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Depth-L Nyquist (M) Filters and Biorthogonal Partners

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ABSTRACT In the past, Nyquist filters have been studied by many researchers. A pair of filters are said to be biorthogonal partners of each other if their cascade forms a Nyquist filter. The theory and design of traditional biorthogonal partners have been studied extensively. In this paper, we extend these results to depth-L Nyquist filters and biorthogonal partners. Two potential applications of them are discussed in fractionally spaced equalizer (FSE) and filter bank multicarrier (FBMC) systems. Moreover, the necessary and sufficient condition is derived for the existence of a finite impulse response (FIR) depth-L biorthogonal partner. We find out that the existence depends on the cardinality of the largest so-called congruous-zero set. In addition, we will show how to design these filters by using the eigenfilter method. Finally, performance comparisons are carried out to demonstrate the advantage of depth-L biorthogonal partners. It is shown that the depth-L version is more robust against inter-symbol interference (ISI) and timing synchronization error compared to the traditional biorthogonal partner.

INDEX TERMS Biorthogonal partner, filter bank multicarrier, fractionally spaced equalizer, Nyquist filter.

I. INTRODUCTION

A digital filter $P(z)$ is called a Nyquist(M) filter¹ if its impulse response $p(n)$ satisfies the property [1], [2]

$$p(Mn) = \delta(n) \quad (\text{Nyquist}(M) \text{ property}), \quad (1)$$

where $\delta(n) = 1$ if $n = 0$ and 0 if $n \neq 0$. Nyquist(M) filters have found many applications in perfect reconstruction filter banks, nonuniform sampling, interpolation, communications and so on. Various design methods for Nyquist(M) filters have been proposed in [3]–[6], [7], [8]. The theory of biorthogonal partners was first developed in [9] for the single-input-single-output (SISO) case. Two digital filters $H(z)$ and $F(z)$ are said to be biorthogonal partners of each other with respect to an integer M if their cascade $H(z)F(z)$ forms a Nyquist(M) filter. Biorthogonal partners have been studied in a variety of digital signal processing techniques, e.g., filterbank theory [1], [10], [11], exact and least-squares digital interpolation [12], sampling theory [13], and

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¹A more general definition of Nyquist(M) filter is $p(Mn - k) = c\delta(n - n_0)$ for some nonzero c , integer n_0 , and integer k , $0 \leq k \leq M - 1$. One can use simple delay and scaling operations to reduce it to the simplified form in (1).

fractionally spaced equalizers (FSE) in digital communications [14]. Then in [15], the idea of Nyquist(M) and biorthogonal partners was extended to the multiple-input-multiple-output (MIMO) case. However, as we will see in this paper, biorthogonal partners can only achieve inter-symbol interference (ISI) free transmission for additive white Gaussian noise (AWGN) channel when applied to FSE systems. When the channel is frequency selective, they suffer from ISI.

As of today, orthogonal frequency division multiplexing (OFDM) systems have been the dominant technology for broadband multicarrier communications [16]. They have the great advantage of being robust against ISI and simple one-tap equalizers can be used at the receivers for channel equalization. However, it is also known that OFDM systems suffer from severe out-of-band spectral leakage and they are sensitive to carrier frequency offset. Therefore, in recent years filter bank multicarrier (FBMC) systems have been promoted as an alternative to OFDM systems [17]–[19]. The transmitting and receiving filters in FBMC systems have good frequency responses inherited from filter bank designs. Nevertheless, when the transmission channel is frequency selective, FBMC systems suffer from ISI and additional equalizers are often needed at the receiver to remove inter-carrier and intra-carrier

interference. Therefore, it is desirable to have FBMC systems that can achieve ISI-free transmission for frequency selective channels with simple one-tap equalizers, like OFDM systems. In this paper, we will explore the connection of depth- L biorthogonal partners with ISI-free FBMC systems.

Our paper offers two main contributions:

1) We extend the concept of Nyquist(M) filters and develop depth- L Nyquist(M) filters. A digital filter $P(z)$ is called a depth- L Nyquist(M) filter if its impulse response $p(n)$ satisfies the property

$$p(Mn - k) = \gamma_k \delta(n), \quad k = 0, 1, \dots, L, \quad (2)$$

where $\{\gamma_k\}$ are constants, not all zeros. We then have the corresponding definition of depth- L biorthogonal partners. When the depth L is equal to 0, depth- L Nyquist(M) filters and biorthogonal partners reduce to traditional Nyquist(M) filters and biorthogonal partners respectively. Moreover, the necessary and sufficient condition is derived for the existence of a finite impulse response (FIR) depth- L biorthogonal partner. It turns out that the existence depends on the zero locations rather than the length of the filter. In addition, the eigenfilter method is adopted for the design of depth- L Nyquist(M) filters and biorthogonal partners. Performance comparisons are provided to demonstrate the advantage of depth- L biorthogonal partners. It is shown that the depth- L version is more robust against ISI and timing synchronization error compared to the traditional biorthogonal partner.

2) Two potential applications of depth- L biorthogonal partners are discussed. One is for FSE systems. We will show that when the transmitting and receiving filter in an FSE system are depth- L biorthogonal partners of each other, then it achieves ISI-free transmission for any FIR channel of order $\leq L$. Another application is for FBMC systems. We first extend the theory of depth- L biorthogonal partners to the N -pair case. Then it is shown that when these N -pair depth- L biorthogonal partners are applied to FBMC systems, they can achieve ISI-free transmission for any FIR channel of order $\leq L$.

The remainder of this paper is organized as follows. In Section II, Nyquist(M) filters and conventional biorthogonal partners are briefly reviewed. In Section III, the concept of Nyquist(M) filters is first extended to that of depth- L Nyquist(M) filters. We then have definitions of depth- L and N -pair depth- L biorthogonal partners. After that, two potential applications of them in communications are described. In Section IV, the concept of congruous zeros is discussed. Then we derive the necessary and sufficient condition for the existence of an FIR depth- L biorthogonal partner. In Section V, we adopt the eigenfilter method for the design of depth- L Nyquist(M) filters and biorthogonal partners. In Section VI, performance comparisons are carried out to demonstrate the advantage of depth- L biorthogonal partners. The conclusion is given in Section VII.

Notations: Boldfaced lower case and upper case letters are used to denote column vectors and matrices, respectively. The notations \mathbf{A}^T denote the transpose of the matrix \mathbf{A} . For any

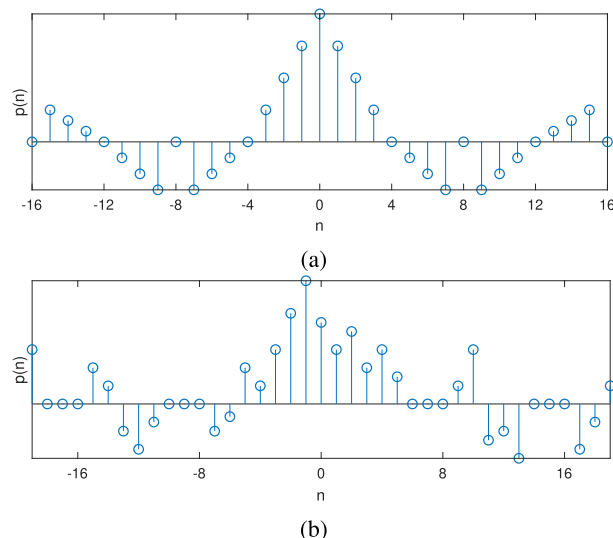


FIGURE 1. (a) Nyquist(4) filter. (b) Depth-2 Nyquist(8) filter.

positive integer M and any integer m , the notation $((m))_M$ represents m modulo M , which is a number between 0 and $M - 1$. $E[X]$ stands for the expected value of the random variable X . The notation $[x(n)]_{\downarrow M}$ denotes the M -fold decimated version of $x(n)$, i.e., $[x(n)]_{\downarrow M} = x(Mn)$, and $[X(z)]_{\downarrow M}$ denotes the z -transform of $[x(n)]_{\downarrow M}$. An empty set is denoted as ϕ .

II. REVIEW: NYQUIST(M) FILTERS AND BIORTHOGONAL PARTNERS

The definition of a Nyquist(M) filter $p(n)$ is given by (1), i.e.,

$$p(Mn) = \delta(n).$$

That is, in the time domain, it has regular zero-crossings at nonzero multiples of M and moreover $p(0) = 1$. Fig. 1(a) shows an example of a Nyquist(4) filter. In other words, a Nyquist(M) filter satisfies $[p(n)]_{\downarrow M} = 1$, i.e., in the z -domain $[P(z)]_{\downarrow M} = 1$. Express $P(z)$ in the polyphase form with respect to M :

$$P(z) = \sum_{k=0}^{M-1} z^k P_k(z^M). \quad (3)$$

Then equivalently, a filter $P(z)$ is said to be Nyquist(M) if its 0-th polyphase component is equal to one, i.e.,

$$P_0(z) = 1.$$

An important property [2] of Nyquist(M) filters is that an M -fold decimated version of the interpolated signal with ratio M using a Nyquist(M) filter returns the original signal without distortion.

Two transfer functions $H(z)$ and $F(z)$ are said to form a biorthogonal pair [9] with respect to an integer M if their cascade $P(z) = H(z)F(z)$ forms a Nyquist(M) filter, i.e.,

$$[P(z)]_{\downarrow M} = [H(z)F(z)]_{\downarrow M} = 1.$$

We say that $H(z)$ is a biorthogonal partner of $F(z)$. Note that if M is changed, the two filters may not remain partners. Also $H(z)$ and $F(z)$ can be interchanged without altering this property. We can regard $H(z)$ and $F(z)$ as any pair that defines a factorization of a Nyquist(M) filter $P(z)$. The theory and design of biorthogonal partners have been studied extensively in [9]. In particular, the necessary and sufficient condition for the existence of an FIR biorthogonal partner is proved therein:

Theorem 1: [9] Suppose $F(z)$ is an FIR filter. Let $F(z) = \sum_{k=0}^{M-1} z^{-k} F_k(z^M)$. Then there exists an FIR filter $H(z)$ such that $[H(z)F(z)]_{\downarrow M} = 1$ if and only if the greatest common divisor $D(z)$ of the M polyphase components $\{F_k(z)\}_{k=0}^{M-1}$ is trivial, i.e., $D(z)$ has the form $D(z) = dz^{-N}$ for some nonzero constant d and integer N . \square

III. DEPTH- L NYQUIST(M) FILTERS AND BIORTHOGONAL PARTNERS

The definition of depth- L Nyquist(M) filters is given by (2), i.e.,

$$p(Mn - k) = \gamma_k \delta(n), \quad k = 0, 1, \dots, L.$$

That is, in the time domain, it has a group of $L + 1$ consecutive zero-crossings separated by M samples. Fig. 1(b) shows an example of a depth-2 Nyquist(8) filter. Note from the figure that the impulse response has zero crossings at $p(8n)$, $p(8n - 1)$ and $p(8n - 2)$ for nonzero n . Express $P(z)$ in the polyphase form as in (3). Then equivalently, a filter $P(z)$ is said to be depth- L Nyquist(M) if its first $L + 1$ polyphase components are constants, not all zeros, i.e.,

$$P_k(z) = \gamma_k, \quad k = 0, 1, \dots, L. \quad (4)$$

Note that (4) is also equivalent to

$$[P(z)z^{-k}]_{\downarrow M} = \gamma_k, \quad k = 0, 1, \dots, L.$$

From the above discussion, it is clear that a depth- L Nyquist(M) filter reduces to a traditional Nyquist(M) filter when $L = 0$.

Having the definition of depth- L Nyquist(M) filters, we are ready to define depth- L biorthogonal partners. Two filters $H(z)$ and $F(z)$ are said to be depth- L biorthogonal partners of each other with respect to an integer M if their cascade $P(z) = H(z)F(z)$ forms a depth- L Nyquist(M) filter, i.e.,

$$[P(z)z^{-k}]_{\downarrow M} = [H(z)F(z)z^{-k}]_{\downarrow M} = \gamma_k \quad (5)$$

for $k = 0, 1, \dots, L$, where $\{\gamma_k\}$ are arbitrary constants, not all zeros.

Extension to the N -pair Case: The definition of depth- L biorthogonal partners can be extended to N pairs of filters. Two sets of filters $\{F_0(z), F_1(z), \dots, F_{N-1}(z)\}$ and $\{H_0(z), H_1(z), \dots, H_{N-1}(z)\}$ are said to form an N -pair depth- L biorthogonal partners if they satisfy

$$[H_j(z)F_i(z)z^{-n}]_{\downarrow M} = \begin{cases} 0, & i \neq j \\ \gamma_{i,n}, & i = j \end{cases} \quad (6)$$

for $i, j = 0, 1, \dots, N - 1$ and $n = 0, 1, \dots, L$, where $\{\gamma_{i,n}\}$ are arbitrary constants, not all zeros for a fixed value

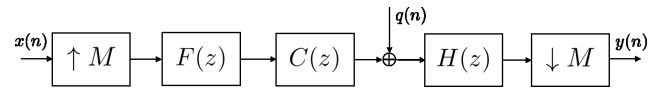


FIGURE 2. The block diagram of the FSE system.

of i . Notice that $H_i(z)$ and $F_i(z)$ form a pair of depth- L biorthogonal partners for each $i = 0, 1, \dots, N - 1$. Below we will describe two potential applications of these filters in communications.

A. APPLICATION OF DEPTH- L BIORTHOGONAL PARTNERS IN FSE

Consider the FSE system shown in Fig. 2. The channel, transmitting filter, and receiving filter are respectively given by $C(z)$, $F(z)$, and $H(z)$. Using the polyphase identity in multirate theory [1], [2], we know that the system from $x(n)$ to $y(n)$ is linear time-invariant (LTI) with transfer function

$$T(z) = [H(z)C(z)F(z)]_{\downarrow M}. \quad (7)$$

For AWGN channel, we have $C(z) = 1$. In this case, one can see that the FSE system is free of ISI if and only if $F(z)$ and $H(z)$ are biorthogonal partners of each other with respect to M . This is because the overall transfer function from $x(n)$ to $y(n)$ is $T(z) = [H(z)F(z)]_{\downarrow M} = 1$ for a biorthogonal pair $\{F(z), H(z)\}$.

On the other hand, when the channel is frequency selective, in order to achieve ISI-free transmission, $H(z)$ needs to be designed as the biorthogonal partner of the product $C(z)F(z)$. In this case, $H(z)$ will become dependent on the channel $C(z)$. However in practice, it is often desirable to have channel-independent transmitting and receiving filters that can achieve ISI-free transmission [20]. Below we will see how this can be achieved.

Let $C(z)$ be an FIR channel of order $\leq L$, i.e., $C(z) = \sum_{n=0}^L c(n)z^{-n}$. Substituting $C(z)$ into (7), we get

$$T(z) = \sum_{n=0}^L c(n)[H(z)F(z)z^{-n}]_{\downarrow M}. \quad (8)$$

Thus the FSE system is ISI-free for any FIR channel $C(z)$ of order $\leq L$ if and only if

$$[H(z)F(z)z^{-n}]_{\downarrow M} = \gamma_n, \quad n = 0, 1, \dots, L, \quad (9)$$

where $\{\gamma_n\}$ are constants, not all zeros. From (5), the ISI-free condition (9) is equivalent to that $H(z)$ and $F(z)$ are depth- L biorthogonal partners of each other. In this case, the transfer function is

$$T(z) = \sum_{n=0}^L c(n)\gamma_n.$$

We conclude that $H(z)$ and $F(z)$ form a pair of depth- L biorthogonal partners if and only if the FSE system is ISI-free for any FIR channel of order $\leq L$.

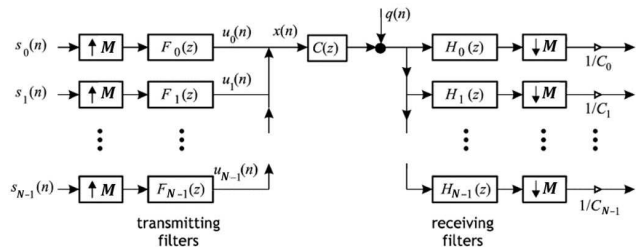


FIGURE 3. The block diagram of the FBMC system.

B. APPLICATION OF N-PAIR DEPTH-L BIORTHOGONAL PARTNERS IN FBMC

Fig. 3 shows the block diagram of the FBMC system with transmitting filters $F_k(z)$ and receiving filters $H_k(z)$ for $k = 0, 1, \dots, N - 1$. In the design of FBMC systems, the number of subcarriers N is usually smaller than the decimation or interpolation ratio M [2], [20]. Let us look at the system from the i -th input to the j -th output. From polyphase identity, we know that it is LTI with transfer function $T_{ji}(z) = [H_j(z)C(z)F_i(z)]_{\downarrow M}$ for $i, j = 0, 1, \dots, N - 1$.

To achieve ISI-free transmission, the transfer function must satisfy

$$T_{ji}(z) = \begin{cases} 0, & i \neq j \\ \alpha_i, & i = j \end{cases}, \quad (10)$$

where α_i is the gain of the i -th subcarrier. In practice, it is often desirable to have channel-independent transmitting and receiving filters that can achieve ISI-free transmission, like OFDM systems. Let $C(z)$ be an FIR channel of order $\leq L$. Then the transfer function from the i -th input to the j -th output can be rewritten as

$$T_{ji}(z) = \sum_{n=0}^L c(n)[H_j(z)F_i(z)z^{-n}]_{\downarrow M}. \quad (11)$$

We would like to have ISI-free transmission for any $c(n)$. This is achieved if and only if the filters $H_j(z)$ and $F_i(z)$ satisfy (6). Substituting (6) into (11) and (10), the i -th subcarrier gain would be

$$\alpha_i = \sum_{n=0}^L c(n)\gamma_{i,n}.$$

In conclusion, the FBMC system achieves ISI-free transmission for any FIR channel $C(z)$ of order $\leq L$ if and only if its transmitting and receiving filters form N -pair depth- L biorthogonal partners.

IV. EXISTENCE OF AN FIR DEPTH-L BIORTHOGONAL PARTNER

In this section, we will discuss the existence of an FIR depth- L biorthogonal partner. It turns out that the existence depends on the zero locations rather than the length of the filter. As we will see below, the existence of an FIR depth- L biorthogonal partner is closely related to the so-called congruous zeros [2], [21].

A. DEFINITIONS AND EXAMPLES

Definition 1 (Congruous zeros): Distinct zeros $\alpha_1, \alpha_2, \dots, \alpha_\rho$ of $F(z)$ are called congruous zeros with respect to an integer M if

$$\alpha_1^M = \alpha_2^M = \dots = \alpha_\rho^M.$$

Congruous zeros are distinct. Their phases differ by an integer multiple of $\frac{2\pi}{M}$ but their magnitudes are identical. They can be considered as rotations of each other. We can express each as a rotation of α_1 :

$$\alpha_k = \alpha_1 W^{nk}, \quad 0 \leq n_k < M,$$

where

$$W = e^{-j\frac{2\pi}{M}}.$$

Because the congruous zeros are distinct by definition, the integers n_k are also distinct. Note that the largest possible number of zeros that are congruous is M .

Definition 2: Let an arbitrary set $\mathcal{S} = \{s_1, s_2, \dots, s_\rho\}$. Define the following notation

$$W^n \mathcal{S} \triangleq \{s_1 W^n, s_2 W^n, \dots, s_\rho W^n\}$$

for some integer n . In other words, each element in $W^n \mathcal{S}$ is a rotated version of the corresponding element in \mathcal{S} .

Let \mathcal{Z} denote the set of all distinct zeros of $F(z)$. We can partition \mathcal{Z} into disjoint subsets $\{\mathcal{Z}^{(k)}\}$ containing congruous zeros with respect to M . That is,

$$\mathcal{Z} = \bigcup_k \mathcal{Z}^{(k)},$$

where the subsets $\{\mathcal{Z}^{(k)}\}$ satisfy

$$\mathcal{Z}^{(i)} \cap W^n \mathcal{Z}^{(j)} = \emptyset$$

for $i \neq j, n = 0, 1, \dots, M - 1$. The subsets $\{\mathcal{Z}^{(k)}\}$ are called congruous-zero sets of $F(z)$ with respect to M . Moreover, a congruous-zero set $\mathcal{Z}^{(k)}$ is said to be *complete* if its cardinality is equal to M , i.e.,

$$\mathcal{Z}^{(k)} = \{\alpha, \alpha W, \dots, \alpha W^{M-1}\}$$

for some $\alpha \in \mathbb{C}$. As we will show later, the necessary and sufficient conditions for the existence of an FIR biorthogonal partner (Theorem 1) and depth- L biorthogonal partner are closely related to congruous zeros. Below we first provide two examples to illustrate the concept of congruous zeros.

Example 1: Suppose the set of all distinct zeros of $F(z)$ is given by

$$\mathcal{Z} = \{3e^{-j\frac{2\pi}{7}}, 2, 2e^{-j\frac{2\pi}{8}3}, 4e^{-j\frac{2\pi}{8}2}, 4e^{-j\frac{2\pi}{8}3}, 4e^{-j\frac{2\pi}{8}4}, 4e^{-j\frac{2\pi}{8}7}\}.$$

Let $M = 8$ and $W = e^{-j\frac{2\pi}{8}}$. Then the zeros of $F(z)$ can be partitioned into the following three congruous-zero sets:

$$\mathcal{Z}^{(1)} = \{3e^{-j\frac{2\pi}{7}}\},$$

$$\begin{aligned} \mathcal{Z}^{(2)} &= \{2, 2W^3\}, \\ \mathcal{Z}^{(3)} &= \{4W^2, 4W^3, 4W^4, 4W^7\}. \end{aligned}$$

One can verify that $\mathcal{Z} = \bigcup_{k=1}^3 \mathcal{Z}^{(k)}$ and $\mathcal{Z}^{(i)} \cap W^n \mathcal{Z}^{(j)} = \phi$, for $i \neq j, n = 0, 1, \dots, 7$.

Example 2: Suppose the set of all distinct zeros of $F(z)$ is given by

$$\begin{aligned} \mathcal{Z} &= \{3e^{-j\frac{2\pi}{7}}, (3e^{-j\frac{2\pi}{7}})e^{-j\frac{2\pi}{3}2} \\ &\quad 3e^{-j\frac{2\pi}{7}}, (3e^{-j\frac{2\pi}{7}})e^{-j\frac{2\pi}{3}3}, (3e^{-j\frac{2\pi}{7}})e^{-j\frac{2\pi}{3}2}\}. \end{aligned}$$

Let $M = 3$ and $W = e^{-j\frac{2\pi}{3}}$. Then the zeros of $F(z)$ can be partitioned into the following two congruous-zero sets:

$$\begin{aligned} \mathcal{Z}^{(1)} &= \{3e^{-j\frac{2\pi}{7}}, (3e^{-j\frac{2\pi}{7}})W^2\}, \\ \mathcal{Z}^{(2)} &= \{3e^{-j\frac{2\pi}{7}}, (3e^{-j\frac{2\pi}{7}})W, (3e^{-j\frac{2\pi}{7}})W^2\}. \end{aligned}$$

One can verify that $\mathcal{Z} = \bigcup_{k=1}^2 \mathcal{Z}^{(k)}$ and $\mathcal{Z}^{(i)} \cap W^n \mathcal{Z}^{(j)} = \phi$, for $i \neq j, n = 0, 1, 2$. Note that $3e^{-j\frac{2\pi}{7}}$ and $3e^{-j\frac{2\pi}{7}}$ do not belong to the same congruous-zero set because their phases do not differ by an integer multiple of $\frac{2\pi}{M}$. Also note that $\mathcal{Z}^{(2)}$ is a complete congruous-zero set because its cardinality is equal to $M = 3$.

B. NECESSARY AND SUFFICIENT CONDITION

Using congruous zeros, we will first derive an equivalent necessary and sufficient condition for Theorem 1.

Theorem 2: Suppose $F(z)$ is an FIR filter. Let $F(z) = \sum_{k=0}^{M-1} z^{-k} F_k(z^M)$. The greatest common divisor $D(z)$ of the M polyphase components $\{F_k(z)\}$ is trivial, i.e., $D(z)$ has the form $D(z) = dz^{-N}$ for some nonzero constant d and integer N if and only if the cardinality of the largest congruous-zero set of $F(z)$ with respect to M is smaller than M . \square

Proof: We first prove the ‘‘only if’’ part. Suppose the cardinality of the largest congruous-zero set of $F(z)$ with respect to M is M . This means that $F(z)$ has M congruous zeros, say $\alpha, \alpha W, \dots, \alpha W^{M-1}$, where $W = e^{-j\frac{2\pi}{M}}$. Thus $F(z)$ has a factor

$$(z^M - \alpha^M) = (z - \alpha)(z - \alpha W) \dots (z - \alpha W^{M-1}),$$

i.e., $F(z) = (z^M - \alpha^M)\hat{F}(z)$ for some FIR $\hat{F}(z)$. Consider the polyphase form $\hat{F}(z) = \sum_{k=0}^{M-1} z^{-k} \hat{F}_k(z^M)$. Multiply both sides by $(z^M - \alpha^M)$ and we obtain

$$F(z) = \sum_{k=0}^{M-1} z^{-k} (z^M - \alpha^M) \hat{F}_k(z^M).$$

One can see that the polyphase components of $F(z)$ are given by $F_k(z) = (z - \alpha^M)\hat{F}_k(z)$. Obviously, these polyphase components $\{F_k(z)\}$ have a nontrivial common factor $(z - \alpha^M)$.

Conversely, suppose the greatest common divisor of $\{F_k(z)\}$ is nontrivial, say $D(z)$, i.e., $F_k(z) = D(z)\hat{F}_k(z)$. Then

$$\begin{aligned} F(z) &= \sum_{k=0}^{M-1} z^{-k} F_k(z^M) = \sum_{k=0}^{M-1} z^{-k} D(z^M)\hat{F}_k(z^M) \\ &= D(z^M) \sum_{k=0}^{M-1} z^{-k} \hat{F}_k(z^M). \end{aligned}$$

Note that $D(z^M)$ has at least one factor of the form $(1 - cz^{-M})$ for some nonzero constant c . This implies that $F(z)$ has M congruous zeros $c^{\frac{1}{M}} e^{-j\frac{2\pi k}{M}}$ for $k = 0, 1, \dots, M - 1$. In other words, the cardinality of the largest congruous-zero set of $F(z)$ is M . \blacksquare

In the following, we will discuss the necessary and sufficient condition for the existence of an FIR depth- L biorthogonal partner. Suppose an FIR $F(z)$ is given. An FIR $H(z)$ is a depth- L biorthogonal partner of $F(z)$ if

$$[H(z)F(z)z^{-k}]_{\downarrow M} = \gamma_k, \quad k = 0, 1, \dots, L, \quad (12)$$

where $\{\gamma_k\}$ are constants, not all zeros. (12) can be formulated into the matrix form as

$$\begin{bmatrix} F(z) \\ z^{-1}F(z) \\ \vdots \\ z^{-L}F(z) \end{bmatrix} H(z) = \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_L \end{bmatrix} \triangleq \boldsymbol{\gamma}. \quad (13)$$

Express $F(z)$ and $H(z)$ in their polyphase form: $F(z) = \sum_{k=0}^{M-1} z^{-k} F_k(z^M)$, $H(z) = \sum_{k=0}^{M-1} z^k H_k(z^M)$. By substituting the polyphase forms of $F(z)$ and $H(z)$ into (13), we obtain

$$\mathbf{F}_L(z)\mathbf{h}(z) = \boldsymbol{\gamma}, \quad (14)$$

where $\mathbf{F}_L(z)$ is shown in (15), as shown at the bottom of the page and $\mathbf{h}(z) = [H_0(z) H_1(z) \dots H_{M-1}(z)]^T$. We can see from above that finding an FIR depth- L biorthogonal partner $H(z)$ is equivalent to solving (14) for an FIR vector $\mathbf{h}(z)$. Note that the existence of an FIR solution for $\mathbf{h}(z)$ is not always guaranteed. Later we will derive the necessary and sufficient condition on $\mathbf{F}_L(z)$ such that an FIR solution for $\mathbf{h}(z)$ exists.

Now suppose the $(L + 1) \times M$ matrix $\mathbf{F}_L(z)$ has an FIR right inverse $\mathbf{G}(z)$, i.e.,

$$\mathbf{F}_L(z)\mathbf{G}_L(z) = \mathbf{I}_L.$$

Then a solution to (14) can be obtained by simply taking a nonzero linear combination of the columns of $\mathbf{G}_L(z)$, given by

$$\mathbf{h}(z) = \mathbf{G}_L(z)\boldsymbol{\gamma}. \quad (16)$$

$$\mathbf{F}_L(z) \triangleq \begin{bmatrix} F_0(z) & F_1(z) & F_2(z) & \dots & F_{M-1}(z) \\ z^{-1}F_{M-1}(z) & F_0(z) & F_1(z) & \dots & F_{M-2}(z) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z^{-1}F_{M-L}(z) & z^{-1}F_{M-L+1}(z) & z^{-1}F_{M-L+2}(z) & \dots & z^{-1}F_{M-L-1}(z) \end{bmatrix} \quad (15)$$

Therefore, a sufficient condition for the existence of an FIR depth- L biorthogonal partner $H(z)$ is that $\mathbf{F}_L(z)$ has an FIR right inverse $\mathbf{G}_L(z)$. The existence of an FIR right inverse of $\mathbf{F}_L(z)$ has been studied in [22] and the result is stated as follows. Let \mathcal{Z} denote the set of all distinct zeros of $F(z)$. It was proven in Corollary 2 [22] that the $(L + 1) \times M$ matrix $\mathbf{F}_L(z)$ with $0 \leq L < M$ has an FIR right inverse if and only if

$$\bigcap_{0 \leq \ell_0 < \ell_1 < \dots < \ell_L \leq M-1} (\mathcal{Z}_{\ell_0} \cup \mathcal{Z}_{\ell_1} \cup \dots \cup \mathcal{Z}_{\ell_L}) = \emptyset, \quad (17)$$

where $\mathcal{Z}_{\ell_k} = W^{\ell_k} \mathcal{Z}$, $k = 0, 1, \dots, L$.

From the above discussions, we conclude that a sufficient condition for the existence of an FIR depth- L biorthogonal partner is given by (17), which says that the intersection of the unions of any $L + 1$ sets of all M rotated zero sets of \mathcal{Z} with angles $\frac{2\pi}{M}\ell$, $\ell = 0, 1, \dots, M - 1$ of $F(z)$ is an empty set. In fact, this seemingly complicated condition is equivalent to a simpler one related to congruous zeros as stated in the following theorem:

Theorem 3: Let \mathcal{Z} denote the set of all distinct zeros of $F(z)$. Let us partition \mathcal{Z} as $\mathcal{Z} = \bigcup_k \mathcal{Z}^{(k)}$, where $\{\mathcal{Z}^{(k)}\}$ are congruous-zero sets of $F(z)$ with respect to M . Then the $(L + 1) \times M$ matrix $\mathbf{F}_L(z)$ in (15) has an FIR right inverse if and only if the cardinality of the largest $\{\mathcal{Z}^{(k)}\}$ is less than $M - L$.

Proof: See Appendix. ■

Using Theorem 3, we conclude that if the cardinality of the largest $\{\mathcal{Z}^{(k)}\}$ is less than $M - L$, then an FIR depth- L biorthogonal partner $H(z)$ exists, i.e., $\mathbf{h}(z)$ is given by (16). It turns out that this sufficient condition is also a necessary condition as stated in the following theorem:

Theorem 4 (Existence of an FIR Depth- L Biorthogonal Partner): Suppose $F(z)$ is an FIR filter. Then there exists an FIR filter $H(z)$ such that $H(z)$ and $F(z)$ are depth- L biorthogonal partners of each other if and only if the cardinality of the largest congruous-zero set of $F(z)$ with respect to M is less than $M - L$. □

Proof: The “if” part has been proved in the previous discussions. Here we prove the “only if” part. Let $\mathcal{Z}' = \{\alpha W^{m_1}, \alpha W^{m_2}, \dots, \alpha W^{m_\rho}\}$ be the congruous-zero set of $F(z)$ with the largest cardinality ρ . Also let $\{n_1, n_2, \dots, n_{M-\rho}\} = \{0, 1, \dots, M-1\} \setminus \{m_1, m_2, \dots, m_\rho\}$. Note that from the discussions in section III-A, we know that $H(z)$ and $F(z)$ are depth- L biorthogonal partners of each other if and only if $[F(z)C(z)H(z)]_{\downarrow M}$ is a nonzero constant for all $C(z)$ of order $\leq L$. Suppose $\rho \geq M - L$. Let us choose an L th order $C(z)$ as $C(z) = (z - \alpha W^{n_1})(z - \alpha W^{n_2}) \dots (z - \alpha W^{n_{M-\rho}})\hat{C}(z)$ for some FIR $\hat{C}(z)$. This is possible because $L \geq M - \rho$. In this case, the cardinality of the largest congruous-zero set of $C(z)F(z)$ will be M . Then by Theorem 2, there does not exist $H(z)$ such that $[F(z)C(z)H(z)]_{\downarrow M}$ is a nonzero constant. Therefore, we conclude that when $\rho \geq M - L$, an FIR depth- L biorthogonal partner does not exist. ■

A Note on the Existence of FIR N -Pair Depth- L Biorthogonal Partners: The necessary and sufficient condition on

the existence of N -pair depth- L biorthogonal partners is still an open problem. Hopefully, our results on the single-pair case will inspire some other breakthroughs in solving this challenging problem.

V. DESIGN METHODS FOR DEPTH- L NYQUIST(M) FILTERS AND BIORTHOGONAL PARTNERS

In the past, the design of Nyquist(M) filters have been studied extensively [3]–[6], [7], [8]. Many of these methods can be extended to the case of depth- L Nyquist(M) filters. In this section, we will adopt the eigenfilter method [5] due to its capability to incorporate various time and frequency-domain constraints easily. Below we will show the design of depth- L Nyquist(M) and biorthogonal partners using the eigenfilter method. As the necessary and sufficient condition for the existence of N -pair case is still unknown, only the single-pair case will be considered.

A. DESIGN OF DEPTH- L NYQUIST FILTERS

Suppose we want to design a depth- L Nyquist(M) low-pass filter $P(z)$. To prevent phase distortion, we constrain $P(z)$ to be a linear-phase filter. Due to the coefficient symmetry, it can be shown that for even L , only Type-1 and Type-3 linear phase filters can satisfy the depth- L Nyquist(M) constraint, whereas for odd L , only Type-2 and Type-4 linear phase filters can satisfy the depth- L Nyquist(M) constraint. Below we will describe our design method for even L and Type-1 linear phase filters, which can easily be modified for other cases.

Suppose L is even and $P(z)$ is a Type-1 linear phase filter of even order $N_p = 2K_p$. For notational simplicity, we consider zero phase filter $P(z) = \sum_{n=-K_p}^{K_p} p(n)z^{-n}$, where $p(n)$ satisfies the symmetry condition $p(n) = p(-n)$. As a result, the frequency response $P(e^{j\omega})$ is the same as the amplitude response $P_R(\omega)$, given by

$$P(e^{j\omega}) = P_R(\omega) = \sum_{n=0}^{K_p} b_n \cos(\omega n), \quad (18)$$

where

$$b_n = \begin{cases} 2p(n), & 1 \leq n \leq K_p \\ p(n), & n = 0 \end{cases}. \quad (19)$$

For the coefficient symmetry and notational simplicity, the depth- L Nyquist(M) constraint is chosen as $p(Mn - k) = \gamma_k \delta(n)$ for $k = -\frac{L}{2}, -\frac{L}{2} + 1, \dots, \frac{L}{2}$. One can obtain the condition in (2) by applying a simple delay operation. Hence, the coefficients $\{b_n\}$ in (19) must satisfy

$$b_{Mn-k} = 0$$

for $n = 1, 2, \dots$ and $k = -\frac{L}{2}, -\frac{L}{2} + 1, \dots, \frac{L}{2}$. Thus the amplitude response in (18) can be expressed as

$$P_R(\omega) = \mathbf{b}^T \mathbf{c}(\omega), \quad (20)$$

where \mathbf{b} and $\mathbf{c}(\omega)$ is shown respectively in (21) and (22), as shown at the bottom of next page.

Consider approximating a low-pass filter with passband edge ω_p and stopband edge ω_s . The cost function for the eigenfilter design can be written as

$$\varepsilon = (1 - \alpha) \int_0^{\omega_p} [P_R(0) - P_R(\omega)]^2 \frac{d\omega}{\pi} + \alpha \int_{\omega_s}^{\pi} [0 - P_R(\omega)]^2 \frac{d\omega}{\pi},$$

where $0 \leq \alpha \leq 1$. The first and second terms on the right hand side represent the passband and stopband error respectively. Substituting (20) into the above equation, we get

$$\varepsilon = \mathbf{b}^T [\alpha \mathbf{C}_s + (1 - \alpha) \mathbf{C}_p] \mathbf{b} \triangleq \mathbf{b}^T \mathbf{C} \mathbf{b}, \quad (23)$$

where the matrices $\mathbf{C}_s = \int_{\omega_s}^{\pi} \mathbf{c}(\omega) \mathbf{c}^T(\omega) \frac{d\omega}{\pi}$, $\mathbf{C}_p = \int_0^{\omega_p} (\mathbf{1} - \mathbf{c}(\omega)) (\mathbf{1} - \mathbf{c}(\omega))^T \frac{d\omega}{\pi}$ and $\mathbf{1}$ is the column vector of all 1's with appropriate size. The matrix \mathbf{C} is real, symmetric, and positive definite, so by Rayleigh's principle [23], the optimal \mathbf{b} which minimizes ε in (23) subject to a unit norm constraint ($\mathbf{b}^T \mathbf{b} = 1$) is simply the eigenvector corresponding to the minimum eigenvalue of \mathbf{C} .

In Fig. 4, the magnitude responses are plotted for depth- L Nyquist(M) low-pass filters $P(z)$, where $L = 0, 2, 4$ and $M = 16$. Here, $P(z)$ is of order $N_p = 120$ with $\omega_p = 0.0563\pi$ and $\omega_s = 0.0688\pi$. Besides, the tradeoff parameter is chosen to be $\alpha = 0.98$. One can observe that as the depth L increases, the magnitude response becomes worse. This is because as L increases, more filter taps are set to zeros.

In Fig. 5, the magnitude responses are plotted for depth- L Nyquist(M) low-pass filters $P(z)$, where $L = 1, 3, 5$ and $M = 16$. Here, $P(z)$ is a Type-2 linear phase filter of order $N_p = 119$ with $\omega_p = 0.0563\pi$ and $\omega_s = 0.0688\pi$. Besides, the tradeoff parameter is chosen to be $\alpha = 0.98$.

B. DESIGN OF DEPTH-L BIORTHOGONAL PARTNERS

Suppose $F(z)$ is a predetermined low-pass filter. Our goal is to find $H(z)$, a depth- L biorthogonal partner of $F(z)$ with respect to M . We will describe our method for even L . To prevent phase distortion and for simplicity, both $F(z)$ and $H(z)$ are constrained to be Type 1 linear-phase filters. Assume that $F(z)$ is of even order $N_f = 2K_f$, $F(z) = \sum_{n=-K_f}^{K_f} f(n)z^{-n}$, and $f(n)$ satisfies the symmetry condition $f(n) = f(-n)$. Also assume that $H(z)$ is of even order $N_h = 2K_h$, $H(z) = \sum_{n=-K_h}^{K_h} h(n)z^{-n}$, and $h(n)$ satisfies the symmetry condition $h(n) = h(-n)$. Then the cascade $P(z) = F(z)H(z)$ is of even order $N_p = N_f + N_h = 2K_p$, given by

$$P(z) = F(z)H(z) = \sum_{n=-K_p}^{K_p} p(n)z^{-n},$$

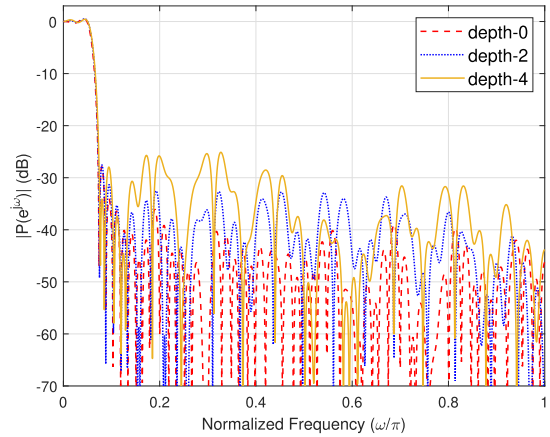


FIGURE 4. Magnitude responses of depth- L Nyquist(M) low-pass filters with $L = 0, 2, 4, M = 16$.

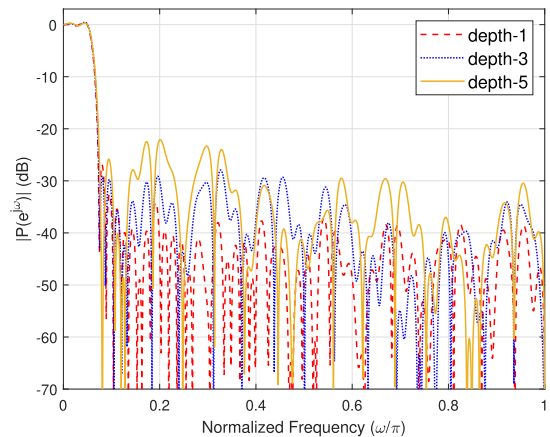


FIGURE 5. Magnitude responses of depth- L Nyquist(M) low-pass filters with $L = 1, 3, 5, M = 16$.

where $p(n)$ can be shown as

$$p(n) = \sum_{k=-K_h}^{K_h} h(k)f(n - k), \quad -K_p \leq n \leq K_p. \quad (24)$$

Our goal is to set $P(z)$ as a depth- L Nyquist(M) filter, i.e., $p(Mn - k) = \gamma_k \delta(n)$, $k = -\frac{L}{2}, -\frac{L}{2} + 1, \dots, \frac{L}{2}$. That is, $p(n)$ must satisfy

$$p(Mn - k) = 0 \quad (25)$$

for $n = 1, 2, \dots$ and $k = -\frac{L}{2}, -\frac{L}{2} + 1, \dots, \frac{L}{2}$. We only consider positive n since $p(n) = p(-n)$. Using (24) and the coefficient symmetry of $h(n)$, the constraint (25) can be expressed in the matrix form as

$$\mathbf{F} \mathbf{h} = \mathbf{0}, \quad (26)$$

$$\mathbf{b} \triangleq [b_0 \ b_1 \ \dots \ b_{M-\frac{L}{2}-1} \ b_{M+\frac{L}{2}+1} \ \dots \ b_{2M-\frac{L}{2}-1} \ b_{2M+\frac{L}{2}+1} \ \dots]^T \quad (21)$$

$$\mathbf{c}(\omega) \triangleq [1 \ \cos \omega \ \dots \ \cos(M - \frac{L}{2} - 1)\omega \ \cos(M + \frac{L}{2} + 1)\omega \ \dots \ \cos(2M - \frac{L}{2} - 1)\omega \ \cos(2M + \frac{L}{2} + 1)\omega \ \dots]^T \quad (22)$$

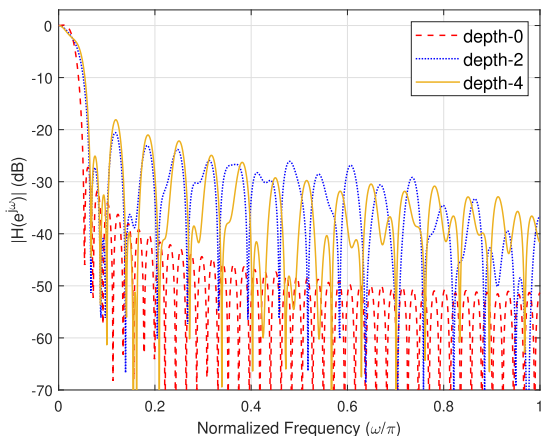


FIGURE 6. Magnitude responses of depth-L biorthogonal partners with $L = 0, 2, 4, M = 32$.

where the matrix \mathbf{F} is given in (27), as shown at the bottom of this page and the $(K_h + 1) \times 1$ vector $\mathbf{h} = [h(0) \ 2h(1) \ \dots \ 2h(K_h)]^T$. Note that in (27) \mathbf{J}_{K_h} is the $K_h \times K_h$ row-reversed identity matrix with its (i, j) th entry given by $[\mathbf{J}_{K_h}]_{ij} = \delta(i + j - K_h - 1)$. Also note that the matrix \mathbf{F} in general has full row rank. Thus for a given FIR filter $F(z)$, a nonzero solution \mathbf{h} to the equation (26) exists only if the number of columns of \mathbf{F} is larger than the number of rows, which is a necessary condition for the existence of an FIR depth-L biorthogonal partner $H(z)$.

Assume that the above necessary condition is satisfied. Then let \mathbf{F}_{null} be a matrix whose columns form a basis for the null space of \mathbf{F} . Thus the filter coefficients \mathbf{h} can be chosen as a nonzero linear combination of the columns of \mathbf{F}_{null} , i.e.,

$$\mathbf{h} = \mathbf{F}_{null} \mathbf{b}, \tag{28}$$

where \mathbf{b} is an arbitrary nonzero vector of appropriate size. Next, having \mathbf{h} in this form, we can apply the eigenfilter method introduced earlier for $h(n)$ to approximate the

low-pass filter. The stopband error ε_s and passband error ε_p are then obtained. The total error is of the form $\varepsilon = \mathbf{h}^T \mathbf{C} \mathbf{h}$ for some positive definite matrix \mathbf{C} . Substituting the constraint of depth-L biorthogonal partners (28), it follows that the total error becomes

$$\varepsilon = \mathbf{b}^T \mathbf{F}_{null}^T \mathbf{C} \mathbf{F}_{null} \mathbf{b}.$$

Since the matrix $\mathbf{F}_{null}^T \mathbf{C} \mathbf{F}_{null}$ is real, symmetric, and positive definite, we can apply Rayleigh’s principle to get \mathbf{b} such that the total error ε is minimized. Finally, the biorthogonal partner $h(n)$ can be easily obtained from (28).

In Fig. 6, we plot the magnitude responses for depth-L biorthogonal partners low-pass filters $H(z)$, where $L = 0, 2, 4$ and $M = 32$. Here, the predetermined low-pass filter $F(z)$ designed by eigenfilter method is of order $N_f = 30$ with $\omega_p = 0.0313\pi$ and $\omega_s = 0.0469\pi$. The filter $H(z)$ is of order $N_h = 90$ with the same ω_p and ω_s as those of $F(z)$. Besides, the tradeoff parameter is chosen to be $\alpha = 0.98$. One can observe that as the depth L increases, the magnitude response becomes worse. This is because as L increases, the number of constraints in (26) increases. Although the magnitude response of depth-L biorthogonal partners becomes worse as L increases, the existence of more zero crossings improves the robustness against ISI for FSE systems with frequency selective channels and timing synchronization error, as we will demonstrate below. Therefore in practice, there is a tradeoff between magnitude response and ISI robustness when we choose the value of L .

VI. PERFORMANCE COMPARISONS

In this section, we will compare the bit error rate (BER) performance of depth-0 and depth-2 biorthogonal partners in the FSE system shown in Fig. 7, which is the same as in Fig. 2 with an equalizer $E(z)$ added at the receiver. To sum up, the advantage of depth-L biorthogonal partners is being more robust against ISI and time synchronization error.

$$\mathbf{F} \triangleq \begin{bmatrix} f(M - \frac{L}{2} + K_h) & f(M - \frac{L}{2} + K_h - 1) & \dots & f(M - \frac{L}{2} - K_h) \\ f(M - \frac{L}{2} + K_h + 1) & f(M - \frac{L}{2} + K_h) & \dots & f(M - \frac{L}{2} - K_h + 1) \\ \vdots & \vdots & \ddots & \vdots \\ f(M + \frac{L}{2} + K_h) & f(M + \frac{L}{2} + K_h - 1) & \dots & f(M + \frac{L}{2} - K_h) \\ f(2M - \frac{L}{2} + K_h) & f(2M - \frac{L}{2} + K_h - 1) & \dots & f(2M - \frac{L}{2} - K_h) \\ \vdots & \vdots & \ddots & \vdots \\ f(2M + \frac{L}{2} + K_h) & f(2M + \frac{L}{2} + K_h - 1) & \dots & f(2M + \frac{L}{2} - K_h) \\ \vdots & \vdots & \ddots & \vdots \end{bmatrix} \begin{bmatrix} \mathbf{0} & \frac{1}{2} \mathbf{J}_{K_h} \\ \hline 1 & \mathbf{0}^T \\ \hline \mathbf{0} & \frac{1}{2} \mathbf{I}_{K_h} \end{bmatrix} \tag{27}$$

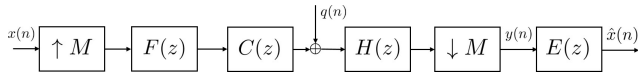


FIGURE 7. The block diagram of the FSE system with an equalizer $E(z)$.

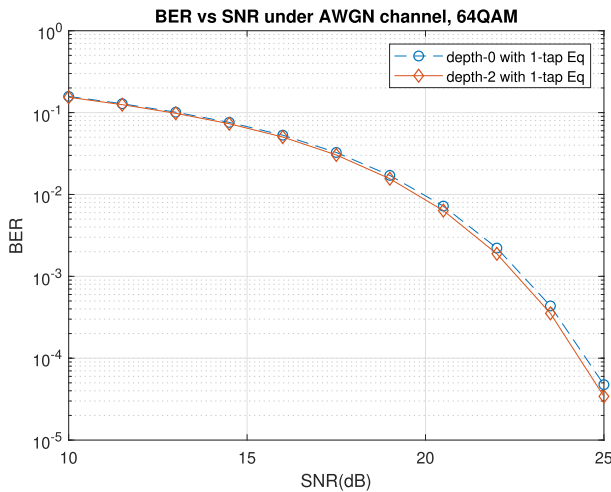


FIGURE 8. BER vs SNR of FSE systems under AWGN channel, using depth- L biorthogonal partners where $L = 0, 2, M = 32$.

A. SYSTEM DESCRIPTION

The system in Fig. 7 is set as follows. The modulation is 64QAM, sampling ratio $M = 32$ and $F(z)$ is the transmitting filter of order $N_f = 30$ given in Section V-B. The receiving filter $H(z)$ is either a depth-0 or a depth-2 biorthogonal partner designed in Section V-B. The channel $C(z)$ will be set as either AWGN or a 3-tap FIR for different purposes of performance comparison. An equalizer $E(z)$ is employed to recover $x(n)$ from $y(n)$. In the following, $E(z)$ will be designed as either a 5-tap minimum mean squared error (MMSE) equalizer or simply an 1-tap equalizer. The BER values are averaged over 6×10^7 bits sent by the transmitter. The signal-to-noise ratio (SNR) is defined as $\frac{P_x}{P_q}$, where P_x and P_q are respectively the powers of the signal $x(n)$ and noise $q(n)$ in Fig. 7. Because the impulse response of the transmitting filter $F(z)$ has unit norm, P_x is also equal to the transmission power.

B. UNDER AWGN CHANNEL

Let the channel in Fig. 7 be AWGN, i.e., $C(z) = 1$. For both depth-0 and depth-2 cases, the transfer functions are $[H(z)F(z)]_{\downarrow M} = \gamma_0$. Therefore, zero-forcing can be achieved by using an 1-tap equalizer, $E(z) = \frac{1}{\gamma_0}$. From Fig. 8, we see that for a fixed transmitting filter $F(z)$, when the receiving filter $H(z)$ is designed as its depth-0 or depth-2 biorthogonal partner, their BER performances are very close to each other. This is because both depth-0 and depth-2 biorthogonal partner are able to achieve ISI-free transmission under AWGN channel.

Next we consider the scenario where the FSE system suffers from timing synchronization error. In Fig. 9, we can

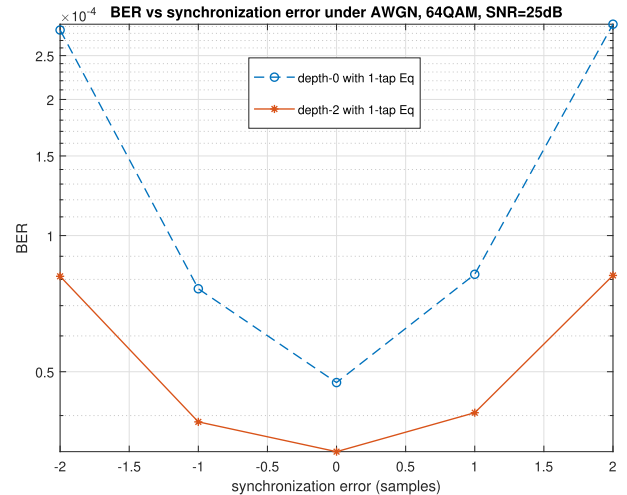


FIGURE 9. BER vs synchronization error of FSE systems under AWGN channel at SNR = 25dB, using depth- L biorthogonal partners where $L = 0, 2, M = 32$.

observe that depth-0 is more sensitive to the timing synchronization error. That is, the BER degradation due to timing synchronization error of the depth-0 case is more severe than that of the depth-2 case.

C. UNDER 3-TAP FIR CHANNEL

Let $C(z)$ in Fig. 7 be a multipath fading channel with three taps, where each tap $c(n)$ is an independently and identically distributed (iid) zero mean complex Gaussian random variable with variance σ_n^2 such that $\sum \sigma_n^2 = 1$. More specifically, we take $E[c(n)c(n-k)] = \sigma_n^2 \delta(k)$, where $\sigma_n^2 = \frac{1}{3}$ for $n = 0, 1, 2$. A total of 10^3 random channels are generated in the simulation. In this case, the depth-0 biorthogonal partner suffers from ISI. Thus two different equalizers $E(z)$ are considered, an 1-tap equalizer and a 5-tap MMSE equalizer. Note that the noise at the input of $E(z)$ is colored and hence the MMSE equalizer is designed for colored noise [2]. On the other hand, the depth-2 biorthogonal partner continues to be able to achieve ISI-free transmission and hence only an 1-tap equalizer is needed.

Fig. 10 shows that the BER performance of the depth-2 case is significantly better than that of the depth-0 case when $E(z)$ is an 1-tap equalizer in both cases. This is because the depth-2 biorthogonal partner can achieve ISI-free transmission under FIR channel of order ≤ 2 , whereas the depth-0 one suffers from ISI. Furthermore, even with the benefit of using a 5-tap MMSE equalizer, the depth-0 biorthogonal partner has a worse BER performance than the depth-2 biorthogonal partner with an 1-tap equalizer.

As for complexity, both depth-0 and depth-2 receiving filters have the same length, so their implementation costs are the same. Please note that depth- L filters can achieve ISI-free transmission for any FIR channels of order $\leq L$, like OFDM systems. Thus when the channel changes, we do not have to redesign depth-2 filters. On the other hand, for depth-0 filters, we need a 5-tap MMSE equalizer to suppress

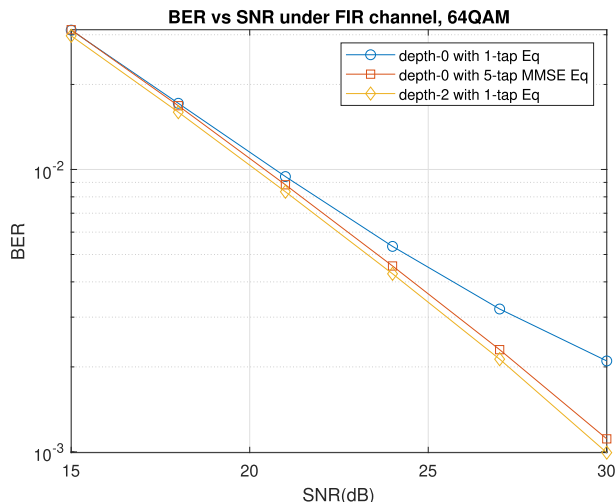


FIGURE 10. BER vs SNR of FSE systems under 3-tap FIR channel, using depth-L biorthogonal partners where $L = 0, 2, M = 32$.

the ISI. When the channel changes, we need to design a new MMSE equalizer for depth-0 filters.

VII. CONCLUSION

In this paper, we extend the traditional Nyquist(M) filters and biorthogonal partners to those of depth-L versions. The applications of them in FSE and FBMC systems are discussed. Using the concept of congruous zeros, we derive the necessary and sufficient condition for the existence of an FIR detph-L biorthogonal partner. We then show how the eigenfilter method can be adopted for the design of depth-L Nyquist(M) filters and biorthogonal partners. Simulation results show that in FSE systems, depth-L biorthogonal partners outperform conventional depth-0 biorthogonal partners when the channel is frequency selective or there is timing synchronization error.

APPENDIX PROOF OF THEOREM 3

From Corollary 2 in [22], we know that $F_L(z)$ has an FIR right inverse if and only if (17) is satisfied. Thus proving Theorem 3 is equivalent to proving that (17) is satisfied if and only if the cardinality of the largest $\{Z^{(k)}\}$ is less than $M - L$. We will first prove this for the special case where Z consists of only one congruous-zero set, i.e., $Z = Z^{(1)}$, where $Z^{(1)}$ is a congruous-zero set. After that, the general case where $Z = \bigcup_k Z^{(k)}$ will be proven.

For the special case, suppose $Z = Z^{(1)} = \{\alpha W^{m_1}, \alpha W^{m_2}, \dots, \alpha W^{m_\rho}\}$ is a congruous-zero set. Without loss of generality, assume that $\alpha = 1$ and thus $Z = \{W^{m_1}, W^{m_2}, \dots, W^{m_\rho}\}$. Define $\Gamma = \{m_1, m_2, \dots, m_\rho\}$ and $\Gamma' = \{0, 1, \dots, M - 1\} \setminus \Gamma = \{n_1, n_2, \dots, n_{M-\rho}\}$. In other words, $\Gamma \cup \Gamma' = \{0, 1, \dots, M - 1\}$ and $\Gamma \cap \Gamma' = \emptyset$. We first prove the "only if" part. Suppose $\rho \geq M - L$, i.e., $M - \rho \leq L$. Below we will first prove that $(Z_{\ell_0} \cup Z_{\ell_1} \cup \dots \cup Z_{\ell_L})$ is the complete congruous-zero set $\{W^0, W^1, \dots, W^{M-1}\}$ for all combinations of $\ell_0, \ell_1, \dots, \ell_L$. The proof is by

contradiction. Suppose $W^i \notin (Z_{\ell_0} \cup Z_{\ell_1} \cup \dots \cup Z_{\ell_L})$ for some i and for some $\ell_0, \ell_1, \dots, \ell_L$. This means that $W^i \notin Z_{\ell_k}$ for $k = 0, 1, \dots, L$, that is, $((i - \ell_k))_M \in \Gamma'$. Note that $((i - \ell_0))_M, ((i - \ell_1))_M, \dots, ((i - \ell_L))_M$ are distinct integers in $\{0, 1, \dots, M - 1\}$. This leads to a contradiction to the fact that Γ' has only $M - \rho$ (which is $\leq L$) distinct integers. Therefore, if $\rho \geq M - L$, $(Z_{\ell_0} \cup Z_{\ell_1} \cup \dots \cup Z_{\ell_L})$ is a complete congruous-zero set $\{W^0, W^1, \dots, W^{M-1}\}$ for all combinations of $\ell_0, \ell_1, \dots, \ell_L$. Taking the intersection of them, we get

$$\bigcap_{0 \leq \ell_0 < \ell_1 < \dots < \ell_L \leq M-1} (Z_{\ell_0} \cup Z_{\ell_1} \cup \dots \cup Z_{\ell_L}) = \{W^0, W^1, \dots, W^{M-1}\}.$$

Conversely, suppose $\rho < M - L$. This implies that Γ' contains at least $L + 1$ distinct integers because $M - \rho > L$. Using this, we will prove that the intersection on the left hand side of (17) is an empty set. To see this, we first take $\ell_0 = ((0 - n_1))_M, \ell_1 = ((0 - n_2))_M, \dots, \ell_L = ((0 - n_L))_M$, then $W^0 \notin (Z_{\ell_0} \cup Z_{\ell_1} \cup \dots \cup Z_{\ell_L})$. Second, if we take $\ell_0 = ((1 - n_1))_M, \ell_1 = ((1 - n_2))_M, \dots, \ell_L = ((1 - n_L))_M$, then $W^1 \notin (Z_{\ell_0} \cup Z_{\ell_1} \cup \dots \cup Z_{\ell_L})$. This procedure can be performed M times. Finally, if we take $\ell_0 = ((M - 1 - n_1))_M, \ell_1 = ((M - 1 - n_2))_M, \dots, \ell_L = ((M - 1 - n_L))_M$, then $W^{M-1} \notin (Z_{\ell_0} \cup Z_{\ell_1} \cup \dots \cup Z_{\ell_L})$. Therefore, for each $W^i, i = 0, 1, \dots, M - 1$, we can find a combination of $\ell_0, \ell_1, \dots, \ell_L$ such that $W^i \notin (Z_{\ell_0} \cup Z_{\ell_1} \cup \dots \cup Z_{\ell_L})$. Taking only the intersection of these M combinations of $\ell_0, \ell_1, \dots, \ell_L$ then we get an empty set. This proves that the intersection on the left hand side of (17) is an empty set.

Now we consider the general case where $Z = \bigcup_k Z^{(k)}$. Since $Z = \bigcup_k Z^{(k)}, Z_{\ell_i} = \bigcup_k Z_{\ell_i}^{(k)}$ for $i = 0, 1, \dots, L$. Then the left hand side of (17) can be expressed as

$$\bigcap_{0 \leq \ell_0 < \ell_1 < \dots < \ell_L \leq M-1} \left(\left(\bigcup_k Z_{\ell_0}^{(k)} \right) \cup \left(\bigcup_k Z_{\ell_1}^{(k)} \right) \cup \dots \cup \left(\bigcup_k Z_{\ell_L}^{(k)} \right) \right),$$

which is equal to

$$\bigcap_{0 \leq \ell_0 < \ell_1 < \dots < \ell_L \leq M-1} \left(\bigcup_k (Z_{\ell_0}^{(k)} \cup Z_{\ell_1}^{(k)} \cup \dots \cup Z_{\ell_L}^{(k)}) \right) \quad (29)$$

because the order of union operations can be exchanged. Moreover, due to the fact that

$$(Z_{\ell_0}^{(i)} \cup Z_{\ell_1}^{(i)} \cup \dots \cup Z_{\ell_L}^{(i)}) \cap (Z_{\ell_0}^{(j)} \cup Z_{\ell_1}^{(j)} \cup \dots \cup Z_{\ell_L}^{(j)}) = \emptyset$$

for $i \neq j$, we can rewrite (29) as

$$\bigcup_k \left(\bigcap_{0 \leq \ell_0 < \ell_1 < \dots < \ell_L \leq M-1} (Z_{\ell_0}^{(k)} \cup Z_{\ell_1}^{(k)} \cup \dots \cup Z_{\ell_L}^{(k)}) \right). \quad (30)$$

Therefore, the original condition in (17) is reduced to (30) being an empty set, which is satisfied if and only if

$$\bigcap_{0 \leq \ell_0 < \ell_1 < \dots < \ell_L \leq M-1} (\mathcal{Z}_{\ell_0}^{(k)} \cup \mathcal{Z}_{\ell_1}^{(k)} \cup \dots \cup \mathcal{Z}_{\ell_L}^{(k)}) = \phi \quad (31)$$

for all k . The condition in (31) is that every congruous-zero set $\{\mathcal{Z}^{(k)}\}$ of $F(z)$ has to satisfy the same condition as in (17) for \mathcal{Z} . Using the above proof for the special case, this holds if and only if the cardinalities of $\{\mathcal{Z}^{(k)}\}$ are less than $M - L$, i.e., the cardinality of the largest $\{\mathcal{Z}^{(k)}\}$ is less than $M - L$.

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