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Static Output Constrained Control for Discrete-Time Hidden Markov Jump Linear Systems

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ABSTRACT This paper studies the static output quadratic control problem of discrete-time Markov jump linear systems (MJLS) with hard constraints on the norm of the state and control variables. Both cases the finite horizon as well as the infinite horizon are considered. Regarding the Markov chain parameter $\theta(k)$, it is assumed that the controller only has access to a detector which emits signals $\widehat{\theta}(k)$ providing information on the parameter $\theta(k)$. The goal is to design a static output feedback linear control using the information provided by detector $\widehat{\theta}(k)$ in order to minimize an upper bound for the quadratic cost and satisfy the hard constraints. For the infinite horizon case it is also imposed that the controller stochastically stabilizes the closed loop system. LMIs (linear matrix inequalities) are formulated in order to obtain a solution for these optimization problems. The cases in which the initial conditions are fixed and when it is desired to maximize an estimate of the domain of an invariant set are also analyzed. Some numerical examples are presented for the purpose of illustrating the results obtained.

INDEX TERMS Constrained control, hybrid systems, Markov processes, LMIs, static output control, stochastic optimal control.

I. INTRODUCTION

In recent years systems subject to sudden changes in their dynamics have been the focus of many researches in engineering and related fields. Faced with this situation, Markov jump linear systems (MJLS) appear as an useful mathematical tool capable of modeling and analyzing these systems, covering several areas of application such as: systems subject to component failures and repairs [3], [27]; active fault-tolerant control systems (AFTCS) [15], [16], [22], [46]; economics [38]; finance [4], [44]; energy planning [18], [31]; reservoir operation [26], [47]; etc.

MJLS have by now an extensive literature in which assumptions, extensions, generalizations and different structures are considered (see, for instance, [30]). As a sample of works in this area we can mention [24], [28], [36] for the continuous-time domain, and [1], [25], [41] for the discrete-time case. Also, we can cite the books [6], [7], which give a solid basis on stability, filtering, and optimal control of MJLS in the discrete-time and continuous-time cases. In the aforementioned works the Markov parameter $\theta(k)$ is assumed to be available to the controller. However in many applications this does not occur, and the only information available to the controller about the system mode is provided by an associated detector $\hat{\theta}(k)$. In order to analyze this situation it is usually assumed that $(\theta(k), \hat{\theta}(k))$ follows a hidden Markov model, as it was adopted, for instance, in [8], [39], yielding to the so-called hidden MJLS (or detector-based MJLS approach). Nowadays, we can find in the literature several works using this framework, also referred to as *asynchronous control* as presented, e.g., in Song et al [34], [35], in which the problems of static output feedback control and sliding mode control of MJLS with hidden observations were considered, and Dong et al [12]–[14], where an asynchronous

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controller is designed based on a hidden Markov model and following a Takagi-Sugeno fuzzy approach under different assumptions. Ogura et al [23] considered an observation process in which the Markov chain is accessed only when some operation modes of a different Markov process are visited, a model that generalizes the mode-dependent, cluster, and mode-independent formulations, as well as the detector approach of Costa et al [8].

In many practical situations in the control of a dynamic system there are physical constraints on the actuators and state variables, so that these constraints have to be taken into consideration in the design of the controller. Regarding the constrained MJLS with the Markov parameter available to the controller we can mention the papers [42], [43], where constraints on the first and second moments for the state and control variables are imposed; [25], which presents a generalization of MJLS, where the subsystems can be non-linear; [21], which deals with a model predictive control (MPC) formulation and adopts uncertainties of the polytopic type in the system matrices as well as in the transition probabilities between modes; [40], which introduces polytopic constraints on the inputs and states of a robust MPC problem for MJLS; and [5], which considers a quadratic state feedback optimization problem for MJLS subject to constraints on the state and control variables.

This paper deals with constrained static output control for discrete-time MJLS with partial information using LMI optimization problems for minimizing an upper bound for the quadratic cost function and/or maximizing the estimate of the domain of an invariant set with a fixed upper bound cost. The static output feedback control is known for its complexity, since even for the case without jumps it can be only formulated as a bilinear matrix inequality (BMI) problem, see, for instance, the survey [29]. For hidden MJLS, it was introduced in [9] a new method for the design of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ static output feedback controllers providing suitable upper bounds for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms under uncertainty in the transition matrix. Within the discrete-time finite horizon setup it was considered in [33] the stochastic boundedness and $\ell_2 - \ell_\infty$ disturbance attenuation performance with guaranteed upper bound costs for hidden MJLS via a static output feedback strategy. Notice that the introduction of constraints on the actuators and state variables brings new challenges for the problem, since the design of the controllers has to take into account not only the uncertainties due to the detector but also the presence of the hard constraints on the state and control variables. These points represent the main technical difficulties for studying this problem. The advantages of the method proposed in this paper is that we develop LMIs optimization problems in order to obtain the desired controllers for the finite and infinite horizon cases. Notice that with the existing LMIs tool boxes the proposed optimization problems can be easily implemented. As far as the authors are aware of, this static-output hidden MJLS problem with constraints on the actuators and state variables had not been considered in previous existing works. The main contributions of this paper are summarized as follows.

- Differently from [5], [21], [25], [40], [42], [43], we introduce the constrained quadratic control for discrete-time MJLS considering that the Markov parameter $\theta(k)$ is not available to the controller and, instead, we only have an estimation for this parameter provided by $\hat{\theta}(k)$ with an associated detection probability matrix, following the hidden MJLS methodology.
- With respect to the works [8] and [9], which study control problems for the hidden MJLS without constraints, it is imposed in this paper hard symmetrical constraints on the norm of the state and control variables when the hidden MJLS framework is adopted.
- The finite and infinite horizon scenarios are tackled in the development of the control law via static output feedback, which can be considered as a generalization of the work introduced in [45], which only addressed the state feedback case within the infinite horizon setup.
- Numerical simulations of an unmanned aircraft system subject to actuators faults and under hard constraints in the control variable are presented as an illustrative example of the derived algorithms.

The work is organized as follows. The notation that will be used in the following sections are presented in Section II. Section III introduces some important definitions such as: structure of the system, time-variant and time-invariant controllers, stabilizability for MJLS and the finite and infinite horizon cases. The main results regarding the finite and infinite cases (Problem 9 and Problem 12) are shown in Section IV and Section V respectively, where we suppose that the initial condition for state x_0 and Markov parameter θ_0 are unknown. Section VI deals with two alternative problems, in which in the first one we consider that the initial state and Markov parameter are known (x_0, θ_0), and in the second problem it is obtained the largest internal ball in a critical region for a fixed cost upper bound. Numerical simulations to illustrate the developed results are presented in Section VII. We conclude this paper with some final comments in Section VIII.

II. NOTATION

For $\mathbb{X}$ and $\mathbb{Y}$ complex Banach spaces, we set $\mathbb{B}(\mathbb{X}, \mathbb{Y})$ the Banach space of all bounded linear operator of $\mathbb{X}$ into $\mathbb{Y}$. For simplicity we set $\mathbb{B}(\mathbb{X}) := \mathbb{B}(\mathbb{X}, \mathbb{X})$. We denote by $\mathbb{R}^n$ the n-dimensional real space, and set $\mathbb{M}(\mathbb{R}^n, \mathbb{R}^m)$ the normed linear space of all m by n real matrices. Whenever $m = n$ we write $\mathbb{M}(\mathbb{R}^n, \mathbb{R}^n) = \mathbb{M}(\mathbb{R}^n)$ for simplicity. The superscript $'$ will indicate transpose. $L \geq 0$ and $L > 0$ will be used if a self-adjoint matrix is positive semi-definite or positive definite respectively and we write $\mathbb{M}(\mathbb{R}^n)^+ = \{L \in \mathbb{M}(\mathbb{R}^n); L = L' \geq 0\}$. We denote by $\|.\|$ either the induced norm in $\mathbb{M}(\mathbb{R}^n)$ or the standard norm in $\mathbb{R}^n$. We set diag$\{Q_s\}$ as the matrix in $\mathbb{M}(\mathbb{R}^{Sn})$ formed by $Q_1, \ldots, Q_S$ in the diagonal, and zero elsewhere.

We define $\mathbb{H}^{m,n}$ the linear space made up of all N-sequence of matrices $V = (V_1, \ldots, V_N)$, $V_i \in \mathbb{M}(\mathbb{R}^m, \mathbb{R}^n)$, $i \in \mathbb{N}$. We set $\mathbb{H}^{n,n} = \mathbb{H}^n$ and $\mathbb{H}^{n+} = \{V = (V_1, \ldots, V_N) \in \mathbb{H}^n;\ V_i \in \mathbb{M}(\mathbb{R}^n)^+, i \in \mathbb{N}\}$. For $H = (H_1, \ldots, H_N)$ and $V = (V_1, \ldots, V_N)$ in $\mathbb{H}^{n+}$ the notation $H \leq L$ ($H < L$) indicates that $H_i \leq L_i$ ($H_i < L_i$) for each $i \in \mathbb{N}$.

On a probability space $(\Omega, \mathcal{P}, \mathcal{F})$ with filtration $\{\mathcal{F}_k\}$ we define $E(.)$ as the expected value operator and ℓ_2^n as the Hilbert space formed by the sequence of second order random vectors $z = (z(0), z(1), \ldots)$ with $z(k) \in \mathbb{R}^n$ and $\mathcal{F}_k$-measurable for each $k = 0, 1, \ldots$, and such that, $\|z\|_2^2 := \sum_{k=0}^{\infty} \|z(k)\|_2^2 < \infty$, where $\|z(k)\|_2^2 := E(\|z(k)\|^2)$.

We have the following results, stated as remarks, which will be useful in the sequel.

Remark 1: $W = \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} > 0$ if and only if $R > 0$, $Q > SR^{-1}S'$. For non-strict inequalities this result can be generalized as follows: $W = \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \geq 0$ if and only if $R \geq 0$, $Q \geq SR^{\dagger}S'$, $S(I - RR^{\dagger}) = 0$, where $R^{\dagger}$ denotes the Moore-Penrose inverse of R (see [2]).

Remark 2: If $Q > 0$ then $U + U' - Q \leq U'Q^{-1}U$.

III. PROBLEMS FORMULATION

On a probabilistic space $(\Omega, \mathcal{P}, \mathcal{F})$ consider the following controlled discrete-time linear system with Markov jumps:

$$x(k+1) = A_{\theta(k)}x(k) + B_{\theta(k)}u(k), \tag{1}$$
$$y(k) = H_{\theta(k)}x(k), \tag{2}$$
$$z(k) = C_{\theta(k)}x(k) + D_{\theta(k)}u(k), \tag{3}$$
$$x(0) = x_0, \quad \theta(0) = \theta_0, \tag{4}$$

where $x(k) \in \mathbb{R}^n$ is the state variable, $y(k) \in \mathbb{R}^p$ the observable output variable, $u(k) \in \mathbb{R}^m$ the control variable and $z(k) \in \mathbb{R}^r$ the controlled output variable. The operation mode of the system is determined by a Markov chain $\theta(k)$ taking values in the set $\mathbb{N} = \{1, \ldots, N\}$ and with transition probability matrix $\mathbf{P} = [p_{ij}]$. We will assume that the controller does not have access to neither $x(k)$ nor $\theta(k)$ but, instead, it can observe the output variable $y(k)$ and a signal $\widehat{\theta}(k) \in \mathbb{M}$. This signal takes values in a finite set $\mathbb{M} = \{1, \ldots M\}$, and is related to the Markov chain $\theta(k)$ in the following way. Let $\widehat{\mathcal{F}}_0$ be the σ-field generated by $\{x(0), u(0), \theta(0)\}$ and $\widehat{\mathcal{F}}_k$ be the σ-field generated by $\{x(0), u(0), \theta(0), \widehat{\theta}(0), \ldots, x(k), u(k-1), \theta(k)\}$ (therefore excluding $\widehat{\theta}(k)$ at time k). We assume that the signal provided by the detector $\widehat{\theta}(k) \in \{1, \ldots, M\}$ is related to $\theta(k)$ in such a way that

$$\begin{aligned} P(\widehat{\theta}(k) = \ell \mid \widehat{\mathcal{F}}_k) &= P(\widehat{\theta}(k) = \ell \mid \theta(k)) \\ &= \alpha_{\theta(k)\ell}, \quad \ell \in \mathbb{M}, \end{aligned} \tag{5}$$

with $\sum_{\ell=1}^{M} \alpha_{i\ell} = 1$ for each $i \in \mathbb{N}$. Roughly speaking, the values of $\widehat{\theta}(k) \in \mathbb{M}$ depends only on the present value of $\theta(k)$, being thus independent of all previous and present values of the other processes, and $\alpha_{i\ell}$ gives the probability of $\widehat{\theta}(k) = \ell$ whenever $\theta(k) = i$. We define for each $i \in \mathbb{N}$,

$$\mathcal{I}_i \doteq \{\ell \in \mathbb{M};\ \alpha_{i\ell} > 0\} = \{k_1^i, \ldots, k_{\tau^i}^i\}$$

and we assume that $\cup_{i=1}^{N} \mathcal{I}_i = \mathbb{M}$. It will be convenient to define $\tau = \tau^1 + \ldots + \tau^N$.

Remark 3: An important question that may be posed is how to implement the model $(\theta(k), \widehat{\theta}(k))$ in real applications. This is an open point in the literature that deserves further investigation, that is, how to satisfactorily tune the probabilities $\alpha_{i\ell}$. A possibility is to follow a similar approach as in [22], presented within the continuous-time framework, and develop a statistical algorithm to estimate the values of $\alpha_{i\ell}$ within the discrete-time setup, or to follow the partly accessible mode detection probabilities approach as presented in [35].

As pointed out in [8], the model for $\widehat{\theta}(k)$ above encompasses the perfect information case ($M = N$ and $\alpha_{ii} = 1$, for $i \in \mathbb{N}$, which would correspond to the situation in which $\widehat{\theta}(k) = \theta(k)$, that is, $\theta(k)$ is known, and $\mathcal{I}_i = \{i\}$, $\mathbb{M} = \mathbb{N}$), the mode-independent case ($M = 1$ and $\alpha_{i1} = 1$ for all $i \in \mathbb{N}$, which corresponds to the situation in which $\widehat{\theta}(k)$ does not provide any information about $\theta(k)$, that is, $\theta(k)$ is totally unknown), and the cluster case (see [11]), which corresponds to the situation such that the state space $\mathbb{N}$ can be decomposed into disjoint sets and it is only known to which of these disjoint sets the Markov chain $\theta(k)$ belongs to.

Remark 4: We are going to study optimization problems for the finite and infinite horizon cases. For the finite horizon case all the matrices A_i, B_i, H_i, C_i, D_i, $\mathbf{P}$, and $[\alpha_{i\ell}]$ could be time-dependent but, for notational simplicity, we will consider them time invariant.

We will consider static output feedback controls using the observed emitted signal $\widehat{\theta}(k)$ instead of the unknown variable $\theta(k)$, that is, $u(k)$ will be of the following form:

$$u(k) = K_{\widehat{\theta}(k)}(k)y(k), \tag{6}$$

for $K_\ell(k) \in \mathbb{M}(\mathbb{R}^p, \mathbb{R}^m)$, $\ell \in \mathbb{M}$. Associated to a control as in (6) set for $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$,

$$A_{i\ell}(k) \doteq A_i + B_iK_\ell(k)H_i, \tag{7}$$
$$C_{i\ell}(k) \doteq C_i + D_iK_\ell(k)H_i. \tag{8}$$

For the infinite horizon case we will need the definition of stochastic stabilizability, presented next. In this case we consider time-invariant controllers

$$u(k) = K_{\widehat{\theta}(k)}y(k), \tag{9}$$

for $K_\ell \in \mathbb{M}(\mathbb{R}^p, \mathbb{R}^m)$, $\ell \in \mathbb{M}$, and set for $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$,

$$A_{i\ell} \doteq A_i + B_iK_\ell H_i, \quad C_{i\ell} \doteq C_i + D_iK_\ell H_i. \tag{10}$$

Definition 5: We say that System (1) is stochastically stabilizable if there exists $K_\ell \in \mathbb{B}(\mathbb{R}^n, \mathbb{R}^p)$, $\ell \in \mathbb{M}$, such that for $u(k)$ as in (6) we have, for every initial condition x_0 with finite second moment and every initial Markov state θ_0, that

$$\|x\|_2^2 = \sum_{k=0}^{\infty} E(\|x(k)\|^2) < \infty. \tag{11}$$

We denote by $\mathcal{K}$ the set of feedback gains $K = \{K_\ell;\, \ell \in \mathbb{M}\}$, such that stochastically stabilizes System (1).

For controllers as in (6), and a set of feedback gains $K(k) = \{K_\ell(k);\, \ell \in \mathbb{M}\}$, define the finite horizon cost, with final time T_f, as:

$$J_{T_f}(K) \doteq \sum_{k=0}^{T_f-1} E(\|z(k)\|^2) + E(\|C^f_{\theta(T_f)} x(T_f)\|^2) \tag{12}$$

with $z = (z(0), \ldots)$ given by (3). Since $u(k) = K_{\widehat{\theta}(k)}(k)y(k) = K_{\widehat{\theta}(k)}(k)H_{\theta(k)}x(k)$ we get that

$$\begin{aligned} z(k) &= C_{\theta(k)}x(k) + D_{\theta(k)}u(k) \\ &= C_{\theta(k)}x(k) + D_{\theta(k)}K_{\widehat{\theta}(k)}(k)H_{\theta(k)}x(k) \\ &= C_{\theta(k)\widehat{\theta}(k)}(k)x(k) \end{aligned} \tag{13}$$

and thus (12) can be re-written as

$$J_{T_f}(K) = \sum_{k=0}^{T_f-1} E(\|C_{\theta(k)\widehat{\theta}(k)}(k)x(k)\|^2) + E(\|C^f_{\theta(T_f)} x(T_f)\|^2). \tag{14}$$

For the infinite horizon case with $K = \{K_\ell;\, \ell \in \mathbb{M}\} \in \mathcal{K}$ and control as in (9), the cost will be given by (see (13))

$$\begin{aligned} J(K) \doteq \|z\|_2^2 &= \sum_{k=0}^{\infty} E(\|z(k)\|^2) \\ &= \sum_{k=0}^{\infty} E(\|C_{\theta(k)\widehat{\theta}(k)}x(k)\|^2). \end{aligned} \tag{15}$$

We present next the hard constraints that will be considered in our optimization problems. The motivation for these restrictions is that in many practical situations there are constraints on the manipulated and controlled variables. As pointed out in [17], constraints on the input are typically hard constraints, since they represent limitations on process equipment (such as valve saturations in industrial processes), and thus cannot be relaxed. On the other hand, constraints on the output are often associated to performance goals in which it is desired to keep the output within some range. With this in mind we will consider the following hard constraints on the state variable $x(k)$ and control variable $u(k)$. For coefficient matrices F_ι of appropriated dimensions and bounds ρ_ι, which are parameters chosen or established according to the nature or design criteria of the system to be controlled, we introduce the constraints

$$\begin{bmatrix} x(k)' & u(k)' \end{bmatrix} F_\iota' F_\iota \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \leq \rho_\iota, \quad \iota = 1, \ldots, t. \tag{16}$$

We define next the control problems we are interested in;

Definition 6 (The Finite Horizon Case): Find $K(k) = \{K_\ell(k);\, \ell \in \mathbb{M}\}$, $k = 0, \ldots, T_f - 1$, and a set $\mathcal{D}_0 \subset \mathbb{R}^n \times \mathbb{N}$ such that whenever $(x_0, \theta_0) \in \mathcal{D}_0$, we have that $J_{T_f}(K) \leq \delta\|x_0\|^2$, (16) (for $k = 0, \ldots, T_f - 1$) and the final constraint (17) (below) are satisfied,

$$x(T_f)'(G^f_\iota)'G^f_\iota x(T_f) \leq \rho^f_\iota, \quad \iota = 1, \ldots, t. \tag{17}$$

In the next definition we consider a variable $\delta > 0$ which is associated to an upper bound value for the cost.

Definition 7 (The Infinite Horizon Case): Find $K \in \mathcal{K}$ and a set $\mathcal{D}_0 \subset \mathbb{R}^n \times \mathbb{N}$ such that whenever $(x_0, \theta_0) \in \mathcal{D}_0$ we have that $J(K) \leq \delta\|x_0\|^2$ and (16) are satisfied for all $k = 0, 1, \ldots$.

We conclude this section with the following condition regarding H_i:

Condition 8: It is assumed that H_i has full row rank for all $i \in \mathbb{N}$.

From Condition 8 there exist non-singular matrices S_i such that for each $i \in \mathbb{N}$,

$$H_i S_i = \begin{bmatrix} I & 0 \end{bmatrix}. \tag{18}$$

IV. THE FINITE HORIZON CASE

In this section we analyze the finite horizon quadratic control problem as posed in Definition 6 through a LMI optimization problem. The goal will be to obtain $K(k) = \{K_\ell(k);\, \ell \in \mathbb{M}\}$, $k = 0, \ldots, T_f - 1$, which minimizes the upper bound value δ at the same time that we get a set $\mathcal{D}_0 \subset \mathbb{R}^n \times \mathbb{N}$ such that whenever $(x_0, \theta_0) \in \mathcal{D}_0$, we have that $J_{T_f}(K) \leq \delta\|x_0\|^2$ and (16), (17), are satisfied.

In order to define the LMI optimization problem, set for $i \in \mathbb{N}$, $\Gamma_i = [p_{i1}^{1/2}\mathbf{I}_1 \ldots p_{iN}^{1/2}\mathbf{I}_N] \in \mathbb{M}(\mathbb{R}^n, \mathbb{R}^{\tau n})$, where $\mathbf{I}_i$ is an $n \times \tau^i n$ matrix formed by τ^i identity matrices of dimension n, and

$$\mathrm{diag}\{R_{s\zeta}\} \doteq \mathrm{diag}\{R_{1k_1^1}, \ldots, R_{1k^1_{\tau^1}}, \ldots, R_{Nk_1^N}, \ldots, R_{Nk^N_{\tau^N}}\},$$

a block-diagonal matrix of dimension $n\tau$ and, for fixed $i \in \mathbb{N}$,

$$\mathrm{diag}\{R_{i\zeta}\} \doteq \mathrm{diag}\{R_{ik_1^i}, \ldots, R_{ik^i_{\tau^i}}\},$$

a block-diagonal matrix of dimension $n\tau_i$. Notice that $\mathrm{diag}\{R_{s\zeta}\} = \mathrm{diag}\{\mathrm{diag}\{R_{1\zeta}\}, \ldots, \mathrm{diag}\{R_{N\zeta}\}\}$. We will consider the following problem:

Problem 9: Find $\delta > 0$; $Q(k) = (Q_1(k), \ldots, Q_N(k)) > 0$; $R_{i\zeta}(k) > 0$; $\Phi_{i\zeta}$ for $i \in \mathbb{N}$, $\zeta \in \mathcal{I}_i$, $k = 0, \ldots, T_f$; and $Y_\ell(\kappa)$, $U_\ell(\kappa)$ for $\ell \in \mathbb{M}$, $\kappa = 0, \ldots, T_f - 1$, such that

$$\min \delta$$

subject to,

$$\begin{bmatrix} \delta I & \mathbf{I}_i \\ \bullet & \mathrm{diag}\{R_{i\zeta}(0)\} \end{bmatrix} \geq 0, \quad \text{for } i \in \mathbb{N}, \tag{19}$$

$$\begin{bmatrix} U_\ell(k)' + U_\ell(k) - \alpha_{i\ell}S_i^{-1}R_{i\ell}(k)(S_i^{-1})' & (A_iT_iU_\ell(k) + B_i\begin{bmatrix} Y_\ell(k)' \\ 0 \end{bmatrix})'\Gamma_i & (C_iS_iU_\ell(k) + D_i\begin{bmatrix} Y_\ell(k)' \\ 0 \end{bmatrix})' \\ \bullet & \mathrm{diag}\{R_{s\zeta}(k+1)\} & 0 \\ \bullet & \bullet & I \end{bmatrix} > 0, \tag{20}$$

$$\begin{bmatrix} \Phi_{i\zeta} + \Phi_{i\zeta}' - R_{i\zeta}(T_f) & \Phi_{i\zeta}'(C_i^f)' \\ \bullet & \tau^i I \end{bmatrix} > 0, \tag{21}$$

for $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$, $k = 0, \dots, T_f - 1$, where,

$$U_\ell(k) = \begin{bmatrix} U_{\ell,1}(k) & 0 \\ U_{\ell,2}(k) & U_{\ell,3}(k) \end{bmatrix}, \quad (22)$$

and

$$\begin{bmatrix} U'_\ell(k) + U_\ell(k) - S_i^{-1} Q_i(k)(S_i^{-1})' & (A_i T_i U_\ell(k) + B_i \begin{bmatrix} Y_\ell(k)' \\ 0 \end{bmatrix})' \\ \bullet & Q_j(k+1) \end{bmatrix} > 0, \quad (23)$$

for $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$, j such that $p_{ij} > 0$, $k = 0, \dots, T_f - 1$, and

$$\begin{bmatrix} \rho_\iota^2 I & F_\iota \begin{bmatrix} S_i U_\ell(k) \\ [Y_\ell(k) \quad 0] \end{bmatrix} \\ \bullet & U_\ell(k)' + U_\ell(k) - S_i^{-1} Q_i(k)(S_i^{-1})' \end{bmatrix} > 0, \quad (24)$$

for $\iota = 1, \dots, t$, $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$, $k = 0, \dots, T_f - 1$, and

$$\begin{bmatrix} \rho_\iota^2 I & G_\iota S_i U_\ell(T_f) \\ \bullet & U_\ell(T_f)' + U_\ell(T_f) - S_i^{-1} Q_i(T_f)(S_i^{-1})' \end{bmatrix} > 0, \quad (25)$$

for $\iota = 1, \dots, t$, $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$.

For $P(k) = (P_1(k), \dots, P_N(k)) > 0$ define the function $P(x, i, k) = x' P_i(k) x$, $i \in \mathbb{N}$, and, for $\gamma > 0$,

$$L_P(\gamma, k) := \left\{(x, i) \in \mathbb{R}^n \times \mathbb{N};\, x' P_i(k) x \le \frac{1}{\gamma}\right\}. \quad (26)$$

We have the following theorem.

Theorem 10: Suppose there is a solution $\delta > 0$, $Q(k) = (Q_1(k), \dots, Q_N(k)) > 0$, $R_{i\zeta}(k) > 0$, $\Phi_{i\zeta}$, $i \in \mathbb{N}$, $\zeta \in \mathcal{I}_i$, $k = 0, \dots, T_f$, $Y_\ell(\kappa)$, $U_\ell(\kappa)$, $\ell \in \mathbb{M}$, $\kappa = 0, \dots, T_f - 1$, for Problem 9. Define $K_\ell(k) = Y_l(k) U_{\ell,1}(k)^{-1}$, $\ell \in \mathbb{M}$ and $P(x, i, k) = x' P_i(k) x$, $P_i(k) = Q_i(k)^{-1}$, $i \in \mathbb{N}$. Then $J_{T_f}(K) \le \delta \|x_0\|^2$, and if $(x_0, \theta_0) \in L_P(1)(0)$ then $(x(k), \theta(k)) \in L_P(1, k)$ for all $k = 0, 1, \dots, T_f$ and the constraints (16), (17), are satisfied.

Proof: From (20) we must have that $U_\ell(k)$ are non-singular. Indeed, if we could find $v \neq 0$ such that $U_\ell(k) v = 0$ then from (20) and pre and pos multiplying by the vector $\begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix}$ we would end up with $v' U_\ell(k)' v + v' U_\ell(k) v - \alpha_{i\ell} v' S_i^{-1} R_{i\ell}(k)(S_i^{-1})' v = -\alpha_{i\ell} v' S_i^{-1} R_{i\ell}(k)(S_i^{-1})' v > 0$ (since $\alpha_{i\ell} > 0$ and $R_{i\ell}(k) > 0$), which is a contradiction. From this and (22) it follows that

$$U_\ell(k)^{-1} = \begin{bmatrix} U_{\ell,1}(k)^{-1} & 0 \\ -U_{\ell,3}(k)^{-1} U_{\ell,2}(k) U_{\ell,1}(k)^{-1} & U_{\ell,3}(k)^{-1} \end{bmatrix}. \quad (27)$$

From Remark 2 and (20) we get that

$$\begin{bmatrix} U_\ell(k)'(S_i^{-1}(\alpha_{i\ell} R_{i\ell}(k))(S_i^{-1})')^{-1} U_\ell(k) & (A_i T_i U_\ell(k) + B_i \begin{bmatrix} Y_\ell(k)' \\ 0 \end{bmatrix} U_\ell(k))' \Gamma_i & (C_i S_i U_\ell(k) + D_i \begin{bmatrix} Y_\ell(k)' \\ 0 \end{bmatrix})' \\ \bullet & \mathrm{diag}\{R_{s\zeta}(k+1)\} & 0 \\ \bullet & \bullet & I \end{bmatrix} > 0, \quad (28)$$

so that by pre and post multiplying (28) by $\mathrm{diag}\{(U'_\ell)(k)^{-1}, I, I\}$ and its transpose, it yields to:

$$\begin{bmatrix} S'_i(\alpha_{i\ell} R_{i\ell}(k))^{-1} S_i & (A_i T_i + B_i \begin{bmatrix} Y_\ell(k)' \\ 0 \end{bmatrix} U_\ell(k)^{-1})' \Gamma_i & (C_i S_i + D_i \begin{bmatrix} Y_\ell(k)' \\ 0 \end{bmatrix} U_\ell(k)^{-1} \\ \bullet & \mathrm{diag}\{R_{s\zeta}(k+1)\} & 0 \\ \bullet & \bullet & I \end{bmatrix} > 0. \quad (29)$$

It follows from (27) and (18) that

$$\begin{aligned} B_i \begin{bmatrix} Y_\ell(k) & 0 \end{bmatrix} U_\ell(k)^{-1} &= B_i \begin{bmatrix} Y_\ell(k) U_{\ell,1}(k)^{-1} & 0 \end{bmatrix} \\ &= B_i \begin{bmatrix} K_\ell(k) & 0 \end{bmatrix} \\ &= B_i K_\ell(k) \begin{bmatrix} I & 0 \end{bmatrix} \\ &= B_i K_\ell(k) H_i S_i, \end{aligned} \quad (30)$$

and similarly, $D_i \begin{bmatrix} Y_\ell(k) & 0 \end{bmatrix} U_\ell(k)^{-1} = D_i K_\ell(k) H_i S_i$. From (30) and pre and post multiplying (29) by $\mathrm{diag}\{(S'_i)^{-1}, I, I\}$ and its transpose, it yields to

$$\begin{bmatrix} (\alpha_{i\ell} R_{i\ell}(k))^{-1} & (A_i + B_i K_\ell(k) H_i)' \Gamma_i & (C_i + D_i K_\ell(k) H_i)' \\ \bullet & \mathrm{diag}\{R_{s\zeta}(k+1)\} & 0 \\ \bullet & \bullet & I \end{bmatrix} > 0. \quad (31)$$

From Remark 1 we get that (31) is equivalent to

$$\begin{aligned} R_{i\ell}(k)^{-1} > \alpha_{i\ell} \Big\{ &(A_i + B_i K_\ell(k) H_i)' \\ &\times \Big(\sum_{j=1}^N p_{ij} \Big(\sum_{\zeta \in \mathcal{I}_j} R_{j\zeta}(k+1)^{-1} \Big) \Big) (A_i + B_i K_\ell(k) H_i) \\ &+ (C_i + D_i K_\ell(k) H_i)'(C_i + D_i K_\ell(k) H_i) \Big\}, \end{aligned} \quad (32)$$

for $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$. Set $V_i(k) = \sum_{\zeta \in \mathcal{I}_i} R_{i\zeta}(k)^{-1}$, $i \in \mathbb{N}$, $V(k) = (V_1(k), \dots, V_N(k))$. From (32) we have that

$$\begin{aligned} V_i(k) > \sum_{\zeta \in \mathcal{I}_i} \alpha_{i\zeta} \Big\{ &(A_i + B_i K_\zeta(k) H_i)' \mathcal{E}_i(V(k+1)) \\ &\times (A_i + B_i K_\zeta(k) H_i) \\ &+ (C_i + D_i K_\zeta(k) H_i)'(C_i + D_i K_\zeta(k) H_i) \Big\}. \end{aligned} \quad (33)$$

Multiplying (33) from the left-hand side by $x(k)'$ and the right-hand by $x(k)$ and taking the expected value, we get that (from the same arguments as in the proof of Proposition 4 in [8]):

$$\begin{aligned} \left\| V_{\theta(k)}(k)^{1/2} x(k) \right\|_2^2 &= E(x(k)' V_{\theta(k)}(k) x(k)) \\ &\ge E(x(k+1)' V_{\theta(k+1)}(k+1) x(k+1)) \\ &\quad + \|z(k)\|_2^2 \\ &= \left\| V_{\theta(k+1)}(k+1)^{1/2} x(k+1) \right\|_2^2 \\ &\quad + \|z(k)\|_2^2 . \end{aligned} \quad (34)$$

Summing up (34) from $k = 0$ to $T_f - 1$, we obtain that

$$\sum_{k=0}^{T_f-1} \|z(k)\|_2^2 \le E(x_0' V_{\theta_0}(0) x_0) - E(x(T_f)' V_{\theta(T_f)}(T_f) x(T_f)) \le \delta \|x_0\|_2^2 - E(x(T_f)' V_{\theta(T_f)}(T_f) x(T_f)), \quad (35)$$

where the last inequality follows from (19) since that from Remark 1 we derive that (19) is equivalent to

$$\delta I \ge \sum_{\zeta \in \mathcal{I}_i} R_{i\zeta}^{-1}(0) = V_i(0). \quad (36)$$

Notice that, from (21) and the same arguments as above, we get that $R_{i\zeta}(T_f)^{-1} > \frac{1}{\tau^i}(C_i^f)'(C_i^f)$, so that $V_i(T_f) = \sum_{\zeta \in \mathcal{I}_i} R_{i\zeta}(T_f)^{-1} > (C_i^f)' C_i^f$ (since $= \sum_{\zeta \in \mathcal{I}_i} 1 = \tau^i$). From this and (35) we obtain that $J_{T_f}(K) \le \delta \|x_0\|_2^2$.

Let us show now that if $(x_0, \theta_0) \in L_P(1)(0)$ then $(x(k), \theta(k)) \in L_P(1, k)$ for all $k = 0, 1, \ldots, T_f$. From (23) and Remark 2, we have that

$$\begin{bmatrix} U_\ell' S_i' Q_i(k)^{-1} S_i U_\ell & (A_i S_i U_\ell(k) + B_i \begin{bmatrix} Y_\ell(k) & 0 \end{bmatrix})' \\ \bullet & Q_j \end{bmatrix} > 0, \quad (37)$$

so that by pre and post multiplying (23) by $\text{diag}\{(U_\ell')(k)^{-1}, I\}$ and its transpose, and repeating the same reasoning as in (30) we get that

$$\begin{bmatrix} S_i' Q_i(k)^{-1} S_i & (A_i S_i + B_i \begin{bmatrix} Y_\ell(k) & 0 \end{bmatrix} U_\ell(k)^{-1})') \\ \bullet & Q_j(k+1) \end{bmatrix} = \begin{bmatrix} S_i' Q_i(k)^{-1} S_i & ((A_i + B_i K_\ell(k) H_i) S_i)') \\ \bullet & Q_j(k+1) \end{bmatrix} > 0. \quad (38)$$

Pre and post multiplying (38) by $\text{diag}\{(S_i')^{-1}, I\}$ and its transpose and applying Remark 1, we get that for $p_{ij} > 0$ (after setting $P_s(k) = Q_s(k)^{-1}$):

$$P_i(k) > (A_i + B_i K_\ell(k) H_i)' P_j(k+1)(A_i + B_i K_\ell(k) H_i). \quad (39)$$

Hence, from (1) and (6), $x(k+1) = (A_{\theta(k)} + B_{\theta(k)} K_{\widehat{\theta}(k)}(k) H_{\theta(k)}) x(k)$, and from (39) we get that

$$x(k)' P_{\theta(k)}(k) x(k) \ge x(k+1)' P_{\theta(k+1)}(k+1) x(k+1). \quad (40)$$

and thus (40) yields to

$$x_0' P_{\theta_0}(0) x_0 \ge x(k)' P_{\theta(k)}(k) x(k) > x(k+1)' P_{\theta(k+1)}(k+1) x(k+1). \quad (41)$$

From (41) we have that if $(x_0, \theta_0) \in L_P(1)(0)$ (that is, $x_0' P_{\theta_0}(0) x_0 \le 1$) then $(x(k), \theta(k)) \in L_P(1, k)$ for every $k = 0, 1, \ldots$ (since from (41) $x(k)' P_{\theta(k)}(k) x(k) \le 1$). Finally let us show that the constraints (16), (17), are satisfied. From (24) and Remark 2 again we get that

$$\begin{bmatrix} \rho_\iota^2 I & F_\iota \begin{bmatrix} S_i U_\ell(k) \\ \begin{bmatrix} Y_\ell(k) & 0 \end{bmatrix} \end{bmatrix} \\ \bullet & U_\ell(k)'(S_i^{-1} Q_i(k)(S_i^{-1})')^{-1} U_\ell(k) \end{bmatrix} > 0. \quad (42)$$

Pre and post multiplying (42) by $\text{diag}\{I, (U_\ell')(k)^{-1}\}$ and its transpose, and repeating the same reasoning as in (30) we have that

$$\begin{bmatrix} \rho_\iota^2 I & F_\iota \begin{bmatrix} S_i U_\ell(k) \\ \begin{bmatrix} Y_\ell(k) & 0 \end{bmatrix} \end{bmatrix} U_\ell^{-1} \\ \bullet & S_i' Q_i(k)^{-1} S_i \end{bmatrix} = \begin{bmatrix} \rho_\iota^2 I & F_\iota \begin{bmatrix} I \\ K_\ell(k) H_i \end{bmatrix} S_i \\ \bullet & S_i' Q_i^{-1}(k) S_i \end{bmatrix} > 0. \quad (43)$$

Pre and post multiplying (43) by $\text{diag}\{I, (S_i')^{-1}\}$ and its transpose, and Remark 1, we derive that

$$\rho_\iota^2 I > F_\iota \begin{bmatrix} I \\ K_\ell(k) H_i \end{bmatrix} Q_i(k) (F_\iota \begin{bmatrix} I \\ K_\ell(k) H_i \end{bmatrix})', \quad (44)$$

so that, from (44), we conclude that

$$\|F_\iota \begin{bmatrix} I \\ K_\ell(k) H_i \end{bmatrix} Q_i(k)^{1/2}\|^2 \le \rho_\iota^2 \quad \text{for all } \ell \in \mathcal{I}_i, i \in \mathbb{N}.$$

Thus we get that

$$\begin{aligned} \|F_\iota \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}\|^2 &= \|F_\iota \begin{bmatrix} I \\ K_{\widehat{\theta}(k)} H_{\theta(k)} \end{bmatrix} x(k)\|^2 \\ &= \|F_\iota \begin{bmatrix} I \\ K_{\widehat{\theta}(k)} H_{\theta(k)} \end{bmatrix} Q_{\theta(k)}(k)^{1/2} \times Q_{\theta(k)}(k)^{-1/2} x(k)\|^2 \\ &\le \|F_\iota \begin{bmatrix} I \\ K_{\widehat{\theta}(k)} H_{\theta(k)} \end{bmatrix} Q_{\theta(k)}(k)^{1/2}\|^2 \times \|Q_{\theta(k)}(k)^{-1/2} x(k)\|^2 \\ &\le \rho_\iota^2 \; x(k)' Q_{\theta(k)}^{-1} x(k) \\ &= \rho_\iota^2 \; x(k)' P_{\theta(k)} x(k) \le \rho_\iota^2 \end{aligned} \quad (45)$$

since $x(k)' P_{\theta(k)} x(k) \le 1$, showing the result for $k = 0, \ldots, T_f - 1$. From (25) and repeating the same arguments as above we get that $\|G_\iota x(T_f)\|^2 \le \rho_\iota^2$, completing the proof. □

V. THE INFINITE HORIZON CASE

For the infinite horizon case we will need to introduce the following operators $\mathcal{E}$, $\mathcal{L}$ in $\mathbb{B}(\mathbb{H}^n)$. For $V = (V_1, \ldots, V_N) \in \mathbb{H}^n$, and $i, j \in \mathbb{N}$, $\mathcal{E}_i(V) = \sum_{j=1}^N p_{ij} V_j$, $\mathcal{L}_i(V) = \sum_{\ell \in \mathcal{I}_i} \alpha_{i\ell} A_{i\ell}' \mathcal{E}_i(V) A_{i\ell}$. The proof of the next result can be found in [8], and it presents conditions for stochastic stabilizability of System (1).

Theorem 11: System (1) is stochastically stabilizable if and only if there exists $K_\ell \in \mathbb{B}(\mathbb{R}^p, \mathbb{R}^m)$, $\ell \in \mathbb{M}$ and $P \in \mathbb{H}^n$, $P > 0$, such that for $A_{i\ell}$ as in (7),

$$P - \mathcal{L}(P) > 0. \quad (46)$$

The next LMI optimization problem aims at obtaining a $K \in \mathcal{K}$ which minimizes the upper bound value δ at the same time that obtains an invariant set $\mathcal{D}_0$ such that whenever $(x_0, \theta_0) \in \mathcal{D}_0$ we have that $(x(k), \theta(k)) \in \mathcal{D}_0$ for all $k = 0, 1, \ldots$ and the constraints (16) are satisfied. In Section VI we present other versions of this problem, either by fixing the initial condition (x_0, θ_0) or by fixing $\delta > 0$ and aiming to find the largest inner ball inside an invariant set $\mathcal{D}_0$.

Problem 12: Find $\delta > 0$; $Q = (Q_1, \dots, Q_N) > 0$; $R_{i\zeta} > 0$ for $i \in \mathbb{N}$, $\zeta \in \mathcal{I}_i$; and Y_ℓ, U_ℓ for $\ell \in \mathbb{M}$, such that

min δ

subject to,

$$\begin{bmatrix} \delta I & \mathbf{I}_i \\ \bullet & \operatorname{diag}\{R_{i\zeta}\} \end{bmatrix} \geq 0, \quad \text{for } i \in \mathbb{N}, \tag{47}$$

$$\begin{bmatrix} U_\ell' + U_\ell - \alpha_{i\ell} S_i^{-1} R_{i\ell} (S_i^{-1})' & (A_i T_i U_\ell + B_i \begin{bmatrix} Y_\ell' \\ 0 \end{bmatrix})' \Gamma_i & (C_i S_i U_\ell + D_i \begin{bmatrix} Y_\ell' \\ 0 \end{bmatrix})' \\ \bullet & \operatorname{diag}\{R_{s\zeta}\} & 0 \\ \bullet & \bullet & I \end{bmatrix} > 0, \tag{48}$$

for $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$,

$$U_\ell = \begin{bmatrix} U_{\ell,1} & 0 \\ U_{\ell,2} & U_{\ell,3} \end{bmatrix}, \tag{49}$$

$$\begin{bmatrix} U_\ell' + U_\ell - S_i^{-1} Q_i (S_i^{-1})' & (A_i T_i U_\ell + B_i \begin{bmatrix} Y_\ell & 0 \end{bmatrix})' \\ \bullet & Q_j \end{bmatrix} > 0, \tag{50}$$

for $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$, j such that $p_{ij} > 0$, and

$$\begin{bmatrix} \rho_\iota^2 I & F_\iota \begin{bmatrix} S_i U_\ell \\ \begin{bmatrix} Y_\ell & 0 \end{bmatrix} \end{bmatrix} \\ \bullet & U_\ell' + U_\ell - S_i^{-1} Q_i (S_i^{-1})' \end{bmatrix} > 0, \tag{51}$$

for $\iota = 1, \dots, t$, $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$.

For $P = (P_1, \dots, P_N) > 0$, set the function $P(x, i) = x' P_i x$, $i \in \mathbb{N}$, and, for $\gamma > 0$,

$$L_P(\gamma) := \left\{ (x, i) \in \mathbb{R}^n \times \mathbb{N};\, x' P_i x \leq \frac{1}{\gamma} \right\}. \tag{52}$$

We have the following result.

Theorem 13: Suppose there is a solution $\delta > 0$, $Q = (Q_1, \dots, Q_N) > 0$, $R_{i\zeta} > 0$, $i \in \mathbb{N}$, $\zeta \in \mathcal{I}_i$, Y_ℓ, U_ℓ, $\ell \in \mathbb{M}$, for Problem 12. Define $K_\ell = Y_l U_{\ell,1}^{-1}$, $\ell \in \mathbb{M}$ and $P(x, i) = x' P_i x$, $P_i = Q_i^{-1}$, $i \in \mathbb{N}$. Then: i) $K \in \mathcal{K}$; ii) $J(K) \leq \delta \|x_0\|^2$. If $(x_0, \theta_0) \in L_P(1)$ then: iii) $(x(k), \theta(k)) \in L_P(1)$ for all $k = 0, 1, \dots$; iv) the constraints (16) are satisfied.

Proof: Following the same steps as in the proof of Theorem 10 we obtain from (48) that

$$\begin{bmatrix} (\alpha_{i\ell} R_{i\ell})^{-1} & (A_i + B_i K_\ell H_i)' \Gamma_i & (C_i + D_i K_\ell H_i)' \\ \bullet & \operatorname{diag}\{R_{s\zeta}\} & 0 \\ \bullet & \bullet & I \end{bmatrix} > 0, \tag{53}$$

and thus, from Remark 1, we get that (53) is equivalent to

$$R_{i\ell}^{-1} > \alpha_{i\ell} \Big\{ (A_i + B_i K_\ell H_i)' \Big(\sum_{j=1}^N p_{ij} \Big(\sum_{\zeta \in \mathcal{I}_j} R_{j\zeta}^{-1} \Big) \Big) \times (A_i + B_i K_\ell H_i) + (C_i + D_i K_\ell H_i)' (C_i + D_i K_\ell H_i) \Big\} \tag{54}$$

for $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$. Set $V_i = \sum_{\zeta \in \mathcal{I}_i} R_{i\zeta}^{-1}$, $i \in \mathbb{N}$, $V = (V_1, \dots, V_N)$. From (54) we have that

$$V_i > \sum_{\zeta \in \mathcal{I}_i} \alpha_{i\zeta} \Big\{ (A_i + B_i K_\zeta H_i)' \mathcal{E}_i(V) (A_i + B_i K_\zeta H_i) + (C_i + D_i K_\zeta H_i)' (C_i + D_i K_\zeta H_i) \Big\} \tag{55}$$

and thus (55) implies that $V - \mathcal{L}(V) > 0$, so that, from Theorem 11, $K \in \mathcal{K}$, showing i). Let us now show ii). Following the same steps as in the proof of Proposition 4 in [8] we get from (55) that

$$\begin{aligned} \left\| V_{\theta(k)}^{1/2} x(k) \right\|_2^2 &= E(x(k)' V_{\theta(k)} x(k)) \\ &\geq E(x(k+1)' V_{\theta(k+1)} x(k+1)) + \|z(k)\|_2^2 \\ &= \left\| V_{\theta(k+1)}^{1/2} x(k+1) \right\|_2^2 + \|z(k)\|_2^2. \end{aligned} \tag{56}$$

Since $K \in \mathcal{K}$ we have, from the stochastic stability of (1), (6), that $\left\| V_{\theta(k)}^{1/2} x(k) \right\|_2^2 \to 0$ as $k \to \infty$. From Remark 1 we derive that (47) is equivalent to

$$\delta I \geq \sum_{\zeta \in \mathcal{I}_i} R_{i\zeta}^{-1} = V_i. \tag{57}$$

Taking the sum in (56) for $k = 0$ to ∞, and using (57) we obtain that

$$J(K) = \|z\|_2^2 = \sum_{k=0}^\infty \|z(k)\|_2^2 \leq E(x_0' V_{\theta_0} x_0) \leq \delta \|x_0\|_2^2. \tag{58}$$

Let us now show iii). From (50) and repeating the same reasoning as in the proof of Theorem 10 we get that (after setting $P_s = Q_s^{-1}$):

$$P_i > (A_i + B_i K_\ell H_i)' P_j (A_i + B_i K_\ell H_i), \quad \text{for } p_{ij} > 0, \tag{59}$$

and recalling that $x(k+1) = (A_{\theta(k)} + B_{\theta(k)} K_{\widehat{\theta}(k)} H_{\theta(k)}) x(k)$, we get from (59) that

$$x(k)' P_{\theta(k)} x(k) \geq x(k+1)' P_{\theta(k+1)} x(k+1). \tag{60}$$

so that (60) yields to

$$x_0' P_{\theta_0} x_0 \geq x(k)' P_{\theta(k)} x(k) \geq x(k+1)' P_{\theta(k+1)} x(k+1). \tag{61}$$

Therefore (61) implies that if $(x_0, \theta_0) \in L_P(1)$ (that is, $x_0' P_{\theta_0} x_0 \leq 1$) then $(x(k), \theta(k)) \in L_P(1)$ for every $k = 0, 1, \dots$ (since from (41) $x(k)' P_{\theta(k)} x(k) \leq 1$), completing the proof of iii). Let us now show iv). Repeating the same reasoning as in the proof of Theorem 10, we get from (51) that

$$\rho_\iota^2 I > F_\iota \begin{bmatrix} I \\ K_\ell H_i \end{bmatrix} Q_i (F_\iota \begin{bmatrix} I \\ K_\ell H_i \end{bmatrix})', \tag{62}$$

and from (62) we conclude that $\|F_\iota \begin{bmatrix} I \\ K_\ell H_i \end{bmatrix} Q_i^{1/2}\|^2 \leq \rho_\iota^2$ for all $\ell \in \mathcal{I}_i$, $i \in \mathbb{N}$. As in the proof of Theorem 10, this implies that

$$\begin{aligned} \|F_\iota \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}\|^2 &\leq \|F_\iota \begin{bmatrix} I \\ K_\ell H_i \end{bmatrix} Q_i^{1/2}\|^2 \|Q_{\theta(k)}^{-1/2} x(k)\|^2 \\ &\leq \rho_\iota^2\, x(k)' Q_{\theta(k)}^{-1} x(k) = \rho_\iota^2\, x(k)' P_{\theta(k)} x(k) \\ &\leq \rho_\iota^2 \end{aligned}$$

since $x(k)' P_{\theta(k)} x(k) \leq 1$, completing the proof. □

VI. ALTERNATIVE PROBLEMS

We have 2 alternative problems associated to Problems 9 (finite horizon case) and 12 (infinite horizon case). For simplicity we will present the results only for the infinite horizon case (Problem 12). These 2 new problems will be referred to as Problem 14 and Problem 15. If the initial condition x_0 and the initial probability for θ_0 are fixed, with $\mu_i = \mathcal{P}(\theta_0 = i)$ for $i \in \mathbb{N}$, we can re-write Problem 12, taking into account these specific initial conditions, as Problem 14 in the following way:

Problem 14: Find $\delta > 0$; $Q = (Q_1, \dots, Q_N) > 0$; $R_{i\zeta} > 0$ for $i \in \mathbb{N}$, $\zeta \in \mathcal{I}_i$; and Y_ℓ, U_ℓ for $\ell \in \mathbb{M}$, such that

$$\min \ \delta$$

subject to,

$$\begin{bmatrix} \delta & x_0' \left[\mu_1^{1/2}\mathbf{I}_1 \quad \cdots \quad \mu_N^{1/2}\mathbf{I}_N \right] \\ \bullet & \operatorname{diag}\{R_{s\zeta}\} \end{bmatrix} \geq 0, \qquad (63)$$

$$\begin{bmatrix} 1 & x_0' \\ \bullet & Q_i \end{bmatrix} \geq 0, \quad \text{for } i \in \mathbb{N} \text{ with } \mu_i > 0, \qquad (64)$$

and the LMIs (48)-(51).

As shown in Corollary 1 in [45], we have that (63) and Remark 1 implies that

$$\delta \geq x_0' \Big(\sum_{i=1}^{N} \mu_i \big(\sum_{\zeta \in \mathcal{I}_\rangle} R_{i\zeta}^{-1} \big) \Big) x_0$$
$$= x_0' \Big(\sum_{i=1}^{N} \mu_i V_i \Big) x_0 = E(x_0' V_{\theta_0} x_0) \qquad (65)$$

so that, from (58), we obtain that $J(K) \leq \delta$. From (64) it follows that $1 \geq x_0' Q_i^{-1} x_0 = x_0' P_i x_0$ for any $i \in \mathbb{N}$ with $\mu_i > 0$, and thus $(x_0, \theta_0) \in L_P(1)$. From (61) and the fact that $(x_0, \theta_0) \in L_P(1)$ we get that $(x(k), \theta(k)) \in L_P(1)$ for every $k = 0, 1, \dots$, and the remaining of the proof of Theorem 13 can be applied.

The second alternative problem would be, for a fixed $\delta > 0$, to get an approximation for the largest inner ball (with radius $\frac{1}{\upsilon}$) $\mathcal{D}_\upsilon \doteq \{x_0 \in \mathbb{R}^n; \|x_0\|^2 \leq \frac{1}{\upsilon}\}$ included in the set $\widehat{L}_P(1) := \{x \in \mathbb{R}^n; (x, i) \in L_P(1) \text{ for some } i \in \mathbb{N}\}$, in other words, to obtain the minimum $\upsilon > 0$ such that $\mathcal{D}_\upsilon \subseteq \widehat{L}_P(1)$. This problem would be too hard to be solved, so that a simplified convex version of this problem would be in the following way. Notice that, with $\upsilon = \max_{i \in \mathbb{N}} \|P_i\|$, we have that $\mathcal{D}_\upsilon \times \mathbb{N} \subset L_P(1)$ since that, if $x_0 \in \mathcal{D}_\upsilon$ then for any $i \in \mathbb{N}$ we have that $x_0' P_i x_0 \leq \|x_0\|^2 \|P_i\| \leq \|x_0\|^2 \upsilon \leq 1$. Thus, by minimizing $\max_{i \in \mathbb{N}} \|P_i\|$ υ we get the largest inner ball as defined in $\mathcal{D}_\upsilon$ with $\upsilon = \max_{i \in \mathbb{N}} \|P_i\|$, included in the set $\widehat{L}_P(1)$. Having this in mind, for $\delta > 0$ fixed, we re-write Problem 12 as follows:

Problem 15: Find $\upsilon > 0$; $Q = (Q_1, \dots, Q_N) > 0$; $R_{i\zeta} > 0$ for $i \in \mathbb{N}$, $\zeta \in \mathcal{I}_i$; and Y_ℓ, U_ℓ for $\ell \in \mathbb{M}$, such that

$$\min \ \upsilon$$

subject to,

$$\begin{bmatrix} \upsilon I & I \\ \bullet & Q_j \end{bmatrix} \geq 0, \qquad (66)$$

and the LMIs (47)-(51).

As shown in Corollary 2 in [45], we have from (66) and Remark 1 that $\upsilon I \geq Q_j^{-1}$, so that $\upsilon \geq \max_{j \in \mathbb{N}} \|Q_j^{-1}\|$. Since we want to minimize υ the optimal solution υ^* will be such that $\upsilon^* = \max_{j \in \mathbb{N}} \|Q_j^{-1}\| = \max_{j \in \mathbb{N}} \|P_j\|$ since, by definition, $P_j = Q_j^{-1}$.

VII. NUMERICAL SIMULATIONS

For the simulations it is considered the linearized model of a small unmanned aerial vehicle in steady flight with some modifications (see [9]). The state variable $x(k)$ is represented by small perturbations on the roll rate, yaw rate, sideslip, and roll angles, and the control variable $u(k)$ corresponds to the aileron and rudder commands. We assume that the aircraft's motion has two operation modes ($N = 2$), with the nominal operation model assigned by $\theta(k) = 1$ and the faulty operation mode by $\theta(k) = 2$. The system parameters for the nominal operation mode are:

$$A_1 = \begin{bmatrix} 0.5637 & 0.1133 & -0.6607 & -0.0062 \\ 0.0198 & 0.8368 & 1.0512 & 0.0089 \\ 0.0033 & -0.0450 & 0.9481 & 0.0159 \\ 0.0381 & 0.0073 & -0.0164 & 0.9999 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 2.9735 & -0.0618 \\ -0.1175 & 0.6414 \\ 0.0112 & -0.0165 \\ 0.0812 & -0.0006 \end{bmatrix}.$$

For the faulty operation mode it is assumed that the aileron command is ineffective, i.e.,

$$A_2 = A_1, \quad B_2 = B_1 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The matrices H_i and S_i are chosen as follows ($i \in \{1, 2\}$):

$$H_i = \begin{bmatrix} 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix}, \quad S_i = \begin{bmatrix} 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}.$$

Note that states $x_1(k)$ and $x_2(k)$ are not included in the output $y(k)$ (Equation (2)). The remaining parameters are set in the canonical form

$$C_i = \begin{bmatrix} I_2 & 0_2 \\ 0_2 & 0_2 \end{bmatrix}, \quad D_i = \begin{bmatrix} 0_2 \\ I_2 \end{bmatrix}, \quad i \in \{1, 2\}.$$

The transition matrix that relates the system operation modes $\mathbf{P}$, detection probability matrix $[\alpha_{i\ell}]$ and initial probability of Markov parameter θ_0 are given by:

$$\mathbf{P} = \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix}, \quad [\alpha_{i\ell}] = \begin{bmatrix} p_{i\ell} & 1 - p_{i\ell} \\ 1 - p_{i\ell} & p_{i\ell} \end{bmatrix},$$
$$\mu_0 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}.$$

For the constrained case it is considered the following hard restriction,

$$|u_2(k)| \leq 0.1,$$

and therefore we have that,

$$F_\iota = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \rho_1 = 0.1.$$

TABLE 1. Performance parameters for different cases.

Case	δ	$J(K)$	K_ℓ
(Unconstrained, state feedback)	36.1063	35.2050	$K_1 = \begin{bmatrix} -0.1699 & -0.0713 & 0.2494 & -0.1401 \\ -0.0783 & -0.7363 & -0.1282 & -0.0680 \end{bmatrix}$ $K_2 = \begin{bmatrix} -0.0632 & 0.0549 & -0.0100 & -0.0654 \\ -0.0833 & -0.7063 & -0.1671 & -0.0743 \end{bmatrix}$
(Constrained, state feedback)	89.2089	81.9124	$K_1 = \begin{bmatrix} -0.1683 & -0.0320 & 0.1795 & -0.2850 \\ -0.0034 & -0.0080 & -0.0177 & -0.0382 \end{bmatrix}$ $K_2 = \begin{bmatrix} -0.0477 & -0.0102 & 0.0695 & -0.1421 \\ -0.0035 & -0.0082 & -0.0168 & -0.0382 \end{bmatrix}$
(Unconstrained, static output)	37.9658	36.1680	$K_1 = \begin{bmatrix} 1.2627 & -0.6897 \\ -12.6984 & -0.2926 \end{bmatrix}$ $K_2 = \begin{bmatrix} 0.7927 & -0.4362 \\ -12.5349 & -0.3704 \end{bmatrix}$
(Constrained, static output)	96.6686	85.8033	$K_1 = \begin{bmatrix} 0.4318 & -2.1775 \\ -0.2322 & -0.4054 \end{bmatrix}$ $K_2 = \begin{bmatrix} 0.2617 & -1.3386 \\ -0.2208 & -0.4059 \end{bmatrix}$

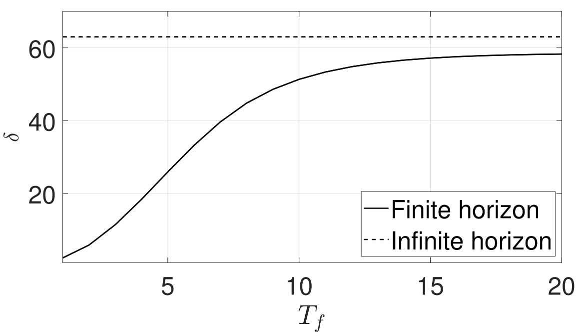

FIGURE 1. The upper bound δ for the finite horizon Case as a function of the final time T_f ($p_{i\ell} = 0.75$), and the upper bound value for the infinite horizon Case (dashed line).

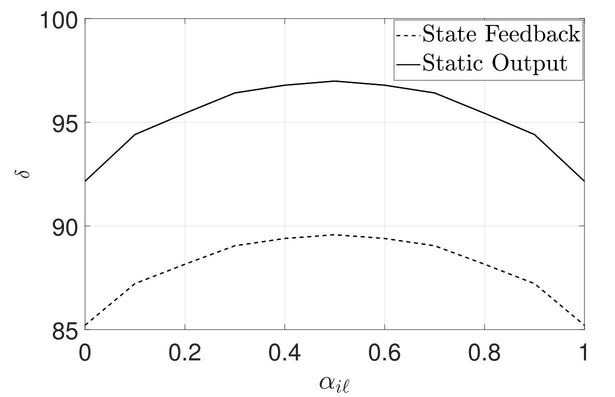

FIGURE 2. The upper bound δ for state feedback and static output cases as a function of the detection probability $\alpha_{i\ell}$.

In the sequel the numerical results will be presented, solving the previous LMI optimization problems, using the YALMIP [20] and SeDuMi [37] numerical tool packages.

Figure 1 shows the convergence of the upper bound δ for the finite horizon case (Theorem 10) as the final time T_f increases, in comparison with the upper bound obtained for the infinite horizon case (Theorem 13). As expected, the limit value for the finite horizon case is lower than for the infinite horizon case, since the finite horizon case allows time varying gains, being thus less restrictive.

The solution of Problem 14 is implemented and presented in Figure 2. We can observe that for both cases in the figure (the state feedback and the static output cases), the upper bound δ is symmetric with respect to the value $\alpha_{i\ell} = 0.5$, and it varies with the degree of information (entropy), attaining its maximum value in the equiprobable scenario ($\alpha_{i\ell} = 0.5$). This situation corresponds to the mode-independent controller since, in this case, the detector does not provide any useful information. We can also corroborate that the upper bound is lower when we have complete observation of the state vector, which results in a better performance of the control system.

For the elaboration of Figure 3 and Figure 4, Montecarlo simulations were performed with 1000 experiments. The mean value of the state (second component, $x_2(k)$) is shown in Figure 3 for the unconstrained and constrained cases. We can observe that even without considering $x_2(k)$ in the output vector (static output case), it is possible to stabilize the closed loop system. Notice that the unconstrained case performs better than the constrained case in terms of oscillations and stabilization time, as expected.

Figure 4 presents the extreme values among all the realizations of $u(k)$ (the second component $u_2(k)$ of $u(k)$ for both cases). The extreme realization values (maximum and minimum) for the constrained case are bounded by the prefixed bound $\rho_1 = 0.1$ (constant lines) as expected in the design for the constrained algorithm. On the other hand, the values for the unconstrained case are outside the region bounded by the hard constraint (shaded zone), showing the

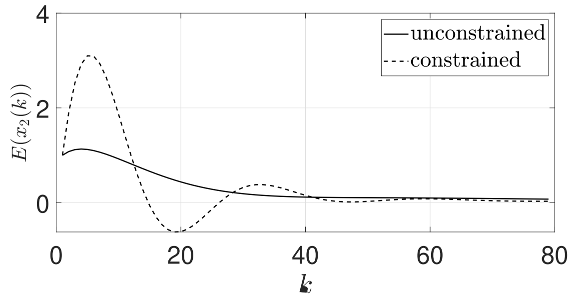

FIGURE 3. Mean value of the state (second component) $E(x_2(k))$ for the unconstrained and constrained cases ($p_{i\ell} = 0.85$).

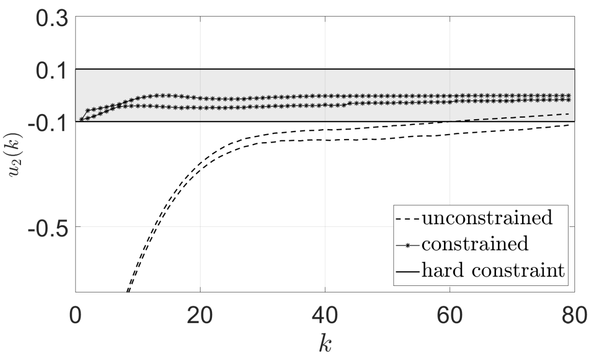

FIGURE 4. Extreme realization values of the control variable (second component) $u_2(k)$ for the unconstrained (dashed line) and constrained (star continuous line) cases ($p_{i\ell} = 0.85$).

importance of including these constraints in the optimization problem.

Some performance parameters are shown in Table 1. We can see that in all cases the real cost $J(K)$ is less than its corresponding upper bound δ. We can also see that the state feedback case has better indices (δ and $J(K)$) than the static output case, as expected. The respective controllers for each case are shown in the fourth column.

VIII. CONCLUSION

In this paper it was studied the constrained static output feedback control problem for discrete-time MJLS considering the finite horizon as well as the infinite horizon cases, and that the Markov parameter $\theta(k)$ is not directly observed. It is assumed that the only information available to the controller with respect to $\theta(k)$ comes from a detector which provides a signal $\hat{\theta}(k)$ (estimation), where it is assumed that $\left(\theta(k), \hat{\theta}(k)\right)$ follows a hidden Markov model. For the infinite horizon case the obtained results in this paper can be seen as a generalization of the state feedback case introduced in [45], by setting the output matrix as the identity matrix. Theorems 10 and 13 show that, by obtaining a solution for the LMI optimization Problems 9 and 12, we can find a static output feedback controller as in (6) for the finite horizon case, stabilizing static output feedback controller as in (9) for the infinite horizon case, such that the hard constraints (16) (and (17) for the finite horizon case) are satisfied and the quadratic costs (14) and (15) are lower than the upper bound $\delta\|x_0\|^2$. Alternative problems in which the initial conditions are fixed and in which it is desired to maximize an estimate of the domain of an invariant set are also analyzed (Problems 14 and 15 respectively). Numerical simulations were implemented showing, as expected, that the case with state feedback has better indices (δ and $J(K)$) than the static output case, and that a more reliable detector yields to a lower value for the upper bound (cost function). A future extension of the present work is to analyze the case with second moment constraints as introduced in [43] instead of hard constraints (which should yield to less conservative controllers), study the so-called positive Markov jump linear systems (PMJLS) (see for instance [19]) in order to apply the derived algorithms to the fields of reservoir control and energy planning, and consider some real applications as in wireless control network systems, in which the burden of data communication loss has to be mitigated (see, for instance, [10], [32]).

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