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Decay Characteristics of User Dynamics in Online Social Networks

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ABSTRACT Users' interests in trends and news that are attracting attention on social media are generally not retained for a long time but decrease with time. The activity dynamics, such as the gradual fading of interests of users in the social network, have been described as the damped oscillation model in previous studies. Previous studies assumed a damping coefficient (which expresses the strength of damping of the oscillation) as a constant independent of the frequency, but it is known that the damping coefficient in the general oscillation phenomena depends on the frequency. In this study, we define two matrices (the Laplacian matrix and its square root matrix) which give the social network structure and consider possible characteristics of the decay rate that can be derived from natural assumptions concerning the network structure.

INDEX TERMS Online social networks, oscillation model, damped oscillation, user dynamics.

I. INTRODUCTION

Information networks have already penetrated deeply into our social life, and situations have arisen in which "user behavior" via information networks has a major impact on the stable operation of information networks. The advent of the Internet of Things (IoT) in which the billions of physical devices around the world are connected to the internet might further increase the impact of user dynamics in online social networks (OSNs) to activities in the real world. Therefore, it is no longer possible to operate the information network stably only by the framework of information network itself, so we should to consider user dynamics that is activated via information networks, and it is an important research subject. In this paper, the structure of OSNs is modeled by a graph. A node in the graph represents a user, and a link represents interaction among users via social networking services (SNSs), and electronic bulletin boards as well as ordinary telephone calls and e-mails. User dynamics in OSNs is modeled by several ways. The propagation of rumors on OSN or dissemination of new SNSs are known to be modeled using the SIR model as an application of infectious disease models [1]. The process of user opinion convergence via OSN is known as a consensus problem and is modeled using a continuous-time Markov chain in OSNs [2]. In both cases, user dynamics is described by first-order differential equations with respect to time. However, first-order differential equations cannot describe explosive user dynamics like flaming.

As a model for describing user dynamics, the oscillation model of networks has been introduced [3]. This model is based on the wave equation on networks, and it is a second-order differential equation with respect to time. In general, the wave equation can describe the propagation of some influence at a finite speed in medium, and it can describe the propagation of the influence of user activity in OSNs in this case. The interaction between users are described by using a concept of minimal model, which is applicable for various user interaction and is as simple as possible. Although, the oscillation model is introduced by purely theoretical way, the oscillation energy of each node can give a generalized node centrality [4]. That is, the oscillation energy of each node gives an evaluation of node centrality in various usage situations, and reproduces the conventional measures of node centrality (degree centrality [5], betweenness centrality [6]) as special cases. Furthermore, the oscillation model can describe explosive user dynamics like flaming as the divergence of the oscillation energy, and predicts that the emergence of low-frequency beat in the intensity of user activity as an omen of flaming phenomena. Such

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low-frequency beat has been observed in actual measurements [7].

It is natural to think that user dynamics on OSNs decay with time unless external stimulation is added. This phenomenon is described using a damped oscillation in the oscillation model in OSNs. In this paper, we investigate the dependence of the damping coefficient representing the strength of the damping on the oscillation frequency in the oscillation model [8]. In particular, the structure of frequency-dependent damping coefficient is considered throught the consistency with the structure of OSNs.

II. PREPARATION

A. DEFINITION OF THE LAPLACIAN MATRIX

Let us consider a loop-free directed graph $\mathcal{G} = \mathcal{G}(V, E)$ with n nodes as the structure of OSNs: $V = \{1, 2, \dots, n\}$ is the set of nodes and E is the set of directed links. The directed link from node i to node j is expressed by $(i \to j) \in E$. In addition, let the link weight for link $(i \to j)$ be $w_{ij} \geq 0$. Then, the (weighted) adjacency matrix $\boldsymbol{\mathcal{A}} = [\mathcal{A}_{ij}]_{1 \leq i,j \leq n}$ is defines as follows:

$$\mathcal{A}_{ij} := \begin{cases} w_{ij}, & ((i \to j) \in E), \\ 0, & ((i \to j) \notin E). \end{cases} \tag{1}$$

If $w_{ij} = w_{ji}$ for all i and j, $\mathcal{G}$ is a undirected graph, and $\boldsymbol{\mathcal{A}}$ is a symmetric matrix.

Next, we define the weighted out-degree d_i of node i ($i = 1, \dots, n$) as

$$d_i := \sum_{j \in \partial i} w_{ij}, \tag{2}$$

where ∂i denotes the set of nodes adjacent to node i. Degree matrix $\boldsymbol{\mathcal{D}}$ of the weighted out-degree is defined as

$$\boldsymbol{\mathcal{D}} := \mathrm{diag}(d_1, d_2, \dots, d_n).$$

Based on the above preparation, we define the Laplacian matrix $\boldsymbol{\mathcal{L}}$ [9] of the directed graph $\mathcal{G}$ as $\boldsymbol{\mathcal{L}} := \boldsymbol{\mathcal{D}} - \boldsymbol{\mathcal{A}}$.

B. SYMMETRIZATION OF LAPLACIAN MATRIX AND THE SCALED LAPLACIAN MATRIX

The Laplacian matrix $\boldsymbol{\mathcal{L}}$ has the left eigenvector ${}^t\boldsymbol{m}$ associated with the left eigenvalue 0, that is,

$${}^t\boldsymbol{m}\, \boldsymbol{\mathcal{L}} = (0, 0, \dots, 0). \tag{3}$$

For each component m_i of the left eigenvector ${}^t\boldsymbol{m} = (m_1, \dots, m_n)$, we assume the following condition:

$$m_i\, w_{ij} = m_j\, w_{ji}, \tag{4}$$

where $m_i > 0$. We call a network satisfying (4) as a symmetrizable directed graph. It is known that there is no flaming phenomena emerged in symmetrizable directed graphs. So, we focus on symmetrizable directed graphs as the structure of OSNs.

Let the Laplacian matrix for a symmetrizable directed graph be $\boldsymbol{\mathcal{L}}_0$. This is decomposed as

$$\boldsymbol{\mathcal{L}}_0 = \boldsymbol{M}^{-1} \boldsymbol{L}, \tag{5}$$

where the matrix $\boldsymbol{M}$ is defined as $\boldsymbol{M} := \mathrm{diag}(m_1, m_2, \dots, m_n)$, and $\boldsymbol{L}$ is the Laplacian matrix of an undirected graph with link weights $k_{ij} := m_i\, w_{ij}$. From (4), link weights are symmetric $k_{ij} = k_{ji}$.

The Laplacian matrix $\boldsymbol{\mathcal{L}}_0$ for the symmetrizable directed graph is converted to real symmetric matrix $\boldsymbol{S}_0$ using $\boldsymbol{M}$ as follows:

$$\boldsymbol{S}_0 := \boldsymbol{M}^{+1/2}\, \boldsymbol{\mathcal{L}}_0\, \boldsymbol{M}^{-1/2}, \tag{6}$$

where $\boldsymbol{M}^{+1/2} := \mathrm{diag}(\sqrt{m_1}, \sqrt{m_2}, \dots, \sqrt{m_n})$. This means that the scaled Laplacian matrix $\boldsymbol{S}_0$ has the same eigenvalues of $\boldsymbol{\mathcal{L}}_0$ and its eigenvector is expressed as $\boldsymbol{y} := \boldsymbol{M}^{1/2}\boldsymbol{x}$ for the eigenvector $\boldsymbol{x}$ of $\boldsymbol{\mathcal{L}}_0$.

III. THE OSCILLATION MODEL FOR DESCRIBING USER DYNAMICS IN OSNs

This section briefly summarizes an outline of the oscillation model.

A. THE OSCILLATION MODEL

Behaviors of users via OSNs are too complex to give complete descriptions. The key of the oscillation model is based on a minimal model. That is, although interactions between users are various and complex, we try to describe characteristics that many of different types of interactions commonly have, as simple model as possible. The minimal model consists of the rule that expresses the user's state and the rule that describes the interaction between users, as follows:

- **Rule representing users' state** Since user's behavior and thinking are complex, multi-dimensional parameters are required to completely describe their degrees of freedom. However, let us introduce here a one-dimensional parameter $x_i(t)$ to describe the state of user i at time t, as the simplest model.
- **Rule representing user interaction** The state of the user defined above is meaningless as it is, and it becomes meaningful only after the interaction between users is specified. Here, as the interaction between users, let us assume that influence acts so that each other's state becomes equal. In addition, we also assume the strength of the interaction is a monotonically increasing function of the absolute value of difference of the state quantities between users. Note that no interaction occurs when all users are in the same state.

Let the strength of the interaction between users i–j be F_{ij}. Since F_{ij} is a function of $\Delta x_{ij} := |x_i(t) - x_j(t)|$, by using the Taylor expansion, we have

$$F_{ij}(\Delta x_{ij}) = w_{ij}\, \Delta x_{ij} + o(\Delta x_{ij}),$$

where $w_{ij} > 0$ is a constant. Therefore, if $F(\Delta x_{ij})$ is a non-linear function of Δx_{ij}, the linear approximation $F(\Delta x_{ij}) = w_{ij}\, \Delta x_{ij}$ is universally valid for at least $\Delta x_{ij} \ll 1$.

Based on the linear interaction, the equation of motion (EoM) of $x_i(t)$ is given by

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\, x_i(t) = -\sum_{j\in\partial i} w_{ij}\,(x_i(t) - x_j(t)). \tag{7}$$

Here, let the constant w_{ij} be the link weight of a directed link $(i \to j)$ of OSNs. The strength of interaction between user is asymmetric in general, $w_{ij} \neq w_{ji}$, but we assume the OSNs are symmetrizable directed graph whose Laplacian matrix is $\boldsymbol{\mathcal{L}}_0$. Then EoM of the state vector $\boldsymbol{x}(t) := {}^t(x_1(t), \ldots, x_n(t))$ for all users is expressed as

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\, \boldsymbol{x}(t) = -\boldsymbol{\mathcal{L}}_0\, \boldsymbol{x}(t). \tag{8}$$

This is a wave equation on networks, and a basic equation of the oscillation model. By multiplying the equation of motion (8) by $\boldsymbol{M}^{1/2}$ from the left of both sides, we get

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\, \boldsymbol{y}(t) = -\boldsymbol{S}_0\, \boldsymbol{y}(t) \tag{9}$$

that is expressed by the scaled Laplacian matrix $\boldsymbol{S}_0$.

B. DAMPED OSCILLATION MODEL

The influence of general oscillation phenomenon will decay with time unless stimulation is applied from the outside. The oscillation model on networks can also incorporate the damping effect. In order to include the damping effect, we consider an EoM in which the resistance force proportional to the change rate of $\boldsymbol{x}(t)$ in addition to the original EoM.

Assuming that the strength of the resistance force between nodes is the same for any link on the network, the equation of motion of the damped oscillation is obtained by

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\, \boldsymbol{x}(t) + \gamma\, \frac{\mathrm{d}}{\mathrm{d}t}\, \boldsymbol{x}(t) = -\boldsymbol{\mathcal{L}}_0\, \boldsymbol{y}(t), \tag{10}$$

where γ (≥ 0) is the damping coefficient. When we rewrite the equation of motion using $\boldsymbol{S}_0$, we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\, \boldsymbol{y}(t) + \gamma\, \frac{\mathrm{d}}{\mathrm{d}t}\, \boldsymbol{y}(t) = -\boldsymbol{S}_0\, \boldsymbol{y}(t). \tag{11}$$

Let λ_μ $(0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1})$ be n eigenvalues of $\boldsymbol{S}_0$, and let $\boldsymbol{v}_\mu$ be the eigenvector associated with the eigenvalue λ_μ. We choose $\{\boldsymbol{v}_\mu\}_{0\leq\mu\leq n-1}$ as the orthonormal basis, that is, $\boldsymbol{v}_\mu\, \boldsymbol{v}_\nu = \delta_{\mu\nu}$. $\boldsymbol{S}_0$ can be diagonalized using the $n \times n$ orthogonal matrix $\boldsymbol{P} := [\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n-1}]$ as

$$\boldsymbol{\Lambda} := {}^t\boldsymbol{P}\, \boldsymbol{S}_0\, \boldsymbol{P} \tag{12}$$

where the diagonal matrix $\boldsymbol{\Lambda}$ is

$$\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}).$$

Multiplying both sides of the EoM (11) by ${}^t\boldsymbol{P}$ from the left,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}({}^t\boldsymbol{P}\, \boldsymbol{y}(t)) + \gamma\, \frac{\mathrm{d}}{\mathrm{d}t}({}^t\boldsymbol{P}\, \boldsymbol{y}(t)) = -{}^t\boldsymbol{P}\boldsymbol{S}_0\, \boldsymbol{P}\, ({}^t\boldsymbol{P}\boldsymbol{y}(t)). \tag{13}$$

When $\boldsymbol{\psi}(t) := {}^t\boldsymbol{P}\, \boldsymbol{y}(t)$, (13) is rearranged as

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\, \boldsymbol{\psi}(t) + \gamma\, \frac{\mathrm{d}}{\mathrm{d}t}\, \boldsymbol{\psi}(t) = -\boldsymbol{\Lambda}\, \boldsymbol{\psi}(t). \tag{14}$$

The transformed EoM (14) is decomposed into n independent differential equations for each oscillation mode. In other words, for each element $\psi_\mu(t)$ $(\mu = 0, 1, \ldots, n-1)$ of the vector $\boldsymbol{\psi}(t) := {}^t(\psi_0(t), \psi_1(t), \ldots, \psi_{n-1}(t))$, we obtain n independent differential equations as follows:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\, \psi_\mu(t) + \gamma\, \frac{\mathrm{d}}{\mathrm{d}t}\, \psi_\mu(t) = -\lambda_\mu\, \psi_\mu(t). \tag{15}$$

IV. DAMPED OSCILLATION MODEL IN THE CASE THAT THE DAMPING COEFFICIENT DEPENDS ON THE FREQUENCY

A. DAMPED OSCILLATION MODEL DEPENDING ON FREQUENCY

The damping coefficient γ appeared in the EoM (15) is a frequency-independent constant. In many oscillation phenomena around us, it is generally known that the damping coefficient of oscillations is not a constant but depends on the frequency. Here, we theoretically investigate how the damping coefficient should depend on the frequency in the oscillation model for OSNs with damping effects.

Starting from the EoM (15) for each oscillation mode μ, we generalize the EoM with frequency-dependent damping coefficient as follows:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\, \psi_\mu(t) + \gamma(\omega_\mu)\, \frac{\mathrm{d}}{\mathrm{d}t}\, \psi_\mu(t) = -\lambda_\mu\, \psi_\mu(t), \tag{16}$$

where ω_μ is the natural frequency for each oscillation mode, and $\omega_\mu := \sqrt{\lambda_\mu}$. Representing the EoM (16) for each oscillation mode simultaneously in the vector form in accordance with the relationship between (14) and (15), we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\, \boldsymbol{\psi}(t) + \boldsymbol{\Gamma}\, \frac{\mathrm{d}}{\mathrm{d}t}\, \boldsymbol{\psi}(t) = -\boldsymbol{\Lambda}\, \boldsymbol{\psi}(t), \tag{17}$$

where

$$\boldsymbol{\Gamma} := \mathrm{diag}\left(\gamma(\omega_0), \gamma(\omega_1), \ldots, \gamma(\omega_{n-1})\right).$$

By using the relation $\boldsymbol{S}_0 = \boldsymbol{P}\, \boldsymbol{\Lambda}\, {}^t\boldsymbol{P}$ of (12), EoM (17) can be transformed to

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\, \boldsymbol{y}(t) + (\boldsymbol{P}\, \boldsymbol{\Gamma}\, {}^t\boldsymbol{P})\, \frac{\mathrm{d}}{\mathrm{d}t}\, \boldsymbol{y}(t) = -\boldsymbol{S}_0\, \boldsymbol{y}(t). \tag{18}$$

The EoM (11) is a special case of (18) where $\boldsymbol{\Gamma} = \gamma\, \boldsymbol{I}$ ($\boldsymbol{I}$ is the $n \times n$ identity matrix). The EoM (18) can be also expressed using the original Laplacian matrix by

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\, \boldsymbol{x}(t) + (\boldsymbol{M}^{-1/2}\, (\boldsymbol{P}\, \boldsymbol{\Gamma}\, {}^t\boldsymbol{P})\, \boldsymbol{M}^{+1/2})\, \frac{\mathrm{d}}{\mathrm{d}t}\, \boldsymbol{x}(t) = -\boldsymbol{\mathcal{L}}_0\, \boldsymbol{x}(t). \tag{19}$$

B. POLICY FOR DECIDING FREQUENCY-DEPENDENCE OF THE DAMPING COEFFICIENT

In order to investigate the dependence offrequency ω for the damping coefficient $\gamma(\omega)$, we focus on the problem how to choose each element of the diagonal matrix $\boldsymbol{\Gamma}$ in the EoM (16) of damped oscillation.

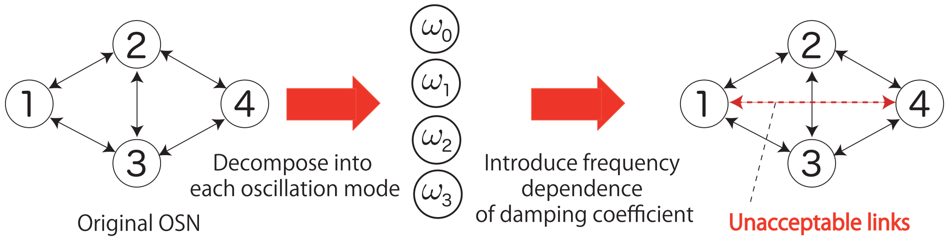

FIGURE 1. Principle for introducing frequency-dependent damping effects in OSNs.

Figure 1 illustrates this problem. This figure shows the operation flow to decompose the EoM with damping effects in the original OSN into EoM for each oscillation mode, to make the damping coefficient dependent the frequency as in (16) and to transform into the EoM corresponding to (19). It is practically unacceptable situations that a components of the matrix $\boldsymbol{M}^{-1/2}\,(\boldsymbol{P}\,\boldsymbol{\Gamma}\,{}^t\!\boldsymbol{P})\,\boldsymbol{M}^{1/2}$ is not 0, despite the corresponding node pair does not have links in OSN. In other words, it is unnatural that some kind of damping force occurs between users who have no direct links. To address this problem, it is natural to adopt the following policy. **Policy 1**: The damping effects between nodes acts only between nodes where links exist in the original OSN structure.

Policy 2: The dependence of the damping coefficient on the natural frequency ω follows the same rule for any OSNs structure.

According to Policy 1, if the non-diagonal elements of $\boldsymbol{P}\,\boldsymbol{\Gamma}\,{}^t\!\boldsymbol{P}$ or $\boldsymbol{M}^{-1/2}\,(\boldsymbol{P}\,\boldsymbol{\Gamma}\,{}^t\!\boldsymbol{P})\,\boldsymbol{M}^{+1/2}$ are nonnegative, the corresponding node pairs should have links in the original OSN. Therefore, for $i \neq j$

$$(\boldsymbol{P}\,\boldsymbol{\Gamma}\,{}^t\!\boldsymbol{P})_{ij} \neq 0 \Rightarrow (\boldsymbol{S}_0)_{ij} < 0, \quad \text{and} \tag{20}$$

$$(\boldsymbol{M}^{-1/2}\,(\boldsymbol{P}\,\boldsymbol{\Gamma}\,{}^t\!\boldsymbol{P})\,\boldsymbol{M}^{+1/2})_{ij} \neq 0 \Rightarrow (\boldsymbol{\mathcal{L}}_0)_{ij} < 0. \tag{21}$$

This means that the damping effects act only between linked nodes in OSNs. According to Policy 2, the condition (20) (or (21)) is required for any network topology.

V. FREQUENCY DEPENDENCE OF DAMPING COEFFICIENT WITH RESPECT TO MATRIX REPRESENTING OSN STRUCTURE

There are two possible ways to define the structure of OSNs. One is by using the Laplacian matrix discussed in the previous sections. The other is by using the square root of the Laplacian matrix. The latter case is related to describe the causality of user dynamics in OSNs. User dynamics changes in accordance with the structure of OSNs. If we want to describe some causal relations that what kind of network structure influences user dynamics as a result, we should use the fundamental equation of user dynamics. In the fundamental equation, the square root of the Laplacian matrix play significant role on behalf of the Laplacian matrix [3]. In this section, we consider the two cases: the structure of OSNs is given whether by the Laplacian matrix or its square root.

A. THE CASE WHERE THE STRUCTURE OF OSNs IS GIVEN BY THE LAPLACIAN MATRIX

When the structure of OSNs is given by the Laplacian matrix, there are two possibilities for deriving the damping matrix $\boldsymbol{\Gamma}$ that satisfies the two conditions in our policy.

- $\boldsymbol{\Gamma}$ is a constant multiple of the identity matrix:

$$\boldsymbol{P}\,\boldsymbol{\Gamma}\,{}^t\!\boldsymbol{P} = \gamma\,\boldsymbol{I}, \quad \text{and}$$
$$\boldsymbol{M}^{-1/2}\,(\boldsymbol{P}\,\boldsymbol{\Gamma}\,{}^t\!\boldsymbol{P})\,\boldsymbol{M}^{+1/2} = \gamma\,\boldsymbol{I},$$

  where γ is a constant (damping coefficient). This corresponds to $\boldsymbol{\Gamma} = \gamma\,\boldsymbol{I}$. In this case, (18) and (19) are equivalent to (11) and (10) of the original damped oscillation, respectively.
- $\boldsymbol{\Gamma}$ is a constant multiple of $\boldsymbol{\Lambda}$ that is diagonalized matrix of the Laplacian matrix:

$$\boldsymbol{P}\,\boldsymbol{\Gamma}\,{}^t\!\boldsymbol{P} = \gamma\,\boldsymbol{S}_0, \quad \text{and}$$
$$\boldsymbol{M}^{-1/2}\,(\boldsymbol{P}\,\boldsymbol{\Gamma}\,{}^t\!\boldsymbol{P})\,\boldsymbol{M}^{+1/2} = \gamma\,\boldsymbol{\mathcal{L}}_0, \tag{22}$$

  where γ is a constant. This means $\boldsymbol{\Gamma} = \gamma\,\boldsymbol{\Lambda}$, and the dependency of the damping coefficient is expressed by

$$\gamma(\omega) = \gamma\,\omega^2. \tag{23}$$

Based on the above discussion so far, for the constants γ_0 and γ_2, the frequency-dependent damping coefficient is expressed as follows:

$$\gamma(\omega) = \gamma_0 + \gamma_2\,\omega^2. \tag{24}$$

When $\boldsymbol{\Gamma} = \gamma_0\,\boldsymbol{I} + \gamma_2\,\boldsymbol{\Lambda}$, it satisfies

$$\boldsymbol{P}\,\boldsymbol{\Gamma}\,{}^t\!\boldsymbol{P} = \gamma_0\,\boldsymbol{I} + \gamma_2\,\boldsymbol{S}_0, \tag{25}$$

$$\boldsymbol{M}^{-1/2}\,(\boldsymbol{P}\,\boldsymbol{\Gamma}\,{}^t\!\boldsymbol{P})\,\boldsymbol{M}^{+1/2} = \gamma_0\,\boldsymbol{I} + \gamma_2\,\boldsymbol{\mathcal{L}}_0, \tag{26}$$

regardless of the network structure.

Here, we demonstrate an example of the frequency dependence of the damping coefficient satisfying the condition (26) using a simple network model. Figure 2 shows an example of OSN model used in the evaluation. The number in each circle denotes node ID, and the number next to each link is the link weight of the corresponding directed link. Note that the directed links ($1 \to 4$) and ($4 \to 1$) do not exist. The directed graph shown in Fig. 2 is the symmetrizable

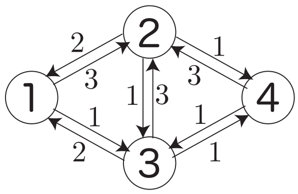

FIGURE 2. An example of the symmetrizable directed graphs.

directed graph, and the Laplacian matrix $\mathcal{L}_0$ is expressed by

$$\mathcal{L}_0 = \begin{bmatrix} 4 & -3 & -1 & 0 \\ -2 & 4 & -1 & -1 \\ -2 & -3 & 6 & -1 \\ 0 & -3 & -1 & 4 \end{bmatrix} \tag{27}$$

and the (1,4)-th and (4,1)-th elements are 0. The diagonal matrix $\mathbf{\Lambda}$ for $\mathcal{L}_0$ is $\mathbf{\Lambda} = \mathrm{diag}(0,\ 4,\ 7,\ 7)$ whose elements are the eigenvalues of $\mathcal{L}_0$.

If the matrix $\mathbf{\Gamma}$ satisfies (25) and (26), we obtain

$$\begin{aligned} &\boldsymbol{M}^{-1/2}\,(\boldsymbol{P}\,\mathbf{\Gamma}\,{}^t\boldsymbol{P})\,\boldsymbol{M}^{+1/2} \\ &= \begin{bmatrix} \gamma_1 + 4\,\gamma_2 & -3\,\gamma_2 & -\gamma_2 & 0 \\ -2\,\gamma_2 & \gamma_1 + 4\,\gamma_2 & -\gamma_2 & -\gamma_2 \\ -2\,\gamma_2 & -3\,\gamma_2 & \gamma_1 + 6\,\gamma_2 & -\gamma_2 \\ 0 & -3\,\gamma_2 & -\gamma_2 & \gamma_1 + 4\,\gamma_2 \end{bmatrix} \end{aligned} \tag{28}$$

The non-diagonal elements having non-zero values completely coincide with the presence of links in OSN.

On the other hand, we demonstrate the case that the matrix $\mathbf{\Gamma}$ does not satisfy (25) and (26), for example, for $\mathbf{\Gamma} = \mathbf{\Lambda}^{1/2}$, we obtain

$$\begin{aligned} &\boldsymbol{M}^{-1/2}\,(\boldsymbol{P}\,\mathbf{\Gamma}\,{}^t\boldsymbol{P})\,\boldsymbol{M}^{+1/2} \\ &= \begin{bmatrix} 1.837 & -0.378 & -1.133 & -0.325 \\ -0.378 & 2.268 & -1.133 & -0.756 \\ -0.378 & -0.378 & 1.512 & -0.756 \\ -0.163 & -0.378 & -1.133 & 1.675 \end{bmatrix} \end{aligned} \tag{29}$$

In this case, the (1, 4)-th and (4, 1)-th components are non-zero, and the condition of Policy 1 is not satisfied. In other words, it means that the damping effect has appeared between nodes where no link exists.

Although the network model used here is very simple, results of the demonstration are not changed even in large scale networks. The structure of OSNs is sparse in general, that is, almost all node pairs does not have link. However, if the conditions (25) and (26) are not satisfied, the matrix $\boldsymbol{M}^{-1/2}\,(\boldsymbol{P}\,\mathbf{\Gamma}\,{}^t\boldsymbol{P})\,\boldsymbol{M}^{+1/2}$ becomes the complete graph in general.

B. THE CASE WHERE THE STRUCTURE OF OSNs IS GIVEN BY THE SQUARE ROOT OF THE LAPLACIAN MATRIX

For a square matrix $\boldsymbol{B}$, a matrix $\boldsymbol{C}$ that satisfies $\boldsymbol{B} = \boldsymbol{C}^2$ is called a square root matrix of the matrix $\boldsymbol{B}$. When the matrix $\boldsymbol{B}$ is a positive semidefinite matrix, the square root matrix $\boldsymbol{C}$ that is a positive semidefinite matrix is unique.

Since the scaled Laplacian matrix $\boldsymbol{S}_0$ of the symmetrizable directed graph is the symmetric and semi-definite matrix, $\boldsymbol{S}_0$ can be diagonalized by using the orthogonal matrix $\boldsymbol{P}$ as shown in (12). Let the square root of $\boldsymbol{S}_0$ and $\mathcal{L}_0$ be $\boldsymbol{H}_0$ and $\mathcal{H}_0$, respectively. By using the square root of $\mathbf{\Lambda}$, that is, $\mathbf{\Omega} := \mathbf{\Lambda}^{1/2}$, the square root of $\boldsymbol{S}_0$ can be obtained as

$$\boldsymbol{H}_0 := \boldsymbol{P}\,\mathbf{\Omega}\,{}^t\boldsymbol{P}.$$

Similarly, the square root of the original Laplacian matrix $\mathcal{L}_0$ can be obtained as

$$\mathcal{H}_0 = \boldsymbol{M}^{-1/2}\,\boldsymbol{H}_0\,\boldsymbol{M}^{+1/2} = \boldsymbol{M}^{-1/2}\,\boldsymbol{P}\,\mathbf{\Omega}\,{}^t\boldsymbol{P}\,\boldsymbol{M}^{+1/2}. \tag{30}$$

This is verified as

$$\begin{aligned} \mathcal{H}_0^2 &= (\boldsymbol{M}^{-1/2}\,\boldsymbol{P}\,\mathbf{\Omega}\,{}^t\boldsymbol{P}\,\boldsymbol{M}^{+1/2})\,(\boldsymbol{M}^{-1/2}\,\boldsymbol{P}\,\mathbf{\Omega}\,{}^t\boldsymbol{P}\,\boldsymbol{M}^{+1/2}) \\ &= \boldsymbol{M}^{-1/2}\,\boldsymbol{P}\,\mathbf{\Lambda}\,{}^t\boldsymbol{P}\,\boldsymbol{M}^{+1/2} = \boldsymbol{M}^{-1/2}\,\boldsymbol{S}_0\,\boldsymbol{M}^{+1/2} = \mathcal{L}_0. \end{aligned}$$

The reason why the square root of the Laplacian matrix gives a more fundamental notion than the Laplacian matrix is as follows. If the structure of OSNs is defined by the Laplacian matrix, most of the non-diagonal components of the Laplacian matrix are zero due to the sparsity of the link structure. However, the square root of the Laplacian matrix is not sparse and all components become non-zero, in general. Despite the structure of OSNs is sparse, the square root of the Laplacian matrix, which gives causal relation of user dynamics, is dense, and the fact is unacceptable. On the other hand, if the structure of OSNs is defined by the square root of the Laplacian matrix, we can choose the square root as a sparse matrix. Then the Laplacian matrix is also sparse.

The square root matrix $\mathcal{H}_0$ given by (30) is equal to $\boldsymbol{M}^{-1/2}\,(\boldsymbol{P}\,\mathbf{\Gamma}\,{}^t\boldsymbol{P})\,\boldsymbol{M}^{+1/2}$ when $\mathbf{\Gamma} = \mathbf{\Omega}$, so both conditions of Policy 1 and Policy 2 are satisfied. Therefore, when the social network structure is given by the square root matrix $\mathcal{H}_0$, the condition for the damping coefficient is

$$\gamma(\omega) = \gamma_0 + \gamma_1\,\omega + \gamma_2\,\omega^2 \tag{31}$$

for the constants γ_0, γ_1, γ_2. This is because as the matrix representation

$$\mathbf{\Gamma} = \gamma_0\,\boldsymbol{I} + \gamma_1\,\mathbf{\Omega} + \gamma_2\,\mathbf{\Lambda},$$

the following relations holds:

$$\boldsymbol{P}\,\mathbf{\Gamma}\,{}^t\boldsymbol{P} = \gamma_0\,\boldsymbol{I} + \gamma_1\,\boldsymbol{H}_0 + \gamma_2\,\boldsymbol{S}_0, \tag{32}$$

$$\boldsymbol{M}^{-1/2}\,(\boldsymbol{P}\,\mathbf{\Gamma}\,{}^t\boldsymbol{P})\,\boldsymbol{M}^{+1/2} = \gamma_0\,\boldsymbol{I} + \gamma_1\,\mathcal{H}_0 + \gamma_2\,\mathcal{L}_0, \tag{33}$$

regardless of the network structure.

VI. CONCLUSION

Many of the oscillation phenomena that occur around us decay with time unless an external stimulus is applied. We have described the damped oscillation model that incorporates the effect of damping into the model that represents user interaction. In addition, considering the case where the damping coefficient in the model depends on the frequency, we discussed the reasonable possibility of the damping coefficient for each method of giving the network structure with the Laplacian matrix and its square root matrix. As a result, we can recognized that the frequency dependence of the damping factor is restricted by the link structure of OSNs.

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