IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received February 19, 2020, accepted March 19, 2020, date of publication March 31, 2020, date of current version April 29, 2020.

Digital Object Identifier 10.1109/ACCESS.2020.2984558

Robust Global Boundary Vibration Control of Uncertain Timoshenko Beam With Exogenous Disturbances

MOHAMED AHMED ESHAG[1,2,3], (Student Member, IEEE), LEI MA[1], (Member, IEEE), YONGKUI SUN[1], AND KAI ZHANG[1] (Member, IEEE)

[1]School of Electrical Engineering, Southwest Jiaotong University, Chengdu 610031, China
[2]PLOP Department, Petro-Energy E&P Company, Ltd., Khartoum 13473, Sudan
[3]School of Electrical and Nuclear Engineering, Sudan University of Sciences and Technology, Khartoum 407, Sudan

Corresponding author: Lei Ma (malei@swjtu.edu.cn)

This work was supported by the NSFC-NSAF Foundation under Grant U1730105.

ABSTRACT In this paper, the dynamics solution problem and the boundary control problem for the Timoshenko beam under uncertainties and exogenous disturbances are addressed. The dynamics of the Timoshenko beam are represented by one non-homogenous hyperbolic partial differential equation (PDE), one homogenous hyperbolic PDE, and three ordinary differential equations (ODEs). The authors suggest a method of lines (MOL) for obtaining the dynamics of the Timoshenko beam in the form of the ODE formula instead of the PDE formula. A global sliding mode boundary control (GSMBC) is designed for vibration reduction of the Timoshenko beam influenced by uncertainties, distributed disturbance, displacement boundary disturbance, and rotation boundary disturbance. Chattering phenomena are avoided by using exponential reaching law reinforced by a relay function. Along the time and position axis: a) the boundary displacements and rotation of the Timoshenko beam are converged to equilibrium; b) the distributed vibrations on both displacements and rotation axis of the Timoshenko are attenuated; c) influence of the exogenous disturbances and system parameters uncertainties are compensated. By using the Lyapunov direct approach, exponential convergence and robustness of the closed-loop system are guaranteed. Finally, simulations are carried out to show that the proposed GSMBC-based MOL scheme is effective for vanishing the vibrations of the Timoshenko beam under uncertainties and external disturbances.

INDEX TERMS Boundary control, distributed parameter system (DPS), global sliding mode control (GSMC), method of lines (MOL), partial differential equation (PDE), Timoshenko beam.

I. INTRODUCTION

Flexible robotic manipulator systems are frequently used in a diversity of industrial fields such as moving strings, variable marine risers, and drill chains [1]–[4]. The Timoshenko beam is a type of flexible robotic manipulator which is an infinite dimensional system composed of one non-homogenous partial differential equation (PDE), one homogenous PDE, and three ordinary differential equations (ODEs). By considering the exogenous disturbances and parameters uncertainties of the Timoshenko beam, the dynamics solution and control implementation of such structure turns out to be hard [5]–[7].

The associate editor coordinating the review of this manuscript and approving it for publication was Jianyong Yao.

Global sliding-mode control (GSMC) has a dynamic sliding surface functions including a linear sliding surface and a nonlinear exponential decay function. These functions have benefits in robustness against the plant uncertainties and exogenous disturbances. The limitations of the sliding-mode control (SMC) are overcome in GSMC by rejecting the attaining motion phase and then guarantees that the closed-loop performance is robust [8]. GSMC plays a vital role in the lumped parameter systems (LPSs), including nonlinear system [9], uncertain nonlinear system, linear time-delay system [10], and the hypersonic slide automobile [11]. For the sake of LPS, GSMC is commonly used to tracking control [9], system parameter uncertainties [10], enhancing the system robustness [11], time-delay, system uncertainties, non-linear input, speed up the convergence of the system [12], and finite-time tracking control [13]. In [14], the robust stability for the position of the servo system is established by GSMC through static and dynamic demonstration. GSMC is suggested for the hypersonic slide-car to enhance the system robustness on the lowest norm pole assignment approach in [11]. Moreover, by using adaptive GSMC, the uncertain nonlinear system's tracking control is accomplished in [9]. Improving the performance of the complex nonlinear system is achieved by a global terminal sliding surface integrated with the reaching law in [12]. However, there has been little consideration of using GSMC for distributed parameter system (DPS). It is due to the existence of PDE form in the DPS dynamics, such that GSMC for DPS become more difficult and complicated In another hand, for the sake of Timoshenko beam, vibration control is achieved under input backlash [15], uncertainties [16], a cooperative control problem [17], input and output constraint [18], input dead zone [19], piezoelectric actuators [20], optimal piezoelectric vibration control [21], output constraint and input backlash [22], contact-force control problem [23].

In production areas, uncertainties are typically considered as a lack of data that could motivate the entire system to fail. It has significant features in the plant implementation as the closed-loop system capable to deal with the worst-case scenario [24]. Thus, the whole plant could work in a safe zone for a high range of parameter uncertainties. The process with uncertainties could be effected by disturbances, faults, inaccuracy, and instability [25]. Therefore, it is vital to tackle it through an effective approach. Until now, the aforementioned researches have studied the uncertainties with GSMC. The uncertain linear system is controlled by GSMC under the control input time delay in [10]. The handling of a chaotic nonlinear plant under various delays and input nonlinearities and uncertainties is achieved through GSMC in [26]. Moreover, by global terminal sliding-mode control, SISO nonlinear system is controlled under uncertainties in [27]. The global robust optimal sliding-mode controller is proposed for an affine nonlinear plant in the presence of uncertainties in [28]. In [29], the GSMC is suggested for Genesio's chaotic plant under nonlinearity and system parameter uncertainties. The vibration control of uncertain nonlinear system is manipulated with GSMC under exogenous disturbances in [30]. In another hand, for addressing the disturbance rejection problem, there are existing works such as the robust output regulation that is achieved by using state feedback regulator combined with a backstepping approach for coupled linear parabolic partial integro-differential equations in [31], [32]. Also, the output regulation problem is accomplished for the distributed parameter system under the backstepping approach in [33], [34]. Meanwhile, for handling the uncertainties and external disturbances, An electro-hydraulic system is controlled by a state-observer based adaptive controller under unknown velocity signal and external disturbance in [35]. The exogenous disturbance and unmeasurable system states are handled for MIMO nonlinear system by output feedback control design in [36]. By using adaptive integral robust control, the robustness against parametric uncertainties and additive disturbance are assured for nonlinear system in [37]. However, to the best of our knowledge, no research suggests global sliding-mode boundary control (GSMBC) to deal with the vibrations and uncertainties of DPS in the presence of the distributed disturbance, displacement boundary disturbance, and rotation boundary disturbance.

To attain a precise solution for the DPSs, a novel solution approach has to be devoted. Otherwise, it is prone to have imprecision, truncation errors, deflection in actual values, and weaknesses among all the data [38]. Various numerical approaches can be applied to get the approximated solution for the DPS dynamics, such as method of line (MOL) [39]–[41], the finite element (FE) [42], [43], and the finite difference (FD) [44]–[46] methods. However, most of the previous studies have been used FD method to get the numerical solution of DPS. Up to now, few results have been applied MOL for the dynamic solution of the Timoshenko beam. Although, MOL has advantages over the FD approach. Hence, in this paper, MOL is used to manipulate the limitations of the FD approach [38].

In this paper, GSMBC-based MOL design is proposed for reducing the vibrations of the Timoshenko beam with uncertainties, distributed disturbance, displacement boundary disturbance, and rotation boundary disturbance and to tackle the disadvantages of FD method. By using an exponential reaching law with a relay function, the chattering phenomena is avoided, then a nonlinear function was added to the sliding surface for increasing the robustness of the Timoshenko beam against the uncertainty and exogenous disturbances. The proposed scheme has an enthusiastic impact on the performance of the closed-loop system with unknown maximum exogenous disturbances and maximum parameters uncertainty, which to our knowledge have not been initially inspected for the Timoshenko beam.

The main contribution compared with the previous studies can be counted as follows:

a) GSMBC is proposed for reducing the vibrations and increase the robustness of the Timoshenko beam against system parameters uncertainties, distributed disturbance, displacement boundary disturbance, and rotation boundary disturbance.
b) MOL is suggested for obtaining an accurate solution for the DPS in form of ODE dynamics instead of PDE dynamics.
c) Robust stability and exponential convergence to equilibrium for the proposed scheme under maximum and minimum uncertainties and maximum exogenous disturbances are assured by Lyapunov direct approach.

The remaining part of this paper is organized as follows. Section II deals with the Timoshenko beam dynamics and MOL solution process. In Section III, proposed the GSMBC for Timoshenko beam for tackling the exogenous disturbances and uncertainties. In Section IV, simulations were carried out to illustrate the performance of GSMBC-based MOL design, while Section V gives the conclusion.

II. PROBLEM FORMULATION

The Timoshenko beam is fixed from one end and free at another end as shown in Fig. 1. Boundary control force and boundary control torque are applied simultaneously at the free end tip to diminish the vibrations induced by time-varying distributed disturbance, boundary disturbances, and the system parameters uncertainties. The Timoshenko beam variables used through this paper are shows in Table 1.

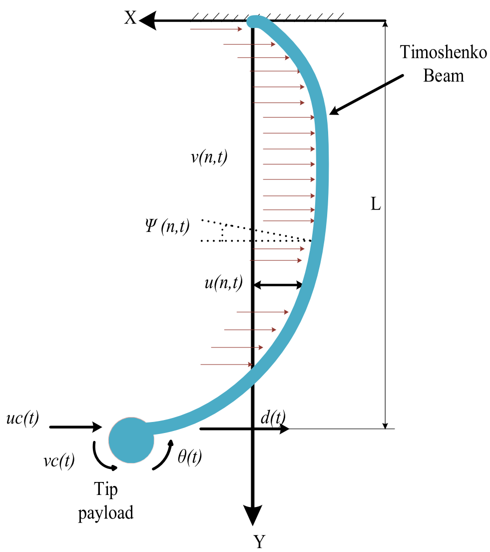

FIGURE 1. Timoshenko beam structure.

A. TIMOSHENKO BEAM MODEL

Define The model of the Timoshenko beam under distributed disturbance, displacement boundary disturbance, and rotation boundary disturbance is represented by two PDEs and three ODEs [47] in the following expressions:

$$\rho\ddot{u}(n,t) + K[\psi'(n,t) - u''(n,t)] = v(n,t) \quad (1)$$

$$I_\rho\ddot{\psi}(n,t) - EI\psi''(n,t) + K[\psi(n,t) - u'(n,t)] = 0 \quad (2)$$

with the three boundary conditions

$$u(0,t) = \psi(0,t) = 0 \quad (3)$$

$$M\ddot{u}(L,t) - K[\psi(L,t) - u'(L,t)] = u_c(t) + d(t) \quad (4)$$

$$J\ddot{\psi}(L,t) - EI\psi'(L,t) = v_c(t) + \phi(t), \quad \forall(0 < t < T, 0 < n < L) \quad (5)$$

Assumption 1: There exist three constants $\bar{F} \in R^+$, $\bar{E} \in R^+$, and $\bar{D} \in R^+$such that unidentified distributed disturbance, boundary disturbance force, and boundary disturbance torque satisfy $|f(n,t)| \le \bar{F}$, $|d(t)| \le \bar{D}$, $|\theta(t)| \le \bar{E}$, $\forall(0 < t < T, 0 < n < L)$.

B. DYNAMICS SOLUTION OF TIMOSHENKO BEAM

MOL has been suggested as an approximated numerical solution approach for the Timoshenko beam dynamic (1)-(5).

TABLE 1. Timoshenko beam variables.

Variable	Description
t	independent time variable
n	independent position variable
$u(n,t)$	displacement of the beam
$\psi(n,t)$	rotation of the beam
$u'(n,t) = \partial u(n,t)/\partial n$	displacement angle of the beam
$u''(n,t) = \partial^2 u(n,t)/\partial n^2$	displacement angular velocity
$\dot{u}(n,t) = \partial u(n,t)/\partial t$	displacement velocity of the beam
$\ddot{u}(n,t) = \partial^2 u(n,t)/\partial t^2$	displacement acceleration of the beam
$\psi'(n,t) = \partial\psi(n,t)/\partial n$	rotational angle of the beam
$\psi''(n,t) = \partial^2\psi(n,t)/\partial n^2$	rotational angular velocity
$\dot{\psi}(n,t) = \partial\psi(n,t)/\partial t$	rotational velocity of the beam
$\ddot{\psi}(n,t) = \partial^2\psi(n,t)/\partial t^2$	rotational acceleration of the beam
$f(n,t)$	time-varying distributed disturbance
$d(t)$	boundary disturbance force
$\theta(t)$	boundary disturbance torque
ρ	beam density
EI	beam bending stiffness
K	kAG
M	beam tip payload mass
I_ρ	mass moment of inertia
J	inertia of the payload
L	beam length
$u_c(t)$	boundary control force
$v_c(t)$	boundary control torque
N	position subdivision points
T	time interval

In this paper, MOL is considered for checking the Timoshenko beam performance under GSMBC. The algorithm of MOL is explained as follows:

Step (1): Discretizing the position axis n of (1) and (2) with N as consistently spaced point $n_j = n_{j-1} + dx$, $n_0 = 0$, $n_N = L$, $j = 1, 2, \ldots, N-1$, $d_x = L/N$, $N = 20$ as shown in Fig. 2. This explains the PDE description region by the

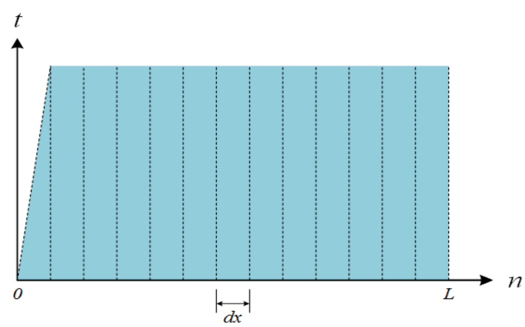

FIGURE 2. MOL definition zone.

painted area and the estimated solution by the dotted lines. Applying Taylor's series expansion for dynamic (1) and (2) results in

$$\ddot{u}(n,t) = \frac{K}{\rho dx^2}[u(n_{j+1}) - 2u(n_j) + u(n_{j-1})] - \frac{K}{\rho dx}[\psi(n_j) - \psi(n_{j-1})] + \frac{1}{\rho}v(n,t) \tag{6}$$

$$\ddot{\psi}(n,t) = \frac{EI}{I_\rho dx^2}[\psi(n_{j+1}) - 2\psi(n_j) + \psi(n_{j-1})] + \frac{K}{I_\rho dx}[\psi(n_j) - \psi(n_{j-1})] - \frac{1}{I_\rho}\psi(n_j)$$
$$j = 1,2,3,\ldots.,N-2, \quad t = 0,1,2,\ldots.,T \tag{7}$$

Step (2) Discretizing the position axis of the boundary conditions (3)-(5). Then, applying Taylor's series expansion results in

$$u(n_0) = \psi(n_0) = 0 \tag{8}$$

$$\ddot{u}(L,t) = \frac{K}{M}\psi(L) + \frac{K}{Mdx}[u(L) - u(L-1)] + \frac{1}{M}u_c(t) + \frac{1}{M}d(t) \tag{9}$$

$$\ddot{\psi}(L,t) = \frac{EI}{Jdx}[\psi(L) - \psi(L-1)] + \frac{1}{J}v_c(t) + \frac{1}{J}\phi(t) \tag{10}$$

Step (3): Applying the initial conditions results in

$$u(n_j,t) = n_j \tag{11}$$
$$\dot{u}(n_j,t) = 0 \tag{12}$$
$$\psi(n_j,t) = n_j \tag{13}$$
$$\dot{\psi}(n_j,t) = 0$$
$$j = 1,2,3,\ldots.,N-1 \tag{14}$$

Step (4): Solving the resulted ODE system by MATLAB to obtain the solution of the Timoshenko beam dynamics.

Through the MOL algorithm, the Timoshenko beam's length was subdivide by N subdivisions, but in FD method, both the length of the beam N_1 and the time interval N_2 are subdivided, which affected the position step size $dx_{FD} = L/N_1$ and the time step size $dt_{FD} = L/N_2$. Hence, the numerical algorithm of FD method depends on both the time and position step size. Meanwhile, in MOL, it depends typically on selecting applicable ODE solver. That apparently has a numerical stability feature compared with the FD approach. Table 2, shows some comparison between FD and MOL approach.

III. BOUNDARY CONTROL DESIGN

A. GLOBAL SLIDING MODE BOUNDARY CONTROL (GSMBC) DESIGN

In this section, GSMBC-based MOL design is proposed for the Timoshenko beam in the presence of parameters uncertainties and exogenous disturbances as in Fig. 3. This aim to improve the performance and confirm the robustness of

TABLE 2. Comparison between MOL and FD.

Comparison Item	MOL	FD
mathematical formulation	quite simple	complex
stability	stable	relatively stable
higher order PDE	suitable	relatively unsuitable
setup time	short	long
accuracy to the final time	not affected	effected when the final time rises
accuracy to the great value of step size	high	poor
accuracy to the short length of beam	relatively high	poor
programming tool	required	not required

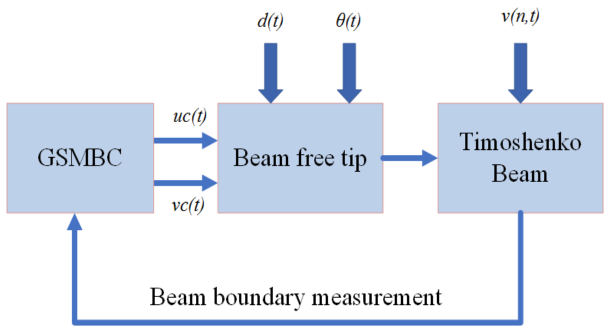

FIGURE 3. Boundary control design.

beam with regards to parameters uncertainties, reducing the unknown vibrations, eliminating the time-varying distributed disturbance, boundary disturbances, and stabilizing the beam free end to equilibrium.

The mathematical representations of GSMBC can be implemented as follow:

Firstly, design the global nonlinear-dynamic sliding function $s_1(L,t)$ for the boundary control force and $s_2(L,t)$ for the boundary control torque as shows below.

$$s_1(L,t) = ce_1(L,t) + \dot{e}_1(L,t) - \gamma_1(t) \tag{15}$$

where

$$e_1(L,t) = u(L,t) - u_d(L,t) \tag{16}$$
$$\dot{e}_1(L,t) = \dot{u}(L,t) - \dot{u}_d(L,t) \tag{17}$$

and

$$s_2(L,t) = ce_2(L,t) + \dot{e}_2(L,t) - \gamma_2(t) \tag{18}$$

where

$$e_2(L,t) = \psi(L,t) - \psi_d(L,t) \tag{19}$$
$$\dot{e}_2(L,t) = \dot{\psi}(L,t) - \dot{\psi}_d(L,t) \tag{20}$$

both $e_1(L,t)$ and $e_2(L,t)$ denotes the displacement and rotation position tracking error respectively, $c > 0$ is a constant, $u(L,t)$and $\psi(L,t)$ represent the boundary and rotation deflection respectively, $u_d(L,t)$ and $\psi_d(L,t)$ denotes the desired displacement and rotation position of the tip beam respectively, $\gamma_1(t) = \gamma_1(0)e^{-\varepsilon t}$, $\gamma_2(t) = \gamma_2(0)e^{-\varepsilon t}$, $\gamma_1(0) = ce_1(L,0) + \dot{e}_1(L,0)$, $\gamma_2(0) = ce_2(L,0) + \dot{e}_2(L,0)$ and $\varepsilon > 0$ is a constant.

By differentiating (15) and (18), one obtains

$$\dot{s}_1(L,t) = c\dot{e}_1(L,t) + \ddot{u}(L,t) - \ddot{u}_d(L,t) - \dot{\gamma}_1(t) \tag{21}$$

$$\dot{s}_2(L,t) = c\dot{e}_2(L,t) + \ddot{\psi}(L,t) - \ddot{\psi}_d(L,t) - \dot{\gamma}_2(t) \tag{22}$$

Substituting (4) and (5) into (21) and (22) respectively, one obtains

$$\begin{aligned}\dot{s}_1(L,t) &= c(\dot{u}(L,t) - \dot{u}_d(L,t)) - \ddot{u}_d(L,t) - \dot{\gamma}_1(t)\\ &\quad + \frac{\hat{K}}{\hat{M}}[\psi(L,t) - u'(L,t)] + \frac{1}{\hat{M}}u_c(t) + \frac{1}{\hat{M}}d(t)\end{aligned} \tag{23}$$

$$\begin{aligned}\dot{s}_2(L,t) &= c(\dot{\psi}(L,t) - \dot{\psi}_d(L,t)) - \ddot{\psi}_d(L,t) - \dot{\gamma}_2(t)\\ &\quad + \frac{\hat{E}I}{\hat{J}}\psi'(L,t) + \frac{1}{\hat{J}}v_c(t) + \frac{1}{\hat{J}}\phi(t)\end{aligned} \tag{24}$$

where $\hat{M}$, $\hat{K}$, $E\hat{I}$ and $\hat{J}$ are the estimated parameters of the Timoshenko beam as given below

$$\hat{M} = \frac{M_{\max} + M_{\min}}{2} \tag{25}$$

$$\Delta M = \frac{M_{\max} - M_{\min}}{2} \tag{26}$$

$$\hat{K} = \frac{K_{\max} + K_{\min}}{2} \tag{27}$$

$$\Delta K = \frac{K_{\max} - K_{\min}}{2} \tag{28}$$

$$\hat{J} = \frac{J_{\max} + J_{\min}}{2} \tag{29}$$

$$\Delta J = \frac{J_{\max} - J_{\min}}{2} \tag{30}$$

$$\hat{E}I = \frac{EI_{\max} + EI_{\min}}{2} \tag{31}$$

$$\Delta EI = \frac{EI_{\max} - EI_{\min}}{2} \tag{32}$$

For boundary displacement control of the Timoshenko beam, we choose the exponential reaching law for ensuring the reachability of sliding surface as given below

$$\dot{s}_1(L,t) = -ks_1(L,t) - \beta\frac{s_1(L,t)}{|s_1(L,t)| + \partial} \tag{33}$$

where the first term is the exponential function and the second term is the relay function, k, β, and ∂ are the positive constants.

Multiplying (15) and (21), one obtains

$$\begin{aligned}\dot{s}_1 s_1 &= s_1(c(\dot{u}(L,t) - \dot{u}_d(L,t)) - \ddot{u}_d(L,t) - \dot{\gamma}_1(t)\\ &\quad + \frac{\hat{K}}{\hat{M}}[\psi(L,t) - u'(L,t)] + \frac{1}{\hat{M}}u_c(t) + \frac{1}{\hat{M}}d(t))\end{aligned} \tag{34}$$

Therefore, for ensuring the reaching condition $\dot{s}_1 s_1 < 0$, we design GSMBC force as shown

$$\begin{aligned}u_c(t) &= -\hat{M}A_1 + \hat{M}B_1 - \hat{K}[\psi(L,t) - u'(L,t)]\\ &\quad - (\Delta M[|A_1| + |B_1|] + D + \Delta K[|\psi(L,t)|\\ &\quad - |u'(L,t)|])(ks_1(L,t) - \beta\frac{s_1(L,t)}{|s_1(L,t)| + \partial})\end{aligned} \tag{35}$$

where

$$A_1 = c\dot{u}(L,t) + \dot{\gamma}_1(t) \tag{36}$$

$$B_1 = c\dot{u}_d(L,t) + \ddot{u}_d(L,t) \tag{37}$$

For boundary rotation control of the Timoshenko beam, we choose the exponential reaching law for ensuring the reachability of sliding surface as shown

$$\dot{s}_2(L,t) = -ks_2(L,t) - \beta\frac{s_2(L,t)}{|s_2(L,t)| + \partial} \tag{38}$$

Multiplying (18) and (22), one obtains

$$\begin{aligned}\dot{s}_2 s_2 &= s_2(c(\dot{\psi}(L,t) - \dot{\psi}_d(L,t)) - \ddot{\psi}_d(L,t) - \dot{\gamma}_2(t)\\ &\quad + \frac{\hat{E}I}{\hat{J}}\psi'(L,t) + \frac{1}{\hat{J}}v_c(t) + \frac{1}{\hat{J}}\phi(t))\end{aligned} \tag{39}$$

Therefore, for ensuring the reaching condition $\dot{s}_2 s_2 < 0$, we design GSMBC rotational torque as shown

$$\begin{aligned}v_c(t) &= -\hat{J}A_2 + \hat{J}B_2 - EI\psi'(L,t) - (\Delta J[|A_2| + |B_2|] + E\\ &\quad + \Delta EI\left|\psi'(L,t)\right|)(ks_2(L,t) - \beta\frac{s_2(L,t)}{|s_2(L,t)| + \partial})\end{aligned} \tag{40}$$

where

$$A_2 = c\dot{\psi}(L,t) + \dot{\gamma}_2(t) \tag{41}$$

$$B_2 = c\dot{\psi}_d(L,t) + \ddot{\psi}_d(L,t) \tag{42}$$

Remark 1: The main challenging aspect of this paper is summarized as follows:

a) *How to tackle the uncertainties and exogenous disturbances:* Dealing with system parameter uncertainty and exogenous disturbance, other schemes have been proposed frequently by GSMC for LPS. It is challenge to accomplish stability for the Timoshenko beam with distributed disturbance, displacement boundary disturbance, and rotation boundary disturbance, as well as the uncertainties. To the author's knowledge, no researches addressed the uncertainties and external disturbances for Timoshenko beam using GSMBC.
b) *How to solve the dynamics of the Timoshenko Beam:* The numerical solutions for Timoshenko beam has been discussed repeatedly by FD approach. To the author knowledge, no researches proposed MOL for the dynamic solution of the Timoshenko beam, which is proven to be more powerful approach than FD method.
c) *How to investigate the GSMBC for Timoshenko beam:* The dynamic of the beam is a function in boundary displacement and boundary rotation. Subsequently,

these two variables are function of time and position, which make the dynamic solution and boundary control design hard to implement. In addition, GSMBC design with boundary control force input and boundary control torque input are suggested to confirm with the boundness and robust stability of the whole system with parameters uncertainty and unknown disturbances.

B. ROBUST STABILITY ANALYSIS

To demonstrate the robust stability for boundary control force design, we can choose Lyapunov function candidate as

$$Z_F(\mathrm{t}) = \frac{1}{2}s_1^2 \tag{43}$$

Substituting (4) and (35) subsequently into (21), after considering $\hat{M}$, one obtains

$$\begin{aligned}\dot{s}_1(L,t) &= c(\dot{u}(L,t) - \dot{u}_d(L,t)) - \ddot{u}_d(L,t) - \dot{\gamma}_1(t)\\ &\quad + \frac{K}{M}[\psi(L,t) - u'(L,t)] + \frac{1}{M}d(t)\\ &\quad + \frac{1}{M}[-\hat{M}A_1 + \hat{M}B_1 - K[\psi(L,t) - u'(L,t)]\\ &\quad - (\Delta M[|A_1| + |B_1|] + D)(ks_1(L,t)\\ &\quad - \beta\frac{s_1(L,t)}{|s_1(L,t)| + \partial})]\end{aligned} \tag{44}$$

Then

$$\begin{aligned}M\dot{s}_1(L,t) &= (M - \hat{M})A_1 - (M - \hat{M})B_1 + d(t)\\ &\quad - (\Delta M[|A_1| + |B_1|] + D)(ks_1(L,t)\\ &\quad - \beta\frac{s_1(L,t)}{|s_1(L,t)| + \partial})]\end{aligned} \tag{45}$$

Multiply both side of (45) by $s_1(L,t)$, one obtains

$$\begin{aligned}&M\dot{Z}_F\\ &= M\dot{s}_1(L,t)s_1(L,t) = (M - \hat{M})A_1 s_1(L,t)\\ &\quad - (M - \hat{M})B_1 s_1(L,t) + d(t)s_1(L,t)\\ &\quad - (\Delta M[|A_1| + |B_1|] + D)\left|ks_1(L,t) - \beta\frac{s_1(L,t)}{|s_1(L,t)| + \partial}\right|\end{aligned} \tag{46}$$

From (25) and (26) we have

$$M - \hat{M} = M - \frac{M_{\max} + M_{\min}}{2} \leq \frac{M_{\max} - M_{\min}}{2} = \Delta M > 0 \tag{47}$$

Therefore

$$\begin{aligned}M\dot{Z}_F &< -s_1(L,t)D\left|ks_1(L,t) - \beta\frac{s_1(L,t)}{|s_1(L,t)| + \partial}\right|\\ &\quad + ds_1(L,t) < 0\end{aligned} \tag{48}$$

$$\dot{Z}_F < 0 \tag{49}$$

Substituting (4) and (35) subsequently into (21), after considering $\hat{K}$, one obtains

$$\begin{aligned}\dot{s}_1(L,t) &= c(\dot{u}(L,t) - \dot{u}_d(L,t)) - \ddot{u}_d(L,t) - \dot{\gamma}_1(t)\\ &\quad + \frac{K}{M}[\psi(L,t) - u'(L,t)] + \frac{1}{M}d(t)\\ &\quad + \frac{1}{M}[-MA_1 + MB_1 - \hat{K}[\psi(L,t) - u'(L,t)]\\ &\quad - (\Delta K[|\psi(L,t)| + \left|u'(L,t)\right|] + D)(ks_1(L,t)\\ &\quad - \beta\frac{s_1(L,t)}{|s_1(L,t)| + \partial})]\end{aligned} \tag{50}$$

Then

$$\begin{aligned}M\dot{s}_1(L,t) &= [K - \hat{K}][\psi(L,t) + u'(L,t)] + d(t)\\ &\quad - (\Delta K[|\psi(L,t)| + \left|u'(L,t)\right|] + D)(ks_1(L,t)\\ &\quad - \beta\frac{s_1(L,t)}{|s_1(L,t)| + \partial})]\end{aligned} \tag{51}$$

Multiply both side of (51) by $s_1(L,t)$, one obtains

$$\begin{aligned}M\dot{Z}_F &= M\dot{s}_1(L,t)s_1(L,t) = [K - \hat{K}][\psi(L,t)\\ &\quad + u'(L,t)]s_1(L,t) + d(t)s_1(L,t) - (\Delta K[|\psi(L,t)|\\ &\quad + \left|u'(L,t)\right|] + D)\left|ks_1(L,t) - \beta\frac{s_1(L,t)}{|s_1(L,t)| + \partial}\right|\end{aligned} \tag{52}$$

From (27) and (28) we have

$$K - \hat{K} = K - \frac{K_{\max} + K_{\min}}{2} \leq \frac{K_{\max} - K_{\min}}{2} = \Delta K > 0 \tag{53}$$

Therefore

$$\begin{aligned}M\dot{Z}_F &< -s_1(L,t)D\left|ks_1(L,t) - \beta\frac{s_1(L,t)}{|s_1(L,t)| + \partial}\right|\\ &\quad + ds_1(L,t) < 0\end{aligned} \tag{54}$$

$$\dot{Z}_F < 0 \tag{55}$$

Subsequently, for demonstrating the robust stability for boundary control torque design, we can choose Lyapunov function candidate as

$$Z_T(\mathrm{t}) = \frac{1}{2}s_2^2 \tag{56}$$

Substituting (5) and (40) subsequently into (22), after considering $\hat{J}$, one obtains

$$\begin{aligned}\dot{s}_2(L,t) &= c\dot{e}_2(L,t) - \dddot{\psi}_d(L,t) - \dot{\gamma}_2(t)\\ &\quad + \frac{EI}{J}\psi'(L,t) + \frac{1}{J}\phi(t)\\ &\quad + \frac{1}{J}[-\hat{J}A_2 + \hat{J}B_2 - EI\psi'(L,t)]\\ &\quad - (\Delta J[|A_2| + |B_2| + E)(ks_2(L,t)\\ &\quad - \beta\frac{s_2(L,t)}{|s_2(L,t)| + \partial})]\end{aligned} \tag{57}$$

Then

$$\begin{aligned}J\dot{s}_2(L,t) &= [J - \hat{J}]A_2 + [J - \hat{J}]B_2 + \phi(t)\\ &\quad - (\Delta J[|A_2| + |B_2| + E)(ks_2(L,t)\\ &\quad - \beta\frac{s_2(L,t)}{|s_2(L,t)| + \partial})]\end{aligned} \tag{58}$$

Multiply both side of (58) by $s_2(L,t)$, one obtains

$$\begin{aligned} J\dot{Z}_T &= J\dot{s}_2(L,t)s_2(L,t) = [J-\hat{J}]A_2 s_2(L,t) \\ &\quad + [J-\hat{J}]B_2 s_2(L,t) + \phi(t)s_2(L,t) \\ &\quad - (\Delta J[|A_2| + |B_2| + E)(ks_2(L,t) \\ &\quad - \beta\frac{s_2(L,t)}{|s_2(L,t)| + \partial})] \end{aligned} \tag{59}$$

From (29) and (30) we have

$$J-\hat{J} = J - \frac{J_{\max}+J_{\min}}{2} \le \frac{J_{\max}-J_{\min}}{2} = \Delta J > 0 \tag{60}$$

Therefore

$$\begin{aligned} M\dot{Z}_T &< -s_2(L,t)E\left|ks_2(L,t) - \beta\frac{s_2(L,t)}{|s_2(L,t)|+\partial}\right| \\ &\quad + \phi(t)s_2(L,t) < 0 \end{aligned} \tag{61}$$

$$\dot{Z}_T < 0 \tag{62}$$

Substituting (5) and (40) subsequently into (22), after considering $\hat{EI}$, one obtains

$$\begin{aligned} \dot{s}_2(L,t) &= c\dot{e}_2(L,t) - \ddot{\psi}_d(L,t) - \dot{\gamma}_2(t) + \frac{EI}{J}\psi'(L,t) \\ &\quad + \frac{1}{J}\phi(t) + \frac{1}{J}[-JA_2 + JB_2 - \hat{E}I\psi'(L,t)] \\ &\quad - (\Delta EI\left|\psi'(L,t)\right| + E)(ks_2(L,t) \\ &\quad - \beta\frac{s_2(L,t)}{|s_2(L,t)|+\partial})] \end{aligned} \tag{63}$$

Then

$$\begin{aligned} J\dot{s}_2(L,t) &= [EI - \hat{E}I]\psi'(L,t) + \phi(t) \\ &\quad - (\Delta EI\left|\psi'(L,t)\right| + E)(ks_2(L,t) \\ &\quad - \beta\frac{s_2(L,t)}{|s_2(L,t)|+\partial})] \end{aligned} \tag{64}$$

Multiply both side of (64) by $s_2(L,t)$, one obtains

$$\begin{aligned} J\dot{s}_2(L,t)s_2(L,t) &= [EI - \hat{E}I]\psi'(L,t)s_2(L,t) + \phi(t)s_2(L,t) \\ &\quad - (\Delta EI\left|\psi'(L,t)\right| + E) \\ &\quad \times \left|ks_2(L,t) - \beta\frac{s_2(L,t)}{|s_2(L,t)|+\partial}\right| \end{aligned} \tag{65}$$

From (31) and (32) we have

$$EI - \hat{E}I = EI - \frac{EI_{\max}+EI_{\min}}{2} \le \frac{EI_{\max}-EI_{\min}}{2} = \Delta EI > 0 \tag{66}$$

Therefore

$$\begin{aligned} M\dot{Z}_T &< -s_2(L,t)E\left|ks_2(L,t) - \beta\frac{s_2(L,t)}{|s_2(L,t)|+\partial}\right| \\ &\quad + \phi(t)s_2(L,t) < 0 \end{aligned} \tag{67}$$

$$\dot{Z}_T < 0 \tag{68}$$

Remark 2: From (49), (56), (62), and (68) and based on the Lyapunov direct method, it can be concluded that the sliding functions $s_1(L,t)$ and $s_2(L,t)$ asymptotically converges to zero. Moreover, when $s_1(L,t) = 0$ and $s_2(L,t) = 0$, the displacement position tracking error $e_1(L,t)$ and rotation position tracking error $e_2(L,t)$ asymptotically converges to zero.

IV. SIMULATION RESULTS

In this section, the vibration attenuating of a Timoshenko beam is accomplished by GSMBC-based MOL. The Timoshenko beam is fixed at one end and free from the other end, effected by time-varying distributed disturbance $f(n,t)$, displacement boundary disturbance $d(t)$, and rotation boundary disturbance $\theta(t)$, initially at rest $u(n,0) = n$, $\dot{u}(n,0) = 0$, $\psi(n,0) = n$, and $\dot{\psi}(n,0) = 0$. Both boundary control force and boundary control torque are applied simultaneously at the free end of the beam for mitigating the vibrations stemmed by the unknown disturbances and system parameters uncertainties. The distributed disturbance is represented by the following equation:

$$f(n,t) = (2 + \sin(1.5n\pi t) + \sin(2n\pi t) + \sin(4n\pi t)) \times \frac{n}{10} \tag{69}$$

The displacement boundary disturbance is represented by the following formula:

$$d(t) = 2 + 0.2\sin(t) + 0.5\sin(0.2t) + 0.5\sin(0.5t) \tag{70}$$

The rotation boundary disturbance is represented by the following formula:

$$\theta(t) = 2 + 0.2\sin(t) + 0.5\sin(0.2t) + 0.5\sin(0.5t) \tag{71}$$

The simulation presentation of the suggested control scheme for the Timoshenko is demonstrated in the following three cases:

a) *Without control:* The Timoshenko beam is simulated with distributed disturbance (69), displacement boundary disturbance (70), and rotation boundary disturbance (71). The displacement and rotation of the Timoshenko beam without control is shown in Fig. 4 and Fig. 5, respectively. It is obvious that the deflections are significantly high for both cases.

b) *With robust adaptive control:* The robust adaptive boundary control force (72) and boundary control torque (73) 47], acts on the Timoshenko beam with the

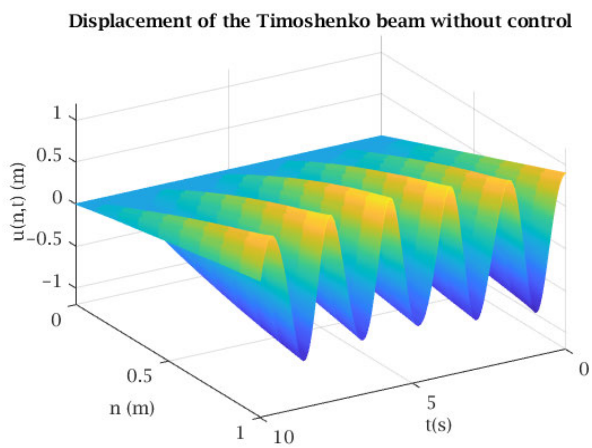

FIGURE 4. Displacement of the timoshenko beam without control.

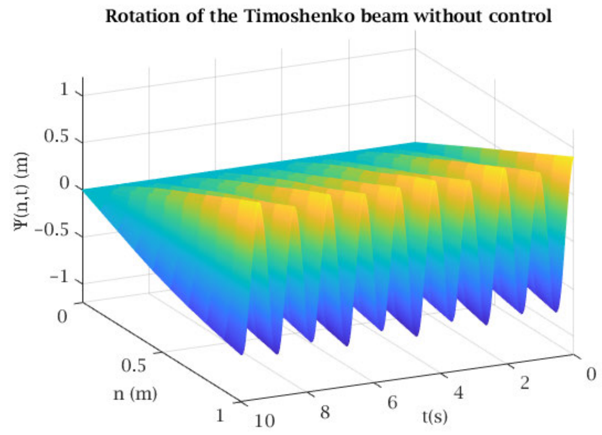

FIGURE 5. Rotation of the timoshenko beam without control.

control parameters $k_1 = 50$ and $k_2 = 50$. The spatial time representation for the displacement and rotation of the Timoshenko beam with robust adaptive control is shown in Fig. 6 and Fig. 7, respectively. It is clear that the deflection of the Timoshenko beam is relatively high for the displacement and rotation representation. This denote that the robust adaptive boundary control is incapable of effectively tackling the vibrations of the Timoshenko beam in the presence of the unknown disturbances and parameters uncertainty.

$$u_c(t) = -\hat{M}\dot{u}'(L,t) + \hat{M}\dot{\psi}(L,t) - \hat{K}\psi(L,t) + \hat{K}u'(L,t) - k_1\lambda_1(t) - d(t) \tag{72}$$

$$v_c(t) = -\hat{J}\dot{\psi}'(L,t) + \hat{E}I\psi'(L,t) - k_2\lambda_2(t) - \theta(t) \tag{73}$$

where

$$\lambda_1(t) = \dot{u}(L,t) + u'(L,t) - \psi(L,t) \tag{74}$$

$$\lambda_2(t) = \dot{\psi}(L,t) - \psi'(L,t) \tag{75}$$

c) *With the proposed GSMBC:* The proposed GSMBC (35) and (40), acts on the Timoshenko beam with the control parameters $k = 25$ and $c = 5$. The spatial time representation for the displacement and rotation of the Timoshenko beam with GSMBC is shown in Fig. 8 and Fig. 9, respectively. It is evident that the deflection of the Timoshenko beam is effectively damped for the displacement and rotation representation. This indicate that the GSMBC is capable

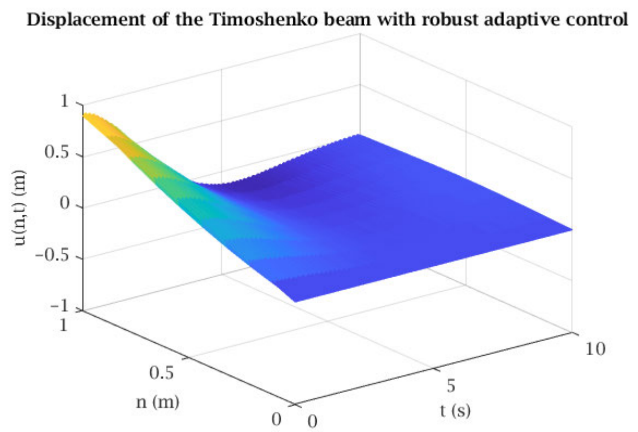

FIGURE 6. Displacement of the timoshenko beam with robust adaptive control.

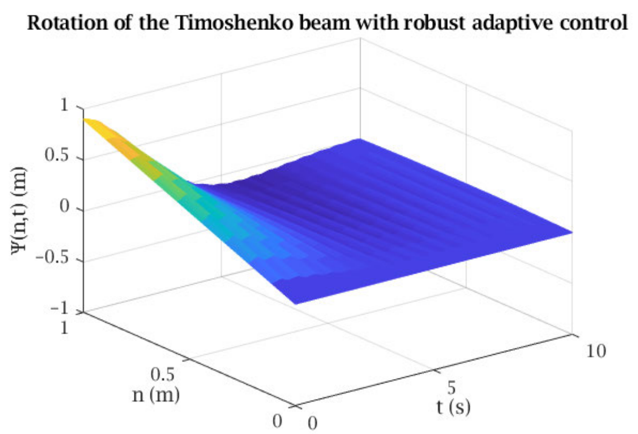

FIGURE 7. Rotation of the timoshenko beam with robust adaptive control.

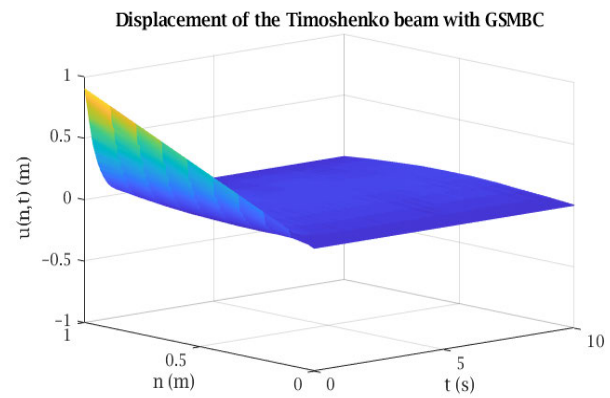

FIGURE 8. Displacement of the timoshenko beam with GSMBC.

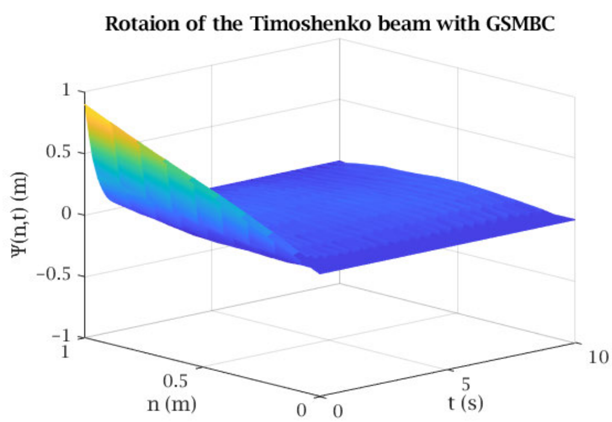

FIGURE 9. Rotation of the timoshenko beam with GSMBC.

of handling the vibrations of the Timoshenko beam efficiently in the presence of the unknown disturbances and parameters uncertainty. Hence, this suggested control scheme is prospered to tackle the vibrations of the Timoshenko beam under unknown disturbances and parameters uncertainty.

The deflection for the boundary displacement and boundary rotation of the Timoshenko beam are shown in Fig. 10 and Fig. 11, respectively. It is clear that the boundary displacement and boundary rotation of the Timoshenko beam with GSMBC are converging to equilibrium faster than robust adaptive control.

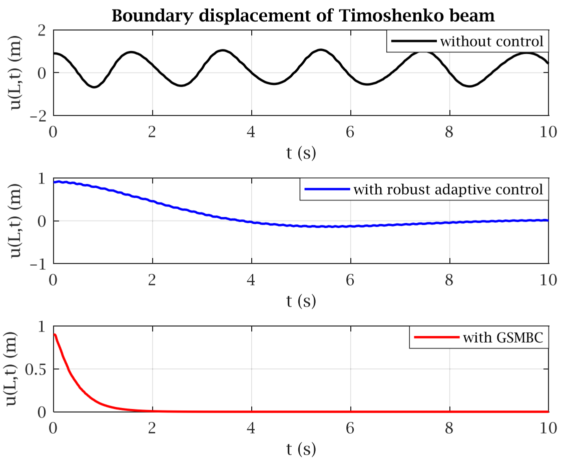

FIGURE 10. Boundary displacement of the timoshenko beam.

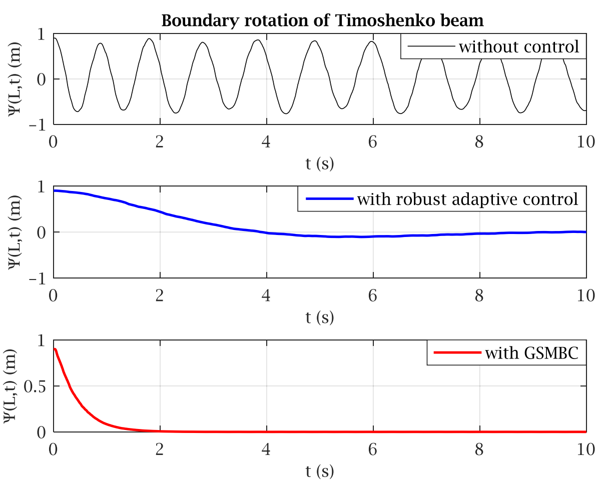

FIGURE 11. Boundary rotation of the Timoshenko beam.

The boundary displacement and boundary rotation of the Timoshenko beam under robust adaptive control with different ranges of uncertainties are shown in Fig. 12 and Fig. 14. It is evident that the deflections converging to equilibrium are relatively at high different time range for both cases. This indicate that robust adaptive boundary control is relatively efficient for the different ranges of uncertainties. Fig. 13 and Fig. 15 show the equivalent robust adaptive control forces and control torques under different uncertainties.

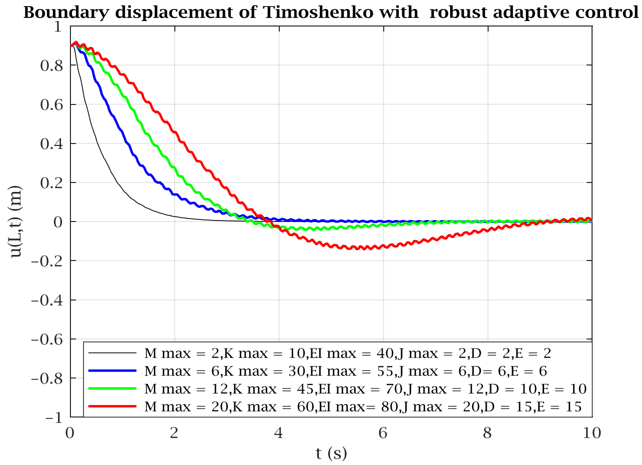

FIGURE 12. Boundary displacement of timoshenko with robust adaptive control.

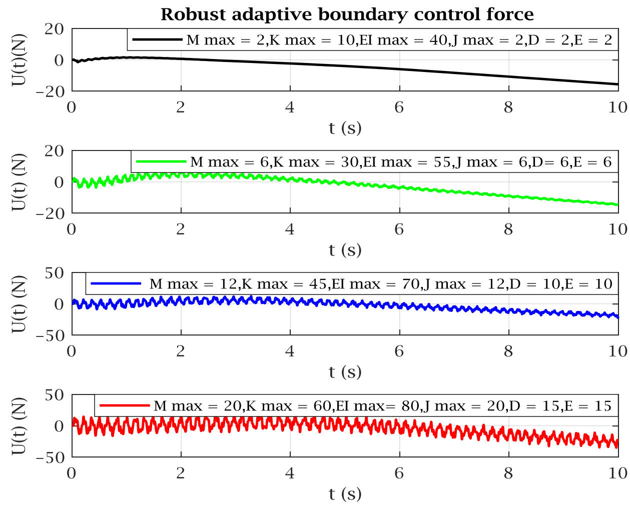

FIGURE 13. Robust adaptive control force.

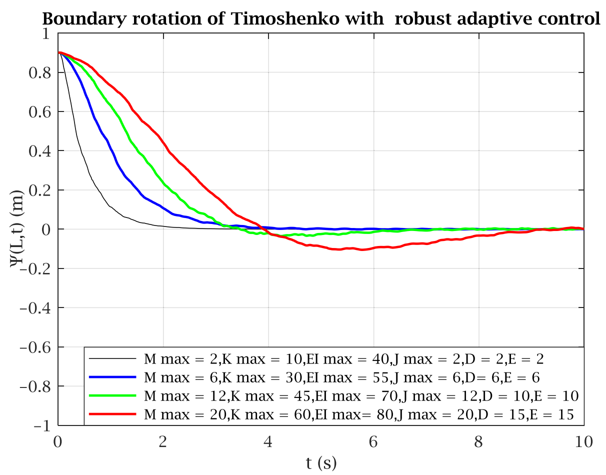

FIGURE 14. Boundary rotation of Timoshenko with robust adaptive control.

The boundary displacement and boundary rotation of the Timoshenko beam under GSMBC with different ranges of uncertainties are shown in Fig. 16 and Fig. 18. It is evident

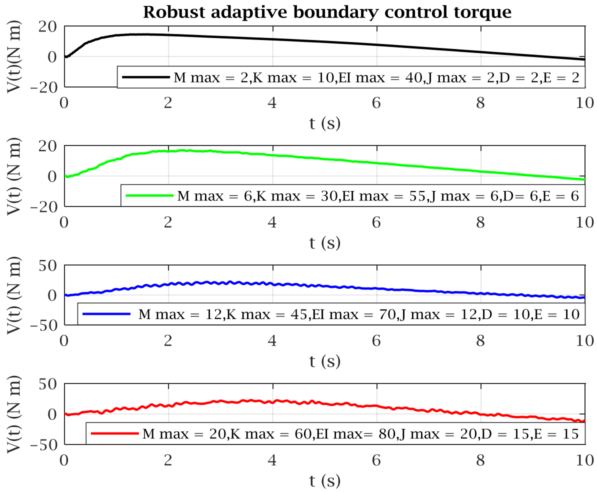

FIGURE 15. Robust adaptive control torque.

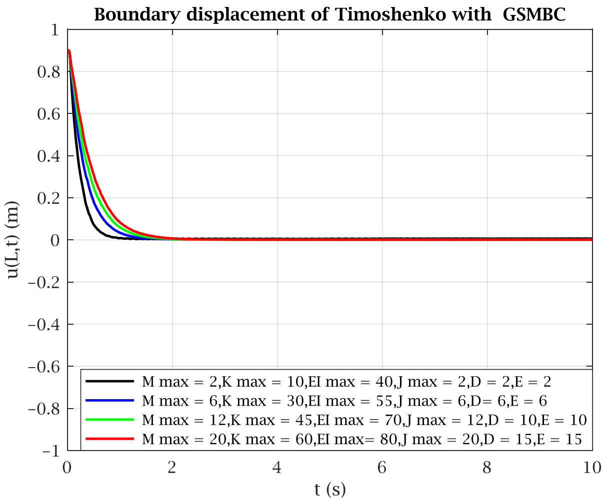

FIGURE 16. Boundary displacement of timoshenko with GSMBC.

that the deflections convergence to equilibrium within small range of time is experienced in both cases. This indicate that the GSMBC is significantly robust for different ranges of uncertainties. Further, we can state that the maximum values of uncertainties that can handle $M_{\max} = 20$, $K_{\max} = 60$, $EI_{\max} = 80$, $D = 15$, $E = 15$, is within 2 seconds, which is approximately 10 seconds in comparison with robust adaptive control performance in Fig. 12 and Fig. 14. After substituting these values in (25), (27), (29), and (31) while considering $M_{\min} = 0$, $K_{\min} = 0N$, $EI_{\min} = 0N$, and $J_{\min} = 0$, we have obtained the maximum estimated parameters uncertainty $\hat{M} = 10$, $\hat{K} = 30$, $\hat{EI} = 40$, $\hat{J} = 10$, $\bar{D} = 15$, $E = 15$. Fig. 17 and Fig. 19 shows the equivalent control forces and control torques under different uncertainties. It is obvious that all the control forces and torques are bounded and saturated. From Fig. 16 and Fig. 18, we have proven that the numerical simulations are compatible with algebraic approach stability analysis.

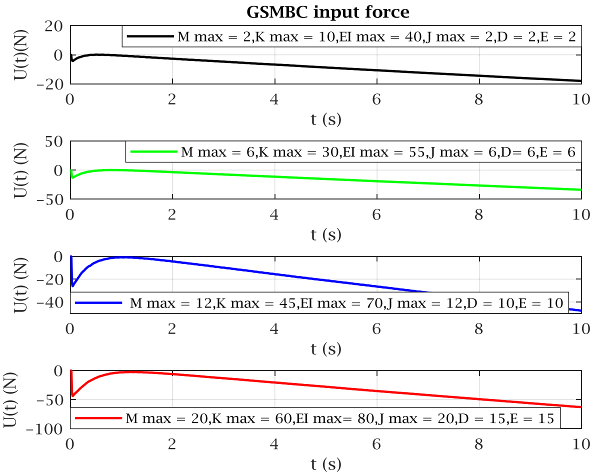

FIGURE 17. GSMBC force.

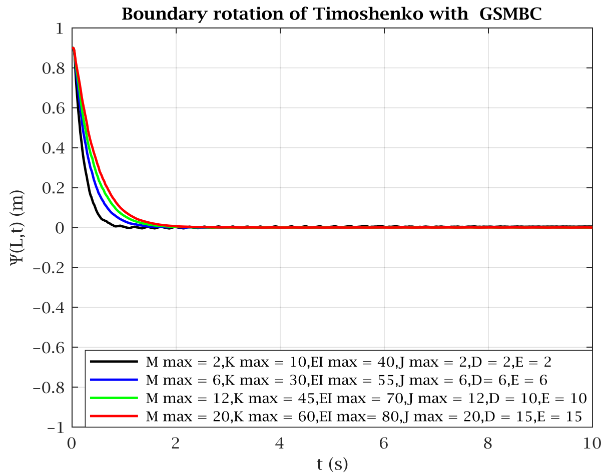

FIGURE 18. Boundary rotation of timoshenko with GSMBC.

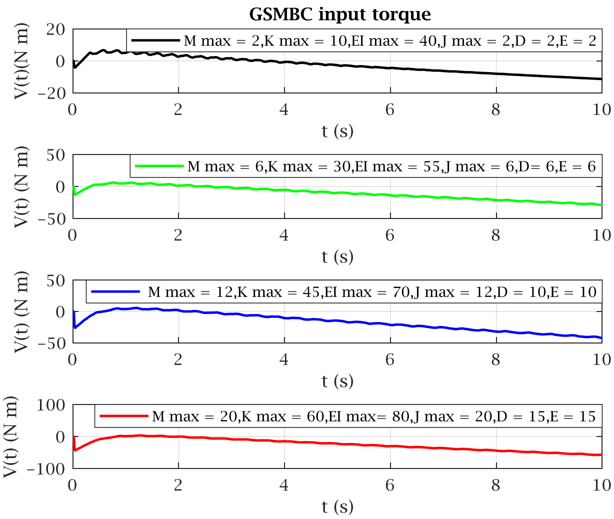

FIGURE 19. GSMBC torque.

V. CONCLUSION

In this paper, a Timoshenko beam with system parameter uncertainties and exogenous disturbances was investigated. GSMBC-based MOL scheme was proposed for reducing the vibrations of Timoshenko beam under distributed disturbance, displacement boundary disturbance, and rotation boundary disturbance and for compensating the effect of the uncertainties. Firstly, MOL is proposed for attaining an accurate approximated solution of the Timoshenko beam. Subsequently, the GSMBC-based MOL scheme has been inspected for Timoshenko beam under maximum parameter uncertainties and maximum boundary disturbances. Moreover, the displacement and rotation convergencies under system parameters uncertainties are assured mathematically. The simulation results are matched with the proved theoretical results, which signify that the GSMBC-based MOL scheme has attenuated the vibrations of the Timoshenko beam efficiently. In future work, a boundary control input constraint will be suggested. Also, it is a challenging topic to use this design for non-linear Timoshenko beam.

ACKNOWLEDGMENT

The authors gratefully acknowledge Chinese scholarship council (CSC) for their full support and offering research tools and good research environment.

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MOHAMED AHMED ESHAG (Student Member, IEEE) received the B.S. degree in electrical engineering (control) and the M.S degree in mechatronics engineering from the Sudan University of Science and Technology, Khartoum, Sudan, in 2009 and 2014, respectively. He is currently pursuing the Ph.D. degree in control theory and control engineering with Southwest Jiaotong University, Chengdu, China.

His current research interests include global sliding mode control, boundary control, robust control, infinite-dimensional system control, and control theory with applications to robotics.

LEI MA (Member, IEEE) received the B.Eng. degree in automatic control from Chongqing University, Chongqing, China, in 1993, the M.Sc. degree in electrical engineering from Southwest Jiaotong University, Chengdu, China, in 1996, and the Dr.Ing. degree in electrical engineering from Ruhr University Bochum, Bochum, Germany, in 2006. From 2006 to 2009, he was an Assistant Professor with the Department of Robotics and Telematics, University of Würzburg, and also the Deputy Director and a member of the Board of Supervisors with the Zentrum für Telematik e.V., Würzburg, Germany. Since September 2009, he has been a Professor with the School of Electrical Engineering, Southwest Jiaotong University. His research interests include control theory with applications to robotics, renewable energy, and rail transit systems.

YONGKUI SUN received the Diploma degree in mechanical engineering from the Hunan Institute of Engineering, Xiantan, China, in 1994, and the M.Sc. degree in mechatronics and the Ph.D. degree in measuring and testing technology and instruments from the University of Electronic Science and Technology of China, Chengdu, China, in 2004 and 2009, respectively. From June 2015 to May 2016, he was a Visiting Scholar with the Department of Electrical and Computer Engineering, University of Alberta, Edmonton, AB, Canada. Since 2009, he has been a Lecturer with the School of Electrical Engineering, Southwest Jiaotong University, Chengdu. His current research interests include modeling, analysis, control of electromechanical system data driven methods for equipment condition monitoring and fault diagnosis, and their applications to industrial problems.

KAI ZHANG (Member, IEEE) received the Ph.D. degree from the Department of Control Science and Engineering, Harbin Institute of Technology, Harbin, China, in 2018. He is currently a Lecturer with Southwest Jiaotong University, Chengdu. His research interests include adaptive control, sliding model control, and spacecraft guidance and control.