Q}$, respectively. To FHF method, it is enlarging the utilization of the fresh measurement \mathbf{z}_k by increasing \mathbf{K}_k in (5). However, comparing the third item of (5) with (9), if the process noise means is biased, the effects of the uncertain vector \mathbf{q}_0 are not fully compensated by $\Phi_k \mathbf{K}_k (\mathbf{z}_k - \bar{\mathbf{H}}_k \hat{\mathbf{x}})$, thus leading to the biased or even divergent solution for FHF with strong robustness.

b) Assume that the original measurement noise $\mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R})$, if the system measurement noise statistics are biased, i.e.

$$\mathbf{v}'_k \sim \mathcal{N}(\mathbf{r}_0, \mathbf{R}_0) \quad (10)$$

where $\mathbf{R}_0 = \mathbf{R} + \Delta\mathbf{R}$ ($\Delta\mathbf{R}$ is the positive definite matrix).

Letting $\gamma_o^2 \rightarrow \infty$, substituting (10) into (5) yields

$$\mathbf{P}''_k = \Phi_{k-1}[(\mathbf{P}''_{k-1})^{-1} + \mathbf{H}_{k-1}^T \mathbf{R}_0^{-1} \mathbf{H}_{k-1}]^{-1} \Phi_{k-1}^T + \mathbf{Q} \quad (11)$$

$$\mathbf{K}''_k = [(\mathbf{P}''_{k-1})^{-1} + \mathbf{H}_{k-1}^T \mathbf{R}_0^{-1} \mathbf{H}_{k-1}]^{-1} \mathbf{H}_{k-1}^T \mathbf{R}_0^{-1} \quad (12)$$

$$\hat{\mathbf{x}}''_{k+1} = \bar{\Phi}_k \hat{\mathbf{x}}_k + \Phi_k \mathbf{K}''_k (\mathbf{z}_k - \bar{\mathbf{H}}_k \hat{\mathbf{x}}_k - \mathbf{r}_0) \quad (13)$$

Similar to the analysis of process noise biased, \mathbf{K}_k and \mathbf{P}_k are definitely increased by adding γ_o in FHF. Comparing (11)-(13) with (5), it is obvious that the excessive use of inaccurate residual term $\mathbf{z}_k - \bar{\mathbf{H}}_k \hat{\mathbf{x}}_k$ is overused by increasing \mathbf{K}_k in FHF method, in which case, it makes the estimated state $\hat{\mathbf{x}}_{k+1}$ far from $\hat{\mathbf{x}}''_{k+1}$ (it is close to the real state \mathbf{x}_{k+1}) when measurements involve uncertain errors, thus leading to the biased or even divergent solution.

c) It is possible that an indefinite quadratic term (4) will not be held, which leading to the filtering failure or poor robustness under the system uncertainty disturbances.

As can be seen from the above analysis, the performance of FHF is decreased since the system noise characteristics or the attenuation value γ_o cannot be updated in real-time. Therefore, it is necessary to construct a novel robust nonlinear adaptive filter, namely the adaptive fitting H-infinity filter (AFHF), which adaptively adjusts the attenuation level γ_o and the statistics of system noises in a practical system.

III. ADAPTIVE FITTING H-INFINITY FILTER

In this section, AFHF as a robust estimation approach is presented for nonlinear discrete-time systems with disturbance

uncertainties. A novel noise estimator is primarily derived from the moving-window technique. The estimated results in each window epoch obtained from the above noise estimator are taken as the statistical characteristics of the process or measurement noises in FHF. Then, the suboptimal attenuation level is designed to adaptively calculate local attenuation level at each step time.

A. THE NOVEL NOISE ESTIMATOR BASED MOVING-WINDOW

According to the nonlinear system (1), the fitting function at time k is as follows:

$$\begin{cases} \mathbf{x}_{k+1} = \bar{\Phi}_k \bar{\mathbf{x}}_k + \mathbf{e}_{x,k-1} + \mathbf{w}_k = \bar{\Phi}_k \bar{\mathbf{x}}_k + \mathbf{w}_{k,o} \\ \mathbf{z}_k = \bar{\mathbf{H}}_k \bar{\mathbf{x}}_k + \mathbf{e}_{z,k-1} + \mathbf{v}_k = \bar{\mathbf{H}}_k \bar{\mathbf{x}}_k + \mathbf{v}_{k,o} \end{cases} \quad (14)$$

Assumed that the process noise $\mathbf{w}_{k,o} \sim \mathcal{N}(\bar{\mathbf{q}}_k, \bar{\mathbf{Q}}_k)$ and measurement noise $\mathbf{v}_{k,o} \sim \mathcal{N}(\bar{\mathbf{r}}_k, \bar{\mathbf{R}}_k)$, and their bounded proof is shown in [31], where $\bar{\mathbf{q}}_k$ and $\bar{\mathbf{r}}_k$ are the mean vectors, $\bar{\mathbf{Q}}_k$ and $\bar{\mathbf{R}}_k$ are the covariance matrices. It is impossible to obtain $\mathbf{w}_{k,o}$ and $\mathbf{v}_{k,o}$ by $\mathbf{w}_{k,o} = \mathbf{x}_{k+1} - \bar{\Phi}_k \bar{\mathbf{x}}_k$ and $\mathbf{v}_{k,o} = \mathbf{z}_k - \bar{\mathbf{H}}_k \bar{\mathbf{x}}_k$ due to unknown the true state vector \mathbf{x}_{k+1} and $\bar{\mathbf{x}}_k$. However, the approximate estimation of nonlinear system noise can be obtained based on the moving-window technique.

Theorem 1: Consider a window of N epochs. Assume that the noise statistics are constant or have minor changes in the window. In order to reduce the uncertainty of nonlinear system noise, the suboptimal unbiased noise estimator can be described as

$$\begin{cases} \hat{\mathbf{r}}_k = \sum_{j=1}^N \mathbf{r}_{k-j}/N \\ \hat{\mathbf{q}}_k = \sum_{j=1}^N \mathbf{q}_{k-j}/N \\ \hat{\mathbf{R}}_k = \sum_{j=1}^N [(\mathbf{r}_{k-j} - \hat{\mathbf{r}})(*)^T - \mathbf{H}_{k-j} \mathbf{P}_{k-j} \mathbf{H}_{k-j}^T]/N \\ \hat{\mathbf{Q}}_k = \sum_{j=1}^N [\mathbf{P}_{k-j+1} + \Phi_{k-j} \mathbf{P}_{k-j} \Phi_{k-j}^T - (\mathbf{q}_{k-j} - \hat{\mathbf{q}})(*)^T]/N \end{cases} \quad (15)$$

where $\mathbf{r}_k = \mathbf{z}_k - \bar{\mathbf{H}}_k \hat{\mathbf{x}}_k$ and $\mathbf{q}_k = \hat{\mathbf{x}}_{k+1} - \bar{\Phi}_k \hat{\mathbf{x}}_k$ are the residual vectors of measurement and process. We write $\mathbf{X}\mathbf{X}^T = \mathbf{X}(\mathbf{*})^T$ and $\mathbf{X}\mathbf{A}\mathbf{X}^T = \mathbf{X}\mathbf{A}(\mathbf{*})^T$ to save space.

Proof: The nonlinear measurement residual is defined by (14) as follows

$$\begin{aligned} \mathbf{r}_k &= \mathbf{z}_k - \bar{\mathbf{H}}_k \hat{\mathbf{x}}_k \\ &= (\bar{\mathbf{H}}_k \bar{\mathbf{x}}_k + \mathbf{v}_{k,o}) - \bar{\mathbf{H}}_k \hat{\mathbf{x}}_k \\ &= [\mathbf{H}_k \mathbf{b}_{m \times 1}] (\mathbf{x}_k^T \mathbf{1})^T - [\hat{\mathbf{x}}_k^T \mathbf{1}]^T + \mathbf{v}_{k,o} \\ &= \mathbf{H}_k (\mathbf{x}_k - \hat{\mathbf{x}}_k) + \mathbf{v}_{k,o} \\ &= \mathbf{H}_k \tilde{\mathbf{x}}_k + \mathbf{v}_{k,o} \end{aligned} \quad (16)$$

where the measurement matrix is $\bar{\mathbf{H}}_k = [\mathbf{H}_k \mathbf{b}_{m \times 1}]$, and the state estimation error is $\tilde{\mathbf{H}}_k = [\mathbf{H}_k \mathbf{b}_{m \times 1}]$.

Similarly, according to (14), the state residual can be defined as

$$\mathbf{w}_{k,o} = \mathbf{x}_{k+1} - \bar{\Phi}_k \bar{\mathbf{x}}_k \quad (17)$$

$$\mathbf{q}_k = \hat{\mathbf{x}}_{k+1} - \bar{\Phi}_k \hat{\mathbf{x}}_k \quad (18)$$

where the state matrix is $\bar{\Phi}_k = [\Phi_k \mathbf{a}_{n \times 1}]$.

The window width is N , namely N measurements within the time interval (t_{k-N}, t_k) . Suppose that system noises $\mathbf{v}_{k-j,o}$ and $\mathbf{w}_{k-j,o}(j = 1, 2, \dots, N)$ are independent sequences, and their means and covariances are both values as $(\bar{\mathbf{r}}_k, \bar{\mathbf{R}}_k)$ and $(\bar{\mathbf{q}}_k, \bar{\mathbf{Q}}_k)$, respectively. The measurement residual \mathbf{r}_k and the state residual \mathbf{q}_k can be used as the approximation of the measurement noise $\mathbf{v}_{k,o}$ and the state noise $\mathbf{w}_{k,o}$. Then, it can be transformed into a simple parameter estimation problem. Assume that $\mathbf{r}_{k-j} \sim \mathcal{N}(\bar{\mathbf{r}}_k, \mathbf{C}_r)$ and $\mathbf{q}_{k-j} \sim \mathcal{N}(\bar{\mathbf{q}}_k, \mathbf{C}_q)$. The residual \mathbf{r}_{k-j} and \mathbf{q}_{k-j} can be obtained by (16) and (18), respectively. Based on statistical knowledge, the expectation of unbiased estimates of $\bar{\mathbf{r}}_k$ and $\bar{\mathbf{q}}_k$ are

$$\begin{aligned} E[\hat{\mathbf{r}}_k] &= E\left(\sum_{j=1}^N \mathbf{r}_{k-j}/N\right) \\ &= \sum_{j=1}^N E(\mathbf{H}_{k-j}\tilde{\mathbf{x}}_{k-j} + \mathbf{v}_{k-j,o})/N \\ &= \sum_{j=1}^N E(\mathbf{H}_{k-j}\tilde{\mathbf{x}}_{k-j} + \bar{\mathbf{r}}_k)/N \\ &= \bar{\mathbf{r}}_k \end{aligned} \quad (19)$$

$$\begin{aligned} E[\hat{\mathbf{q}}_k] &= E\left(\sum_{j=1}^N \mathbf{q}_{k-j}/N\right) \\ &= \sum_{j=1}^N E(\hat{\mathbf{x}}_{k+1-j} - \bar{\Phi}_{k-j}\hat{\mathbf{x}}_{k-j})/N \\ &= \sum_{j=1}^N E(\bar{\Phi}_{k-j}\hat{\mathbf{x}}_{k-j} + \mathbf{q}_{k-j} - \bar{\Phi}_{k-j}\hat{\mathbf{x}}_{k-j})/N \\ &= \bar{\mathbf{q}}_k \end{aligned} \quad (20)$$

where denote that $\hat{\mathbf{r}}_k = \sum_{j=1}^N \mathbf{r}_{k-j}/N$ and $\hat{\mathbf{q}}_k = \sum_{j=1}^N \mathbf{q}_{k-j}/N$ are the unbiased estimates of $\bar{\mathbf{r}}_k$ and $\bar{\mathbf{q}}_k$, respectively.

The covariances \mathbf{C}_r and \mathbf{C}_q can be estimated as

$$\hat{\mathbf{C}}_r = \sum_{j=1}^N (\mathbf{r}_{k-j} - \hat{\mathbf{r}}_k)(\mathbf{r}_{k-j} - \hat{\mathbf{r}}_k)^T / N \quad (21)$$

$$\hat{\mathbf{C}}_q = \sum_{j=1}^N (\mathbf{q}_{k-j} - \hat{\mathbf{q}}_k)(\mathbf{q}_{k-j} - \hat{\mathbf{q}}_k)^T / N \quad (22)$$

Substituting (16) into (21) gives the estimation expectation of \mathbf{C}_r . Primarily, getting the expectation of $(\mathbf{r}_{k-j} - \hat{\mathbf{r}}_k)(\mathbf{r}_{k-j} - \hat{\mathbf{r}}_k)^T$, that is

$$\begin{aligned} &E[(\mathbf{r}_{k-j} - \hat{\mathbf{r}}_k)(*)^T] \\ &= E[\mathbf{r}_{k-j}\mathbf{r}_{k-j}^T - \mathbf{r}_{k-j}\hat{\mathbf{r}}_k^T - \hat{\mathbf{r}}_k\mathbf{r}_{k-j}^T + \hat{\mathbf{r}}_k\hat{\mathbf{r}}_k^T] \\ &= E[(\mathbf{H}_{k-j}\tilde{\mathbf{x}}_{k-j} + \mathbf{v}_{k-j,o})(*)^T - (\mathbf{H}_{k-j}\tilde{\mathbf{x}}_{k-j} + \mathbf{v}_{k-j,o})\hat{\mathbf{r}}_k^T \\ &\quad - \hat{\mathbf{r}}_k(\mathbf{H}_{k-j}\tilde{\mathbf{x}}_{k-j} + \mathbf{v}_{k-j,o})^T + \hat{\mathbf{r}}_k\hat{\mathbf{r}}_k^T] \\ &= E[(\mathbf{H}_{k-j}\tilde{\mathbf{x}}_{k-j})(*)^T + (\mathbf{v}\mathbf{v}_{k-j,o} - \hat{\mathbf{r}}_k)(*)^T \\ &\quad + \mathbf{H}_{k-j}\tilde{\mathbf{x}}_{k-j}\mathbf{v}_{k-j,o}^T + \mathbf{v}_{k-j,o}\tilde{\mathbf{x}}_{k-j}^T\mathbf{H}_{k-j}^T \\ &\quad - \mathbf{H}_{k-j}\tilde{\mathbf{x}}_{k-j}\hat{\mathbf{r}}_k^T - \hat{\mathbf{r}}_k\tilde{\mathbf{x}}_{k-j}^T\mathbf{H}_{k-j}^T] \\ &= E[\mathbf{H}_{k-j}\tilde{\mathbf{x}}_{k-j}\tilde{\mathbf{x}}_{k-j}^T\mathbf{H}_{k-j}^T + (\mathbf{v}_{k-j,o} - \hat{\mathbf{r}}_k)(*)^T] \\ &= \mathbf{H}_{k-j}\mathbf{P}_{k-j}\mathbf{H}_{k-j}^T + \bar{\mathbf{R}}_{k-j} \end{aligned} \quad (23)$$

where $\mathbf{P}_{k-j} = E[\tilde{\mathbf{x}}_{k-j}\tilde{\mathbf{x}}_{k-j}^T]$ is the covariance at time $k-j$.

The estimation expectation of \mathbf{C}_r is

$$\begin{aligned} E[\hat{\mathbf{C}}_r] &= \sum_{j=1}^N E[(\mathbf{r}_{k-j} - \hat{\mathbf{r}}_k)(\mathbf{r}_{k-j} - \hat{\mathbf{r}}_k)^T] / N \\ &= \sum_{j=1}^N (\mathbf{H}_{k-j}\mathbf{P}_{k-j}\mathbf{H}_{k-j}^T + \bar{\mathbf{R}}_{k-j}) / N \end{aligned}$$

$$\begin{aligned} &= \sum_{j=1}^N \mathbf{H}_{k-j}\mathbf{P}_{k-j}\mathbf{H}_{k-j}^T / N + \bar{\mathbf{R}}_k \\ &\neq \bar{\mathbf{R}}_k \end{aligned} \quad (24)$$

The noise statistic estimation of \mathbf{C}_r is biased in (24). However, it provides a way to find the unbiased estimation of measurement noise. It can be rewritten as

$$\begin{aligned} E[\hat{\mathbf{C}}_r] &= \sum_{j=1}^N E[(\mathbf{r}_{k-j} - \hat{\mathbf{r}}_k)(\mathbf{r}_{k-j} - \hat{\mathbf{r}}_k)^T] / N \\ &= \sum_{j=1}^N \mathbf{H}_{k-j}\mathbf{P}_{k-j}\mathbf{H}_{k-j}^T / N + \bar{\mathbf{R}}_k \end{aligned} \quad (25)$$

From (25), the suboptimal unbiased estimation of the measurement variance $\bar{\mathbf{R}}_k$ is

$$\hat{\bar{\mathbf{R}}}_k = \sum_{j=1}^N [(\mathbf{r}_{k-j} - \hat{\mathbf{r}}_k)(\mathbf{r}_{k-j} - \hat{\mathbf{r}}_k)^T - \mathbf{H}_{k-j}\mathbf{P}_{k-j}\mathbf{H}_{k-j}^T] / N \quad (26)$$

Similarly, the estimation of the state covariance $\bar{\mathbf{Q}}_k$ can be obtained. Subtracting (18) from (17) yields

$$\begin{aligned} \mathbf{w}_{k-j,o} - \mathbf{q}_{k-j} &= (\mathbf{x}_{k-j+1} - \hat{\mathbf{x}}_{k-j+1}) - \bar{\Phi}_{k-j}(\tilde{\mathbf{x}}_{k-j} - \hat{\mathbf{x}}_{k-j}) \\ &= \tilde{\mathbf{x}}_{k-j+1} - ([\Phi_{k-j} \mathbf{a}_{n \times 1}]) \left([\mathbf{x}_{k-j}^T \ 1]^T - [\hat{\mathbf{x}}_{k-j}^T \ 1]^T \right) \\ &= \tilde{\mathbf{x}}_{k-j+1} - \Phi_{k-j}\tilde{\mathbf{x}}_{k-j} \end{aligned} \quad (27)$$

Substitute (27) into the covariance of $\mathbf{w}_{k-j,o} - \mathbf{q}_{k-j}$, i.e.

$$\begin{aligned} &E[(\mathbf{w}_{k-j,o} - \mathbf{q}_{k-j})(*)^T] \\ &= E[(\tilde{\mathbf{x}}_{k-j+1} - \Phi_{k-j}\tilde{\mathbf{x}}_{k-j})(\tilde{\mathbf{x}}_{k-j+1} - \Phi_{k-j}\tilde{\mathbf{x}}_{k-j})^T] \\ &= E[\tilde{\mathbf{x}}_{k-j+1}\tilde{\mathbf{x}}_{k-j+1}^T - \tilde{\mathbf{x}}_{k-j+1}\tilde{\mathbf{x}}_{k-j}^T\Phi_{k-j}^T \\ &\quad - \Phi_{k-j}\tilde{\mathbf{x}}_{k-j}\tilde{\mathbf{x}}_{k-j+1}^T + \Phi_{k-j}\tilde{\mathbf{x}}_{k-j}\tilde{\mathbf{x}}_{k-j}^T\Phi_{k-j}^T] \\ &= E[\tilde{\mathbf{x}}_{k-j+1}\tilde{\mathbf{x}}_{k-j+1}^T + \Phi_{k-j}\tilde{\mathbf{x}}_{k-j}\tilde{\mathbf{x}}_{k-j}^T\Phi_{k-j}^T] \\ &= \mathbf{P}_{k-j+1} + \Phi_{k-j}\mathbf{P}_{k-j}\Phi_{k-j}^T \end{aligned} \quad (28)$$

where the estimation errors $\tilde{\mathbf{x}}_{k-j}$ and $\tilde{\mathbf{x}}_{k-j+1}$ are independent of each other, namely $E[\tilde{\mathbf{x}}_{k-j+1}\tilde{\mathbf{x}}_{k-j}^T] = E[\tilde{\mathbf{x}}_{k-j}\tilde{\mathbf{x}}_{k-j+1}^T] = \mathbf{0}$.

The relationship between $\bar{\mathbf{Q}}_k$ and \mathbf{C}_q needs to be constructed. Assuming that $\mathbf{w}_{k-j,o}$ and \mathbf{q}_{k-j} are independent of each other, then $E(\mathbf{w}_{k-j,o})$ and $E(\mathbf{q}_{k-j})$ are equal. Add or subtract $\hat{\mathbf{q}}_k$ from (27) to get the covariance of $\mathbf{w}_{k-j,o} - \mathbf{q}_{k-j}$, that is

$$\begin{aligned} &E[(\mathbf{w}_{k-j,o} - \mathbf{q}_{k-j})(*)^T] \\ &= E[(\mathbf{w}_{k-j,o} - \hat{\mathbf{q}}_k) - (\mathbf{q}_{k-j} - \hat{\mathbf{q}}_k)][*]^T \\ &= E[(\mathbf{w}_{k-j,o} - \bar{\mathbf{q}}_k)(*)^T + (\mathbf{q}_{k-j} - \bar{\mathbf{q}}_k)(*)^T \\ &\quad - (\mathbf{w}_{k-j,o} - \hat{\mathbf{q}}_k)(\mathbf{q}_{k-j} - \hat{\mathbf{q}}_k)^T \\ &\quad - (\mathbf{q}_{k-j} - \hat{\mathbf{q}}_k)(\mathbf{w}_{k-j,o} - \hat{\mathbf{q}}_k)^T] \\ &= \bar{\mathbf{Q}}_{k-j} + \mathbf{C}_q - \mathbf{S}_k \end{aligned} \quad (29)$$

where $E[\mathbf{q}_{k-j}] = \bar{\mathbf{q}}_k$ and $E[\hat{\mathbf{q}}_k] = \bar{\mathbf{q}}_k$. Meanwhile, \mathbf{S}_k can be written as

$$\begin{aligned} \mathbf{S}_k &= E[(\mathbf{w}_{k-j,o} - \hat{\mathbf{q}}_k)(\mathbf{q}_{k-j} - \hat{\mathbf{q}}_k)^T \\ &\quad + (\mathbf{q}_{k-j} - \hat{\mathbf{q}}_k)(\mathbf{w}_{k-j,o} - \hat{\mathbf{q}}_k)^T] \end{aligned}$$

$$\begin{aligned}
 &= (E[\mathbf{w}_{k-j,o} \mathbf{q}_{k-j}^T] - E[\mathbf{w}_{k-j,o}] \bar{\mathbf{q}}_k^T - \bar{\mathbf{q}}_k E[\mathbf{q}_{k-j}^T] + \bar{\mathbf{q}}_k \bar{\mathbf{q}}_k^T) \\
 &\quad + (E[\mathbf{q}_{k-j} \mathbf{w}_{k-j,o}^T] - \bar{\mathbf{q}}_k E[\mathbf{w}_{k-j,o}^T] - E[\mathbf{q}_{k-j}] \bar{\mathbf{q}}_k^T + \bar{\mathbf{q}}_k \bar{\mathbf{q}}_k^T) \\
 &= \mathbf{0} \tag{30}
 \end{aligned}$$

From (28) to (30), it is known that

$$\mathbf{C}_q = \mathbf{P}_{k-j+1} + \Phi_{k-j} \mathbf{P}_{k-j} \Phi_{k-j}^T - \bar{\mathbf{Q}}_{k-j} \tag{31}$$

According to (22) and (31), the estimation expectation of \mathbf{C}_q is

$$\begin{aligned}
 E[\hat{\mathbf{C}}_q] &= \sum_{j=1}^N E[(\mathbf{q}_{k-j} - \hat{\mathbf{q}}_k)(\mathbf{q}_{k-j} - \hat{\mathbf{q}}_k)^T] / N \\
 &= \sum_{j=1}^N (\mathbf{P}_{k-j+1} + \Phi_{k-j} \mathbf{P}_{k-j} \Phi_{k-j}^T - \bar{\mathbf{Q}}_{k-j}) / N \\
 &= \sum_{j=1}^N (\mathbf{P}_{k-j+1} + \Phi_{k-j} \mathbf{P}_{k-j} \Phi_{k-j}^T) / N - \bar{\mathbf{Q}}_k \\
 &\neq \bar{\mathbf{Q}}_k \tag{32}
 \end{aligned}$$

The noise statistic estimation of \mathbf{C}_q is biased in (32). However, it provides a way to find the unbiased estimation of state noise. It can be rewritten as

$$\begin{aligned}
 E[\hat{\mathbf{C}}_q] &= \sum_{j=1}^N E[(\mathbf{q}_{k-j} - \hat{\mathbf{q}}_k)(\mathbf{q}_{k-j} - \hat{\mathbf{q}}_k)^T] / N \\
 &= \sum_{j=1}^N (\mathbf{P}_{k-j+1} + \Phi_{k-j} \mathbf{P}_{k-j} \Phi_{k-j}^T) / N - \bar{\mathbf{Q}}_k \tag{33}
 \end{aligned}$$

Finally, according to (33), the suboptimal unbiased estimation of the covariance $\bar{\mathbf{Q}}_k$ can be obtained as

$$\begin{aligned}
 \hat{\bar{\mathbf{Q}}}_k &= \sum_{j=1}^N [(\mathbf{P}_{k-j+1} + \Phi_{k-j} \mathbf{P}_{k-j} \Phi_{k-j}^T \\
 &\quad - (\mathbf{q}_{k-j} - \hat{\mathbf{q}}_k)(*)^T] / N \tag{34}
 \end{aligned}$$

The suboptimal unbiased statistical estimation of noise can be obtained from (16) to (33), and the proof of Theorem 1 is completed. It is used to reduce the uncertainty of nonlinear system noise. Meanwhile, the window width N is an important parameter for the moving-window estimation. The larger N is selected, the higher precision of nonlinear estimation but with increasing of its computational complexity. In this paper, the selection of N will be achieved through computational tests.

Remark 3: In this novel noise estimator, the process and measurement noise statistics are $(\hat{\mathbf{q}}_k, \hat{\bar{\mathbf{Q}}}_k)$ and $(\hat{\mathbf{r}}_k, \hat{\mathbf{R}}_k)$, respectively. The estimated process and measurement noise are the unbiased mean estimation since the equals $E[\hat{\mathbf{r}}_k] = \bar{\mathbf{r}}_k$ and $E[\hat{\mathbf{q}}_k] = \bar{\mathbf{q}}_k$ are satisfied in (19)-(20). The estimation in (24) and (32) is biased for noise covariance, so the unbiased estimation of noise covariance is designed by (26) and (34). Further, this estimator can be used to provide the noise means and covariances for state estimation with uncertain system disturbances.

B. THE DESIGN OF THE ADAPTIVE FITTING H-INFINITY FILTER

The AFHF algorithm is designed as follows.

Step1. Given the state estimation $\hat{\mathbf{x}}_k$ and the covariance \mathbf{P}_k at step time k .

Step2. Update the state and measurement parameters by Algorithm 1.

$$\begin{cases} \hat{\mathbf{x}}_k = [\hat{\mathbf{x}}_k^T \ \mathbf{1}_{1 \times n}]^T \\ \bar{\Phi}_k \leftarrow \text{FT}(f(\cdot), \hat{\mathbf{x}}_k, \mathbf{P}_k) \\ \bar{\mathbf{H}}_k \leftarrow \text{FT}(h(\cdot), \hat{\mathbf{x}}_k, \mathbf{P}_k) \end{cases} \tag{35}$$

where $\bar{\Phi}_k$ and $\bar{\mathbf{H}}_k$ are numerical fitting matrices of $f(\cdot)$ and $h(\cdot)$ obtained by fitting transformation as shown in Table 1.

Step3. According to (57) in Appendix B, the local attenuation level $\hat{\gamma}_k$ is set as

$$\hat{\gamma}_k = \eta_k \sqrt{\lambda((\mathbf{P}_k^{-1} + \mathbf{H}_k^T \hat{\mathbf{R}}_k^{-1} \mathbf{H}_k)^{-1})} \tag{36}$$

where $\hat{\gamma}_k$ as a positive real number is the local minimum of the attenuation level γ_o , and $\lambda(\mathbf{A})$ denotes the maximum eigenvalue of \mathbf{A} . As it is related to the innovation errors \mathbf{e}_k , its correction coefficient η_k is shown as

$$\eta_k = 1 + \eta_{k-1} * \sqrt{\text{tr}(\mathbf{e}_k^T \mathbf{e}_k)} \tag{37}$$

where $\mathbf{e}_k = \mathbf{z}_k - \bar{\mathbf{H}}_k \hat{\mathbf{x}}_k - \hat{\mathbf{r}}_k$ and $\eta_{k-1} > 0$.

Step4. Update the gain matrix $\bar{\mathbf{K}}_k$.

$$\bar{\mathbf{K}}_k = (\mathbf{P}_k^{-1} - \gamma_k^{-2} \bar{\mathbf{S}}_k + \mathbf{H}_k^T \hat{\mathbf{R}}_k^{-1} \mathbf{H}_k)^{-1} \mathbf{H}_k^T \hat{\mathbf{R}}_k^{-1} \tag{38}$$

Step5. Update the state estimation $\hat{\mathbf{x}}_{k+1}$ and the covariance \mathbf{P}_{k+1} .

$$\begin{cases} \hat{\mathbf{x}}_{k+1} = \bar{\Phi}_k \hat{\mathbf{x}}_k + \Phi_k \bar{\mathbf{K}}_k \mathbf{e}_k + \hat{\mathbf{q}}_k \\ \mathbf{P}_{k+1} = \Phi_k (\mathbf{P}_k^{-1} - \hat{\gamma}_k^{-2} \mathbf{I}_n + \mathbf{H}_k^T \hat{\mathbf{R}}_k^{-1} \mathbf{H}_k)^{-1} \Phi_k^T + \hat{\bar{\mathbf{Q}}}_k \end{cases} \tag{39}$$

where Φ_k and \mathbf{H}_k denote the matrices obtained by removing the last column of $\bar{\Phi}_k$ and $\bar{\mathbf{H}}_k$, respectively.

Step6. Estimate the mean and covariance matrix of noise.

$$\begin{cases} \mathbf{r}_{k-j} = \mathbf{z}_{k-j} - \bar{\mathbf{H}}_k \hat{\mathbf{x}}_{k-j} \\ \mathbf{q}_{k-j} = \hat{\mathbf{x}}_{k-j+1} - \bar{\Phi}_k \hat{\mathbf{x}}_{k-j}, \quad j = 0, 2, \dots, N-1 \end{cases} \tag{40}$$

Then, $\hat{\mathbf{r}}_{k+1}$, $\hat{\mathbf{q}}_{k+1}$, $\hat{\mathbf{R}}_{k+1}$, and $\hat{\bar{\mathbf{Q}}}_{k+1}$ can be obtained by substituting (40) into (15) based on Theorem 1. These estimations are based on the residual sequence $\hat{\mathbf{r}}_{k-j}$ and $\hat{\mathbf{q}}_{k-j}$, which are obtained through FHF and assumed to be statistically independent and identically distributed.

Remark 4: For the performance analysis of AFHF, there are several explanations as follows:

a) The statistical characteristics of the process and measurement noise are estimated by the novel noise estimator, respectively. Specifically speaking, the mean vectors in (40) are regarded as the input vectors of the noise estimator in Theorem 1 to obtain the noise estimated results, i.e. the process noise statistics $(\hat{\mathbf{q}}_{k+1}, \hat{\bar{\mathbf{Q}}}_{k+1})$ and the measurement noise statistics $(\hat{\mathbf{r}}_{k+1}, \hat{\mathbf{R}}_{k+1})$. The noise statistics can be taken as the input noise property in (36), (38) and (39). That is, by making full use of the above noise information, AFHF is dynamically applied to improve the precision of FHF algorithm.

b) Instead of conventional H-infinity filter design, which has a constant attenuation level, the structure of AFHF is changed according to the time-varying attenuation level. The square of attenuation level represents the upper bound of H-infinity norm for estimation error in (2), and its local minimum at time k is $\hat{\gamma}_k$. In the conventional design, the smaller fixed- γ_o causes unavailability of filtering, i.e., $\mathbf{P}_k^{-1} - \hat{\gamma}_o^{-2} \mathbf{I}_n + \mathbf{H}_k^T \hat{\mathbf{R}}_k^{-1} \mathbf{H}_k$ in (38)-(39) is not positive definite. In contrast, the larger fixed γ_o could cause poor robustness, which takes large estimation errors. It indicates that the fixed mode is very inflexible for filtering design. Thus, the adaptive adjustment in AFHF is aimed at estimate the suboptimal upper at each instant time. However, there is no absolute optimal upper bound in (36). To ensure the selection of suboptimal value $\hat{\gamma}_k$, we design an adaptive correction coefficient η_k , which changes over time with the innovation errors e_k . Moreover, sometimes the proposed noise estimator does not ensure the truth of the estimated noise statistics. To deal with this case, AFHF is adaptive to select the suboptimal upper bound of estimation errors by local attenuation level described in (36)-(37), which is eventually to improve the filtering flexibility and robustness.

IV. NUMERICAL SIMULATIONS AND EXPERIMENTS

In this section, the superior of the proposed AFHF is assessed utilizing the univariate nonstationary growth model (UNGM) simulation, the Reentry vehicle tracking system(RVTS) simulation and INS/GPS integrated navigation experiment. The UNGM and RVTS are mainly applied to evaluate high precision and strong robustness of AFHF with noise statistics estimator for the nonlinear system with uncertain disturbances. Finally, the navigation experiment is shown to evaluate its performance in the practical system.

A. SIMULATION IN THE UNGM

The dynamic model for UNGM [5] can be denoted as

$$\begin{cases} x_{k+1} = 0.5x_k + 25x_k/(1 + x_k^2) \\ \quad + 8\cos[1.2(k - 1)] + w_k \\ z_k = x_k^2/20 + v_k \end{cases} \quad (41)$$

Initial state estimation and error covariance are $x_0 = \hat{x}_0 = 0.1$ and $\hat{\mathbf{P}}_0 = 1$, respectively. AFHF in this section is used to compare with FHF, UHKF [25] and adaptive robust unscented Kalman filter (AUKF) [34] for 150 Monte-Carlo runs, and the window width is set to $N = 15$. The attenuation level γ is set as $\gamma^{fhf} = \gamma^{uhkf} = \gamma_0^{afhf} = 10$. Its performance metrics is the root mean squared errors (RMSEs).

Case 1 (Process Noise Estimation): In order to evaluate the performance of AFHF with the estimation process noise, assume that the statistics of the measurement noise is exactly known. It is selected as

$$r_k = 0, \quad \hat{r}_k = 0, \quad R_k = 1, \quad \hat{R}_k = 1$$

The true value and initial estimates of mean and covariance for process noise are set as

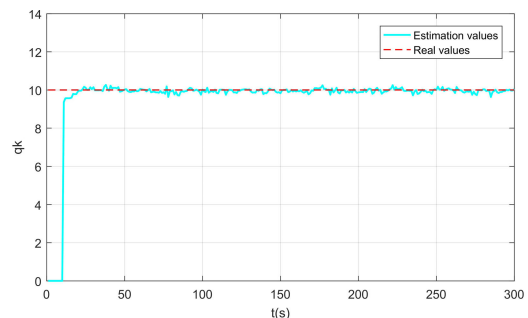
$$q_k = 10, \quad \hat{q}_0 = 0, \quad Q_k = 20, \quad \hat{Q}_0 = 4$$

Case 2 (Measurement Noise Estimation): In order to evaluate the performance of AFHF in the estimation of measurement noise, assume that the statistics of the process noise is exactly known. It is selected as

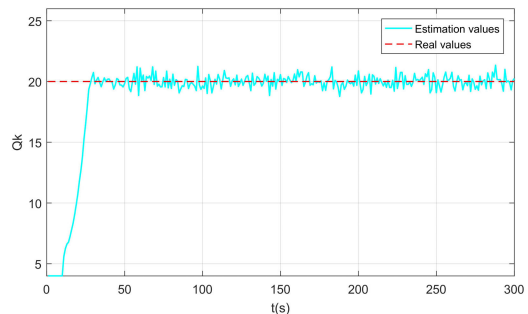
$$q_k = 0, \quad \hat{q}_k = 0, \quad Q_k = 5, \quad \hat{Q}_k = 5$$

The initial estimates of the measurement noise are $\hat{r}_0 = 0$ and $\hat{R}_0 = 1$. The true mean of the measurement noise is $r_k = 10$. Its covariance can be set as

$$R_k = \begin{cases} 5 & k \leq 100 \\ 30 & 100 < k \leq 200 \\ 15 & 200 < k \leq 300 \end{cases}$$



(a) Process noise mean q_k



(b) Process noise covariance Q_k

FIGURE 1. Estimation of the process noise statistics by AFHF in case 1.

For case 1, the estimation of process noise statistics by AFHF in the last Monte-Carlo run is shown in Fig. 1. It is obvious that the values of q_k and Q_k are equal to the initial values \hat{q}_0 and \hat{Q}_0 at the time $k \leq 15$, respectively. The estimation of noise statistics starts running as $k > 15$. This design can be helpful to observe the function of the process noise estimator. As can be seen from Fig. 1, the mean and covariance of process noise gotten by AFHF are closed to the true values after 25s, despite the initial value of process noise is biased. The maximal errors between estimated noise statistics and true statistics (q_k and Q_k) are 0.428 and 1.20, respectively. The RMSE of x_k by AUKF, UHKF, FHF, and AFHF is shown in Fig. 2, which means that the estimation precision

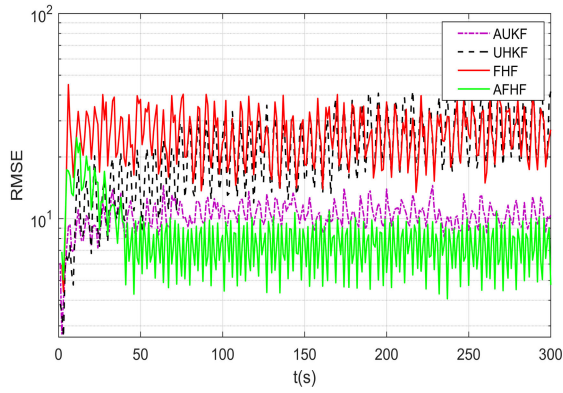


FIGURE 2. RMSE of x_k by different filters in case 1.

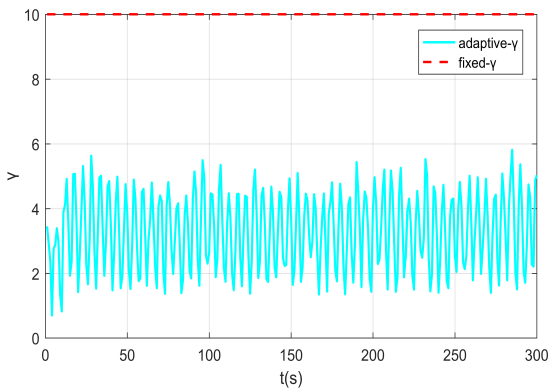
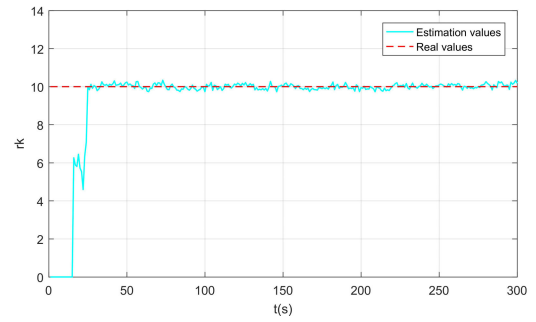


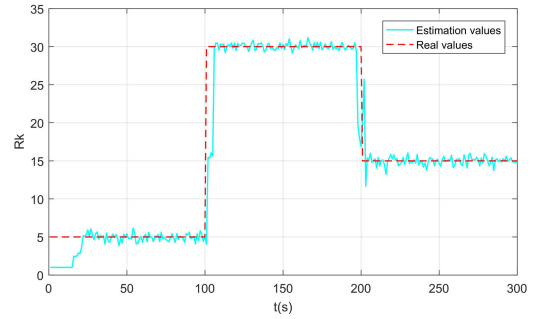
FIGURE 3. Time-varying results of γ in case 1.

of different filters varies with each other. In the stable phase (after 40s), the median values of RMSE for the above filters are 10.25, 21.46, 22.62 and 7.96, respectively. Obviously, the precision of AFHF is higher than other compared algorithms. Further, the time-varying result of γ_k in the last Monte-Carlo run is shown in Fig. 3. It shows that the value of attenuation level in AFHF is adjusted between 0.712 to 5.749, which is different from the fixed solution, i.e., UHKF or FHF based on the highly nonlinear and bimodal in UNGM. Hence, the attenuation level adaptation is to enhance the robustness and flexibility of the fixed factor solutions as other filters in this work.

For case 2, the estimated result of the measurement noise statistics by AFHF in the last Monte-Carlo run is shown in Fig. 4. The estimator of r_k and R_k does not start running until at time $k > 15$, so they are equal to the initial values \hat{r}_0 and \hat{R}_0 as $k \leq 15$, respectively. By observing Fig. 4, despite the initial estimation of measurement noise is biased, the mean and covariance of measurement noise obtained by AFHF are closed to the true values after 25s. The maximal error between estimated noise mean \hat{r}_k and true mean r_k is 0.34 after 25s. The error for covariance R_k is large when the covariance changes greatly, which indicates that the estimation effect is not good at the mutation moments, i.e., 100s and 200s. The estimated errors at 100s and 200s



(a) Measurement noise mean r_k



(b) Measurement noise covariance R_k

FIGURE 4. Estimation of the measurement noise statistics by AFHF in case 2.

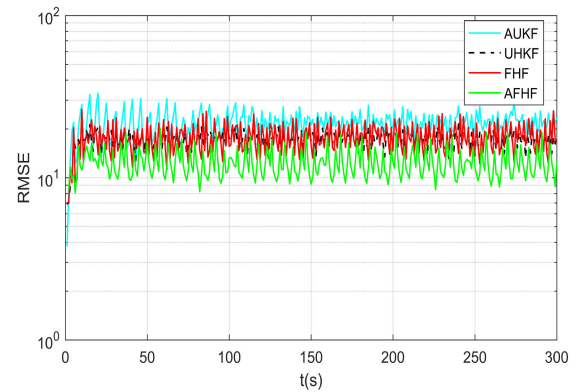


FIGURE 5. RMSEs of x_k by different filters in case 2.

are 13.97 and 10.71, respectively. However, the estimation of measurement statistics could availablely track changes in real statistics. Then, The RMSE of x_k by AUKF, UHKF, FHF, and AFHF is shown in Fig. 5 for case 2. In the stable phase (after 25s), the median values of their RMSE are 21.61, 17.05, 16.1 and 12.91, respectively. It demonstrates that the precision of AFHF is obviously superior to other filters. AUKF may not apply where uncertain disturbance is a gradual change over time like case 2. As seen in Fig. 5, the robustness and stability of AFHF are better than other algorithms, and then its strong robustness is due to the adaptive attenuation level γ_k . Moreover, the time-varying result of attenuation level in the last Monte-Carlo run is shown in Fig. 6. The adjustment of γ_k in Fig. 6 is divided into three stages, namely i) its value is within (2.82, 18.89) at (0s, 100s);

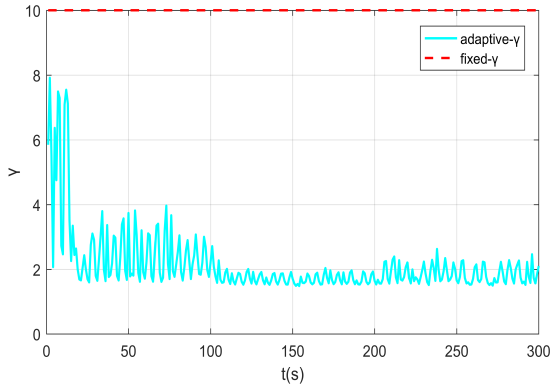


FIGURE 6. Time-varying results of γ in case 2.

ii) its value is within (2.68, 3.51) at (100s, 200s); iii) its value is within (2.76, 4.85) at (200s, 300s). Comparing i) and ii), the factor range in (100s, 200s) is narrowed by increasing R_k . Similarly, compared with ii), the range of γ in (200s, 300s) is enlarged by decreasing R_k . Meanwhile, throughout the whole filtering shown in Fig. 3, the selection of this factor is more flexibility than the fixed mode. Hence, by seeing Fig. 5-6, the attenuation level adaptation is to improve the robustness and flexibility of the fixed factor solutions in this work.

In summary, noise statistics estimator in AFHF can be effective statistics noise information, which improves the estimation precision of the dynamic system. By making full use of the information obtained from the noise estimator, AFHF can be used to improve the filtering precision of nonlinear systems with uncertainty disturbances. Furthermore, adaption of attenuation level in AFHF is applied to enhance the robustness in the estimation process, and then as a time-varying factor is more flexible than the conventional constant mode.

B. SIMULATION IN THE REENTRY VEHICLE TRACKING SYSTEM

A general description of this system is that an aircraft enters the atmosphere at a high altitude and velocity [32]. The position of the vehicle is tracked by radar to measure the distance and bearing angle of the vehicle. The system model is the same as (1). The state vector is defined as

$$\mathbf{x}_k = [x_k^1, x_k^2, x_k^3, x_k^4, x_k^5]^T \quad (42)$$

where (x_k^1, x_k^2) and (x_k^3, x_k^4) denote the position and velocity of the vehicle, and x_k^5 denotes the aerodynamic parameter (aero-param). In order to implement the filters, the continuous process function in [33] should be discretized firstly. In this paper, the nonlinear process function discretized by Euler integration scheme is defined as

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k) + \mathbf{w}_k = \begin{bmatrix} x_k^1 + x_k^3 dt \\ x_k^2 + x_k^4 dt \\ x_k^3 + D_k x_k^3 dt + G_k x_k^1 dt \\ x_k^4 + D_k x_k^4 dt + G_k x_k^2 dt \\ x_k^5 \end{bmatrix} + \mathbf{w}_k \quad (43)$$

where the step size between time steps was dt , $D_k = \beta_k \exp[(R_0 - R_k)/H_0]$ V_k is drag-related force item, $G_k = GM_0/R_k^3$ is gravity-related force item, and $\beta_k = \beta_0 \exp(x_k^5)$, where $R_k = \sqrt{(x_k^1)^2 + (x_k^2)^2}$ is the distance from earth's core to aircraft and $V_k = \sqrt{(x_k^3)^2 + (x_k^4)^2}$ is the total speed. R_0, H_0, β_0 and GM_0 are constant and they are set as

$$R_0 = 6374, \quad H_0 = 13.406, \quad \beta_0 = -0.59783, \\ GM_0 = 3.986 \times 10^5$$

The reference covariances of \mathbf{w}_k is

$$\mathbf{Q} = \text{diag}([10^{-6}, 10^{-6}, 10^{-5}, 10^{-5}, 10^{-6}])$$

The measurement vector is

$$\mathbf{z}_k = [d_k, \theta_k] \quad (44)$$

where d_k is the distance between the vehicle and radar, and θ_k is the bearing angle of the vehicle. The nonlinear measurement function is

$$\mathbf{z}_k = \mathbf{h}(\mathbf{x}_k) + \mathbf{v}_k = \begin{bmatrix} \sqrt{(x_k^1 - x_s)^2 + (x_k^2 - y_s)^2} \\ \tan^{-1}[(x_k^2 - y_s)/(x_k^1 - x_s)] \end{bmatrix} + \mathbf{v}_k \quad (45)$$

where (x_s, y_s) denotes the location of the radar.

The reference covariances of \mathbf{v}_k is

$$\mathbf{R} = \text{diag}([(10^{-3})^2; (0.17 \times 10^{-3})^2])$$

For comparison analysis, simulation trials were conducted by AUKF [34], UHKF [25], FHF and AFHF under the same conditions, i.e.

The attenuation level γ is set as $\gamma^{fhf} = \gamma^{uhkf} = \gamma_0^{afhf} = 5$. Initial state estimate and covariance are

$$\hat{\mathbf{x}}_0 = [6500.4, 349.14, -1.80, 6.79, 0.69]$$

$$\hat{\mathbf{P}}_0 = \text{diag}([10, 10, 10, 10, 0.1])$$

The window size of AFHF is $N = 20$. Then, RMSEs and Averaged RMSEs (ARMSEs) of position, speed and aero-param are selected as the performance standard in this work.

Case 1 (Process With Uncertain Terms): Assume that the unknown term $\Delta \mathbf{x}_k$ is presented in a practical process system after 60s, where $\Delta \mathbf{x} = [-0.1; -0.1; -0.02; -0.02; 0]$. The actual noises of process and measurement are $\mathbf{w}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{Q})$ and $\mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R})$ respectively.

Case 2 (Measurement With Uncertain Terms): We assume that there is the unknown random term \mathbf{v}_0 in the practical measurement system after 30s, where $\mathbf{v}_0 \sim \mathcal{N}(\Delta \mathbf{z}, 4\mathbf{R})$ and $\Delta \mathbf{z} = [0.02; 0.002]$. The other settings are the same as the above situation.

For case 1, the uncertain process term with process noise is considered as a new process noise $\mathbf{w}_{o,k}$. The true uncertain noise $\mathbf{w}_{o,k}$ obeys $\mathbf{w}_{o,k} \sim \mathcal{N}(\mathbf{0}, \mathbf{Q})$ without uncertainty at first 60s, otherwise, it obeys $\mathbf{w}_{o,k} \sim \mathcal{N}(\Delta \mathbf{x}, \mathbf{Q})$ after 60s. The noise statistics estimator in AFHF is to estimate the mean and covariance of the new process noise.

TABLE 1. The mean results of estimated process noise statistics during the time period 60s to 180s.

method	\hat{q}'	\hat{Q}'					
AFHF	$\begin{bmatrix} -0.0439 \\ -0.0414 \\ -0.0184 \\ -0.0206 \\ 0.0011 \end{bmatrix}$	$\begin{bmatrix} 1.0932 \times 10^{-6} & -3.4746 \times 10^{-8} & 1.3145 \times 10^{-7} & -3.3191 \times 10^{-8} & 6.3534 \times 10^{-8} \\ -3.4746 \times 10^{-8} & 1.0118 \times 10^{-6} & -1.1834 \times 10^{-8} & 3.1713 \times 10^{-8} & 2.2898 \times 10^{-9} \\ 1.3145 \times 10^{-7} & -1.1834 \times 10^{-8} & 1.0078 \times 10^{-5} & -2.1357 \times 10^{-8} & 3.0363 \times 10^{-8} \\ -3.3191 \times 10^{-8} & 3.1713 \times 10^{-8} & -2.1357 \times 10^{-8} & 1.0047 \times 10^{-5} & 1.8278 \times 10^{-8} \\ 6.3534 \times 10^{-8} & 2.2898 \times 10^{-9} & 3.0363 \times 10^{-8} & 1.8278 \times 10^{-8} & 8.8554 \times 10^{-7} \end{bmatrix}$					

Table 1 shows the mean results of estimated noise mean and covariance by AFHF during the time periods 60s~180s. The estimated mean of noise mean statistics is \hat{q}' , and it is close to the true noise mean Δx . The estimated mean of noise covariance is \hat{Q}' , and its diagonalization matrix is $\text{diag}([1.04, 1.01, 10.1, 10.1, 0.886] \times 10^{-6})$, which is close to the true noise covariance Q . Hence, comparing Table 1 with the above real noise statistics, it is obvious that the statistical results by AFHF are close to the true statistics. Overall, AFHF with the noise statistics estimator provides the statistics of process uncertain term for state estimation system with case 1. The RMSEs of filters with process uncertain term by 100 Monte Carlo runs are presented in Fig 7.

The median values of RMSEs for AUKF, UHKF, FHF and AFHF after 60s are (11.4m, 115.5m/s, 1.348), (36.4m, 63.6m/s, 0.32), (37.6m, 56.9m/s, 0.258) and (8.2m, 40.9m/s, 0.192), respectively. It is obvious that the precision of the first three filters is affected due to the uncertain process term after 60s, resulting in poor estimation precision. By observing Fig. 7, the precision of AFHF is clearly better than AUKF, UHKF and FHF since AFHF can obtain statistical information of uncertain disturbances with H-infinity based filter.

Moreover, the attenuation level γ in AFHF can be adaptively adjusted according to (36), and then its time-varying results during the last Monte-Carlo run are shown in Fig. 8. Its value is adjusted between 0.007 to 3.293, which means that the robustness of conventional filtering, i.e., FHF or UHKF, is affected by a fixed factor that greater than 3.293 and its availability is affected by a fixed factor that less than 3.293. From Fig. 7-8, the unknown process term is added to the state system at 60s. The robustness of AFHF is stronger than other compared filters since the RMSEs at 61s are smaller than other filters. In Fig. 8, the selection of this factor is more flexibility than the fixed mode. It is evident that the adaption of attenuation level for AFHF is to improve the robustness and flexibility of the fixed attenuation level solutions.

For case 2, the uncertain random term with measurement noise is considered as new measurement noise $v_{o,k}$. The true uncertain noise $v_{o,k}$ obeys $v_{o,k} \sim \mathcal{N}(0, R)$ without uncertainty at first 30s, otherwise, as a mixed Gaussian noise obeys $v_{o,k} \sim \mathcal{N}(\Delta y, 5R)$ after 30s. The noise statistics estimator in AFHF is to estimate the mean and variance of the new measurement noise. Table2 shows the mean value of estimated unknown measurement statistics obtained by AFHF during the time periods 30s~180s. The estimated mean of measurement noise mean statistics is \hat{r}' , and it is

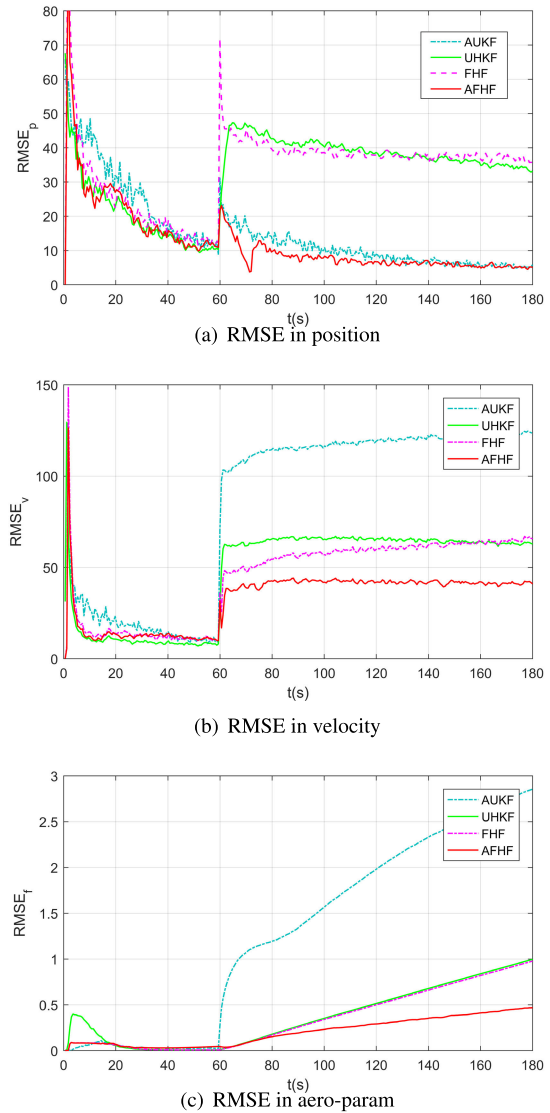


FIGURE 7. Estimated RMSEs of different filters in case 1.

close to the true noise mean Δy . The estimated mean of noise covariance is \hat{R}' , and its diagonalization matrix is $\text{diag}([1.04 \times 10^{-6}, 1.7445 \times 10^{-7}])$, which is close to the true noise covariance $5R$. Its results are similar to case 1, that is, the statistical results by AFHF is close to the true measurement noise statistics. AFHF with the noise statistics

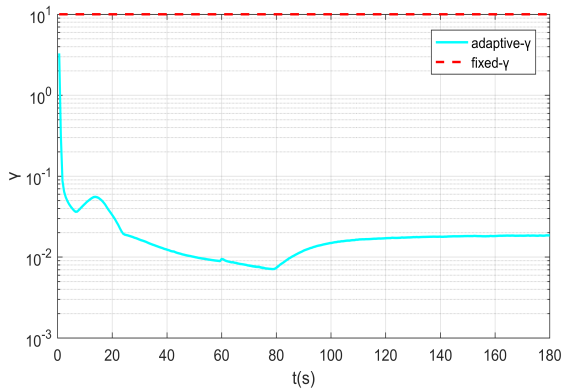


FIGURE 8. Time-varying results of γ in case 1.

TABLE 2. The mean estimation value of uncertainty measurement statistics during the time period 30s to 180s.

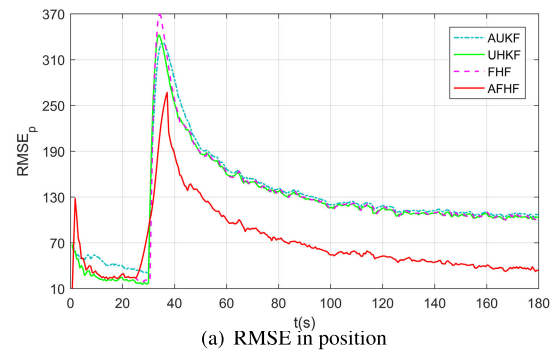
method	$\hat{\gamma}'$	\hat{R}'
AFHF	$\begin{bmatrix} 1.891 \times 10^{-2} \\ 2.140 \times 10^{-3} \end{bmatrix}$	$\begin{bmatrix} 4.9715 \times 10^{-6} & -1.0991 \times 10^{-8} \\ -1.0991 \times 10^{-8} & 1.7445 \times 10^{-7} \end{bmatrix}$

provides the statistics of measurement uncertain term for FHF in case 2.

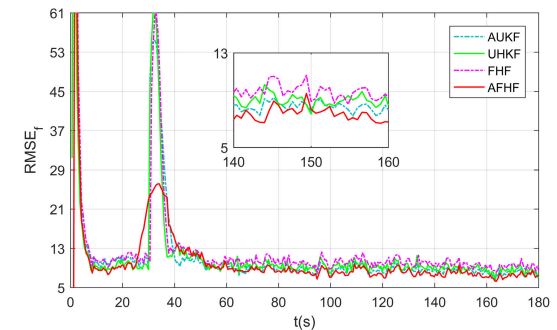
The RMSEs of filters with measurement uncertain term by 100 Monte Carlo runs are presented in Fig. 9. The median RMSEs after 30s of AUKF, UHKF, FHF and AFHF are (116.9m, 8.98m/s, 0.017), (112.7m, 9.33m/s, 0.017), (113.2m, 10.22m/s, 0.017) and (53.5m, 8.4m/s, 0.013), respectively. It is shown that the precision of the first three methods is greatly affected due to the uncertain term in the measurement system, resulting in poor estimation precision. The precision of AFHF is higher than the other filters during the time period 30s to 180s. Moreover, by observing Fig. 9, the peak of mutated RMSEs for AFHF after 30s are (267.1m, 26.12m/s, 0.032), which means that its robustness is stronger than other compared filters. Meanwhile, the estimated errors of AFHF are large during the times (30s, 50s), and then stabilizes at the relatively smaller values.

Further, the time-varying results of attenuation level during the last Monte-Carlo run in case 2 are shown in Fig. 10. Its value has a slow increase after 30s and remains stable at $\gamma = 0.081$ after 60s. From Fig. 9-10, the unknown measurement term is added to the measurement at 30s. The robustness of AFHF is stronger than other compared filters since the RMSEs after 30s are smaller than other filters. Meanwhile, the tendency of RMSEs for AFHF is similar to that of the attenuation level in AFHF. Throughout the filtering process shown in Fig. 10, the selection of this factor is more flexibility than the fixed mode. Hence, the adaption of the attenuation level in AFHF is to modify the robustness and flexibility of the fixed factor solutions, i.e., UHKF or FHF.

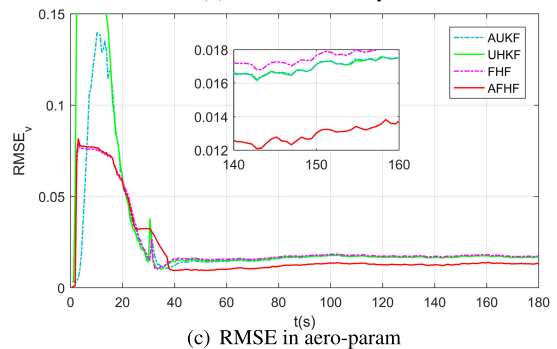
In summary, noise statistics estimator in AFHF is an effective noise estimator and can improve the estimation precision of the dynamic system. Meanwhile, the adaption of



(a) RMSE in position



(b) RMSE in velocity



(c) RMSE in aero-param

FIGURE 9. Estimated RMSEs of filters in case2.

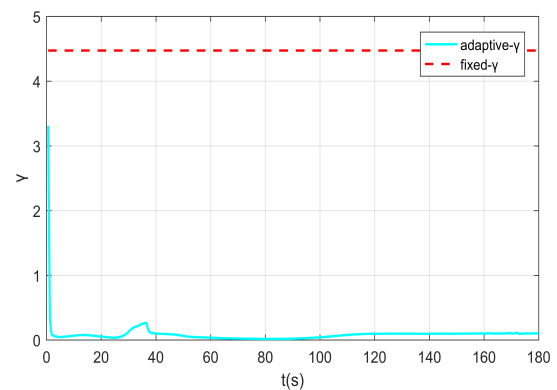


FIGURE 10. Time-varying results of γ in case 2.

the attenuation level in AFHF is employed to increase the robustness in the estimation process. This time-varying factor is more flexible than the conventional constant mode.

C. EXPERIMENT AND ANALYSIS

For the performance evaluation of the proposed AFHF, the practical experiment was also conducted to observe UAV navigation in Xi'an, Shaanxi, China. The UAV applied an INS/GPS integrated system for navigation and location, and its model system can be reference literature [6]. The INS/GPS integrated system consists of an MPU 9250 inertial measurement unit (IMU) and a geo-m8 GPS. The position data (its precision is less than 0.1m) obtained off-line calibration of the camera (SONY ILCE-7R) with the ground control points was taken as the reference values to evaluate the position errors of UAV navigation. The sampling frequency of IMU, GPS and camera were set as 10Hz, 1Hz and 1Hz, respectively. The UAV trajectory is shown in Fig. 11. The window size was set to 20. The other parameter settings of INS/GPS integrated system were displayed in Table 3.

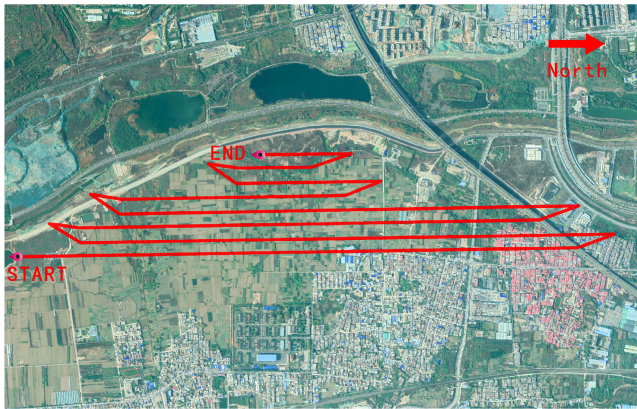


FIGURE 11. The UAV trajectory.

TABLE 3. Experiments parameters.

Initial position	East longitude	109.051°
	North latitude	34.180°
	Altitude	652m
Initial velocity	East	1m/s
	North	18 m/s
	Up	0m/s
Initial attitude	Pitch	-5.28°
	Roll	6.43°
	Yaw	14.59°
Initial position error	East longitude	5m
	North latitude	5m
	Altitude	5m
Initial velocity error	East	0.2 m/s
	North	0.2m/s
	Up	0.4 m/s
Initial attitude error	Pitch	0.3°
	Roll	0.3°
	Yaw	0.5°
Gyro parameters	In-run bias stability	0.05°/s
	Resolution	0.0076°/s
Accelerometer parameters	In-run bias stability	0.01g
	Resolution	6.1 × 10 ⁻⁵ g
GPS parameters	Horizontal position error	5~10m
	Altitude error	<15m
	Sampling frequency	1Hz

Fig.12 illustrates the position errors of the UAV by AUKF, UHKF, FHF and AFHF with period time 900s. During the

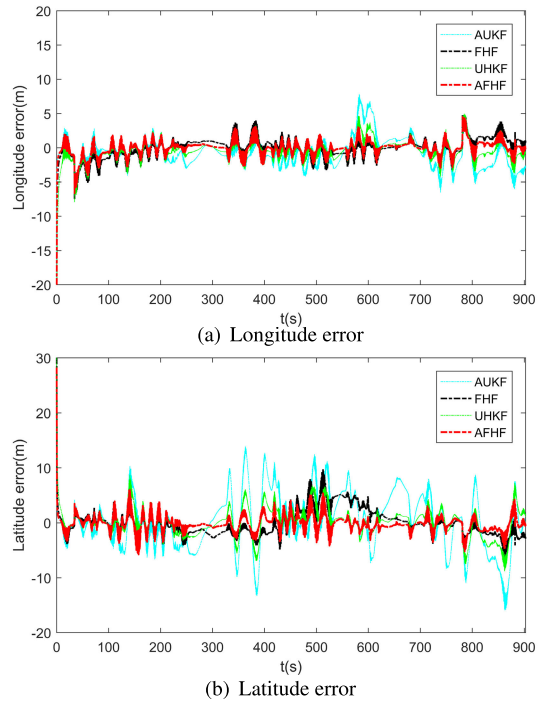


FIGURE 12. The positions errors for the UAV navigation.

TABLE 4. MAE and STD of the position errors by four filters for the UAV navigation.

Filtering methods		position	
		longitude	latitude
AUKF	MAE(m)	1.5563	3.9783
	STD(m)	2.1543	5.0750
UHKF	MAE(m)	1.0701	1.8813
	STD(m)	1.4993	2.3862
FHF	MAE(m)	1.0709	1.8326
	STD(m)	1.4977	2.3622
AFHF	MAE(m)	0.7395	0.9978
	STD(m)	1.0865	1.3779

testing process, the system noise statistics involve uncertainties due to disturbances in the dynamic environment. The position errors of AUKF is within (-7.54m, 7.88m) and (-13.45m, 13.82m), and the obvious oscillations still exist in the filtering curve of AUKF. The position errors of UHKF are within (-7.87m, 4.96m) and (-8.85m, 8.96m). Meanwhile, the position errors by FHF are within (-7.32m, 4.86m) and (-6.79m, 9.61m), which are close to UHKF. They are still disturbed by the uncertainties of noise statistics, resulting in the large magnitude of oscillations in the filtering curve. In contrast, the position errors by the AFHF are within (-6.19m, 4.41m) and (-5.79m, 5.65m). They are smaller than those by the AUKF, UHKF and FHF. The mean absolute errors (MAEs) and standard deviations (STDs) of the position errors by the compared algorithms are listed in Table 4. The MAE and STD of position errors by the proposed AFHF are also much smaller than other methods.

V. CONCLUSION

This study devotes to solving the estimation problems in a nonlinear system with uncertain disturbances. The proposed AFHF is to improve FHF by combining a noise statistic estimator. Firstly, the related theory of noise estimator is presented based on the moving window technique. Secondly, based on the results of the noise estimator in the previous N epochs the AFHF handles the previous problems in the case of uncertain systems. Therefore, by estimating the statistical characteristics of uncertain noises adaptively, AFHF can be employed for the state estimation of nonlinear systems with uncertain disturbances. Then, compared with the conventional constant mode, AFHF is adaptive to calculate the local attenuation level at each time step thus to change the structure of AFHF. Furthermore, it can also guarantee the precision, robustness and flexibility of state estimation in the whole filtering process. Therefore, the AFHF proposed in this paper is a novel and robust adaptive nonlinear filter.

VI. APPENDIXES

APPENDIX A

In algorithm 1, $\ell(\cdot)$ denotes the analytic function ($f(\cdot)$ or $h(\cdot)$). If $f(\cdot)$ or $h(\cdot)$ is known, it can be taken as the input function of FT algorithm, and then the fitting matrix $\bar{\Phi}_k$ or $\bar{\mathbf{H}}_k$ of process or measurement system can be gotten, respectively. If state function $f(\cdot)$ is unknown, we can obtain its fitting matrix through the following steps.

Step 1. Assume that state vectors \mathbf{x}_0 and \mathbf{x}_1 at initial and next time respectively are known with $\mathbf{w}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{Q})$ in advance. According to (1), it can be obtained as [29]

$$\begin{aligned} \mathbf{x}_1^T &= \mathbf{f}^T(\mathbf{x}_0^T) + \mathbf{w}_0^T \\ &= \mathbf{x}_0^T \Phi_0^T + \mathbf{u}_{x,0}^T + (\mathbf{e}_{x,0}^T + \mathbf{w}_0^T) \\ &= \bar{\mathbf{x}}_0^T \bar{\Phi}_0^T + (\mathbf{e}_{x,0}^T + \mathbf{w}_0^T) \end{aligned} \quad (46)$$

where $\bar{\mathbf{x}}_0 = [\mathbf{x}_0^T, 1]^T$, $\bar{\Phi}_0 = [\Phi_0, \mathbf{u}_{x,0}]$ is the approximate coefficient matrix, $\mathbf{u}_{x,0}$ is the constant terms, and $\mathbf{e}_{x,k} \sim \mathcal{N}(\mathbf{0}, \Omega)$ is the approximate error of the state function.

Applying the least square method, one obtains

$$\hat{\Phi}_0 = \left[(\bar{\mathbf{x}}_0 \bar{\mathbf{x}}_0^T)^{-1} \bar{\mathbf{x}}_0^T \bar{\mathbf{x}}_1^T \right]^{-1} \quad (47)$$

The fitting matrix $\bar{\mathbf{H}}_0$ of measurement function $h(\cdot)$ is gotten by the fitting transformation (FT) algorithm, and the state estimation $\hat{\mathbf{x}}_1$ and its covariance \mathbf{P}_1 can be presented by substituting the fitting matrices $\bar{\Phi}_0$ and $\bar{\mathbf{H}}_0$ into (5).

Then, the coefficient matrix is rewritten as

$$\bar{\Phi}_0 = \left[(\bar{\mathbf{x}}_0 \bar{\mathbf{x}}_0^T)^{-1} \bar{\mathbf{x}}_0^T \bar{\mathbf{x}}_1^T \right]^{-1} \quad (48)$$

Thus, the state function is $f(\hat{\mathbf{x}}_1) = \bar{\Phi}_0 \hat{\mathbf{x}}_1 + \mathbf{e}_{x,1}$ by (1).

Step 2. At the next time, $f(\cdot)$, $\hat{\mathbf{x}}_1$ and \mathbf{P}_1 are regarded as inputs of FT algorithm, $\bar{\mathbf{X}}_1$ and \mathbf{Z}_1 are given by

$$\bar{\mathbf{X}}_1 = \left[\hat{\mathbf{x}}_1^1 \cdots, \hat{\mathbf{x}}_1^{2n} \right] = \left[\hat{\mathbf{x}}_1 + \sqrt{\mathbf{P}_1} \xi \right]_{(n+1) \times 2n} \quad (49)$$

$$\begin{aligned} \mathbf{Z}_1 &= \left[f(\hat{\mathbf{x}}_1^1) \cdots, f(\hat{\mathbf{x}}_1^{2n}) \right] \\ &= \left[\bar{\Phi}_0 \bar{\mathbf{X}}_1 + \sqrt{\mathbf{P}_1} \xi + \mathbf{E}_{x,1} \right]_{n \times 2n} \end{aligned} \quad (50)$$

where the unit points are $\xi = [\sqrt{n}\mathbf{I}_n, -\sqrt{n}\mathbf{I}_n]$ and the linear error matrix $\mathbf{E}_{x,1} = [\mathbf{e}_{x,1}, \cdots, \mathbf{e}_{x,1}]_{n \times 2n}$. Therefore, based on weighted least squares (WLS), the coefficient matrix of state function is written as

$$\bar{\Phi}_1 = [(\bar{\mathbf{X}}_1 \mathbf{W} \bar{\mathbf{X}}_1^T)^{-1} \bar{\mathbf{X}}_1 \mathbf{W} \mathbf{Z}_1^T]^T \quad (51)$$

where $\mathbf{W} = \frac{1}{2n} \mathbf{I}_{2n}$ represents the weight coefficients. Similar to step 1, the state estimation $\hat{\mathbf{x}}_2$ and its covariance \mathbf{P}_2 can be given by substituting the fitting matrices $\bar{\Phi}_1$ and $\bar{\mathbf{H}}_1$ into (5).

Step 3. By analogy with the above steps, the state function is $f(\hat{\mathbf{x}}_k) = \bar{\Phi}_{k-1} \hat{\mathbf{x}}_k + \mathbf{e}_{x,k}$ at time k , and then $f(\cdot)$, $\hat{\mathbf{x}}_k$ and \mathbf{P}_k are regarded as inputs of FT algorithm, $\bar{\mathbf{X}}_k$ and \mathbf{Z}_k are presented as

$$\bar{\mathbf{X}}_k = \left[\hat{\mathbf{x}}_k^1 \cdots, \hat{\mathbf{x}}_k^{2n} \right] = \left[\hat{\mathbf{x}}_k + \sqrt{\mathbf{P}_k} \xi \right]_{(n+1) \times 2n} \quad (52)$$

$$\begin{aligned} \mathbf{Z}_k &= \left[f(\hat{\mathbf{x}}_k^1) \cdots, f(\hat{\mathbf{x}}_k^{2n}) \right] \\ &= \left[\bar{\Phi}_{k-1} \bar{\mathbf{X}}_k + \sqrt{\mathbf{P}_k} \xi + \mathbf{E}_{x,k} \right]_{n \times 2n} \end{aligned} \quad (53)$$

where the linear error matrix $\mathbf{E}_{x,k} = [\mathbf{e}_{x,k}, \cdots, \mathbf{e}_{x,k}]_{n \times 2n}$; the mean of $\mathbf{e}_{x,k}$ is the zero vector but its covariance Ω is unknown in practical application, and then $\mathbf{e}_{x,k}$ can be ignored.

Therefore, based on weighted least squares (WLS), the coefficient matrix of a state function is written as

$$\bar{\Phi}_k = [(\bar{\mathbf{X}}_k \mathbf{W} \bar{\mathbf{X}}_k^T)^{-1} \bar{\mathbf{X}}_k \mathbf{W} \mathbf{Z}_k^T]^T \quad (54)$$

Next, the state estimation $\hat{\mathbf{x}}_{k+1}$ and its covariance \mathbf{P}_{k+1} can be gotten by substituting the fitting matrices $\bar{\Phi}_k$ and $\bar{\mathbf{H}}_k$ into (5) in revised manuscript, where $\bar{\mathbf{H}}_k$ can be given by the FT algorithm.

To sum up, if the function $\ell(\cdot)$ at time k is unknown function in (3) of the revised manuscript, it should be the unknown state function $f(\cdot)$, i.e.,

$$\ell(\cdot) = f(\cdot) = \bar{\Phi}_{k-1}(\cdot) + \mathbf{e}_{x,k} \quad (55)$$

where the input of $f(\cdot)$ is $\hat{\mathbf{x}}_k$ at time k ; the mean of $\mathbf{e}_{x,k}$ is the zero vector but its covariance Ω is unknown in practical application, and then $\mathbf{e}_{x,k}$ can be ignored in this paper.

APPENDIX B

Based on the matrix inversion lemma, \mathbf{M}_k in (4) can be written as

$$\mathbf{M}_k = \mathbf{P}_k^{-1} + \mathbf{H}_k^T \mathbf{R}_{k,o}^{-1} \mathbf{H}_k - \gamma_o^{-2} \mathbf{I}_n > 0 \quad (56)$$

where the attenuation level γ_o is to adjust the filtering robustness. For the conventional H-infinity filter, it is a constant value determined by the experience of engineering practices, which is generally not an optimal solution. Meanwhile, there is no closed solution to this optimal problem.

To solve above problem, we design $\gamma_o > \gamma_k$, where γ_k as a suboptimal local solution is presented with adaptive adjustment to improving the filtering performance. Thus, (56) can be obtained as

$$\begin{aligned} \gamma_k^2 &> (\mathbf{P}_k^{-1} + \hat{\mathbf{H}}_k^T \mathbf{R}_{k,o}^{-1} \hat{\mathbf{H}}_k)^{-1} \\ \gamma_k &> \sqrt{\lambda((\mathbf{P}_k^{-1} + \hat{\mathbf{H}}_k^T \mathbf{R}_{k,o}^{-1} \hat{\mathbf{H}}_k)^{-1})} \end{aligned} \quad (57)$$

where $\lambda(\mathbf{A})$ denotes the maximum eigenvalue of \mathbf{A} .

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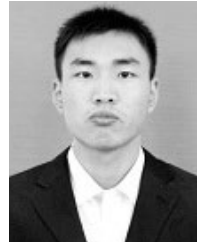
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