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Complex Social Contagions on Weighted Networks Considering Adoption Threshold Heterogeneity

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ABSTRACT Many real-world phenomena can be described as complex contagions, which has attracted much attention in the field of network science. However, the effects of the heterogeneous adoption thresholds on complex contagions in weighted networks have not been systematically investigated. In this paper, we propose a heterogeneous complex contagion model on the weighted network, in which individuals have different adoption thresholds. For individuals with a relatively small adoption threshold, they are more likely to adopt the contagion and act as activists. An edge-weight based compartmental theory is developed to unveil spreading dynamics. Through extensive numerical simulations and theoretical analysis, we find that, for any weight distribution heterogeneity, with the increase of the activist fraction, the growth pattern of the final adoption size versus the information spreading probability changes from hybrid phase transition to a second-order continuous phase transition. Meanwhile, increasing the activist fraction can promote behavior spreading. Through bifurcation analysis, we discover that changing the heterogeneity of the weight distribution will not change the type of phase transition. Besides, reducing weight distribution heterogeneity can facilitate behavior spreading. Extensive numerical simulations verify that the theoretical solutions coincide with the numerical results very well.

INDEX TERMS Complex contagions, heterogeneous adoption, weighted networks, threshold model, compartmental theory.

I. INTRODUCTION

As an important media for information spreading, social network [1]–[6] facilitate people to transmit information, such as to recommend commodities, to forward news, to exchange information, and so on [7]–[10]. Subsequently, information spreading arouses the spreading of behavior related to the information. Because of the remarkable function of the social network, researchers based on the complex network theory explored its spreading mechanism and paid more attention to its network structure [11]–[14]. Among various factors, the edges representing the social interconnection relations [15] are responsible for reliable and useful information transmission but always modeled with binary state [16]. In reality, social ties may be intimate or distant. Even for the close relation, the strength can be strong or weak. Therefore, weighted edges in a weighted network can reasonably model the ties in social networks [17], [18], such as denoting the strength of reputation in scientific network [19], [20], number of calls in communication networks [21], the public cooperation on interdependent networks [22], the number of passengers between two airports in the airline network [23].

Traditional researches leverage the epidemic process [24]–[28] and virus propagation [29]–[34] to investigate the simple contagions on social networks [35]–[37], in which a single contact between the infected and susceptible nodes is

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enough to trigger the contagion. However, for some high-risk or high-cost innovation contagions, the single contact is not enough to trigger the contagions. Because in the cases of high risk or high-cost innovation contagions, one contact merely brings in limited information and only multiple contacts can eliminate people's doubts. As a result, multiple contacts are always necessary, and social reinforcement is needed. For the contagion process with social reinforcement, researchers called it complex contagions, which is precisely depicted as a threshold model for in-depth and comprehensive exploration [38]. Furthermore, social reinforcement is mostly induced by the effect of non-redundant information memory [39], [40], which should also be involved in the threshold model on weighted networks.

Besides the complicated network structure, the widely existed heterogeneity of the behavioral agents also affects the information spreading [41], [42]: Because of different position [43], social users have different opinions; Because of discrepant action capabilities [44], [45], they show different waiting and response time; Because of diverse acceptance willingness [46], [47], they usually exhibit distinctive adoption thresholds to mimic the same activity; and so on. The heterogeneity alters the effect of spreading dynamics and induces sophisticated statistical physical phenomena. Statistical physicists found that social heterogeneity can engender a kind of hybrid phase transition [48], [49], which simultaneously possesses the traditional first-order discontinuous phase transition and the second-order continuous phase transition. Because of different knowledge levels, practical experiences, personal attitudes, and social emotions, social users have distinguishing desires to adopt informed behavior with heterogeneous adoption thresholds. We regard the enthusiastic individual with a low adoption threshold as an activist and the passive individual with a high adoption threshold as conservative. For example, an investor with sufficient practical experience may more easily adopt investment advice than a freshman.

In this paper, we articulate a general complex contagion model to investigate the aforementioned heterogeneous adoption on weighted social networks, emphasizing the social reinforcement effect derived from the memory of non-redundant information. To analyze the impact of complex contagion with heterogeneous adoption on weighted the social network, we develop a unified edge-weight based compartmental theory. Based on our proposed model and theory, under any edge weight distribution, if we increase the fraction of the activist, the growth pattern of the final adoption size versus the information spreading probability will change from the hybrid phase transition to the continuous phase transition. At the same time, if we enlarge the activist fraction, the behavior spreading will be expedited. Moreover, the behavior spreading can also be promoted if we reduce the weight distribution heterogeneity. However, the type of phase transition will not be altered. Extensive simulations and analysis suggest that our theoretical predictions coincide with numerical simulations very well.

The rest of the paper organizes as follows. In Sec. II, we describe the spreading dynamics on the weighted network. In Sec. III, we develop the edge-weight based compartmental theory. In Sec. IV, we introduce the numerical simulation method and parameter settings; In Sec. V, we give the theoretical predictions as well as simulation results and make further discussion in terms of our results. In Sec. VI, we draw the conclusions.

II. HETEROGENOUS COMPLEX CONTAGION MODEL ON WEIGHTED NETWORK

To investigate the social behavior spreading, we conceive a complex network to mimic the situation of contagion with N nodes. For avoiding the unnecessary obstruction of degree-degree correlation, a configuration model [50] is applied here to model such network with degree distribution $p(k)$. In the network, nodes and edges stand for the individuals and their relationships, respectively. Because of the inherent characteristic of social reinforcement [51], an individual i adopt a behavior when he successfully received m_i pieces of non-redundant behavior information [39] exceed his or her adoption threshold T_i, which reflects the willingness of acceptance. A smaller adoption threshold corresponds to a higher willingness of behavior acceptance. In the model, to depict the heterogenous willingness of acceptance, an individual i as an activist is assigned with adoption threshold $T_i = 1$ according to the probability q and as a conservative is assigned with adoption threshold $T_i = T$ according to the probability $1 - q$. To describe the intimate relationships, we use the weighted network, in which an edge between node i and j has a specific weight ω_{ij} to indicate the extent of intimacy in terms of a weight distribution $g(\omega)$ independent of the degree distribution $p(k)$. Because the information transmission probability from individual i to j positively correlates with their relationship weight ω_{ij}, individual i can transmit behavior information to j with probability

$$\lambda(\omega_{ij}) = 1 - (1 - \beta)^{\omega_{ij}}, \tag{1}$$

where β is the unit transmission probability. Obviously, given β, increasing ω_{ij} will augment $\lambda(\omega_{ij})$, conforming with the positive correlation between the weight of relationship and transmission preference.

Traditional model SIR (susceptible-infected-recovered) depicts the state-transition of individuals in the scene of epidemic spreading without memory of epidemic contagion. Since social reinforcement originated from the memory of non-redundant information transmission, we use a generalized susceptible-adopted-recovered (SAR) model to describe the state-transition of individuals in the scene of behavior spreading with the memory of information transmission. At each time step, an individual simultaneously can only stay in one of the three states: S-state (Susceptible), A-state (Adopted) or R-state (Recovered). An individual in the S-state can receive the behavior information from neighbors but has not adopted the behavior. An individual in the A-state adopts the behavior and can transmit the behavior information to the susceptible neighbors. Besides, an individual in the R-state abandons the behavior and refuses to transmit behavior information further.

Furthermore, in the model, social behavior spreads according to the following process. Initially, a vanishingly small fraction of individuals are randomly selected as seeds (in the adopted state). At each time step, each adopted individual i tries to deliver information to his or her susceptible neighbor j alongside an edge of weight ω_{ij} with the probability $\lambda(\omega_{ij})$. Once the information is successfully delivered through an edge, the edge will disallow for the repeated information transmission, i.e., one edge only allows for non-redundant information transmission. Noticeably, an adopted individual can try many times before the success of information transmission. After successfully receiving information, the susceptible neighbor j will add one to his or her cumulative number m_j of information pieces, i.e., $m_j \to m_j + 1$. Subsequently, the susceptible neighbor j compares the new value of m_j with his or her adoption threshold T_j. If $m_j \geq T_j$, he or she will adopt the behavior and enter into the A-state, and stay in the S-state otherwise. Because the cumulative number of received information pieces from distinct neighbors determines the adoption of behavior, the non-Markovian effect is undoubtedly generated from the behavior spreading with social reinforcement. When finishing information transmission, the adopted individual i may lose interest in the behavior and turn into the R-state with probability γ. Once the individual falls into the R-state, he or she will never participate in further behavior spreading. At last, the spreading dynamics will terminate when there are no adopted individuals left.

III. HETEROGENEOUS EDGE-WEIGHT COMPARTMENTAL THEORY

To theoretically analyze the spreading process, we develop a heterogenous edge-weight compartmental approach, which is derived from [39], [52], [53].

Notations $S(t)$, $A(t)$, and $R(t)$ are provided to respectively represent the densities of the susceptible, adopted and recovered nodes at time step t. As depicted in the model, an adopted individual i try to diffuse the information to the susceptible neighbor j through an edge owning randomly assigned weight ω_{ij} with the probability $\lambda(\omega_{ij})$. After the successful transmission of information, the susceptible neighbor j will add one to the cumulative number m_j of information pieces from distinct adopted informers, since multiple transmission through the same edge is prohibited. When m_j exceeds the adoption threshold T_j, the susceptible individual j will adopt the behavior and enter into A-state. Because of the heterogeneous behavior adoption, the adoption threshold $T_j = 1$ with probability q and $T_j = T$ with probability $1 - q$. Therefore, the probability that individual j is not informed by a neighbor i by time step t can be denoted by

$$\theta(t) = \sum_{w} g(w)\theta_w(t), \tag{2}$$

where $g(\omega)$ is the weight distribution and $\theta_w(t)$ conveys the probability that, by time step t, j remains not informed by an edge with weight ω.

Then, the individual j of degree k remains susceptible after successfully receiving m pieces of information at time step t with probability

$$\phi_m(k, t) = \binom{k}{m}[\theta(t)]^{k-m}[1 - \theta(t)]^m, \tag{3}$$

Subsequently, with regard to an individual of degree k, no matter an activist with probability q or conservative with probability $1 - q$, the individual keeps in the S-state at time step t with probability

$$\phi(k, t) = q[\theta(t)]^k + (1 - q)\sum_{m=0}^{T-1}\phi_m(k, t). \tag{4}$$

Thus, the density of the susceptible nodes at time step t is

$$S(t) = \sum_{k=0} p(k)\phi(k, t), \tag{5}$$

which also indicates the probability that an susceptible individual with arbitrary degree has not adopted the behavior by time step t.

Here, the critical quantity to solve $S(t)$ is $\theta(t)$, which depends on $\theta_w(t)$ in Eq. (2). Accordingly, we concentrate on the variable $\theta_w(t)$, which is composed of three parts, $\xi_w^S(t)$, $\xi_w^A(t)$, and $\xi_w^R(t)$, and denoted by

$$\theta_w(t) = \xi_w^S(t) + \xi_w^A(t) + \xi_w^R(t), \tag{6}$$

where $\xi_w^S(t)$ [$\xi_w^A(t)$, $\xi_w^R(t)$] indicates the probability that a neighbor i in the S-state (A-state, R-state) has not transmitted the behavior information to individual j through an edge with weight ω by time step t. After inducing $\xi_w^S(t)$ [$\xi_w^A(t)$, $\xi_w^R(t)$], we can educe the density of susceptible nodes $S(t)$ at any time step t by substituting them into Eqs. (2)-(5).

Let us first calculate the probability $\xi_w^S(t)$. According to the assumption in the model, there are no correlations between the degrees of nodes and their neighbors in uncorrelated networks. Therefore, a randomly selected neighbor j of individual i can possess the k-degree with probability $kp(k)/\langle k\rangle$, where $\langle k\rangle$ represents the mean-degree of the network. Based on the mean-field approximation, $\xi_w^S(t)$ denotes the probability that an arbitrary neighbor of individual j stays in the S-state by time step t. Thus, $\xi_w^S(t)$ can be expressed as

$$\xi_w^S(t) = \frac{1}{\langle k\rangle}\sum_{k} kp(k)\phi(k - 1, t). \tag{7}$$

In the Eq. (7), at time step t, the neighbor j of the susceptible individual i has degree k and $\phi(k - 1, t)$ denotes the probability that j has not adopted the behavior by this time.

Afterward, if an adopted neighbor j has not informed the individual i via their edge of weight ω with probability $1 - \lambda_\omega$ and meanwhile turns into the R-state with probability γ, we can leverage

$$\frac{d\xi_w^R(t)}{dt} = \gamma(1 - \lambda_w)\xi_w^A(t) \tag{8}$$

to compute $\xi_w^R(t)$.

Once the behavior information is transmitted via an edge of weight ω with probability λ_ω, the decrease of the fraction $\theta_w(t)$ can be acquired by

$$\frac{d\theta_w(t)}{dt} = -\lambda_w \xi_w^A(t), \tag{9}$$

Combining Eqs. (8) and (9), we can obtain ξ_w^R as follows

$$\xi_w^R(t) = \frac{\gamma[1-\theta_w(t)](1-\lambda_w)}{\lambda_w}. \tag{10}$$

Substituting Eqs. (7) and (10) into Eq. (6), we induce $\xi_w^A(t)$ as

$$\xi_w^A(t) = \theta_w(t) - \frac{\sum_k kp(k)\phi(k-1,t)}{\langle k\rangle} - \frac{\gamma[1-\theta_w(t)][1-\lambda_w]}{\lambda_w}. \tag{11}$$

Further substituting Eq. (11) into Eq. (9), we can obtain

$$\frac{d\theta_w(t)}{dt} = \frac{\lambda_w \sum_k kp(k)\phi(k-1,t)}{\langle k\rangle} - (1-\gamma)\lambda_w\theta_w(t) + \gamma[1-\lambda_w-\theta_w(t)]. \tag{12}$$

At last, from Eq. (12), the probability $\theta_w(t)$ can be derived.

The densities concerning A-state, and R-state can be acquired by

$$\frac{dR(t)}{dt} = \gamma A(t), \tag{13}$$

and

$$A(t) = 1 - R(t) - S(t). \tag{14}$$

Based on Eqs. (5) and (12)-(14), we can obtain the densities of individuals in S-state, A-state, and R-state at any time step t.

When setting $t \to \infty$ and $d\theta_w(t)/dt = 0$ in Eq. (12), the probability that an edge with weight ω did not spread the information in the whole contagion process can be induced by

$$\theta_w(\infty) = \frac{\langle k\rangle\gamma[1-\lambda_w] + \lambda_w \sum_k kp(k)\phi(k-1,\infty)}{\langle k\rangle[(1-\gamma)\lambda_w+\gamma]}. \tag{15}$$

Substituting $\theta_w(\infty)$ into Eq. (2), we can obtain $\theta(\infty)$ as

$$\theta(\infty) = \sum_w g(w)\frac{\langle k\rangle\gamma[1-\lambda_w] + \lambda_w \sum_k kp(k)\phi(k-1,\infty)}{\langle k\rangle[(1-\gamma)\lambda_w+\gamma]}. \tag{16}$$

Furthermore, according to Eqs.(3) - (5) and Eq. (16), we can obtain the $S(\infty)$ in the steady state. Since in the steady state $A(\infty) = 0$, the final adoption size $R(\infty) = 1 - S(\infty)$. The physical meaning of Eq. (16) indicates the roots that can be obtained in the simulated process. In the information spreading process, with the increase of information transmission rate, the final adoption size $\theta(\infty)$ monotonically decreases. Therefore, if there are more stable roots, the spreading process will stabilize at the maximum root instead of the minimum root [47].

Besides the densities of nodes in S-state, A-state, and R-state, we are also interested in the critical transmission probability β_c, which results in the final outbreak of behavior spreading. The critical transmission probability β_c appears when the function

$$f(\theta(\infty)) = \sum_w g(w)\frac{\langle k\rangle\gamma[1-\lambda_w] + \lambda_w \sum_k kp(k)\phi(k-1,\infty)}{\langle k\rangle[(1-\gamma)\lambda_w+\gamma]} - \theta(\infty) \tag{17}$$

is tangent to the horizontal axis at critical $\theta_c(\infty)$, where

$$\phi(k-1,\infty) = q[\theta(\infty)]^{(k-1)} + (1-q)\sum_{m=0}^{T-1}\phi_m(k-1,\infty). \tag{18}$$

Therefore, the critical spreading condition can be acquired by

$$\frac{df(\theta(\infty))}{d\theta(\infty)}|_{\theta_c(\infty)} = 0. \tag{19}$$

From Eq. (19), we can obtain the critical β_c. After a profound investigation, we find the final growth pattern of $R(\infty)$ as well as the situations of the critical outbreak are closely related to the fraction of activists q. Then we continue to analyze the influence of q on the critical spreading conditions in heterogeneous complex contagion.

IV. NUMERICAL METHOD

Furthermore, we perform numerical simulations to verify the above-mentioned theoretical analysis. Unless otherwise specified, based on Erdös-Rényi (ER) random model with Poisson degree distribution $p(k) = e^{-\langle k\rangle}\frac{\langle k\rangle^k}{k!}$, we build the basic network with $N = 10^4$ nodes, mean-degree $\langle k\rangle = 10$, and recovery probability $\gamma = 1.0$. Besides, network weight distribution follows $g(\omega) \sim \omega^{-\alpha_\omega}$ with $\omega_{max} \sim N^{\frac{1}{\alpha_\omega - 1}}$ and mean-weight $\langle\omega\rangle = 8$. At least 2×10^3 independent dynamical realizations on a fixed network are applied to compute the pertinent average values, which are further averaged over 100 network realizations. The complexity of running time at each time step is $O(E)$, where E is the number of edges.

In numerical verification, the *relative variance* χ [54] is adopted to numerically determine the critical conditions, such as β_c^I and β_c^{II} as follows

$$\chi = N\frac{\langle R(\infty)^2\rangle - \langle R(\infty)\rangle^2}{\langle R(\infty)\rangle}, \tag{20}$$

where $\langle R(\infty)^2\rangle$ and $\langle R(\infty)\rangle$ are the averaged values of $R(\infty)^2$ and $R(\infty)$, and N denotes the network size.

V. RESULTS AND DISCUSSION

The section discusses the results based on parameters as in Section IV. With the increase of q, the critical conditions of β_c differs, we first discuss the situation with a small q, e.g., $q = 0.25$. From Eq. (17), we find $f(\theta(\infty)) = 0$ has only one trivial

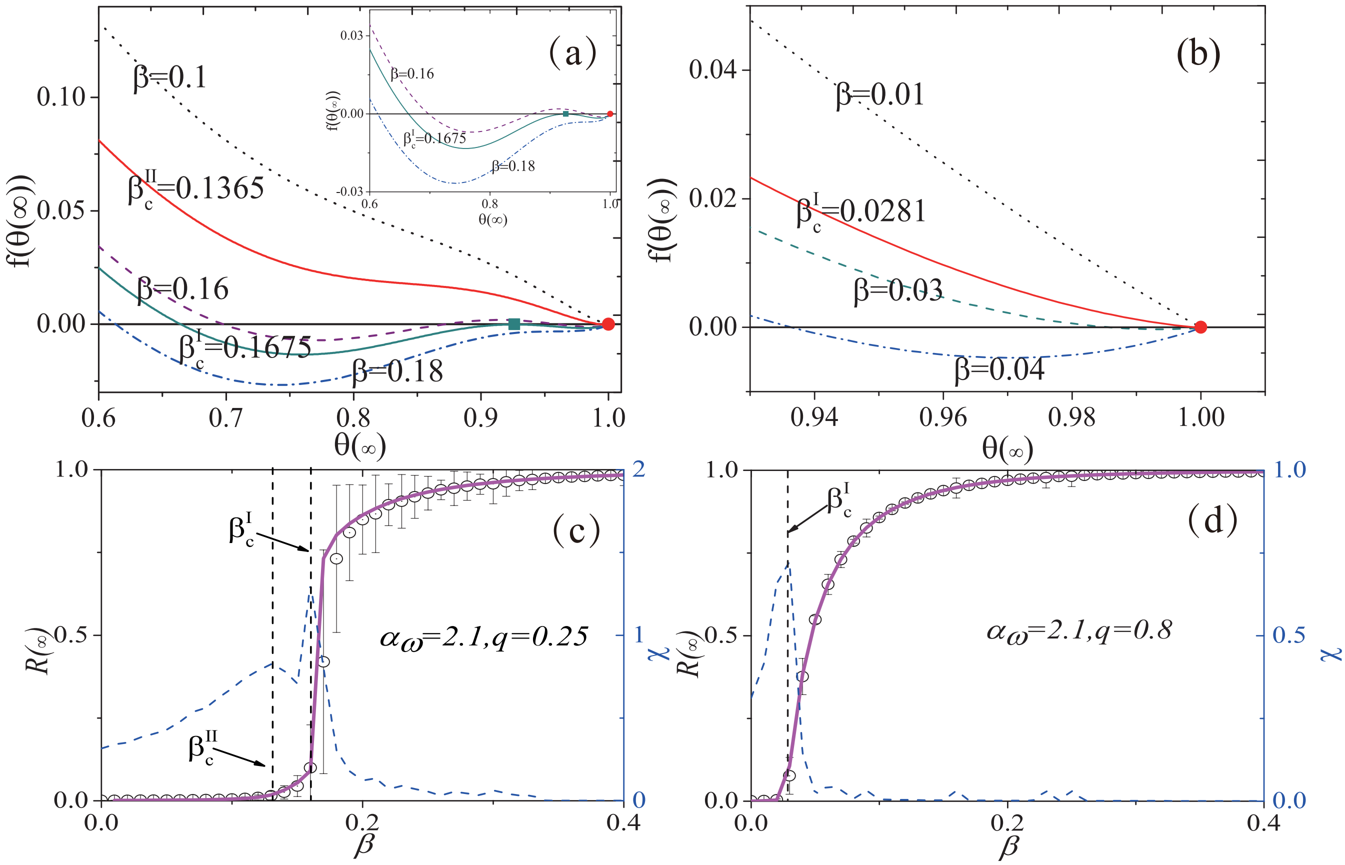

FIGURE 1. **(Color online) Graphical analysis of critical conditions and the dependence of the final adoption size $R(\infty)$ on information transmission probability β. In the top panel, the roots of $f(\theta(\infty))$ with (a) $q = 0.25$ and (b) $q = 0.8$. The final adoption size $R(\infty)$ versus β with (c) $q = 0.25$ and (d) $q = 0.8$. In (c) and (d), the solid lines indicate the theoretical solutions and the symbols represent the numerical simulations and they agree with each other very well. Moreover, in (c) and (d), the relative variance χ versus β with $q = 0.25$ and 0.8 respectively. The parameter settings are network size $N = 10^4$, mean-degree $\langle k\rangle = 10$, recovery probability $\gamma = 1.0$, $\rho_0 = 0.001$, mean-weight $\langle\omega\rangle = 8$, and network weight distribution exponent $\alpha_\omega = 2.1$.**

solution $\theta(\infty) = 1$ when β is small. With the increase of β, $f(\theta(\infty)) = 0$ continuously keeps only one trivial solution $\theta(\infty) = 1$. Until Eq. (17) is first tangent at $\theta(\infty) = 1$ (see the red dot in Fig. 1(a)), $R(\infty)$ always increases continuously (see example in Fig. 1(a)), which means there is a second-order (continuous) phase transition. From setting $\theta_c(\infty) = 1$ in Eq. (19), we can derive

$$\sum_\omega g(\omega)\frac{q\lambda_\omega[\langle k^2\rangle - \langle k\rangle]}{\langle k\rangle[(1-\gamma)\lambda_\omega + \gamma]} = 1, \tag{21}$$

where $\langle k\rangle$ and $\langle k^2\rangle$ are the first and second moments of degree distribution, respectively. Combining Eqs. (21) and (1), β_c^{II} can be obtained (see Fig. 1(a) for $\beta_c^{II} = 0.1365$), which only causes local behavior adoption.

Since Fig. 1 is intended to analyze critical conditions and the dependence of the final adoption size on information transmission probability, the two values of q are adequate for the purpose and comprehensively represent the total two phenomena. Specifically, $q = 0.25$ illustrates the case of hybrid phase transition with two critical transmission rates β_c^I and β_c^{II} of continuous growth pattern and discontinuous growth pattern, and $q = 0.8$ illustrates the case of the second order continuous phase transition with only one critical transmission rate β_c^I. As shown in Fig. 1(a), when β is large enough, three nontrivial roots of $f(\theta(\infty)) = 0$ come out (see Fig. 1(a) for $\beta = 0.16$). In this case, the largest stable root is meaningful. For $\beta = \beta_c^I = 0.1675$ (causing global behavior adoption), the tangent point $\theta_s(\infty)$ is the solution (see the green square in Fig. 1(a)). For $\beta > \beta_c^I$, the meaningful solution is the only stably fixed root. The meaningful solution of $f(\theta(\infty)) = 0$ changes abruptly from a relatively large value to a relatively small value, when $\beta > \beta_c^I$ (see Fig. 1(a) from $\beta = \beta_c^I = 0.1675$ to $\beta = 0.18$), leading to a discontinuous growth of $R(\infty)$. Based on the bifurcation theory [55], we can obtain the discontinuous critical information transmission probability by setting $\theta_c(\infty) = \theta_s(\infty)$ in Eq. (19). We can obtain β_c^I from

$$\sum_\omega g(\omega)\frac{q\lambda_\omega\sum_k kp(k)\Delta(k)}{\langle k\rangle[(1-\gamma)\lambda_\omega + \gamma]} = 1, \tag{22}$$

where

$$\Delta(k) = \frac{d\phi(k-1,\infty)}{d\theta(\infty)}|_{\theta_s(\infty)}.$$

Here $\theta_s(\infty) < 1$ is the fixed point (double roots) in Eq. (16) and combining Eq. (18) and (3) can deduce

$$\Delta(k) = (k-1)q\theta_s(\infty)^{k-2} + (1-q)\Psi(k), \tag{23}$$

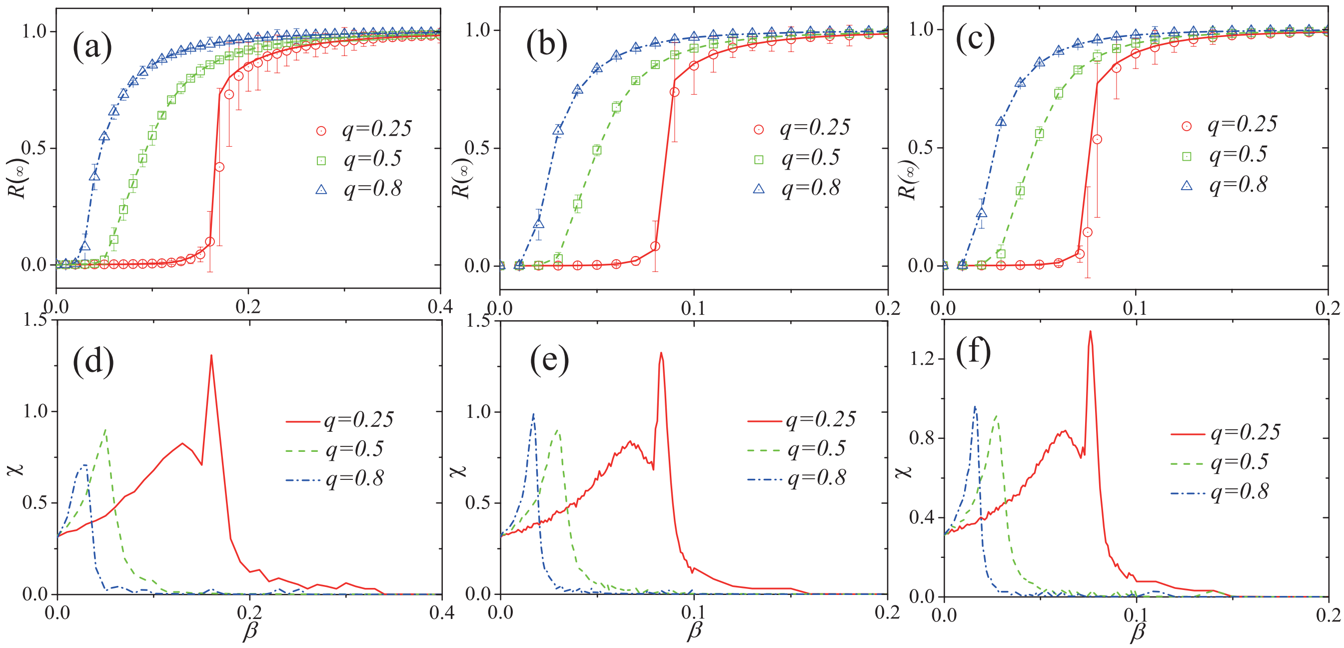

FIGURE 2. (Color online) For different weight distribution exponents α_ω, the final adoption size $R(\infty)$ versus β with (a) $\alpha_\omega = 2.1$, (b) $\alpha_\omega = 3$ and (c) $\alpha_\omega = 4$. The lines from theoretical solutions coincide with the symbols from numerical simulations very well. The relative variance χ versus β with (d) $\alpha_\omega = 2.1$, (e) $\alpha_\omega = 3$ and (f)$\alpha_\omega = 4$. The basic parameters are $N = 10^4$, $\langle k \rangle = 10$, $\langle \omega \rangle = 8$, $\rho_0 = 0.001$, and $\gamma = 1.0$.

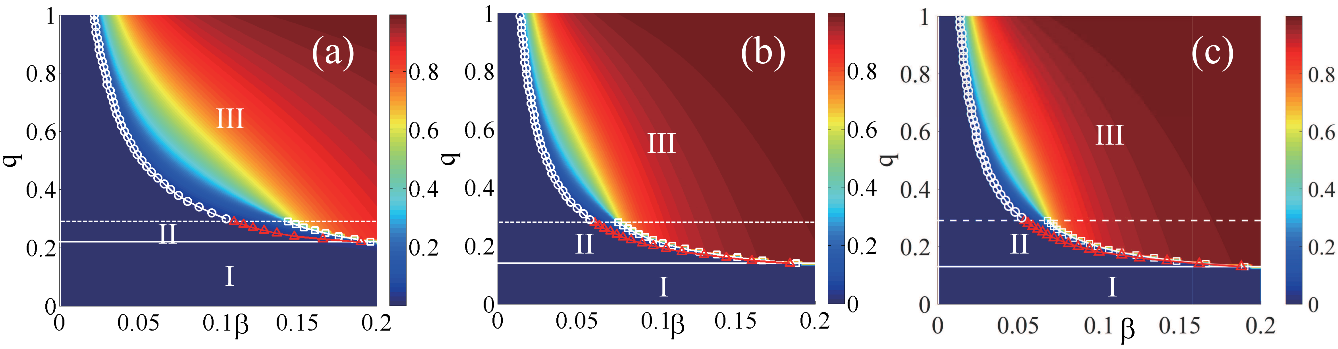

FIGURE 3. (Color online) Demonstrations of phase transition in the (β, q) parameter plane. The final adoption size $R(\infty)$ versus parameter pair (β, q) with (a) $\alpha_\omega = 2.1$, (b) $\alpha_\omega = 3$ and (c) $\alpha_\omega = 4$. Each plane is divided into three areas. In area ***I***, complex contagion can not outbreak and disappear eventually. In the area, ***II***, the growth of the final adoption size $R(\infty)$ follows hybrid phase transition. There exist critical transition probability β_c^{II} marked by a red triangle and β_c^{I} marked by white square, which respectively trigger local and global behavior adoption. In the area, ***III***, the growth of the final adoption size $R(\infty)$ embodies the second-order continuous transition and it possesses the critical transition probability β_c^{I} marked by the white circle, which results in global adoption. The basic parameters are $N = 10^4$, $\langle k \rangle = 10$, $\langle \omega \rangle = 8$, $\rho_0 = 0.001$, and $\gamma = 1.0$.

where

$$\Psi(k) = \sum_{m=0}^{T-1} \binom{k-1}{m} \Big\{ (k-1-m)\theta_s(\infty)^{k-2-m}[1-\theta_s(\infty)]^m - m\theta_s(\infty)^{k-1-m}[1-\theta_s(\infty)]^{m-1} \Big\}.$$

According to the above analysis, we discover that for small q, $R(\infty)$ versus β first continuously increases and then obeys a discontinuous pattern. And the continuous and discontinuous growth of $R(\infty)$ is induced by the activists and conservatives, respectively, which can be regarded as hybrid phase transition [55] in the view of statistical physics, because of mixing the traditional first-order and second-order transitions.

Then, we proceed to study the case of a large q, e.g., $q = 0.8$. As shown in Fig. 1(b), for any β, the Eq. (17) can only be tangent at $\theta(\infty) = 1$ when $\beta_c^I = 0.0281$. Otherwise, $f(\theta(\infty))$ does not have any other tangent point $\theta(\infty) < 1$. Moreover, the Eq. (17) has only one nontrivial solution. With the increase of β, $\theta(\infty)$ decreases continuously to a nontrivial solution (see example Fig. 1(b)), which implies that $R(\infty)$ follows continuous growth fashion all the time. From the above analysis, for large q, $R(\infty)$ grows continuously as a traditional second-order continuous phase transition. In this case, we can set $\theta_c(\infty) = 1$ in Eq. (19) and combine Eq. (18) to obtain β_c^I.

Moreover, the simulation results about influences of activist fraction q and weight distribution exponent α_ω on

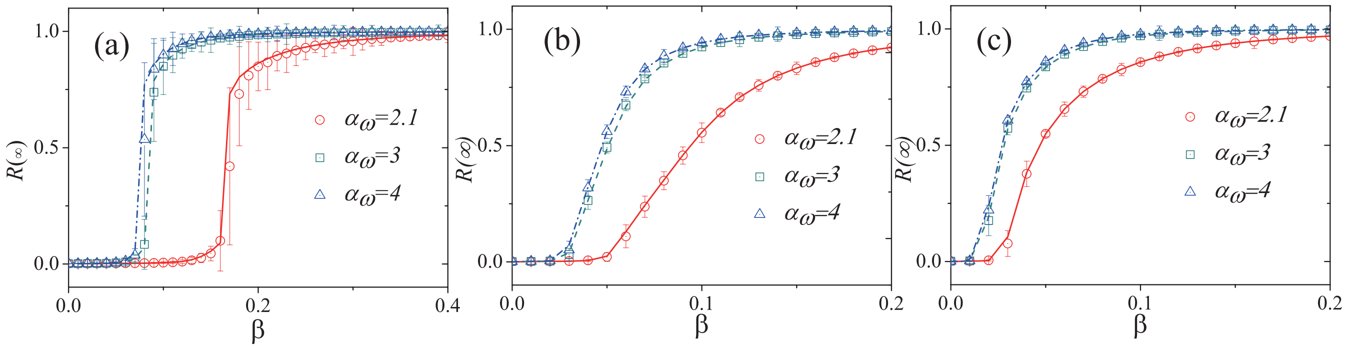

FIGURE 4. (Color online) The final adoption size $R(\infty)$ versus β with (a) $q = 0.25$, (b) $q = 0.5$ and (c) $q = 0.8$. In each subgraph, with the increase of the weight distribution exponent from 2.1 to 4, $R(\infty)$ always grows in the pattern of hybrid phase transition and can reduce the critical transmission probability, facilitating behavior spreading.

heterogeneous complex contagions will be discussed. Given other parameters as in Section IV, increasing q implies enlarging the fraction of activists; moreover, increasing α_ω means reducing the heterogeneity of weight distribution. Here we provide the results and further discussions.

Let us first investigate the impact of q on social behavior spreading. Apart from the analysis of Fig. 1(a) and (b), we provide the growth pattern of $R(\infty)$ versus β via both theoretical solutions (red solid line) and numerical method (black circle) in Figs. 1(c) and (d). In Fig. 1(c) with small $q = 0.25$, $R(\infty)$ first grows continuously with local adoption at the critical transmission probability $\beta_c^{II} = 0.1365$, and then grows discontinuously with global adoption at $\beta_c^I = 0.1675$. In Fig. 1(d) with large $q = 0.8$, $R(\infty)$ grows continuously with global adoption at the critical transmission probability $\beta_c^I = 0.0281$. In both subgraphs, the peaks of the relative variance of χ point out the critical transmission probabilities. In Fig. 1, the theoretical solutions conform to the numerical results very well. We find that given weight distribution exponent α_ω, increasing activist fraction q changes the growth pattern of $R(\infty)$ from the hybrid phase transition to second-order continuous transition, moreover, it reduces the critical transition probability β_c^I, promoting global behavior adoption.

Furthermore, under different weight distribution exponents $\alpha_\omega = 2.1, 3, 4$, we continue to investigate the impact of q on the social behavior spreading in Fig. 2. In Figs. 2(a)-(c), we still exhibit the growths of $R(\infty)$ versus β via both theoretical solutions (denoted by lines) and numerical simulation results (denoted by symbols). We find that no matter what the α_ω is, with the increase of q (from 0.25 to 0.5), the growth of $R(\infty)$ versus β consistently changes from the hybrid phase transition to the second-order continuous transition. It is evident that increasing q consistently reduces the critical transition probability under all α_ω. Moreover, in Figs. 2(d)-(f), relative variance χ points out the critical transition probabilities corresponding to Figs. 2(a), (b) and (c). Noticeably, given α_ω, increasing q can decrease the critical transition probability. The results from Fig. 2 suggest that increasing the fraction of activists can promote information spreading and change the type of phase transition.

Globally, we study the change of growth pattern of the final adoption size $R(\infty)$ with q ranging in [0,1]. In Fig. 3, we plot the final adoption size $R(\infty)$ on the (β, q) plane according to the theoretical method in Section III. In subgraph (a)($\alpha_\omega = 2.1$), (b)($\alpha_\omega = 3$) and (c)($\alpha_\omega = 4$), based on the growth pattern under different q, the plane is consistently divided into three areas. In region I, social behavior spreading can not outbreak. In region II, $R(\infty)$ first grows continuously with local adoption at critical spreading probability β_c^{II} (denoted by red triangle) and then increases discontinuously with global adoption at critical spreading probability β_c^I (denoted by the white square), showing the hybrid phase transition. In region III, $R(\infty)$ keeps growing continuously with global adoption at critical spreading probability β_c^I (denoted by a white circle), which embodies the second-order continuous transition. Moreover, we find the critical spreading probability consistent decreases with the increase of q in all subgraphs. Fig. 3 confirms again that changing weight distribution heterogeneity (i.e., changing α_ω) does not alter the growth pattern, and increasing q can promote the behavior spreading. Besides, we further find that, given q, increasing α_ω (from 2.1 in subgraph (a) to 4 in (c)) also can reduce the critical spreading probability, which means that increasing α_ω and reducing weight distribution heterogeneity can promote the behavior spreading, too.

For emphasizing the promotion of reducing weight distribution heterogeneity on the behavior spreading, we purposely give the growth pattern of $R(\infty)$ versus β with α_ω from 2.1 to 4 in Fig. 4. The lines obtained from theoretical solutions agree with the symbols from numerical results very well. In Fig. 4(a), we find $R(\infty)$ always grows in the pattern of hybrid phase transition regardless of α_ω. Analogously, in Figs. 4(b) and (c), $R(\infty)$ all grows continuously in the pattern of second-order transition. Noticeably, given q, increasing α_ω can decrease the critical transition probability, but will not alter the type of phase transition, which can also be proved by the bifurcation theory as in Fig. 1. The phenomena in Fig. 4 suggest that reducing weight distribution heterogeneity can promote the behavior spreading; however, it cannot alter the type of phase transition.

VI. CONCLUSION

In complex contagion, researchers ignore the heterogeneous adoption effect on weighted social networks with the consideration of social reinforcement derived from the memory of non-redundant information. In this paper, at first, concerning the heterogeneous adoption, we divide the population into activists and conservatives with a tunable fraction, moreover, assign heterogeneous weights to the complex network using weight distribution. Then, we propose a general complex contagion model to describe the heterogenous adoption effect on weighted social networks with heterogeneous weight distribution. Then, we adopted an edge-weight compartmental theory to predict the contagion effect. Our findings suggest that the theoretical predictions agree with the simulation results very well.

Combing the numerical simulations and the theoretical analysis, we find that even the weight distribution varies, the final adoption size grows versus the transmission probability, with the pattern changing consistently from the hybrid phase transition to the second-order continuous phase transition. At the same time, increasing the fraction of activists will spur the behavior spreading, i.e., more activists can accelerate information spreading and further promote conservatives to adopt the behavior. Moreover, diminishing the heterogeneity of the weight distribution can also promote the behavior spreading; however, it cannot alter the type of phase transition. Our work enriches the studies about phase transition phenomenon, moreover, provides insights into deeply understanding the influence of heterogeneous adoption behaviors and weighted network structures on social complex contagions.

The research on the complex contagions with heterogeneous adoption thresholds on weighted the network can offer the heuristic idea to the investigations on other spreading dynamics such as epidemic spreading, innovation spreading, marketing, and diffusion of computer virus. Moreover, our work can stimulate further works to design better control strategies to optimize the behavior spreading in the social network. Besides, the behavior spreading on weighted multiplex networks has not been investigated and deserves to be studied in the future. Additionally, this work mainly aims at innovating theoretical methods and unveiling the mechanism of social contagions on weighted complex networks considering adoption threshold heterogeneity. We will further explore the influence of the heterogeneous degree distribution in our future work with experiments on BA scale-free networks.

REFERENCES

[1] T.-Y. Chen, M.-C. Tsai, and Y.-M. Chen, "A user's personality prediction approach by mining network interaction behaviors on Facebook," *Online Inf. Rev.*, vol. 40, no. 7, pp. 913–937, Nov. 2016.

[2] Z. Wang, A. Szolnoki, and M. Perc, "Optimal interdependence between networks for the evolution of cooperation," *Sci. Rep.*, vol. 3, no. 1, Dec. 2013, Art. no. 2470.

[3] B. J. Jansen, M. Zhang, K. Sobel, and A. Chowdury, "Twitter power: Tweets as electronic word of mouth," *J. Amer. Soc. Inf. Sci. Technol.*, vol. 60, no. 11, pp. 2169–2188, Nov. 2009.

[4] J. A. Baggio, S. B. BurnSilver, A. Arenas, J. S. Magdanz, G. P. Kofinas, and M. De Domenico, "Multiplex social ecological network analysis reveals how social changes affect community robustness more than resource depletion," *Proc. Nat. Acad. Sci. USA*, vol. 113, no. 48, pp. 13708–13713, Nov. 2016.

[5] F. L. Pinheiro, F. C. Santos, and J. M. Pacheco, "Linking individual and collective behavior in adaptive social networks," *Phys. Rev. Lett.*, vol. 116, no. 12, Mar. 2016, Art. no. 128702.

[6] Y. Zhuge, "Video consumption, social networking, and influence," *Commun. ACM*, vol. 61, no. 11, pp. 76–81, Oct. 2018.

[7] L. Wu, Y. Ge, Q. Liu, E. Chen, R. Hong, J. Du, and M. Wang, "Modeling the evolution of users' preferences and social links in social networking services," *IEEE Trans. Knowl. Data Eng.*, vol. 29, no. 6, pp. 1240–1253, Jun. 2017.

[8] D. Li, S. Zhang, X. Sun, H. Zhou, S. Li, and X. Li, "Modeling information diffusion over social networks for temporal dynamic prediction," *IEEE Trans. Knowl. Data Eng.*, vol. 29, no. 9, pp. 1985–1997, Sep. 2017.

[9] M. Del Vicario, F. Zollo, G. Caldarelli, A. Scala, and W. Quattrociocchi, "Mapping social dynamics on Facebook: The Brexit debate," *Social Netw.*, vol. 50, pp. 6–16, Jul. 2017.

[10] L. Pan, W. Wang, S. Cai, and T. Zhou, "Optimal interlayer structure for promoting spreading of SIS model in two-layer networks," 2019, *arXiv:1905.09176*. [Online]. Available: http://arxiv.org/abs/1905.09176

[11] Z. Wang, L. Wang, and M. Perc, "Degree mixing in multilayer networks impedes the evolution of cooperation," *Phys. Rev. E, Stat. Phys. Plasmas Fluids Relat. Interdiscip. Top.*, vol. 89, no. 5, May 2014, Art. no. 052813.

[12] M. E. J. Newman, "The structure and function of complex networks," *SIAM Rev.*, vol. 45, no. 2, pp. 167–256, 2003.

[13] G. Rapisardi, A. Arenas, G. Caldarelli, and G. Cimini, "Multiple structural transitions in interacting networks," *Phys. Rev. E, Stat. Phys. Plasmas Fluids Relat. Interdiscip. Top.*, vol. 98, no. 1, Jul. 2018, Art. no. 012302.

[14] Z. Wang, A. Szolnoki, and M. Perc, "Self-organization towards optimally interdependent networks by means of coevolution," *New J. Phys.*, vol. 16, no. 3, 2014, Art. no. 033041.

[15] A. Arenas and M. De Domenico, "Nonlinear dynamics on interconnected networks," *Phys. D, Nonlinear Phenomena*, vols. 323–324, pp. 1–4, Jun. 2016.

[16] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang, "Complex networks: Structure and dynamics," *Phys. Rep.*, vol. 424, nos. 4–5, pp. 175–308, 2006.

[17] A. Barrat, M. Barthelemy, R. Pastor-Satorras, and A. Vespignani, "The architecture of complex weighted networks," *Proc. Nat. Acad. Sci. USA*, vol. 101, no. 11, pp. 3747–3752, Mar. 2004.

[18] I. Leyva, I. Sendiña-Nadal, J. A. Almendral, A. Navas, S. Olmi, and S. Boccaletti, "Explosive synchronization in weighted complex networks," *Phys. Rev. E, Stat. Phys. Plasmas Fluids Relat. Interdiscip. Top.*, vol. 88, no. 4, Oct. 2013, Art. no. 042808.

[19] M. E. J. Newman, "The structure of scientific collaboration networks," *Proc. Nat. Acad. Sci. USA*, vol. 98, no. 2, pp. 404–409, Jan. 2001.

[20] H. Liao, R. Xiao, G. Cimini, and M. Medo, "Network-driven reputation in online scientific communities," *PLoS ONE*, vol. 9, no. 12, 2014, Art. no. e112022.

[21] J.-P. Onnela, J. Saramäki, J. Hyvönen, G. Szabó, M. A. D. Menezes, K. Kaski, A.-L. Barabási, and J. Kertész, "Analysis of a large-scale weighted network of one-to-one human communication," *New J. Phys.*, vol. 9, no. 6, p. 179, 2007.

[22] Z. Wang, A. Szolnoki, and M. Perc, "Evolution of public cooperation on interdependent networks: The impact of biased utility functions," *Europhys. Lett.*, vol. 97, no. 4, p. 48001, Feb. 2012.

[23] L. E. C. da Rocha, "Structural evolution of the Brazilian airport network," *J. Stat. Mech., Theory Exp.*, vol. 2009, no. 4, 2009, Art. no. P04020.

[24] W. Goffman and V. A. Newill, "Generalization of epidemic theory: An application to the transmission of ideas," *Nature*, vol. 204, no. 4955, pp. 225–228, Oct. 1964.

[25] H.-F. Zhang, P.-P. Shu, Z. Wang, M. Tang, and M. Small, "Preferential imitation can invalidate targeted subsidy policies on seasonal-influenza diseases," *Appl. Math. Comput.*, vol. 294, pp. 332–342, Feb. 2017.

[26] P. Kim and C. H. Lee, "Epidemic spreading in complex networks with resilient nodes: Applications to FMD," *Complexity*, vol. 2018, pp. 1–9, Mar. 2018.

[27] J. Rashidinia, M. Sajjadian, J. Duarte, C. Januário, and N. Martins, "On the dynamical complexity of a seasonally forced discrete SIR epidemic model with a constant vaccination strategy," *Complexity*, vol. 2018, pp. 1–11, Dec. 2018.

[28] X. Chen, W. Wang, S. Cai, H. E. Stanley, and L. A. Braunstein, "Optimal resource diffusion for suppressing disease spreading in multiplex networks," *J. Stat. Mech., Theory Exp.*, vol. 2018, no. 5, 2018, Art. no. 053501.
[29] C. Gan, X. Yang, W. Liu, Q. Zhu, and X. Zhang, "An epidemic model of computer viruses with vaccination and generalized nonlinear incidence rate," *Appl. Math. Comput.*, vol. 222, pp. 265–274, Oct. 2013.
[30] C. Gan, X. Yang, Q. Zhu, J. Jin, and L. He, "The spread of computer virus under the effect of external computers," *Nonlinear Dyn.*, vol. 73, no. 3, pp. 1615–1620, Aug. 2013.
[31] C. Gan, X. Yang, W. Liu, Q. Zhu, J. Jin, and L. He, "Propagation of computer virus both across the Internet and external computers: A complex-network approach," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 19, no. 8, pp. 2785–2792, Aug. 2014.
[32] L.-X. Yang, X. Yang, J. Liu, Q. Zhu, and C. Gan, "Epidemics of computer viruses: A complex-network approach," *Appl. Math. Comput.*, vol. 219, no. 16, pp. 8705–8717, Apr. 2013.
[33] C. Gan, X. Yang, W. Liu, and Q. Zhu, "A propagation model of computer virus with nonlinear vaccination probability," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 19, no. 1, pp. 92–100, Jan. 2014.
[34] C. Gan, X. Yang, W. Liu, Q. Zhu, and X. Zhang, "Propagation of computer virus under human intervention: A dynamical model," *Discrete Dyn. Nature Soc.*, vol. 2012, pp. 1–8, Jul. 2012.
[35] D. J. Daley and D. G. Kendall, "Epidemics and rumours," *Nature*, vol. 204, no. 4963, p. 1118, Dec. 1964.
[36] X. Liu, C. Liu, and X. Zeng, "Online social network emergency public event information propagation and nonlinear mathematical modeling," *Complexity*, vol. 2017, pp. 1–7, Jun. 2017.
[37] L. Huo, F. Ding, C. Liu, and Y. Cheng, "Dynamical analysis of rumor spreading model considering node activity in complex networks," *Complexity*, vol. 2018, pp. 1–10, Nov. 2018.
[38] D. J. Watts, "A simple model of global cascades on random networks," *Proc. Nat. Acad. Sci. USA*, vol. 99, no. 9, pp. 5766–5771, Apr. 2002.
[39] W. Wang, M. Tang, H.-F. Zhang, and Y.-C. Lai, "Dynamics of social contagions with memory of nonredundant information," *Phys. Rev. E, Stat. Phys. Plasmas Fluids Relat. Interdiscip. Top.*, vol. 92, no. 1, Jul. 2015, Art. no. 012820.
[40] W. Wang, M. Cai, and M. Zheng, "Social contagions on correlated multiplex networks," *Phys. A, Stat. Mech. Appl.*, vol. 499, pp. 121–128, Jun. 2018.
[41] D. J. Watts and P. S. Dodds, "Influentials, networks, and public opinion formation," *J. Consum. Res.*, vol. 34, no. 4, pp. 441–458, Dec. 2007.
[42] M. Gutiérrez-Roig, O. Sagarra, A. Oltra, J. R. B. Palmer, F. Bartumeus, A. Díaz-Guilera, and J. Perelló, "Active and reactive behaviour in human mobility: The influence of attraction points on pedestrians," *Roy. Soc. Open Sci.*, vol. 3, no. 7, Jul. 2016, Art. no. 160177.
[43] R. Amato, N. E. Kouvaris, M. S. Miguel, and A. Díaz-Guilera, "Opinion competition dynamics on multiplex networks," *New J. Phys.*, vol. 19, no. 12, 2017, Art. no. 123019.
[44] H.-H. Jo, J. I. Perotti, K. Kaski, and J. Kertész, "Analytically solvable model of spreading dynamics with non-Poissonian processes," *Phys. Rev. X*, vol. 4, no. 1, Mar. 2014, Art. no. 011041.
[45] N. H. Sweilam, S. M. Al-Mekhlafi, and T. A. R. Assiri, "Numerical study for time delay multistrain tuberculosis model of fractional order," *Complexity*, vol. 2017, pp. 1–14, 2017.
[46] M. Karsai, G. Iñiguez, K. Kaski, and J. Kertész, "Complex contagion process in spreading of online innovation," *J. Roy. Soc. Interface*, vol. 11, no. 101, Dec. 2014, Art. no. 20140694.
[47] W. Wang, M. Tang, P. Shu, and Z. Wang, "Dynamics of social contagions with heterogeneous adoption thresholds: Crossover phenomena in phase transition," *New J. Phys.*, vol. 18, no. 1, 2016, Art. no. 013029.
[48] Y. Hu, B. Ksherim, R. Cohen, and S. Havlin, "Percolation in interdependent and interconnected networks: Abrupt change from second- to first-order transitions," *Phys. Rev. E, Stat. Phys. Plasmas Fluids Relat. Interdiscip. Top.*, vol. 84, no. 6, Dec. 2011, Art. no. 066116.
[49] C. Wu, S. Ji, R. Zhang, L. Chen, J. Chen, X. Li, and Y. Hu, "Multiple hybrid phase transition: Bootstrap percolation on complex networks with communities," *Europhys. Lett.*, vol. 107, no. 4, p. 48001, Aug. 2014.
[50] M. Catanzaro, M. Boguñá, and R. Pastor-Satorras, "Generation of uncorrelated random scale-free networks," *Phys. Rev. E, Stat. Phys. Plasmas Fluids Relat. Interdiscip. Top.*, vol. 71, no. 2, Feb. 2005, Art. no. 027103.
[51] W. Wang, M. Tang, H. E. Stanley, and L. A. Braunstein, "Unification of theoretical approaches for epidemic spreading on complex networks," *Rep. Prog. Phys.*, vol. 80, no. 3, Mar. 2017, Art. no. 036603.
[52] J. C. Miller and E. M. Volz, "Model hierarchies in edge-based compartmental modeling for infectious disease spread," *J. Math. Biol.*, vol. 67, no. 4, pp. 869–899, Oct. 2013.
[53] J. C. Miller and E. M. Volz, "Incorporating disease and population structure into models of SIR disease in contact networks," *PLoS ONE*, vol. 8, no. 8, 2013, Art. no. e69162.
[54] P. Shu, W. Wang, M. Tang, and Y. Do, "Numerical identification of epidemic thresholds for susceptible-infected-recovered model on finite-size networks," *Chaos*, vol. 25, no. 6, Jun. 2015, Art. no. 063104.
[55] S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, "Critical phenomena in complex networks," *Rev. Mod. Phys.*, vol. 80, no. 4, pp. 1275–1335, Oct. 2008.

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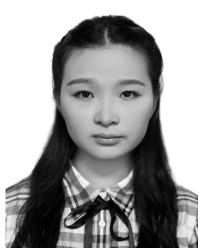

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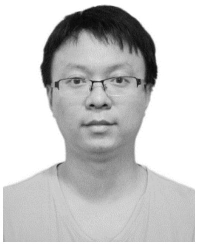

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