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Solution of a Non-Classical Integral Equation Modeling a Rotating Thin Rod Under Some Constraints and Technical Feasibility

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ABSTRACT In this paper, a mechanical system composed of a thin rotating rod under some constraints is considered. For this system, the total torque of the gravity forces is fixed and the unknown function to be determined is the mass density of the rod. This kind of problem is faced in several engineering applications as in aerospace. The resulting problem is formulated as a non classical integral equation, where the conventional methods of resolution do not apply. Therefore, a special treatment is required to solve the obtained integral equation. First, the obtained integral equation is transformed into a system of mixed integral and linear differential equations with two unknown functions. The latter transformation allows the inspiration of the general expression of the requested functions. Consequently, a highly non linear system with several unknowns is obtained. During the resolution of the latter system several mathematical technics are used. After applying all these technics an analytical solution of the studied integral equation is obtained. Finally, the technical feasibility from an engineering viewpoint of the production of a thin rod with the obtained mass density function is briefly discussed. In this context, the Functionally Graded Material is proposed as a material satisfying the obtained mass density function.

INDEX TERMS Thin rod, total torque, Integral equation, linear differential equation system, non-linear system, functionally graded material.

I. INTRODUCTION

Several encountered problems in different scientific areas and engineering are modeled as integral equations [33]. In this context, authors in [1] modeled a problem of fluid mechanic using the integral equations. The problem of diffraction by an elongated body was addressed and formulated as integral equation by [29]. In financial sector the problem of pricing puttable convertible bonds is investigated and modeled as an integral equation by [37]. An acoustic problem arising from Geosciences was investigated by [19] and the mathematical formulation yielded an integral equation. In [34] authors modeled a Convection–diffusion problem using the integral equation. The crack problem of poroelasticity was modeled as an integral equation in [12]. Authors in [9] studied the elastic wave scattering problem with integral equation modelling. The thin plate stability in subsonic flow problem was studied by [9]. In addition, the integral equation mathematical theory has attracted a lot of researchers and an abundant literature was provided ([15]– [17]).

Several kinds of integral equations has been identified as Fredholm equations (first and second kind)([24], [32]), Volterra (first and second kind) [18] equations, and other ones [35]. The resolution of these integral equations is the topic of a plenty of researches, and few of them are solved exactly [35] (closed form). Therefore, the attention was focused on numerical solution ([1], [6], [11], [18], and [25]), or on studying the qualitative behaviour of the solution. In this context, the meshless numerical methods are used in recent works for solving integral equations ([38]–[40]). It is worth noting that these methods do not need any background mesh.

In this work we are interested on a mechanical problem. This problem consists on studying a thin rod that can turn around a moving axis. In this problem, the total torque of

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the rod at the horizontal position about the axis is a given function. In addition, the total torque also depends on the mass density function of the rod. The problem is the determination of the mass density function of the rod satisfying all the required conditions. A more details and figures clarifying the problem will be presented in the problem statement section. The above mentioned problem is modeled using a non classical linear integral equation, that is not already referenced in the related literature (to the best of our knowledge). The resolution of the obtained integral equation is the main objective of this work. For that aim, several advanced mathematical technics are used. These technics allow the resolution of the studied integral equation.

The main advantage of the proposed method is to obtain the exact solution in a closed form for a complex integral equation. This exact solution is obtained after several mathematical transformations which are intended to simplify the initial integral equation. These transformations lead to a partial linear system which is considered as a relatively simple problem to be solved.

The integral equations are mainly solved using two different kind of methods. The first one is the approximate methods or numerical methods, in which the unknown function is replaced by an approximate function. This allows to transform the integral equations into other simple equations. For these methods, the accuracy and reliability of the methods should be studied by analyzing the error. The second kind of methods is the exact resolution which is encountered for a few types of integral equations as Volterra equation. To the best of our knowledge the current integral equation is not previously studied and no resolution methods are proposed.

It is worth noting that each integral equation is dictating the way to be solved with: exact or numerical methods. In this context, the current developed method for this special integral equation could be applied to other problems. This will be the topic of further investigations.

This paper is organized as follows. In section 2, the studied problem is introduced. Section 3 is intended to the resolution of the obtained integral equation. Section 4 is devoted to the technical feasibility of the obtained solution. Finally, a conclusion summarizing the elaborated work and giving some new future research directions, is presented.

II. PROBLEM STATEMENT

Consider the mechanical system presented in figure 1. In this simple system, a thin rod $[OL]$ with length equal to one is allowed to rotate around an axis which is placed in the point A.

The only external force acting on the rod is the gravity force (this force can be replaced by an electrostatic force, or electromagnetic force depending in the application). The distance from A to O is denoted x. The magnitude of the elementary torque at the point M of the gravitational force about the origin A, when the rod is in the horizontal position, is expressed as:

$$d\tau = G \times dm \times (x - t)$$

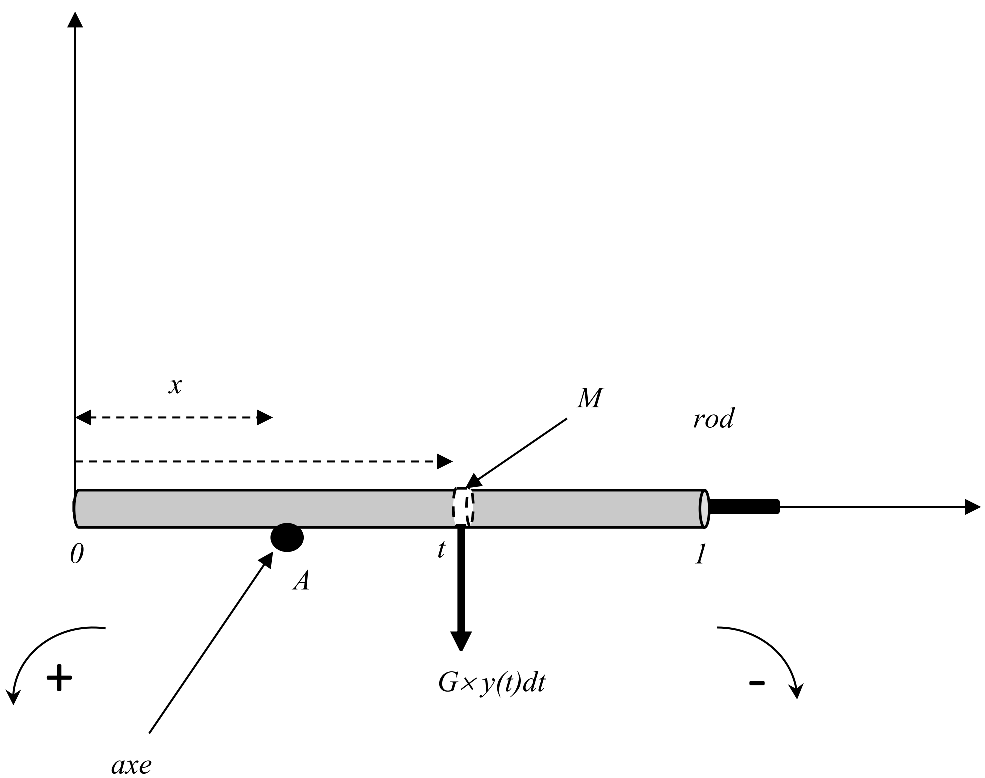

FIGURE 1. The studied mechanical system.

where G is the gravity acceleration, and dm the elementary mass of the point M. It is worth noting that the negative sense is the one of the clockwise direction (as indicated in figure 1), therefore, the algebraic distance from A to M is $(x - t)$. Furthermore, the elementary mass dm is expressed as $dm = y(t)dt$ where $y(t)$ is the mass density function of the rod at point M, then the total torque about the origin A is expressed as follows.

$$T\tau(x) = \int_0^1 Gy(t)(x-t)\,dt \tag{1}$$

The value of the total torque at the horizontal position is important in some engineering applications (mechanic, mechatronics, aerospace [36]), and controlling it is a necessity. The ways controlling this torque's value is the mass density function $y(t)$ and the position of the axis (A). An important property for the total torque is presented in the following remark.

Remark 1: If $y(t)$ is a continuous function on the interval $[0, 1]$, then:

$$T\tau(x) = cx + d \tag{2}$$

with $c = \int_0^1 Gy(t)\,dt$ and $d = \int_0^1 -Gty(t)\,dt$.

Proof: Since the function $y(.)$ is continuous on the compact $[0, 1]$, then $\int_0^1 y(t)\,dt < +\infty$ and $\int_0^1 -ty(t)\,dt < +\infty$. Consequently, the presented integral in (1) $T\tau(x) = \int_0^1 Gy(t)(x-t)\,dt$, is rewritten as $T\tau(x) = x\left(\int_0^1 Gy(t)\,dt\right) + \left(\int_0^1 -Gty(t)\,dt\right) = cx + d$, with $c = \int_0^1 Gy(t)\,dt$ and $d = \int_0^1 -Gty(t)\,dt$. ■

It is worth noting that the mass density function $y(t)$, which is requested in this work, is satisfying an integral equation that will be introduced later, and that has the following expression:

$$\int_0^1 Gy(t)(x-t)\,dt = g(x)$$

In this integral equation, the left side $\int_0^1 Gy(t)(x-t)\,dt$ is representing the total torque (1), and the right hand side $g(x)$ is a kind of constraints or preferences to be satisfied by the unknown mass density function $y(t)$.

According to remark 1, one could observe that:

1) the expression (2) is another representation of (1), and it is not yet an integral equation.
2) the last remark restricts $g(x)$ to be linear (i.e $g(x) = cx + d$).
3) the introduction of any constraints or preferences that should be satisfied by the mass density function $y(t)$ is achieved only throughout the selection of the parameters c and d. Therefore, first we should select the parameters c and d, and after that solving the integral equation.

The selection of c and d is performed in a way to keep track of the mass density function $y(t)$ variation. Since remark 1 states that the total torque is linear (i.e $cx + d$), then the simplest way doing that is to consider the linear function joining the points $C(0, y(0))$ and $D(1, y(1))$. These two points are the extreme points of the graph of $y(t)$, as indicated in figure 2. In other terms, c and d are selected such that $g(x)$ is a linear interpolation of the mass density function $y(t)$.

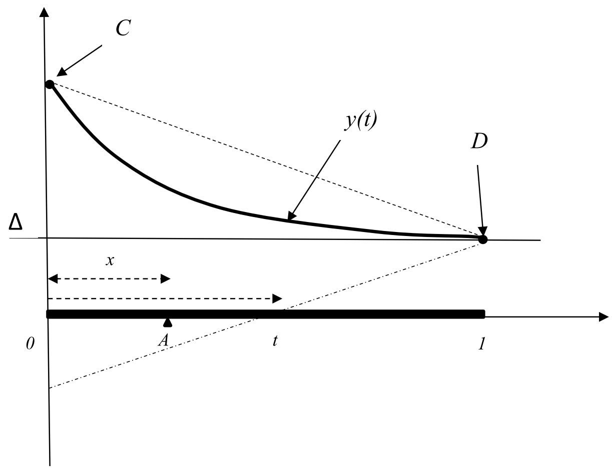

FIGURE 2. Selection of the coefficients *c* and *d*.

According to figure 2, the line connecting the two extreme points C and D of the graph of $y(t)$ has the following equation:(CD): $s = \frac{y(1) - y(0)}{1-0}t + (2y(1) - y(0))$ (t and s are the coordinates). If in addition we assume that $y(t)$ is a decreasing curve then the slope of (CD) is $sl = \frac{y(1) - y(0)}{1-0}$ is negative. From the other side, $T\tau(x)$ is an increasing function then the symmetric of (CD) relatively to the line Δ with equation: $s = y(1)$ has to be consider and not (CD). In this case we have:

$$c = \frac{y(0) - y(1)}{1-0} > 0 \tag{3}$$

$$d = 2y(1) - y(0) \tag{4}$$

Therefore, the problem is the determination of a decreasing and continuous function $y(t)$ on $[0, 1]$ that satisfies the following integral equation.

$$\int_0^1 y(t)\,G \times (x-t)\,dt = (y(0) - y(1))\,x + (2y(1) - y(0)) \tag{5}$$

Without loss of generality, in the remaining of this paper, G is assumed to be equal to one $(G = 1)$. This is achieved in an appropriate unit system, in order to simplify the calculus.

To the best of our knowledge, equation (5) is not a classic integral equation and it has the following general expression:

$$\int_0^1 K(x,t)\, y(t)\, dt = \Phi_1(y)\, x + \Phi_2(y) \tag{6}$$

where:

- The kernel $K(x,t) = (x-t)$, for $(x,t) \in [0,1] \times [0,1]$.
- $y(.)$ the unknown decreasing and continuous function in $[0,1]$.
- $\Phi_1(.)$ and $\Phi_2(.)$ are linear forms defined on the space of continuous functions $C([0,1])$ and their respective expressions are $\Phi_1(y) = y(0) - y(1)$ and $\Phi_2(y) = 2y(1) - y(0)$.

Remark 2: The solutions' space of the integral equation (5) is an IR sub-vector space of $C([0,1])$.

Proof: Indeed, let $y_1(.)$ and $y_2(.)$ be two different solutions of (6) and let $a \in IR$. Thus, $\int_0^1 K(x,t)\, y_1(t)\, dt = \Phi_1(y_1)\, x + \Phi_2(y_1)$ and $\int_0^1 K(x,t)\, y_2(t)\, dt = \Phi_1(y_2)\, x + \Phi_2(y_2)$. Therefore, $\int_0^1 K(x,t)\,(y_1(t) + a y_2(t))dt = \Phi_1(y_1 + a y_2)\, x + \Phi_2(y_1 + a y_2)$ since $\Phi_1(.)$ and $\Phi_2(.)$ are linear forms. Consequently, $y_1 + a y_2$ is a solution of (6) and the result is proved. ∎

Remark 3: The uniform mass distribution is not a solution for (6).

Proof: Assume $y(t) = a$, $\forall t \in [0,1]$ with $a > 0$ is a solution of (6) then $a \int_0^1 (x-t)\, dt = ax - \frac{1}{2}a = 0x + a$ which involves $a = 0$. This contradicts the fact that $a > 0$. ∎

Remark 4: The special expression of the kernel $K(x,t) = x - t$ suggests that $\int_0^1 y(t)(x-t)$ is a convolution product of $y(t)$ and the identity function (i.e. $f(x) = x$). In this case some technics including the inverse Fourier transformation are used to explicitly solve the integral equation. Unfortunately, the right hand side in equation (6), $\Phi_1(y)\, x + \Phi_2(y)$ depends on $y(.)$ and the latter technics do not apply.

Remark 5: The function $y(t) = 0$ is a solution of (6) that we do not consider. To avoid such a case the unknown function $y(t)$ is assumed to satisfy the following condition.

$$y(0) = 1 \tag{7}$$

This choice does not impact the final solution as it will be shown. In addition, since $y(t)$ represents a mass density function then

$$y(t) \geq 0 \text{ for all } t \in [0,1]. \tag{8}$$

III. PROBLEM RESOLUTION

In this section the integral equation, which is not a classical one, will be solved explicitly using successive transformations. These transformations allow the simplification of the proposed problem until reaching solving nonlinear system. For that aim, and as a first step the following proposition (proposition 1) is presented.

Proposition 1: Solving the system (9 and 10) where the unknown is the function $y(.)$ is equivalent to solving the integral equation (6).

$$y(0) - y(1) = \int_0^1 y(t)\, dt \tag{9}$$

and

$$2y(1) - y(0) = \int_0^1 -t y(t)\, dt \tag{10}$$

Proof: $\int_0^1 y(t)\, dt < +\infty$ this is because the function $y(t)$ is continuous on the compact $[0,1]$. The same argument holds for the function $-ty(t)$ then $\int_0^1 -ty(t)\, dt < +\infty$. Consequently, the expression $\int_0^1 y(t)(x-t)\, dt$ is rewritten as follows: $x \int_0^1 y(t)\, dt + \int_0^1 -ty(t)\, dt$. The identification term by term with $(y(0) - y(1))\, x + (2y(1) - y(0))$ leads to the result. ∎

At this stage the following classical result is recalled.

Remark 6: Any continuous function $y(t)$ on the interval $[0,1]$, admits a unique primitive $Y(t)$ satisfying: $Y(0) = 1$.

The choice of $Y(0) = 1$ is arbitrary and has no impact on the obtained solution as it will be shown in the rest of this paper.

The two results contained respectively in (9) and remark 6 allow to obtain the following obvious equation (11).

$$y(0) - y(1) = Y(1) - Y(0) \tag{11}$$

The latter equation (11) is expressing the relationship between the function $Y(t)$ and its derivative $y(t)$.

This is the beginning of the transformation of the integral equation to a system of linear differential equations. It is worth recalling that a system of linear differential equations is a linear system, involving derivative operators and where the unknowns are functions. The reader is referred to [22] for a detailed introduction and results on system of linear differential equations topic. In the next subsection, the details of transforming the integral equation into a system of linear differential equations, are presented.

A. FORMULATION USING SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS

Recall that equation (11) is the main motivation to add another unknown function $Y(t)$, which is the primitive of $y(t)$ (i.e. $\dot{Y}(t) = y(t)$), and this primitive satisfies $Y(0) = 1$.

The partial transformation of the integral equation (6) into a system of linear differential equations is the content of the next proposition.

Proposition 2: Solving the system (12-17) is equivalent to solve the integral equation (6).

$$\dot{Y}(t) = y(t), t \in [0, 1] \quad (12)$$

$$y(0) - y(1) = Y(1) - Y(0) \quad (13)$$

$$y(0) = 1 \quad (14)$$

$$Y(0) = 1 \quad (15)$$

$$\dot{Y}(0) = 1 \quad (16)$$

$$\int_0^1 -ty(t)\,dt = 2y(1) - y(0) \quad (17)$$

Proof: Let $y(t)$ be a solution of (6), therefore $y(t)$ satisfies (9) and (10). Based on the fact that the requested solution $y(t)$ is a continuous function on $[0, 1]$ then $y(t)$ has a unique primitive $Y(t)$ satisfying $Y(0) = 1$. Taking into account (9), we have $y(0) - y(1) = \int_0^1 y(t)\,dt = Y(1) - Y(0)$. In addition, $\dot{Y}(0) = y(0) = 1$.

Reversely, let $y(t)$ and $Y(t)$ be the solutions of the system (12-17). Thus, $Y(t)$ is a primitive of $y(t)$ and $\int_0^1 y(t)\,dt = Y(1) - Y(0)$. Based on (13) we have $y(0) - y(1) = Y(1) - Y(0) = \int_0^1 y(t)\,dt$, and equation (9) is satisfied. Furthermore, $y(t)$ satisfies equation (17) and consequently equation (10). ■

At this stage, it is worth noting that the system (12-17) requires the determination of two unknown functions $Y(t)$ and $y(t)$.

Inspired by the basics of the solutions for the system of linear differential equations, $Y(t)$ and $y(t)$ could have the following form.

$$\begin{cases} Y(t) = \sigma_{11}\exp(\alpha_1 t) + \sigma_{12}\exp(\alpha_2 t) \\ y(t) = \sigma_{21}\exp(\alpha_1 t) + \sigma_{22}\exp(\alpha_2 t) \end{cases} \quad (18)$$

where the real constants $\sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}, \alpha_1$, and α_2 are to be determined. The exploration of this way requires the determination of the unknowns which are the functions $Y(t)$ and $y(t)$ if they exist and have the form presented in (18). The determination of $Y(t)$ and $y(t)$ is now transformed in searching the constants $\sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}, \alpha_1$, and α_2. The substitution of the conditions satisfied by $Y(t)$ and $y(t)$ in the equations (18) will results in a system of equations where the unknowns are $\sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}, \alpha_1, \alpha_2$, this is the topic of the next subsection.

B. NON-LINEAR SYSTEM FORMULATION

Equation (10) is expressed according to the values of α_1 and α_2, and its expression is contained in the following remark.

Remark 7: Based on (10) and (18) the four following expressions are obtained.

1) If $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$ then equation (10) is:

$$\Delta_1 = \sigma_{21}\frac{\left(2\alpha_1^2 + \alpha_1 - 1\right)}{\alpha_1^2}\exp(\alpha_1) + \sigma_{22}\frac{\left(2\alpha_2^2 + \alpha_2 - 1\right)}{\alpha_2^2}\exp(\alpha_2) + \sigma_{21}\frac{\left(1 - \alpha_1^2\right)}{\alpha_1^2} + \sigma_{22}\frac{\left(1 - \alpha_2^2\right)}{\alpha_2^2} = 0. \quad (19)$$

2) If $\alpha_1 = 0$ and $\alpha_2 \neq 0$, then equation (10) is:

$$\Delta_2 = \frac{3}{2}\sigma_{21} + \sigma_{22}\frac{\left(2\alpha_2^2 + \alpha_2 - 1\right)}{\alpha_2^2}\exp(\alpha_2) + \sigma_{22}\frac{\left(1 - \alpha_2^2\right)}{\alpha_2^2} = 0. \quad (20)$$

3) If $\alpha_1 \neq 0$ and $\alpha_2 = 0$, then equation (10) is:

$$\Delta_3 = \frac{3}{2}\sigma_{22} + \sigma_{21}\frac{\left(2\alpha_1^2 + \alpha_1 - 1\right)}{\alpha_1^2}\exp(\alpha_1) + \sigma_{21}\frac{\left(1 - \alpha_1^2\right)}{\alpha_1^2} = 0. \quad (21)$$

4) if $\alpha_1 = 0$ and $\alpha_2 = 0$, then equation (10) is:

$$\Delta_4 = \sigma_{21} + \sigma_{22} = 0 \quad (22)$$

Proof: The proof is based on the explicit expression of the integral $\Omega(\lambda) = \int_0^1 t\exp(\lambda t)\,dt$ where $\lambda \in IR$. The calculation of $\Omega(\lambda)$ requires the consideration of two cases. In the first case $\lambda \neq 0$, and an integration by parts is used to calculate $\Omega(\lambda)$. For that aime we denote $f(t) = t$ and $g'(t) = \exp(\lambda t)$, ($g'(t)$ is the derivative of $g(t)$) then $f'(t) = 1$ and $g(t) = \frac{1}{\lambda}\exp(\lambda t)$. Therefore, $\Omega(\lambda) = \left[\frac{t}{\lambda}\exp(\lambda t)\right]_0^1 - \frac{1}{\lambda}\int_0^1 \exp(\lambda t)\,dt = \frac{1}{\lambda}\exp(\lambda) - \frac{1}{\lambda^2}(\exp(\lambda) - 1) = \frac{(\lambda-1)}{\lambda^2}\exp(\lambda) + \frac{1}{\lambda^2}$. In the second case $\lambda = 0$ then $\Omega(0) = \int_0^1 tdt = \frac{1}{2}$. Consequently, four cases have to be considered while simplifying (10). Indeed, using equations (10) and (18) leads to the following expressions.

1) If $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$ then $2y(1) - y(0) = \int_0^1 -ty(t)\,dt$ is equivalent to $2\sigma_{21}\exp(\alpha_1) + 2\sigma_{22}\exp(\alpha_2) - \sigma_{21} - \sigma_{22} = \int_0^1 -t(\sigma_{21}\exp(\alpha_1 t) + \sigma_{22}\exp(\alpha_2 t))dt = -\sigma_{21}\Omega(\alpha_1) - \sigma_{22}\Omega(\alpha_2) = -\sigma_{21}(\frac{(\alpha_1-1)}{\alpha_1^2}\exp(\alpha_1) + \frac{1}{\alpha_1^2}) - \sigma_{22}(\frac{(\alpha_2-1)}{\alpha_2^2}\exp(\alpha_2) + \frac{1}{\alpha_2^2})$. Then, $\left(2 + \frac{(\alpha_1-1)}{\alpha_1^2}\right)\sigma_{21}\exp(\alpha_1) + \left(2 + \frac{(\alpha_2-1)}{\alpha_2^2}\right)\sigma_{22}\exp(\alpha_2) + \sigma_{21}\left(\frac{1}{\alpha_1^2} - 1\right) + \sigma_{22}\left(\frac{1}{\alpha_2^2} - 1\right) = 0$. Hence, $\sigma_{21}\frac{(2\alpha_1^2+\alpha_1-1)}{\alpha_1^2}\exp(\alpha_1) + \sigma_{22}\frac{(2\alpha_2^2+\alpha_2-1)}{\alpha_2^2}\exp(\alpha_2) + \sigma_{21}\frac{(1-\alpha_1^2)}{\alpha_1^2} + \sigma_{22}\frac{(1-\alpha_2^2)}{\alpha_2^2} = 0 = \Delta_1$.

2) If $\alpha_1 = 0$ and $\alpha_2 \neq 0$, then $2y(1) - y(0) = \int_0^1 -ty(t)\,dt$ is equivalent to $2\sigma_{21} + 2\sigma_{22}\exp(\alpha_2) - \sigma_{21} - \sigma_{22} = -\sigma_{21}\Omega(0) - \sigma_{22}\Omega(\alpha_2) = -\frac{1}{2}\sigma_{21} - \sigma_{22}(\frac{(\alpha_2-1)}{\alpha_2^2}\exp(\alpha_2) + \frac{1}{\alpha_2^2})$. Consequently, $\frac{3}{2}\sigma_{21} + \left(2 + \frac{(\alpha_2-1)}{\alpha_2^2}\right)\sigma_{22}\exp(\alpha_2) + \sigma_{22}\left(\frac{1}{\alpha_2^2} - 1\right) = 0$ which yields $\frac{3}{2}\sigma_{21} + \sigma_{22}\frac{(1-\alpha_2^2)}{\alpha_2^2} + \sigma_{22}\frac{(2\alpha_2^2+\alpha_2-1)}{\alpha_2^2}\exp(\alpha_2) = 0 = \Delta_2$.
3) The proof for the case $\alpha_1 \neq 0$ and $\alpha_2 = 0$, is the same as the previous proof with swapping the roles of α_1 and α_2.
4) If $\alpha_1 = 0$ and $\alpha_2 = 0$ then (10) is equivalent to $2\sigma_{21} + 2\sigma_{22} - \sigma_{21} - \sigma_{22} = -\sigma_{21}\Omega(0) - \sigma_{22}\Omega(0) = -\frac{1}{2}\sigma_{21} - \frac{1}{2}\sigma_{22}$ which involves $\sigma_{21} + \sigma_{212} = 0 = \Delta_4$.

■

Remark 8: The case $\alpha_1 = \alpha_2 = 0$ is not considered. Indeed, based on (18) we have $Y(t) = \sigma_{11} + \sigma_{12}$ and $\dot{Y}(t) = y(t) = 0$ for all $t \in [0, 1]$. This contradicts the fact that $y(0) = 1$.

In the sequel, the system of equations involving the unknowns $\sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}, \alpha_1$, and α_2 is derived.

Firstly, including the conditions $\dot{Y}(0) = 1$ and $Y(0) = 1$ respectively in (18) will results in the following system.

$$\begin{cases} \sigma_{11} + \sigma_{12} = 1 \\ \sigma_{21} + \sigma_{22} = 1 \end{cases} \tag{23}$$

Using equations (18) and $\dot{Y}(t) = y(t)$ one could deduce that $(\alpha_1\sigma_{11} - \sigma_{21})\exp(\alpha_1 t) + (\alpha_2\sigma_{12} - \sigma_{22})\exp(\alpha_2 t) = 0\ \forall\ t \in [0, 1]$. Since the latter expression holds for all $t \in [0, 1]$, then an identification allows to obtain the following system.

$$\begin{cases} \alpha_1\sigma_{11} - \sigma_{21} = 0 \\ \alpha_2\sigma_{12} - \sigma_{22} = 0 \end{cases} \tag{24}$$

Substituting equations (18) into equation (11) yields the following equation.

$$(\sigma_{11} + \sigma_{21})\exp(\alpha_1) + (\sigma_{12} + \sigma_{22})\exp(\alpha_2) - 2 = 0 \tag{25}$$

All the previous obtained equations (19, 20, 21, 23, 24, and 25) are collected and a set of three systems $S_i (i = 1, 2, 3)$ are obtained. It is worth noting that each system S_i depends on Δ_i $(i = 1, 2, 3)$. The obtained systems $S_i (i = 1, 2, 3)$ with unknowns $\sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}, \alpha_1$ and α_2, are displayed as follows.

$$S_i(i = 1, 2, 3): \begin{cases} \alpha_1\sigma_{11} - \sigma_{21} = 0 \\ \alpha_2\sigma_{12} - \sigma_{22} = 0 \\ \sigma_{11} + \sigma_{12} = 1 \\ \sigma_{21} + \sigma_{22} = 1 \\ \alpha_1\sigma_{11} + \alpha_2\sigma_{12} = 1 \\ (\sigma_{11} + \sigma_{21})\exp(\alpha_1) + \\ (\sigma_{12} + \sigma_{22})\exp(\alpha_2) - 2 = 0 \\ \Delta_i = 0 (i = 1, 2, 3) \end{cases} \tag{26}$$

Clearly, the system (26) presents a strong nonlinearity. Indeed, terms as $\alpha_1\sigma_{11}$, $\sigma_{11}\exp(\alpha_1)$, and Δ_i are contained in these systems.

Remark 9: If systems (26) have a solution the it should satisfy the following condition.

$$\alpha_1 \neq \alpha_2 \tag{27}$$

Proof: Assume that $\alpha_1 = \alpha_2$, therefore $\alpha_1\sigma_{11} + \alpha_2\sigma_{12} = \alpha_1(\sigma_{11} + \sigma_{12}) = 1$ (because $\sigma_{11} + \sigma_{12} = 1$), then $\alpha_1 = 1$. In addition, $\exp(\alpha_1)(\sigma_{11} + \sigma_{12} + \sigma_{21} + \sigma_{22}) - 2 = 0$ which gives $2\exp(\alpha_1) = 2$, then $\exp(\alpha_1) = 1$, and consequently $\alpha_1 = 0$. This involves that $\alpha_1 = 1$ and $\alpha_1 = 0$ at the same time which is impossible and consequently $\alpha_1 \neq \alpha_2$. ■

Based on the latter remark (Remark 9), we assume in the remaining of this paper that $\alpha_1 \neq \alpha_2$.

At this point, the resolution of the system (12-17) is transformed into a resolution of the nonlinear system (26). The resolution of the system (26) is the subject of the next subsection.

C. NON-LINEAR SYSTEM RESOLUTION

The resolution of the system (26) requires a special treatment, since it is strongly non linear. Indeed, the system (26) contains the terms Δ_i $(i = 1, 2, 3)$ (19-21) and $\sigma_{ij}\exp(\alpha_i)$ $(i = 1, 2$ and $j = 1, 2)$.

The adopted strategy to solve the system (26) is to subdivided it into more easy partial subsystem. In this context, we first consider the following subsystem which is the least complicated part in (26):

$$\begin{cases} \alpha_1\sigma_{11} - \sigma_{21} = 0 \\ \alpha_2\sigma_{12} - \sigma_{22} = 0 \\ \sigma_{11} + \sigma_{12} = 1 \\ \sigma_{21} + \sigma_{22} = 1 \end{cases} \tag{28}$$

By fixing α_1 and α_2, the system (28) is transformed into a linear system expressed as follows.

$$\begin{pmatrix} \alpha_1 & 0 & -1 & 0 \\ 0 & \alpha_2 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{21} \\ \sigma_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \tag{29}$$

Once the system (29) is solved then the unknowns $\sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}$ are obtained and expressed with respect to α_1 and α_2.

Based on Remark 9, we have $\alpha_1 \neq \alpha_2$, and the solution of system (29) is as follows.

$$\sigma_{11} = -\frac{\alpha_2 - 1}{\alpha_1 - \alpha_2}, \quad \sigma_{12} = \frac{\alpha_1 - 1}{\alpha_1 - \alpha_2}, \tag{30}$$

$$\sigma_{21} = -\frac{\alpha_1(\alpha_2 - 1)}{\alpha_1 - \alpha_2}, \quad \sigma_{22} = \frac{\alpha_2(\alpha_1 - 1)}{\alpha_1 - \alpha_2}. \tag{31}$$

Recall that we have three systems $S_i (i = 1, 2, 3)$ depending on the already quoted cases in (19-22). According to these cases, we explore the rest of the equations in system (26) as follows.

1) CASE $\alpha_1 \neq 0$ AND $\alpha_2 \neq 0$

Denoting $Z = \exp(\alpha_1)$, $T = \exp(\alpha_2)$ and substituting the obtained expressions of σ_{11},σ_{12},σ_{21},σ_{22} (in (30)) in equations (25) and (19: ($\Delta_1 = 0$)). Therefore, a system with the unknowns Z and T is obtained and expressed as follows.

$$\begin{cases} \Gamma_{11}(\alpha_1,\alpha_2) Z + \Gamma_{12}(\alpha_1,\alpha_2) T = \Pi_1(\alpha_1,\alpha_2) \\ \Gamma_{21}(\alpha_1,\alpha_2) Z + \Gamma_{22}(\alpha_1,\alpha_2) T = \Pi_2(\alpha_1,\alpha_2) \end{cases} \tag{32}$$

where:

$\Gamma_{11}(\alpha_1,\alpha_2) = -\frac{(\alpha_2-1)(\alpha_1+1)(2\alpha_1-1)}{(\alpha_1-\alpha_2)\alpha_1}$

$\Gamma_{12}(\alpha_1,\alpha_2) = \frac{(\alpha_1-1)(\alpha_2+1)(2\alpha_2-1)}{(\alpha_1-\alpha_2)\alpha_2}$

$\Gamma_{21}(\alpha_1,\alpha_2) = -\frac{(\alpha_2-1)(\alpha_1+1)}{(\alpha_1-\alpha_2)}$

$\Gamma_{22}(\alpha_1,\alpha_2) = \frac{(\alpha_1-1)(\alpha_2+1)}{(\alpha_1-\alpha_2)}$

$\Pi_1(\alpha_1,\alpha_2) = \frac{(\alpha_2-1)(\alpha_1-1)}{\alpha_1\alpha_2}$

$\Pi_2(\alpha_1,\alpha_2) = 2.$

It is worth noting that system (32) is well defined since $\alpha_1 \neq \alpha_2$, $\alpha_1 \neq 0$, and $\alpha_2 \neq 0$. In addition, the determinant of the linear system in Z and T (32) is: $\Delta = \Gamma_{11}(\alpha_1,\alpha_2) \times \Gamma_{22}(\alpha_1,\alpha_2) - \Gamma_{21}(\alpha_1,\alpha_2) \times \Gamma_{12}(\alpha_1,\alpha_2) = \frac{(\alpha_2-1)(\alpha_1+1)(\alpha_1-1)(\alpha_2+1)}{(\alpha_1-\alpha_2)\alpha_2\alpha_1}$. This determinant $\Delta = 0$ for $\alpha_2 = 1$, or $\alpha_1 = -1$, or $\alpha_1 = 1$, or $\alpha_2 = -1$. These cases have to be considered separately.

1) case $(\alpha_2 = 1, \alpha_1 \neq \alpha_2, \alpha_1 \neq 0)$: If $\alpha_2 = 1$ then $\Gamma_{12}(\alpha_1,\alpha_2) = \Gamma_{21}(\alpha_1,\alpha_2) = \Pi_1(\alpha_1,\alpha_2) = 0$ and the system (32) becomes $\begin{cases} 2T = 0 \\ 2T = 2 \end{cases}$ thus $T = 0$ and $T = 1$. Therefore, $\exp(\alpha_2) = 0$ and $\exp(\alpha_2) = 1$ which is impossible.
2) case $(\alpha_1 = -1, \alpha_1 \neq \alpha_2, \alpha_2 \neq 0)$: If $\alpha_1 = -1$ then $\Gamma_{11}(\alpha_1,\alpha_2) = \Gamma_{21}(\alpha_1,\alpha_2) = 0$ and $\Pi_1(\alpha_1,\alpha_2) = \frac{2(\alpha_2-1)}{\alpha_2}$ the system (32) becomes $\begin{cases} \frac{2(2\alpha_2-1)}{\alpha_2} T = \frac{2(\alpha_2-1)}{\alpha_2} \\ 2T = 2 \end{cases}$ thus $T = \frac{(\alpha_2-1)}{(2\alpha_2-1)}$ and $T = 1$. Therefore, $\exp(\alpha_2) = \frac{(\alpha_2-1)}{(2\alpha_2-1)}$ and $\exp(\alpha_2) = 1$ which involves $\alpha_2 = 0$, this is impossible since $\alpha_2 \neq 0$.
3) case $(\alpha_1 = 1, \alpha_1 \neq \alpha_2, \alpha_2 \neq 0)$: If $\alpha_1 = 1$ then $\Gamma_{11}(\alpha_1,\alpha_2) = \Gamma_{22}(\alpha_1,\alpha_2) = \Pi_1(\alpha_1,\alpha_2) = 0$ and the system (32) becomes $\begin{cases} 2Z = 0 \\ 2Z = 2 \end{cases}$ thus $Z = 0$ and $Z = 1$. Therefore, $\exp(\alpha_1) = 0$ and $\exp(\alpha_1) = 1$ which is impossible.
4) case $(\alpha_2 = -1, \alpha_1 \neq \alpha_2, \alpha_1 \neq 0)$: If $\alpha_2 = -1$ then $\Gamma_{12}(\alpha_1,\alpha_2) = \Gamma_{22}(\alpha_1,\alpha_2) = 0$ and $\Pi_1(\alpha_1,\alpha_2) = \frac{2(\alpha_1-1)}{\alpha_1}$ the system 32 becomes $\begin{cases} \frac{2(2\alpha_1-1)}{\alpha_1} Z = \frac{2(\alpha_1-1)}{\alpha_1} \\ 2Z = 2 \end{cases}$ thus $Z = \frac{(\alpha_1-1)}{(2\alpha_1-1)}$ and $Z = 1$. Therefore, $\exp(\alpha_1) = \frac{(\alpha_1-1)}{(2\alpha_1-1)}$ and $\exp(\alpha_1) = 1$ which involves $\alpha_1 = 0$, this is impossible since $\alpha_1 \neq 0$.

Therefore, for the above mentioned cases the system (32) has no solution.

The resolution of the linear system regarding Z and T (32), when $\Delta \neq 0$ yields the following solution.

$$Z = \frac{-\alpha_1 + 3\alpha_1\alpha_2 + \alpha_2 - 1}{(\alpha_2 - 1)(\alpha_1 + 1)} \text{ and } T = \frac{\alpha_1 + 3\alpha_1\alpha_2 - \alpha_2 - 1}{(\alpha_1 - 1)(\alpha_2 + 1)}$$

which are expressed in other terms as follows.

$$\exp(\alpha_1) = \frac{-\alpha_1 + 3\alpha_1\alpha_2 + \alpha_2 - 1}{(\alpha_2 - 1)(\alpha_1 + 1)} \tag{33}$$

and

$$\exp(\alpha_2) = \frac{\alpha_1 + 3\alpha_1\alpha_2 - \alpha_2 - 1}{(\alpha_1 - 1)(\alpha_2 + 1)} \tag{34}$$

The equations (33) and (34), allow to express α_2 and α_1 respectively as follows.

$$\alpha_2 = \frac{(\alpha_1 + 1)\exp(\alpha_1) - \alpha_1 - 1}{(\alpha_1 + 1)\exp(\alpha_1) - 3\alpha_1 - 1} \tag{35}$$

$$\alpha_1 = \frac{(\alpha_2 + 1)\exp(\alpha_2) - \alpha_2 - 1}{(\alpha_2 + 1)\exp(\alpha_2) - 3\alpha_2 - 1} \tag{36}$$

The equations (35-36) are the conditions that should be satisfied by the unknowns α_1 and α_2.

The existence of α_1 and α_2 satisfying (35-36) is the objective of the next study. For that aim the following functions are defined.

- $h(t) = (t + 1)\exp(t) - 3t - 1$, $t \in IR$.
- $l(t) = (t + 1)\exp(t) - t - 1$, $t \in IR$.
- $k(t) = \frac{l(t)}{h(t)}$, $t \in IR$ such that $h(t) \neq 0$.

Based on the latter notations the equations (35-36) are rewritten as follows.

$$\alpha_2 = k(\alpha_1) \text{ and } \alpha_1 = k(\alpha_2) \tag{37}$$

The variation of the function $h(t)$, is performed over the sign of its derivative function $\dot{h}(t) = (t + 2)\exp(t) - 3$. The sign of $\dot{h}(t)$ is out of reach due to its complexity. In order to overcome this drawback, the derivative of $\dot{h}(t)$ is calculated and we have $\ddot{h}(t) = (t + 3)\exp(t)$. Thus, $\ddot{h}(t) = (t + 3)\exp(t) = 0$ for $t = -3$ and the sign of $\ddot{h}(t)$ is the sign of the function $(t + 3)$. Moreover, $\dot{h}(0.3) = 0.1$ and $\dot{h}(0) = -1$ which involves the existence of a unique $\alpha \in [0, 0.3]$ such that $\dot{h}(\alpha) = 0$.

Based on the variation table of $h(t)$ (see figure 3), there exists a unique $\beta \in [0.50, 0.55]$ such that $h(\beta) = 0$ (since $h(0.5) \times h(0.6) \simeq (-0.02) \times (0.03) < 0$).

At this stage we start the study of the function $k(t)$, and we first observe that $l(0) = h(0) = 0$. Thus, a special treatment is required for the function $k(t)$ at the point $t = 0$. The $\lim_{t\to 0} k(t) = \frac{\frac{l(t)-l(0)}{t-0}}{\frac{h(t)-h(0)}{t-0}} = \frac{\lim_{t\to 0} \frac{l(t)-l(0)}{t-0}}{\lim_{t\to 0} \frac{h(t)-h(0)}{t-0}} = \frac{l'(0)}{h'(0)} = \frac{1}{1} = 1$, the function $k(t)$ is well defined at the point $t = 0$ and $k(0) = 1$. Furthermore, $l(\beta) = (\beta + 1)\exp(\beta) - \beta - 1 = (\beta + 1)\exp(\beta) - 3\beta - 1 + 2\beta = h(\beta) + 2\beta = 2\beta > 0$. According to the variation table of $h(t)$ which is displayed in figure 3, the function $h(t)$ satisfies the following condition: $(h(t) < 0)$ for $t < \beta$ and $h(t) > 0$ for $t < \beta$ (t enough close to β). Therefore, $\lim_{t\to\beta^+} k(t) = +\infty$ and

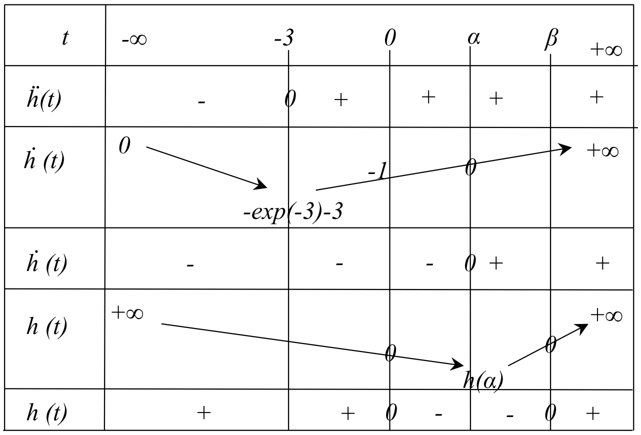

FIGURE 3. Variation table of the function $h(t)$.

$\lim_{t\to\beta^-} k(t) = -\infty$, which involves that the line $\Pi : t = \beta$ is a vertical asymptote for the curve of $k(t)$.

In addition, $\lim_{t\to+\infty} k(t) = 1$ and $\lim_{t\to-\infty} k(t) = \frac{1}{3}$, which means that lines $\Lambda : s = 1$ and $\Psi : s = \frac{1}{3}$ are horizontal asymptotes for the curve of $k(t)$ at $+\infty$ and at $-\infty$, respectively.

The variation of $k(x)$ is given throughout the sign of its derivative $\dot{k}(t) = -\frac{2\left(\left(t^2+t-1\right)\exp(t)+1\right)}{\left((t+1)\exp(t)-3t-1\right)^2}$. The function $\dot{k}(t)$ has the opposite sign of function $m(t) = \left(t^2+t-1\right)\exp(t)+1$. The sign of $m(t)$ is determined over the sign of its derivative $\dot{m}(t) = t\,(t+3)\exp(t)$. The variation table of the function $k(t)$ is presented in figure 4.

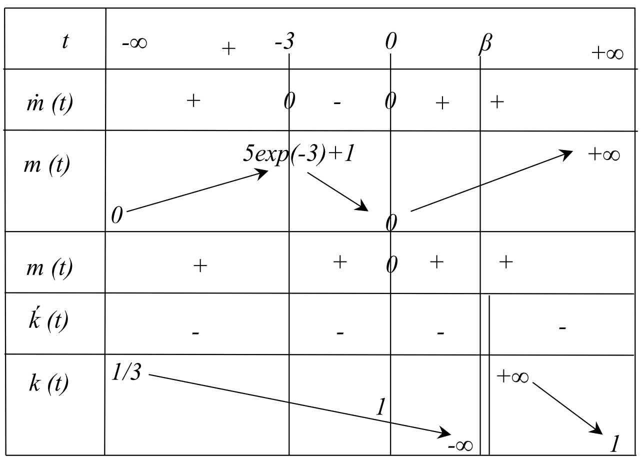

FIGURE 4. Variation table of the function $k(t)$.

Based on the variation table, which is presented in figure 4, the function $k(t)$ is strictly decreasing. This involves that the function $k(t)$ has an inverse function $k^{-1}(t)$. An immediate consequence is that the equations (35-36) are rewritten as:

$$\alpha_2 = k(\alpha_1) \text{ and } \alpha_2 = k^{-1}(\alpha_1) \tag{38}$$

The figure 5 displays the respective curves of the functions $k(t)$ and $k^{-1}(t)$.

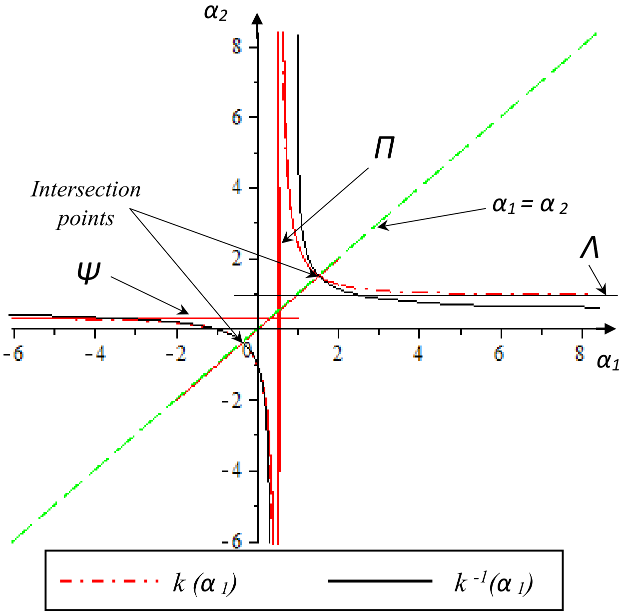

FIGURE 5. Curves of the function $k(t)$ and its inverse.

Since the only intersection points of the curves of $k^{-1}(t)$ and $k(t)$ should satisfy $s = t$ then $\alpha_1 = \alpha_2$. This cannot occur since this case is excluded from the beginning (according to Remark 9). Thus, the system (26) **has no solution** for the case $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$.

2) CASE $\alpha_1 \neq 0$ AND $\alpha_2 = 0$

The expressions of $\sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}$ which are already presented in (30) become:

$$\sigma_{11} = \frac{1}{\alpha_1}, \sigma_{12} = \frac{\alpha_1 - 1}{\alpha_1}, \sigma_{21} = 1, \sigma_{22} = 0 \tag{39}$$

When equations (39) are substituted in equations (25) and (21), we obtain the following system.

$$\begin{cases} d_1(\alpha_1) Z = e_1(\alpha_1) \\ d_2(\alpha_1) Z = e_2(\alpha_1) \end{cases} \tag{40}$$

where

$Z = \exp(\alpha_1)$

$d_1(\alpha_1) = \frac{(\alpha_1+1)(2\alpha_1-1)}{\alpha_1^2}$

$d_2(\alpha_1) = \frac{(\alpha_1+1)}{\alpha_1}$

$e_1(\alpha_1) = \frac{(\alpha_1^2-1)}{\alpha_1^2}$

$e_2(\alpha_1) = \frac{(\alpha_1+1)}{\alpha_1}$

According to the expressions of $d_1(\alpha_1)$, $d_2(\alpha_1)$, $e_1(\alpha_1)$, and $e_2(\alpha_1)$, three cases have to be examined.

- If $\alpha_1 = \frac{1}{2}$, then the first equation $d_1(\alpha_1) Z = e_1(\alpha_1)$ of system (40) is transformed into $0 \times Z = -3$, which is impossible.
- If $\alpha_1 = -1$, then the second equation $d_2(\alpha_1) Z = e_2(\alpha_1)$ of system (40) becomes $0 = 0$,

- If $\alpha_1 \notin \left\{\frac{1}{2}, -1\right\}$ then equation $d_1(\alpha_1) Z = e_1(\alpha_1)$ is solved and the solution is $Z = \frac{(\alpha_1-1)}{(2\alpha_1-1)}$. This solution Z is substituted in the equation $d_2(\alpha_1) Z = e_2(\alpha_1)$. In other term $\frac{(\alpha_1+1)}{\alpha_1}\frac{(\alpha_1-1)}{(2\alpha_1-1)} + \frac{(\alpha_1-1)}{\alpha_1} = 2$, and consequently $\frac{(\alpha_1+1)}{\alpha_1}\frac{(\alpha_1-1)}{(2\alpha_1-1)} + \frac{(\alpha_1-1)(2\alpha_1-1)}{\alpha_1(2\alpha_1-1)} = 2$. In other term, $\frac{\alpha_1^2-1+2\alpha_1^2-3\alpha_1+1}{\alpha_1(2\alpha_1-1)} = 2$, which is expressed as $\frac{3\alpha_1^2-3\alpha_1}{\alpha_1(2\alpha_1-1)} = 2$, and $\frac{3\alpha_1-3}{(2\alpha_1-1)} = 2$. Therefore, $3\alpha_1 - 3 = 4\alpha_1 - 2$ which gives $\alpha_1 = -1$. This contradicts the assumption of $\alpha_1 \notin \left\{\frac{1}{2}, -1\right\}$.

Among the three latter cases, only the one with $\alpha_1 = -1$ holds. Thus, the solution of the system (26) is presented as follows.

$$\sigma_{11} = -1, \sigma_{12} = 2, \sigma_{21} = 1, \sigma_{22} = 0, \alpha_1 = -1, \alpha_2 = 0. \tag{41}$$

Therefore, the requested solution of the system (12-17).

$$\begin{aligned} Y(t) &= -\exp(-t) + 2 \\ y(t) &= \exp(-t) \end{aligned} \tag{42}$$

3) CASE $\alpha_2 \neq 0$ AND $\alpha_1 = 0$

This case is the symmetric of the one with $\alpha_1 \neq 0$ and $\alpha_2 = 0$ and

$$\sigma_{11} = 2, \sigma_{12} = -1, \sigma_{21} = 0, \sigma_{22} = 1, \alpha_1 = 0, \alpha_2 = -1. \tag{43}$$

In this case we have

$$\begin{aligned} Y(t) &= 2 - \exp(-t) \\ y(t) &= \exp(-t) \end{aligned} \tag{44}$$

4) CASE $\alpha_2 = 0$ AND $\alpha_1 = 0$

Based on remark 8, this case is rejected.

Clearly, a solution of the integral equation (6) is:

$$y(t) = \exp(-t), t \in [0, 1]. \tag{45}$$

Seeking simplicity, the above complex technical part about solving the non-linear system is summarized in the following paragraph. The encountered non-linear system (26) is composed of seven equations and six unknowns. First, a simple non-linear subsystem composed of the four first equations, is selected. This subsystem contains only addition and multiplication operations between the variables. The other subsystems contain in addition the exponential of some variables. This is the main reason of selecting the first subsystem. The non-linearity of the first subsystem is overcome by fixing two variables involved in multiplication. This allows to obtain a partial linear subsystem, which is solved easily. The result is four variables expressed using only the two fixed ones. All these four variables are replaced in the two last equations of system (26). Consequently, a system of two equations with two variables (35-36) is obtained. This system contains the exponential function. A technical part, using the study of particular functions is performed. This allows to obtain the values of these two variables and consequently all the other variables.

D. TECHNICAL FEASIBILITY OF THE OBTAINED SOLUTION

At this stage and after obtaining an analytical (exact) solution to the addressed integral equation, the question to be raised is: is it possible technically to produce a rod with an exponential mass density? Indeed, the response is yes and the Functionally Graded Material (FGM) is an example of such a requested material. The FGM's are nonhomogeneous material composed of more than two different homogeneous materials (such as ceramic and metal). The proportions of these materials are continuously and smoothly varying in a given direction as indicated in figure 6.

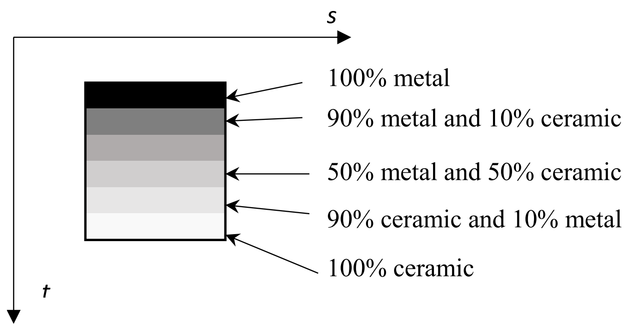

FIGURE 6. Functionally Graded Material composition.

The thick of the presented layers in figure 6 is almost a nanometer ($10^{-9}m$), thus the FGM material is considered as a nano material. In addition, if t is denoting the thickness direction distance then a physical propriety $p(t)$ is expressed as: $p(t) = p_0 \exp(-at)$ with p_0 and a positive parameters. These physical proprieties are for example: the heat conductivity, the shear modulus and the thermal expansion coefficient ([7], [8]). This is also the case of the mass density function $y(t)$ solution of the studied integral equation, where $p_0 = a = 1$.

The FGM's are widely used in different engineering fields as for the aerospace industry, where they serve as a coating of the space shuttles against the thermal shock. The car industry is also using the FGM's material. The Biomedical area is benefiting from these materials where some human body parts as bones and teeth are replaced by FGM materials ([30], [31]). In the Electrical/Electronics area the FGM material is also used ([13]).

A dramatic advances in the fabrication of the FGM during the two latter decades had been observed and a lot of technical procedures had been proposed and used as: Chemical vapour deposition/infiltration, Thermal spray, Surface reaction process, and Laser deposition in addition to others ([21]).

IV. CONCLUSION

In this paper, a non classical integral equation modeling a rotating thin rod under some constraints is derived. The unknown function to be determined is the mass density function of the rod. Conventional methods solving the classical integral equations do not apply therefore a special treatment of the proposed problem is carried out. In order to solve the obtained integral equations, several mathematical and technical steps have been performed. These steps start by transforming the studied integral equation into a system of linear differential and integral with two unknown functions. The next transformation allows the suggestion of a particular solution with unknown constants to be determined. After including the suggested solution, a strong non-linear system is obtained. Deriving the solutions of the latter system requires some mathematical technics as studying some encountered functions. A solution for the latter system is obtained and consequently the addressed integral equation is solved. The obtained solution (mass density of the thin rod) is an exponential one. Finally, an engineering solution for producing such a mass distribution is proposed. This solution requires the utilisation of the Functionally Graded Material which is a wide used and produced material.

More research need to be provided to study the uniqueness of the solution of the studied integral equation. In addition, the resolution of the proposed integral equation in the general case, where the two linear forms $\Phi_1(.)$ and $\Phi_2(.)$ have other expressions. In the latter case new methods have to be explored.

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