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Hesitant Fuzzy Concept Lattice and Its Application

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ABSTRACT The hesitant fuzzy set (HFS) can reflect the hesitation and uncertainty of decision-makers for the reason that it uses some possible values instead of a certain value. Considering that there is still no research on concept lattice or fuzzy formal concept analysis (FFCA) in a hesitant fuzzy environment, in this paper, we propose the definition and the related theory of hesitant fuzzy concept lattice. Firstly, we propose the hesitant fuzzy formal concept analysis theory and study the definition of hesitant fuzzy concept lattice. Secondly, we provide two effective reduction methods of hesitant fuzzy formal context and discuss the differences of them. Thirdly, we propose an incremental construction algorithm to construct the hesitant fuzzy concept lattice. After that, we study the similarity calculation method of hesitant fuzzy concepts. Finally, we design a practical application to validate the hesitant fuzzy concept lattice theory, which is proven to be correct and effective.

INDEX TERMS Hesitant fuzzy set, fuzzy formal concept analysis, concept lattice, hesitant fuzzy element, similarity.

I. INTRODUCTION

Concept lattice was first proposed by Wille [1] in 1982, which is a mathematical tool for data analysis and knowledge processing. It mainly describes the relationships between objects and attributes by a poset with concepts. Each node of the concept lattice is called a formal concept, which contains the intent and the extent [2]. At present, the concept lattice theory has been widely used in data mining [3]–[5], software engineering [6], [7], semantic retrieval [8], [9], ontology construction [10]–[13], and other application fields. In order to improve the retrieval efficiency of the library, Yu [14] introduced concept lattice into the user sequence mining of the digital library. The experiments showed that this method could improve the performance of mining. To deal with the problem of ontology construction of heterogeneous information, Kiu and Lee [15] proposed an ontology merging method-FCA merge, which combined formal concept analysis, concept lattice, and natural language processing methods, then the merged ontology was finally generated.

In a classical 0-1 formal context, the formal concepts and the concept lattice are all precise. However, in practical applications, most of the relationships between objects and properties are fuzzy and uncertain. The classical real number concept lattice theory cannot reflect the degrees of attributes. Since the fuzzy sets theory [16] is a valid theory about uncertainty, many researchers began to combine fuzzy set theory with concept lattice, using membership to express the fuzzy relationship between objects and attributes, and thus fuzzy formal concept analysis [17] is proposed, and fuzzy concept lattice [18] has become a new research field. Quan and Siu [17] proposed the concept of fuzzy formal concept analysis (FFCA), and they also showed that FFCA had superior performance in dealing with information with fuzzy attributes. Based on the theory of concept lattice, Bêlohlávek [19] raised the definition of the fuzzy concept in the fuzzy formal context, and he also proved that the fuzzy concept theory satisfied almost all the properties of the concept in the classical real number formal context. Krupka and Laštovička [20] established the fuzzy formal context and then studied the corresponding concept lattice construction algorithm.

With the development of society, some uncertain information is facing increasingly complex situations, so the fuzzy set theory has been extended to a series of generalized types of fuzzy sets, such as type-2 fuzzy set [21], interval-valued fuzzy set [22], intuitionistic fuzzy set [23]–[25], and interval-valued intuitionistic fuzzy set [26]. At the same time, many scholars have studied the fuzzy concept lattice theory when the 0-1 membership is replaced by the generalized types of fuzzy set.

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Djouadi and Prade [27] use a sub-interval from the scale L to build the formal context, and then get the interval-valued fuzzy formal concepts. Lei [28] studied the formal concept analysis in an intuitionistic fuzzy formal context, and the notion of crisp-intuitionistic fuzzy concept was proposed. Based on the linguistic value intuitionistic fuzzy algebra and intuitionistic fuzzy formal context, Zou *et al.* [29] put forward the linguistic value intuitionistic fuzzy formal context and established the linguistic value intuitionistic fuzzy concept lattice, then some related properties were discussed.

When the experts evaluate the membership of an attribute, they are usually hesitant and irresolute, which may make it hard to come to a definite conclusion. In order to permit the membership having different possible values, Torra [30] introduced the concept of hesitant fuzzy set (HFS), which allows the membership to be a set of several possible values. When dealing with these complex evaluation problems, the hesitant fuzzy theory may be more practical and appropriate than other extended fuzzy set theories [31], [32]. When the hesitant fuzzy relation between object and attribute is described by hesitant fuzzy set, the further analysis of hesitant fuzzy concept has important theoretical significance and practical application value. What is more, a large number of scholars continued to study the related theory of hesitant fuzzy set. Based on these reasons, in this paper, we use hesitant fuzzy set to describe the hesitant fuzzy relationship between objects and attributes. The hesitant fuzzy formal concept analysis is further carried out, and the simplification and the construction method of hesitant fuzzy concept lattice is discussed.

The remainder of this paper is organized as follows: Section 2 reviews the concepts of FFCA and HFS. In Section 3, some definitions and methods of hesitant fuzzy formal concept analysis are proposed. The reduction method and the construction algorithm of hesitant fuzzy concept lattice are carried out, and the similarity calculation method of hesitant fuzzy concept lattice is discussed. In Section 4, a practical application example is provided to support our studies. Finally, the paper ends with some concluding remarks in Section 5.

II. PRELIMINARIES

A. FFCA

Now we give some related concepts of Fuzzy Concept Lattice theory, which are defined as follows:

Definition 1 [17]: The triple $K = (G, M, I = \psi(G \times M))$ is called a fuzzy formal context, where G and M represent a finite set of objects and a finite set of attributes respectively, I is a fuzzy relationship subset which is defined on $G \times M$, and each fuzzy relationship $(g, m) \in I$ has a corresponding membership $v(g, m) \in [0, 1]$.

We can represent the fuzzy formal context as a two-dimensional table, as shown in Table 1. Each fuzzy number in the table expresses a fuzzy relationship between the object and the attribute. Especially, when the fuzzy numbers are 0 or 1, it becomes a classical 0-1 formal context.

TABLE 1. Fuzzy formal context.

	m_1	m_2	m_3	m_4	m_5
g_1	0.1	0.2	0.7	0.6	0.4
g_2	0.2	0.0	0.3	0.6	0.9
g_3	0.8	0.2	0.1	0.4	1.0
g_4	0.2	1.0	0.1	0.5	0.0

Definition 2 [17]: For a fuzzy formal context $K = (G, M, I)$, let $X \subseteq G$ and $Y \subseteq M$. When we give a membership confidence threshold $\lambda \in [0, 1]$, then there are two mappings defined as follows:

1) $X^* = \{m \in M \,|\forall g \in X, v(g, m) \geq \lambda\}$;
2) $Y^* = \{g \in G \,|\forall m \in Y, v(g, m) \geq \lambda\}$.

Definition 3 [17]: Given a fuzzy formal context $K = (G, M, I)$. Let $X \subseteq G$ and $Y \subseteq M$. If $X^* = Y$ and $Y^* = X$, then (X, Y) can be called a fuzzy formal concept. Each attribute $m \in Y$ has a membership v_m that satisfies $v_m = \min_{g \in X} v(g, m)$.

Because of the completeness of concept lattice theory, for any fuzzy formal context, its formal concepts are fixed as long as the simplification method of the fuzzy formal context is unchanged.

Definition 4 [17]: Given a fuzzy formal context $K = (G, M, I)$, the membership confidence threshold λ, and two fuzzy formal concepts $C_1 = (X_1, Y_1)$ and $C_2 = (X_2, Y_2)$. If $X_1 \in X_2$ and $(X_1, Y_1) \leq (X_2, Y_2)$, then we can describe the complete lattice constructed by this partial order relation as the fuzzy concept lattice of the fuzzy formal context K at the confidence threshold.

If $X_1 \in X_2$ and $(X_1, Y_1) \leq (X_2, Y_2)$, then the complete lattice constructed by this partial order relation is called the fuzzy concept lattice of the fuzzy formal context K at the confidence threshold λ.

Generally speaking, compared with the classical 0-1 formal context, the size of fuzzy concept lattice may become very large. Therefore, we can simplify the fuzzy concept lattice by calculating the similarities between fuzzy concepts.

Definition 5 [17]: Let two fuzzy formal concepts $C_1 = (X_1, Y_1)$ and $C_2 = (X_2, Y_2)$, then their similarity degree is defined as:

$$\rho(C_1, C_2) = \frac{\int \left(Y_1^{Y_1 \cup Y_2} \cap Y_2^{Y_1 \cup Y_2}\right)}{\int \left(Y_1^{Y_1 \cup Y_2} \cup Y_2^{Y_1 \cup Y_2}\right)} \tag{2.1}$$

B. HESITANT FUZZY SETS

When we evaluate the attribute characteristics of an object, lots of experts might be needed to express their individual opinions on the same problem. However, the experts usually have different opinions when they are considering the degree that an alternative satisfies a standard. Therefore, different values might be assigned to the alternatives. To solve this problem, Torra [30] introduced the concept of hesitant fuzzy set (HFS). In the HFS, the membership of an element to a set is expressed as a hesitant fuzzy set, which contains some possible values between [0,1].

To be easily understood, Xia and Xu [32] expressed the HFS as follows:

$$A = \{< x,\ h_A(x) > | x \in X\} \tag{2.2}$$

where $h_A(x)$ is a set of some different values in [0,1], denoting the possible memberships of the element $x \in X$ to the set A. Besides, $h = h_A(x)$ is called a hesitant fuzzy element.

For example, let $X = \{x_1, x_2, x_3\}$ be a set, $h_A(x_1) = \{0.1, 0.2, 0.5\}$, $h_A(x_2) = \{0.2, 0.6\}$, and $h_A(x_3) = \{0.3, 0.4, 0.5\}$. Then we can express the HFS A as:

$$A = \{< x_1, (0.1, 0.2, 0.5) >, < x_2, (0.2, 0.6) >, < x_3, (0.3, 0.4, 0.5) >\}$$

For three hesitant fuzzy elements h_1, h_2 and h_3, Torra and Narukawa [33] defined the operations of the union, intersection, and complement as follows:

1) $h^c = \bigcup_{r\in h} \{1 - r\}$
2) $h_1 \cup h_2 = \bigcup_{r_1\in h_1, r_2\in h_2} \max\{r_1, r_2\}$
3) $h_1 \cap h_2 = \bigcap_{r_1\in h_1, r_2\in h_2} \min\{r_1, r_2\}$

Definition 6 [34]: Given a hesitant fuzzy element h, $s(h) = \frac{1}{l_h}\sum_{r\in h} r$ is the score function of h, where l_h is the number of numbers in h; $\sigma(h)$ is called the deviation function of h:

$$\sigma(h) = \left[\frac{1}{l_h}\sum\nolimits_{r\in h}(r - s(h))^2\right]^{\frac{1}{2}} \tag{2.3}$$

And then we can compare two hesitant fuzzy elements based on $s(h)$ and $\sigma(h)$:

Definition 7 [34]: Let h_1 and h_2 be two hesitant fuzzy elements, $s(h_1)$ and $s(h_2)$ be the score function values of h_1 and h_2, $\sigma(h_1)$ and $\sigma(h_2)$ be the deviation function values of h_1 and h_2, then

1) If $s(h_1) < s(h_2)$, then $h_1 \prec h_2$;
2) If $s(h_1) > s(h_2)$, then $h_1 \succ h_2$;
3) If $s(h_1) = s(h_2)$, then:
   (i) If $\sigma(h_1) = \sigma(h_2)$, then $h_1 \sim h_2$;
   (ii) If $\sigma(h_1) < \sigma(h_2)$, then $h_1 \succ h_2$;
   (iii) If $\sigma(h_1) > \sigma(h_2)$, then $h_1 \prec h_2$.

III. HESITANT FUZZY CONCEPT LATTICE

A. FORMAL CONCEPT ANALYSIS OF HESITANT FUZZY INFORMATION

When the relationships in the fuzzy formal context change from the fuzzy real numbers to the hesitant fuzzy sets, the formal context is called a hesitant fuzzy formal context.

Definition 8: $K = (U, A, V, f)$ is called a hesitant fuzzy formal context, in which U represents a nonempty finite set of objects, A represents a nonempty finite set of attributes, V is a set of hesitant fuzzy elements representing the possible values of the memberships, and the information function f is a mapping from $U \times A$ to V, $\forall y \in A, f(x, y) \in V$, where $f(x, y)$ is a hesitant fuzzy element.

For example, there is a hesitant fuzzy formal context $K = (U, A, V, f)$, where $U = \{x_1, x_2, x_3, x_4\}, A = \{y_1, y_2, y_3, y_4\}$,

TABLE 2. Hesitant fuzzy formal context.

	m_1	m_2	m_3	m_4	m_5
g_1	(0.5,0.7,0.9)	(0.2,0.3,0.5,0.7)	(0,0.2)	(0,0.1,0.2,0.4)	(0.1,0.3)
g_2	(0.2,0.3,0.5)	(0.1,0.3,0.4,0.7)	(0.7,1)	(0,0.1,0.2,0.3)	(0.2,0.6)
g_3	(0.1,0.5,0.6)	(0.1,0.2,0.3,0.4)	(0.1,0.5)	(0.5,0.6,0.8,1)	(0.9,1)
g_4	(0.1,0.2,0.3)	(0.2,0.3,0.5,0.7)	(0,0.2)	(0,0.1,0.2,0.4)	(0.1,0.3)

and the hesitant fuzzy formal context can be represented as a two-dimensional table, as shown in Table 2:

When the hesitant fuzzy relationship is a number between 0 and 1, it is a general fuzzy formal context. Therefore, the general fuzzy context can be regarded as a special case of a hesitant fuzzy formal context.

Definition 9: Given a hesitant fuzzy formal context $K = (U, A, V, f)$. Let $X \subseteq U$, $Y \subseteq A$. If $X^* = Y$ and $Y^* = X$, then (X, Y) can be called a hesitant fuzzy formal concept. Each attribute $y \in Y$ has a hesitant membership f_a, which satisfies $f_y = \bigcap_{u\in X} f(u, y)$. The structure of all concepts in the hesitant fuzzy formal context and the partial order relationships between them is called the hesitant fuzzy concept lattice. Each hesitant fuzzy concept is regarded as a node in the hesitant fuzzy concept lattice.

Take Table 2 as an example, let $X = \{x_1, x_3\}$, $Y = \{y_1(0.3, 0.4, 0.5), y_2(0.2, 0.3, 0.5, 0.6), y_5(0.1, 0.3)\}$, then we can get the following results according to Definition 9.

$$X^* = \left\{\begin{matrix} y_1(0.1, 0.5, 0.6), y_2(0.1, 0.2, 0.3, 0.4), \\ y_3(0, 0.2), y_4(0, 0.1, 0.2, 0.4), y_5(0.1, 0.3) \end{matrix}\right\},$$
$$Y^* = \{x_1\}.$$

Proposition 10: Given a hesitant fuzzy formal context $K = (U, A, V, f)$. Let $X_1, X_2, X \subseteq U$, $Y_1, Y_2, Y \subseteq A$. Then we can obtain the following properties:

1) $X_1 \subseteq X_2 \Rightarrow X_2^* \subseteq X_1^*$, $Y_1 \subseteq Y_2 \Rightarrow Y_2^* \subseteq Y_1^*$;
2) $X \subseteq X^{**}$, $Y \subseteq Y^{**}$;
3) $X^* = X^{***}$, $Y^* = Y^{***}$;
4) $X \subseteq Y^* \Leftrightarrow Y \subseteq X^*$.

Proof:

(1) Let $X_1^* = Y_1, X_2^* = Y_2$, then for any $y \in A$, we can get the formulas as follows:
$Y_1(y) = \bigcap\limits_{\forall x\in X_1} f(x, y)$, $Y_2(y) = \bigcap\limits_{\forall x\in X_2} f(x, y)$.
Since $X_1 \subseteq X_2$, then we can get $Y_1(y) \geq Y_2(y)$, so $Y_2 \subseteq Y_1$, that is $X_2^* \subseteq X_1^*$.
For any $f(x, y) \in V$, we can get the formulas as follows:

$$Y_1^* = \{x \in U | f(x, y) \geq Y_1(y), \forall y \in A\},$$
$$Y_2^* = \{x \in U | f(x, y) \geq Y_2(y), \forall y \in A\}.$$

Since $Y_1 \subseteq Y_2$, then we can get $Y_1(y) \leq Y_2(y)$, so if $x \in Y_2^*$, there must be $x \in Y_1^*$, that is, $Y_2^* \subseteq Y_1^*$.

(2) Let $X^* = Z$, then
$X^{**} = Z^* = \{x \in U | f(x, y) \geq Z(y), \forall y \in A\}$,
in which $Z(y) = \bigcap\limits_{\forall x\in X} f(x, y)$.
If $x \in X$, then $f(x, y) \geq \bigcap\limits_{\forall x\in X} f(x, y) = Z(y)$, that is, $x \in X^{**}$. Therefore, $X \subseteq X^{**}$.

For any $f(x,y) \in V$, we can get $Y^* = \{x \in U \,|f(x,y) \geq Y(y), \forall y \in A\}$. That is, for any $x \in Y^*$ and $y \in Y$, we can get $f(x,y) \geq Y(y)$, so $Y^{**}(y) = \bigcap_{\forall x \in Y^*} f(x,y) \geq \bigcap_{\forall x \in X} f(x,y) = Y(y)$.

Therefore, $Y \subseteq Y^{**}$.

(3) Because $X \subseteq X^{**}$ and $X_1 \subseteq X_2 \Rightarrow X_2^* \subseteq X_1^*$, we can easily get $X^{***} \subseteq X^*$ and $X^{***} \subseteq X^*$, that is, $X^{***} \subseteq X^* \subseteq X^{***}$. Therefore, $X^* = X^{***}$.

The same as before, we can easily prove $Y^* = Y^{***}$.

(4) Because $X_1 \subseteq X_2 \Rightarrow X_2^* \subseteq X_1^*$, we can get $X \subseteq Y^* \Rightarrow Y^{**} \subseteq X^*$. And because $Y \subseteq Y^{**}$, we can get $Y \subseteq Y^{**} \subseteq X^*$. So $X \subseteq Y^* \Rightarrow Y \subseteq X^*$;

Because $Y_1 \subseteq Y_2 \Rightarrow Y_2^* \subseteq Y_1^*$, we can get $Y \subseteq X^* \Rightarrow X^{**} \subseteq Y^*$. And because $X \subseteq X^{**}$, we can get $X \subseteq X^{**} \subseteq Y^*$. So $Y \subseteq X^* \Rightarrow X \subseteq Y^*$.

Therefore, $X \subseteq Y^* \Leftrightarrow Y \subseteq X^*$.

This completes the proof.

B. REDUCTION OF HESITANT FUZZY FORMAL CONTEXT

When we analyze a general formal concept, we usually simplify it by setting a membership confidence threshold (see Definition 2), and then the quantity of fuzzy concept lattice is reduced, and the construction process of the fuzzy concept lattice is simplified. In the same way, we set a membership confidence threshold for the hesitant formal context. However, in practical application, we usually have different tolerance for different attributes of a thing. For instance, when a bank recruits some employees, it may require men whose heights are over 170cm, but when a clothing company recruits models, it might require men over 180cm. Therefore, we replace the unique attribute membership confidence threshold with a set of membership confidence threshold $T = \{t_1, t_2, \ldots, t_N\}$, so as to meet the different requirements of different attributes, where N is the number of attributes in the hesitant fuzzy formal context.

When we use the confidence threshold to simplify the hesitant fuzzy formal context, we cannot directly compare a real number with a hesitant fuzzy element. In this situation, we propose two methods to simplify the formal context.

1) Score Function Reduction Method (SF Method): Simplify the hesitant fuzzy formal context by comparing the score function value of the hesitant fuzzy element with the confidence threshold value. If the score function value $s(h)$ of the hesitant fuzzy element h is greater than or equal to the confidence threshold of the attribute, then the hesitant fuzzy element is reserved. Otherwise, it is assigned as 0.

2) *Score Function and Deviation Reduction Method (SFD Method)*: Because the uncertainty of hesitant fuzzy elements represents the instability of its attribute evaluation. In practical application, we may hope to obtain a more reliable and stable result while retaining the complete information of hesitant fuzzy elements. In this method, both the score function value $s(h)$ and the standard deviation $\sigma(h)$ affect the simplification result.

TABLE 3. Score function values of hesitant fuzzy formal context.

	a_1	a_2	a_3	a_4	a_5
x_1	0.7	0.425	0.1	0.175	0.2
x_2	0.33	0.375	0.85	0.15	0.4
x_3	0.4	0.25	0.3	0.725	0.95
x_4	0.2	0.425	0.1	0.175	0.2

TABLE 4. Deviation degrees of hesitant fuzzy formal context.

	a_1	a_2	a_3	a_4	a_5
x_1	0.040	0.049	0.020	0.029	0.020
x_2	0.023	0.063	0.045	0.017	0.080
x_3	0.070	0.017	0.080	0.049	0.005
x_4	0.010	0.049	0.020	0.029	0.020

TABLE 5. Simplified hesitant fuzzy formal context by SF method.

	a_1	a_2	a_3	a_4	a_5
x_1	{0.5,0.7,0.9}	{0.2,0.3,0.5,0.7}	0	0	0
x_2	0	{0.1,0.3,0.4,0.7}	{0.7,1}	0	{0.2,0.6}
x_3	{0.1,0.5,0.6}	0	{0.1,0.5}	{0.5,0.6,0.8,1}	{0.9,1}
x_4	0	{0.2,0.3,0.5,0.7}	0	0	0

TABLE 6. Simplified hesitant fuzzy formal context by SFD method (k = 1).

	a_1	a_2	a_3	a_4	a_5
x_1	{0.5,0.7,0.9}	{0.2,0.3,0.5,0.7}	0	0	0
x_2	0	{0.1,0.3,0.4,0.7}	{0.7,1}	0	0
x_3	0	0	{0.1,0.5}	{0.5,0.6,0.8,1}	{0.9,1}
x_4	0	{0.2,0.3,0.5,0.7}	0	0	0

If $s(h) - k\sigma(h)$ is greater than or equal to the confidence threshold of the attribute, then the hesitant fuzzy element is reserved. Otherwise, it is assigned as 0.

Note: k is a constant. In general, $k \geq 0$. The value of k depends on the decision maker's preference for specific things. When $k = 0$, the SFD Method becomes the SF Method. That is, SF Method is a special case of the SFD Method.

For example, when we give a membership confidence threshold set $T = \{0.4, 0.3, 0.2, 0.2, 0.4\}$, then we simplify the hesitant fuzzy formal context in Table 2 by two methods respectively:

First, we calculate the score function values and deviation degrees of the hesitant fuzzy elements in the formal context, which is shown as Table 3 and Table 4:

Then we can get the simplified hesitant fuzzy formal contexts as shown in Table 5 and Table 6:

Comparing Table 5 with Table 6, it can be found that the SFD Method which considers deviation degree has a higher standard for memberships, and the obtained formal context will be simpler.

Definition 11: For a hesitant fuzzy formal context $K = \{U, A, V, f\}$, given a membership confidence threshold set T, in which each confidence threshold is between 0 and 1. Let $X \in U$ and $Y \in A$, then there are two mappings defined as follows:

1) $X^* = \{a_i \in A \,|\forall x \in U, f(x, a_i) \geq t_i\}$;

2) $Y^* = \{x \in U \mid \forall a_i \in A, f(x, a_i) \geq t_i\}$;

For example, in Table 5, let $X = \{x_2, x_3\}$ and $Y = \{a_3, a_5\}$. Then we can get the results as follows:

$X^* = \{a_3, a_5\}$, $Y^* = \{x_2, x_3\}$, and $(\{x_2, x_3\}, \{a_3, a_5\})$ is called a hesitant fuzzy formal concept.

$$\begin{aligned} f(\{x_2, x_3\}, a_3) &= f(x_2, a_3) \cap f(x_3, a_3) \\ &= \{0.7, 1\} \cap \{0.1, 0.5\} = \{0.1, 0.5\}, \\ f(\{x_2, x_3\}, a_5) &= f(x_2, a_5) \cap f(x_3, a_5) \\ &= \{0.2, 0.6\} \cap \{0.9, 1\} = \{0.2, 0.6\}. \end{aligned}$$

Definition 12: Given two hesitant fuzzy formal concepts $C_1 = (X_1, Y_1)$ and $C_2 = (X_2, Y_2)$ in the hesitant fuzzy formal context $K = \{U, A, V, f\}$, and the membership confidence threshold set T. If $X_1 \in X_2$ and $(X_1, Y_1) \leq (X_2, Y_2)$, then the complete lattice constructed by this partial order relation is called the fuzzy concept lattice of the fuzzy formal context K at the confidence threshold set T.

C. CONSTRUCTION OF HESITANT FUZZY CONCEPT LATTICE

1) FUZZY CONCEPT LATTICE CONSTRUCTION ALGORITHMS

In fact, the algorithmic construction ideas of the classical 0-1 concept lattice and the fuzzy concept lattice are consistent. The most significant difference only lies in the calculation of memberships when dealing with binary relationships. Therefore, we can explore the construction process of the hesitant fuzzy concept lattice by studying the construction method of classical 0-1 concept lattice.

Due to the completeness of the concept lattice, even for a moderate amount of data, the complexity of the algorithm is significantly increased, resulting in a huge data structure, so the construction of concept lattice is also very time-consuming. Since the concept lattice was proposed, the construction algorithms and their improvement methods [35]–[38] have been intensely studied at home and abroad. Whatever construction algorithm is used, for the same formal context, the concept lattice constructed is unique and not affected by the order in which the data or attributes are arranged. So far, the concept lattice construction algorithms can be mainly divided into three types: batch processing algorithms [39], [40], incremental algorithms [36], [41], [42], and distributed algorithms [43].

Among these three types of algorithms, incremental construction algorithms have high efficiency and have been widely used. Incremental algorithms construct concept lattices by gradually adding new nodes. When a new node is inserted, the entire data set does not need to be recalculated. It only needs to intersect the node to be inserted and each concept in the concept lattice, then execute relevant actions according to the result of the intersection, such as the Godin's algorithm [42]. The incremental algorithm is very convenient for the maintenance of the concept lattice. Therefore, in this paper, we draw lessons from the classical incremental algorithm to construct the hesitant fuzzy concept lattice.

2) HESITANT FUZZY CONCEPT LATTICE INCREMENTAL CONSTRUCTION ALGORITHM

In the process of constructing the concept lattice with hesitant fuzzy information, we apply Godin's incremental algorithm to the formal context with hesitant fuzzy information. Besides, we add a simplification process of the hesitant fuzzy formal context. In the algorithm, we mainly observe two principles:

1) In order to reduce the scale of the concept lattice and extract essential knowledge, the simplification process of the hesitant fuzzy formal context ought to be arranged before generating the concept lattice nodes.
2) Calculating the memberships should be placed after all nodes are generated, which can avoid repeated calculation of memberships in the process of node update.

The algorithm steps are as follows:

Step 1. Set a group of membership confidence thresholds, and then the attributes which are below the corresponding membership confidence threshold should set to zero (using the SF Method or the SFD Method), indicating that one object does not have this attribute; otherwise, we can think that the object has this attribute. After this step, we divide the hesitant fuzzy formal context into two parts: the classical real number formal context and those hesitant fuzzy sets.

Step 2. Simplify the classical real number formal context, that is, remove redundant rows and columns, and mark them.

Step 3. Select one concept node, and add a new concept node in turn, then compare the new concept node with the existing ones, take corresponding measures by analyzing the relationships between them. Their relationships can be divided into the following situations.

Case 1 (New node): For a newly generated concept node, if there is no node in the original concept lattice with the same intent as the concept node, then the concept node is called a new node;

Case 2 (Update node): For a newly generated concept node, if its intent is equivalent to the intent of a node in the original concept lattice, and there is an intersection between their extent, then we need to update this original node. The extent of the update node is the union of two nodes' extent, and the intent does not change.

Case 3 (Child inheritance node): For a newly generated concept node, if the intent of a node in the original concept node contains all of the attributes of the new concept node, then the original concept node is called the inheritance node of the new concept node.

Case 4 (Invariant node): For a newly generated concept node, if its intent is equal to that of a node of the original concept lattice, and the extent is a subset of the original concept node's extent, then the node of the concept lattice remain unchanged.

Repeat Step 3 until all the nodes are traversed.

Step 4. Add the redundant objects and attributes removed into the concept lattice.

TABLE 7. Hesitant fuzzy formal context.

	a_1	a_2	a_3	a_4	a_5	a_6
x_1	{0.5,0.7,0.9}	{0.2,0.3,0.5,0.7}	{0,0.2}	{0,0.1,0.2,0.4}	{0.1,0.3}	{0.1,0.2}
x_2	{0.2,0.3,0.5}	{0.1,0.3,0.4,0.7}	{0.7,1}	{0,0.1,0.2,0.3}	{0.2,0.6}	{0.8,1}
x_3	{0.1,0.5,0.6}	{0.1,0.2,0.3,0.4}	{0.1,0.5}	{0.5,0.6,0.8,1}	{0.9,1}	{0.2,0.3}
x_4	{0.1,0.2,0.3}	{0.2,0.3,0.5,0.7}	{0,0.2}	{0,0.1,0.2,0.4}	{0.1,0.3}	{0.1,0.2}
x_5	{0.1,0.2,0.4}	{0.2,0.4,0.5,0.6}	{0.6,0.8}	{0,0.1,0.2,0.4}	{0.3,0.5}	{0.6,0.9}

TABLE 8. Simplified hesitant fuzzy formal context.

	a_1	a_2	a_3	a_4	a_5	a_6
x_1	1	1	0	0	0	0
x_2	0	1	1	0	0	1
x_3	0	0	1	1	1	1
x_4	0	1	0	0	0	0
x_5	0	1	1	0	0	1

Step 5. Draw the Hasse diagram and calculate the hesitant fuzzy memberships of new nodes' attributes.

Let us give an example to illustrate the flow of the algorithm. The hesitant formal context is shown as Table 7.

Step 1. Let the membership confidence threshold set $T = \{0.4,0.3,0.2,0.2,0.4,0.2\}$, then we simplify the hesitant fuzzy formal context in Table 7 by the SF Method, which is shown as Table 8:

Step 2. Remove redundant rows and columns, and mark them:

TABLE 9. Simplest hesitant fuzzy formal context.

	a_1	a_2	a_3	a_4
x_1	1	1	0	0
x_2	0	1	1	0
x_3	0	0	1	1
x_4	0	1	0	0

where $x_2 \sim x_5$, $a_3 \sim a_6$, $a_4 \sim a_5$.

Step 3. Select one concept node $(\{x_1\}, \{a_1, a_2\})$, and add a new concept node, such as $(\{x_2\}, \{a_2, a_3\})$, then compare the two nodes, we can find that the new node $(\{x_1, x_2\}, \{a_2\})$ is generated. For this new node, $(\{x_1\}, \{a_1, a_2\})$ and $(\{x_2\}, \{a_2, a_3\})$ are its child inheritance nodes.

Repeat step 3 until all the nodes are traversed. Then we can generate all the concept nodes, which are shown as follows:

Layer 0: $C_0(\{x_1, x_2, x_3, x_4\}, \{\emptyset\})$;

Layer 1: $C_1(\{x_1, x_2, x_4\}, \{a_2\})$, $C_2(\{x_2, x_3\}, \{a_3\})$.

Layer 2: $C_3(\{x_1\}, \{a_1, a_2\})$, $C_4(\{x_2\}, \{a_2, a_3\})$, $C_5(\{x_3\}, \{a_3, a_4\})$;

Layer 3: $C_6(\{\emptyset\}, \{a_1, a_2, a_3, a_4\})$.

Step 4. Add the redundant objects and attributes removed into the concept lattice, so the concept nodes are adjusted as follows:

Layer 0: $C_0(\{x_1, x_2, x_3, x_4, x_5\}, \{\emptyset\})$;

Layer 1: $C_1(\{x_1, x_2, x_4, x_5\}, \{a_2\})$, $C_2(\{x_2, x_3, x_5\}, \{a_3, a_6\})$

Layer 2: $C_3(\{x_1\}, \{a_1, a_2\})$, $C_4(\{x_2, x_5\}, \{a_2, a_3, a_6\})$, $C_5(\{x_3\}, \{a_3, a_4, a_5, a_6\})$;

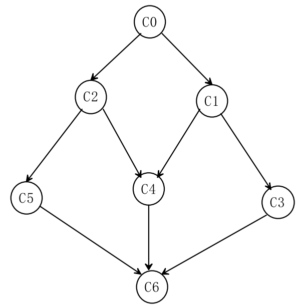

FIGURE 1. Hesitant fuzzy concept lattice Hasse diagram.

Layer 3: $C_6(\{\emptyset\}, \{a_1, a_2, a_3, a_4, a_5, a_6\})$.

Step 5. Generate the Hasse diagram as Fig.1 and calculate the hesitant fuzzy memberships of new nodes' attributes as follows.

Layer 0: $C_0(\{x_1, x_2, x_3, x_4, x_5\}, \{\emptyset\})$;

Layer 1: $C_1(\{x_1, x_2, x_4, x_5\}, \{a_2\{0.1, 0.2, 0.3, 0.4\}\})$, $C_2(\{x_2, x_3, x_5\}, \{a_3\{0.1, 0.5\}, a_6\{0.2, 0.3\}\})$

Layer 2: $C_3(\{x_1\}, \{a_1\{0.5, 0.7, 0.9\}, a_2\{0.2, 0.3, 0.5, 0.7\}\})$, $C_4(\{x_2, x_5\}, \{a_2\{0.1, 0.3, 0, 4, 0.6\}, a_3\{0.6, 0.8\}, a_6\{0.6, 0.9\}\})$, $C_5(\{x_3\}, \{a_3\{0.1, 0.5\}, a_4\{0.5, 0.6, 0.8, 1\}, a_5\{0.9, 1\}, a_6\{0.2, 0.3\}\})$;

Layer 3: $C_6(\{\emptyset\}, \{a_1, a_2, a_3, a_4, a_5, a_6\})$.

The flow chart of the algorithm is shown in Figure 2:

From the program flow chart, we can see that since we separate the calculation process of membership from the process of generating nodes, the core process is equivalent to the construction of the classical real number formal context, and the fuzzy memberships only need to be calculated once. Besides, we added the process of formal context simplification steps before the core process, which can significantly reduce the number of calling commands circularly in the core process.

D. SIMILARITY CALCULATION OF HESITANT FUZZY CONCEPTS

When we calculate the similarities between hesitant fuzzy concepts, sometimes different attributes have different influences on specific circumstances. Therefore, in this part, we assume that the weights of the attributes are different, then the similarity between hesitant fuzzy concepts is defined as follows:

Definition 13: Let two hesitant fuzzy formal concepts $C_1 = (X_1, Y_1)$ and $C_2 = (X_2, Y_2)$, V_1 and V_2 be the weights of Y_1 and, Y_2 then their similarity degree (considering weights) is defined as:

$$\rho(C_1, C_2) = \frac{\int \left(V_1 s\left(Y_1^{Y_1 \cup Y_2}\right) \cap V_2 s\left(Y_2^{Y_1 \cup Y_2}\right)\right)}{\int \left(V_1 s\left(Y_1^{Y_1 \cup Y_2}\right) \cup V_2 s\left(Y_2^{Y_1 \cup Y_2}\right)\right)} \tag{3.1}$$

where $s(\bullet)$ is the score function of hesitant fuzzy set.

In practical application, we often calculate the similarity between concepts according to different goals, and then the impact of different attributes on decision-makers may be different. Considering the preferences of decision-makers for

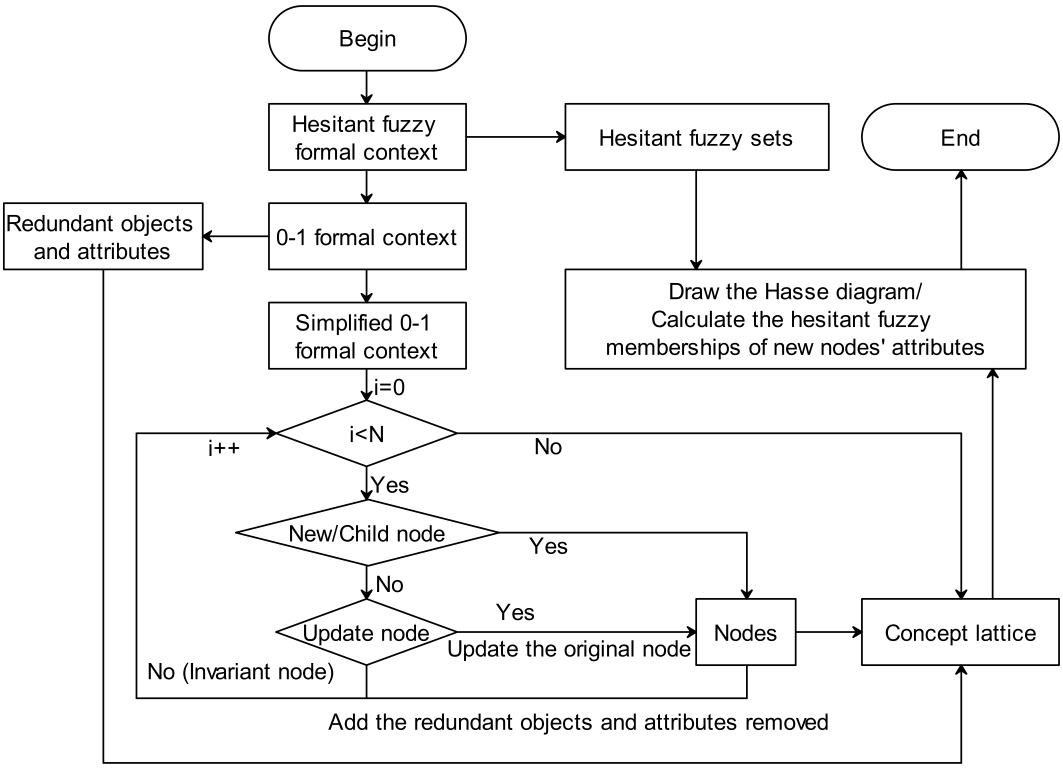

FIGURE 2. Flow chart of hesitant fuzzy concept lattice incremental construction algorithm.

different attributes, we need to calculate the weight of each attribute first. At present, there are many methods of calculating the weights, including subjective weighting methods, objective weighting methods [26], and combinatorial weighting methods [27]. In this section, we take into account the characteristics of the data and the decision-makers' subjective preference of different attributes. Therefore, we design a combinatorial weighting method to calculate the weights of the hesitant fuzzy formal context.

Definition 14: Given a hesitant fuzzy formal context $K = (U, A, V, f)$, for each attribute $a \in A, X \subset U$, let the importance degree $p_a > 0$, then the weight v_a of the attribute a is defined as:

$$v_a = p_a \cdot e^{\min_{x\in X} s(f(x,a)) - \max_{x\in X} s(f(x,a))} \tag{3.2}$$

where $s(\bullet)$ is the score function of hesitant fuzzy set.

Formula 3.2 indicates the following characteristics:

1) The importance of attribute is proportional to weight.
2) If the volatility of an attribute is great, then it can be considered that the attribute is not stable and might have a small impact on the perceptions of decision-makers, so the volatility of the attribute is in inverse proportion to the weight.
3) There is a possibility that $\max_{x\in X} s(f(x,a)) = \min_{x\in X} s(f(x,a))$. In this case, $v_a = p_a$. Otherwise, $v_a < p_a$.

Let us take Table 7 as an example, if the confidence threshold set $T = \{0.4,0.3,0.2,0.2,0.4,0.2\}$, the importance degree set $P = \{2,1,3,1,1,2\}$, then the score function values are shown as Table 3, and we can get the weights as follows:

$$v_1 = 2^* e^{0.2-0.7} = 2e^{-0.5};$$
$$v_2 = 1^* e^{0.25-0.425} = e^{-0.175};$$
$$v_3 = 3^* e^{0.1-0.85} = 3e^{-0.75};$$
$$v_4 = 1^* e^{0.15-0.725} = e^{-0.575};$$
$$v_5 = 1^* e^{0.2-0.95} = e^{-0.75};$$
$$v_6 = 2^* e^{0.15-0.9} = 2e^{-0.75}.$$

Next, we can calculate the similarity degrees of the concepts. The similarity degree of $C_1(\{x_1, x_2, x_4, x_5\}, \{a_2\{0.1, 0.2, 0.3, 0.4\}\})$ and $C_3(\{x_1\}, \{a_1\{0.5, 0.7, 0.9\}, a_2\{0.2, 0.3, 0.5, 0.7\}\})$ is:

$$\rho_v(C_1, C_3) = \frac{v_2 * s(\{0.1, 0.2, 0.3, 0.4\})}{v_1 * s(\{0.5, 0.7, 0.9\}) + v_2 * s(\{0.2, 0.3, 0.5, 0.7\})} = 0.1337.$$

IV. AN APPLICATION OF HESITANT FUZZY CONCEPT LATTICE

In this part, an example is provided to show the practical application of the hesitant fuzzy concept lattice. Nowadays, there are many new energy cars in the market, which not only enrich the choice of consumers but also cause many consumers' choice difficulties. Therefore, we introduce a new

TABLE 10. New energy car evaluation data.

	Security	Comfort	Resale value	Engine power	Cost performance
Car 1	{0.89,0.90,0.92}	{0.82,0.86}	{0.45,0.55}	{0.78,0.80,0.83}	{0.66,0.71}
Car 2	{0.35,0.37,0.39}	{0.87,0.90}	{0.36,0.39}	{0.72,0.76,0.81}	{0.82,0.87}
Car 3	{0.91,0.92,0.94}	{0.88,0.91}	{0.52,0.55}	{0.82,0.83,0.87}	{0.69,0.72}
Car 4	{0.77,0.78,0.82}	{0.83,0.85}	{0.50,0.59}	{0.85,0.88,0.90}	{0.66,0.72}
Car 5	{0.45,0.49,0.51}	{0.69,0.73}	{0.37,0.42}	{0.89,0.90,0.91}	{0.77,0.81}
Car 6	{0.80,0.82,0.83}	{0.59,0.60}	{0.72,0.77}	{0.66,0.68,0.75}	{0.83,0.85}
Car 7	{0.41,0.43,0.47}	{0.82,0.85}	{0.66,0.70}	{0.87,0.89,0.90}	{0.58,0.63}
Car 8	{0.49,0.53,0.56}	{0.77,0.80}	{0.75,0.81}	{0.57,0.60,0.63}	{0.78,0.85}
Car 9	{0.78,0.82,0.89}	{0.68,0.73}	{0.69,0.73}	{0.52,0.56,0.58}	{0.55,0.58}
Car 10	{0.44,0.46,0.49}	{0.72,0.82}	{0.66,0.68}	{0.79,0.83,0.85}	{0.51,0.57}
Car 11	{0.81,0.84,0.85}	{0.80,0.81}	{0.33,0.39}	{0.56,0.62,0.66}	{0.83,0.85}
Car 12	{0.39,0.41,0.43}	{0.83,0.85}	{0.38,0.42}	{0.78,0.79,0.81}	{0.81,0.84}
Car 13	{0.87,0.88,0.93}	{0.87,0.91}	{0.56,0.58}	{0.79,0.81,0.85}	{0.72,0.73}
Car 14	{0.74,0.76,0.77}	{0.80,0.82}	{0.56,0.60}	{0.82,0.84,0.86}	{0.68,0.72}
Car 15	{0.55,0.58,0.63}	{0.82,0.83}	{0.72,0.77}	{0.53,0.55,0.59}	{0.72,0.78}
Car 16	{0.77,0.78,0.80}	{0.65,0.69}	{0.69,0.70}	{0.57,0.62,0.64}	{0.58,0.60}

TABLE 11. Simplified hesitant fuzzy formal context.

	Security	Comfort	Resale value	Engine power	Cost performance
Car 1	{0.89,0.90,0.92}	{0.82,0.86}	{0.45,0.55}	{0.78,0.80,0.83}	{0.66,0.71}
Car 2	{0.35,0.37,0.39}	{0.87,0.90}	{0.36,0.39}	{0.72,0.76,0.81}	{0.82,0.87}
Car 3	{0.91,0.92,0.94}	{0.88,0.91}	{0.52,0.55}	{0.82,0.83,0.87}	{0.69,0.72}
Car 4	{0.77,0.78,0.82}	{0.83,0.85}	{0.50,0.59}	{0.85,0.88,0.90}	{0.66,0.72}
Car 5	{0.45,0.49,0.51}	{0.69,0.73}	{0.37,0.42}	{0.89,0.90,0.91}	{0.77,0.81}
Car 6	{0.80,0.82,0.83}	{0.59,0.60}	{0.72,0.77}	{0.66,0.68,0.75}	{0.83,0.85}
Car 7	{0.41,0.43,0.47}	{0.82,0.85}	{0.66,0.70}	{0.87,0.89,0.90}	{0.58,0.63}
Car 8	{0.49,0.53,0.56}	{0.77,0.80}	{0.75,0.81}	{0.57,0.60,0.63}	{0.78,0.85}
Car 9	{0.78,0.82,0.89}	{0.68,0.73}	{0.69,0.73}	{0.52,0.56,0.58}	{0.55,0.58}
Car 10	{0.44,0.46,0.49}	{0.72,0.82}	{0.66,0.68}	{0.79,0.83,0.85}	{0.51,0.57}
Car 11	{0.81,0.84,0.85}	{0.80,0.81}	{0.33,0.39}	{0.56,0.62,0.66}	{0.83,0.85}
Car 12	{0.39,0.41,0.43}	{0.83,0.85}	{0.38,0.42}	{0.78,0.79,0.81}	{0.81,0.84}
Car 13	{0.87,0.88,0.93}	{0.87,0.91}	{0.56,0.58}	{0.79,0.81,0.85}	{0.72,0.73}
Car 14	{0.74,0.76,0.77}	{0.80,0.82}	{0.56,0.60}	{0.82,0.84,0.86}	{0.68,0.72}
Car 15	{0.55,0.58,0.63}	{0.82,0.83}	{0.72,0.77}	{0.53,0.55,0.59}	{0.72,0.78}
Car 16	{0.77,0.78,0.80}	{0.65,0.69}	{0.69,0.70}	{0.57,0.62,0.64}	{0.58,0.60}

energy car quick evaluation and recommendation method based on the hesitant fuzzy concept lattice theory.

We take all kinds of new energy cars as the extent of the hesitant fuzzy formal context and evaluate the key attributes of these cars, which are called the intent of the hesitant fuzzy formal context. Suppose that a customer wants to buy a new energy car with a price between 50000-70000 dollars. In this price range, there are 16 types of new energy cars, such as Mercedes Benz EQC, Wei Lai ES8, Jaguar I-PACE, Volvo XC60, Volvo S60L, Eulogize MDX, etc. For convenience, we mark them Car 1, Car 2, Car 3,. . ., Car 15, and Car 16.

A. METHOD FLOW AND DATA PROCESSING

Now we begin to evaluate some critical criteria of these car types: A = {Security, Fuel-efficient, Resale value, Engine power, Cost performance}. Sometimes it is impossible to get the crisp values of the criteria, so we can use hesitant fuzzy elements to describe vague information of these car types. Generally speaking, the evaluation and recommendation of new energy cars need to investigate the experience-based data to present the preference information with hesitant fuzzy values, which contains critical criteria of different new energy car types. The new energy car evaluation data are listed as Table 10.

Table 10 is also a hesitant fuzzy formal context. For the reason that different consumer groups usually focus on different indicators when choosing cars, we need to make targeted analysis according to specific consumer groups, so we need to set the importance degrees of the attributes: $P = \{3,1,2,1,\ 2\}$. Next, we need to follow these steps:

Step 1. Build the hesitant fuzzy concept lattice by using the incremental construction algorithm put forward in this paper:

1) According to the memberships in the hesitant fuzzy formal context, we set a confidence threshold set $T = \{0.6,0.75,0.56,0.74,0.72\}$, and apply the SF Method (the SFD Method: $k = 0$) to filter out unimportant information.

Besides, we need to divide the hesitant fuzzy formal context into two parts: the classical real number formal context (see Table 12) and those hesitant fuzzy sets.

2) Remove redundant rows and columns, and then the formal context is as follows

3) Generate all the nodes, and add the redundant objects and attributes removed into the concept lattice, and draw the Hasse diagram, which is depicted as Fig. 3.

TABLE 12. Classical real number formal context.

	Security	Comfort	Resale value	Engine power	Cost performance
Car 1	1	1	0	1	0
Car 2	0	1	0	1	1
Car 3	1	1	0	1	0
Car 4	1	1	0	1	0
Car 5	0	0	0	1	1
Car 6	1	0	1	0	1
Car 7	0	1	1	1	0
Car 8	0	1	1	0	1
Car 9	1	0	1	0	0
Car 10	0	1	1	1	0
Car 11	1	1	0	0	1
Car 12	0	1	0	1	1
Car 13	1	1	1	1	1
Car 14	1	1	1	1	0
Car 15	0	1	1	0	1
Car 16	1	1	1	0	0

TABLE 13. Simplest classical real number formal context.

	SECURITY	COMFORT	RESALE VALUE	ENGINE POWER	COST PERFORMANCE
CAR 1	1	1	0	1	0
CAR 2	0	1	0	1	1
CAR 5	0	0	0	1	1
CAR 6	1	0	1	0	1
CAR 7	0	1	1	1	0
CAR 8	0	1	1	0	1
CAR 9	1	0	1	0	0
CAR 11	1	1	0	0	1
CAR 13	1	1	1	1	1
CAR 14	1	1	1	1	0
CAR 16	1	1	1	0	0

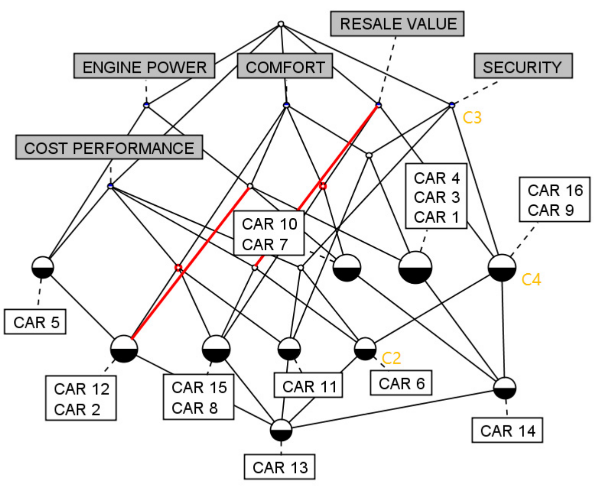

FIGURE 3. Hasse diagram of hesitant fuzzy concept lattice.

4) Calculate the hesitant fuzzy memberships of new nodes' attributes.

It can be shown that the memberships are still hesitant fuzzy elements. Compared with the real number fuzzy concept lattice, hesitant fuzzy concept lattice can maximumly retain the complex information, and thus can better express the fuzzy relationship between objects and attributes in uncertain environment.

After the construction of concept lattice, the new energy vehicles in the same hesitant fuzzy concept node have similar attributes. Through the obtained hesitant fuzzy concepts, we can easily choose out all the types of new energy vehicles with specific attribute information. Take Concept C2 as an example, its extent is (6; 13), and its intent is (security $\{0.80,0.82,0.83\}$; resale value $\{0.56,0.58\}$; cost performance $\{0.72,0.73\}$). It shows that both Car 6 and Car 13 are safe, value preserving, and cost-effective. Besides, the hierarchical relationship between concepts is helpful for consumers to adjust their purchase targets. In a chain of the hesitant fuzzy concept lattice, the new energy vehicles contained in the lower concepts must be evaluated higher than that contained in the upper concepts. This rule tells us that when we want to look for new energy vehicles with higher performance based on the existing requirements, we can search down in the Hasse diagram of hesitant fuzzy concept lattice. For example, at first, a buyer chose the new energy vehicle types in C3 according to his preference, but then he has higher requirements for the performance, so he looks for the lower concepts and then gets C2 and C4, which have more attributes than C3, and the membership of the original attribute (security) remains unchanged or increases. Therefore, it is no doubt that the concept lattice theory can help consumers to select products more appropriately and quickly.

However, the hesitant fuzzy concept lattice contains much more information than the classical concept lattice. When the generated concept lattice becomes large, it is not conducive to the discovery of knowledge, so we need to simplify the hesitant fuzzy concept lattice and then extract more refined information.

Step 2. According to a customer's purchase preference, the importance degree set $P = \{3,1,2,1,2\}$, then the weights of indices can be obtained by Definition 14:

$$V = \{1.73, 0.74, 1.31, 0.70, 1.47\}$$

and the similarity matrix can be calculated as Table 15:

TABLE 14. Hesitant fuzzy concepts.

CONCEPT	EXTENT	INTENT
C0	CAR1-16	∅
C1	CAR1;3;4;13;14	SECURITY{0.74,0.76,0.77};COMFORT{0.80,0.82};ENGINE POWER{0.78,0.80,0.83}
C2	CAR16;13	SECURITY{0.80,0.82,0.83};RESALE VALUE{0.56,0.58};COST PERFORMANCE{0.72,0.73}
C3	CAR1;3;4;6;9;11;13;14;16	SECURITY{0.74,0.76,0.77}
C4	CAR6;9;13;14;16	SECURITY{0.74,0.76,0.77};RESALE VALUE{0.56,0.58}
C5	CAR11;13	SECURITY{0.81,0.84,0.85};COMFORT{0.80,0.81};COST PERFORMANCE{0.72,0.73}
⋮	⋮	⋮
C19	CAR6;8;13;15	RESALE VALUE{0.56,0.58};COST PERFORMANCE{0.72,0.73}
C20	CAR2;5;12;13	ENGINE POWER{0.72,0.76,0.81};COST PERFORMANCE{0.72,0.73}
C21	CAR1;2;3;4;5;7;10;12;13;14	ENGINE POWER{0.72,0.76,0.81}
C22	∅	U

TABLE 15. Similarity matrix.

CONCEPT	C1	C2	C3	C4	…	C18	C19	C20	C21
C1	100.0%	29.9%	53.1%	40.8%	…	0.0%	0.0%	15.0%	21.5%
C2	29.9%	100.0%	40.6%	63.7%	…	23.1%	56.2%	28.5%	0.0%
C3	53.1%	40.6%	100.0%	63.8%	…	0.0%	0.0%	0.0%	0.0%
C4	40.8%	63.7%	63.8%	100.0%	…	36.2%	23.8%	0.0%	0.0%
C5	52.2%	64.6%	42.3%	34.1%	…	0.0%	27.8%	29.5%	0.0%
C6	77.4%	34.3%	68.7%	49.4%	…			0.0%	0.0%
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
C16	16.5%	28.1%	0.0%	0.0%	…	0.0%	44.6%	49.0%	0.0%
C17	18.2%	19.5%	0.0%	28.2%	…	0.0%	0.0%	0.0%	0.0%
C18	0.0%	23.1%	0.0%	36.2%	…	100.0%	41.0%	0.0%	0.0%
C19	0.0%	56.2%	0.0%	23.8%	…	41.0%	100.0%	45.6%	0.0%
C20	15.0%	28.5%	0.0%	0.0%	…	0.0%	45.6%	100%	0.0%
C21	21.5%	0.0%	0.0%	0.0%	…	0.0%	0.0%	0.0%	100%

By calculating the similarity between these hesitant fuzzy concepts, those new energy vehicles with similar evaluation can be easily found and compared with each other.

Step 3. To make it easier to get hidden rules, we need to set a similarity threshold, and then merge the hesitant fuzzy concepts with high similarity, to reduce the size of hesitant fuzzy concept lattice.

If we let the similarity threshold = 0.7, then the simplified hesitant fuzzy concepts are sh own as Table 16:

The merged concept lattice makes the classification rougher, but it is conducive to knowledge discovery and selection for the customers.

In this application, based on the theory and methods proposed in this paper, we finally extract the hesitant fuzzy concepts and construct the hesitant fuzzy concept lattice. Therefore, it is proven that although the hesitant fuzzy information makes the formal context more complex, we can simplify the data processing process by effective simplification methods, and finally get the fuzzy concepts with uncertain information. Besides, the membership expressed by hesitant fuzzy element expresses more potential information. The more significant the variance of the hesitant fuzzy element is, the worse the stability of the attribute might be. When the decision-maker tends to the conservative choice, the final decision-making can be adjusted by turning to the SFD simplification method of the hesitation fuzzy formal context,

	SECURITY	COMFORT	RESALE VALUE	ENGINE POWER	COST PERFORMANCE
CAR 1	{0.89,0.9,0.92}	{0.82,0.86}	0	{0.78,0.80,0.83}	0
CAR 2	0	{0.87,0.90}	0	k=0.7 {0.72,0.76,0.81}	{0.82,0.87}
CAR 3	{0.91,0.92,0.94}	{0.88,0.91}	0	{0.82,0.83,0.87}	0
CAR 4	{0.77,0.78,0.82}	{0.83,0.85}	0	{0.85,0.88,0.90}	0
CAR 5	0	0	0	{0.89,0.90,0.91}	{0.77,0.81}
CAR 6	{0.80,0.82,0.89}	0	{0.72,0.77}	0	{0.83,0.85}
CAR 7	0	{0.82,0.85}	{0.66,0.70}	{0.87,0.83,0.85}	0
CAR 8	0	{0.77,0.80}	{0.75,0.81}	0	{0.78,0.85}
CAR 9	{0.78,0.82,0.89}	0	{0.69,0.73}	0	0
CAR 10	0	k=0.7 {0.72,0.82}	{0.66,0.68}	{0.79,0.83,0.85}	0
CAR 11	{0.81,0.84,0.85}	{0.80,0.81}	0	0	{0.83,0.85}
CAR 12	0	{0.83,0.85}	0	{0.78,0.79,0.81}	{0.81,0.84}
CAR 13	{0.87,0.88,0.93}	{0.87,0.91}	k=1 {0.56,0.58}	{0.79,0.81,0.85}	{0.72,0.73}
CAR 14	{0.74,0.76,0.77}	{0.80,0.82}	{0.56,0.60}	{0.82,0.84,0.86}	0
CAR 15	0	{0.82,0.83}	{0.72,0.77}	0	k=1 {0.72,0.78}
CAR 16	{0.77,0.78,0.80}	{0.65,0.69}	{0.69,0.70}	0	0

FIGURE 4. Changes of the simplified hesitant fuzzy formal context.

which takes the variance of the hesitant fuzzy element into account.

B. DISPERSION ANALYSIS OF HESITANT FUZZY SETS

In certain circumstances, the discreteness of attributes affects the decisions of people. Take a clothing designer as an example. If the clothes he designs are sometimes trendy, but sometimes unsalable, then we can think that his works always have significant volatility and instability, which affects the

TABLE 16. Simplified hesitant fuzzy concepts.

CONCEPT	EXTENT	INTENT
C0	CAR1-16	∅
C1	CAR1;3;4;13;14	SECURITY{0.74,0.76,0.77};COMFORT{0.80,0.82};ENGINE POWER{0.78,0.80,0.83}
C2	CAR16;13	SECURITY{0.80,0.82,0.83};RESALE VALUE{0.56,0.58};COST PERFORMANCE{0.72,0.73}
C3	CAR1;3;4;6;9;11;13;14;16	SECURITY{0.74,0.76,0.77}
C4	CAR6;9;13;14;16	SECURITY{0.74,0.76,0.77};RESALE VALUE{0.56,0.58}
C5	CAR11;13	SECURITY{0.81,0.84,0.85};COMFORT{0.80,0.81};COST PERFORMANCE{0.72,0.73}
⋮	⋮	⋮
C19	CAR6;8;13;15	RESALE VALUE{0.56,0.58};COST PERFORMANCE{0.72,0.73}
C20	CAR2;5;12;13	ENGINE POWER{0.72,0.76,0.81};COST PERFORMANCE{0.72,0.73}
C21	CAR1;2;3;4;5;7;10;12;13;14	ENGINE POWER{0.72,0.76,0.81}
C22	∅	U

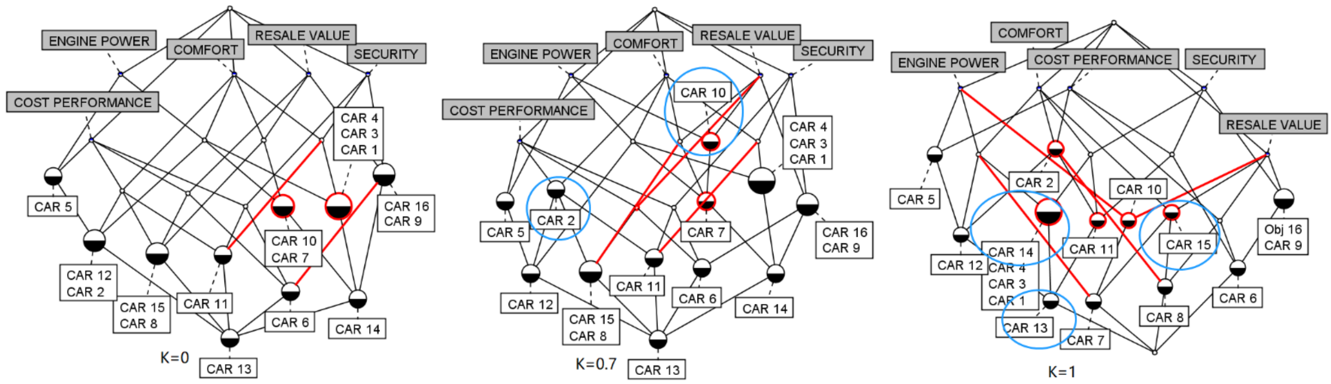

FIGURE 5. Changes of hesitant fuzzy concept lattice.

industry's evaluation of him to some extent. Therefore, in this section, we adjust the relevant parameter of the experiment appropriately, and then discuss the influence on the experiment by observing the changes in the results.

According to the research above, we can use the SFD method (the membership threshold of the attribute is set to $s(h) - k\sigma(h)$) to simplify the hesitant fuzzy formal context, in which the standard deviation of hesitant fuzzy sets is considered. If the decision-makers pay more attention to the stability of attributes, then the value of k can be increased; On the contrary, the value of k can be reduced. Therefore, we can set $k = 0$, $k = 0.7$, and $k = 1$, and then observe the differences in experimental results.

1) The changes of the simplified hesitant fuzzy formal context are shown in Figure 4:

We can see from Figure 4 that when k changes from 0 to 0.7, some attributes of Car 13, Car 14, and Car 15 are filtered out, and when k increases to 1, some attributes of Car 2 and Car 10 are filtered out. This phenomenon shows that the higher the requirement for the stability of attributes, the simpler the hesitant fuzzy formal context is.

2) The changes of hesitant fuzzy concept lattice are shown in Figure 5:

When the value of k increases gradually, we can find the following phenomena from Figure 5:

Firstly, the number of concept nodes has changed, and new nodes appear. For example, when k increases to 0.7, two fuzzy concept nodes emerged for the reason that the attributes of Car 2 and Car10 have changed. Secondly, the extent and intent of some concepts have altered. What is more, the hierarchical relationship of concept lattice has also changed. When k = 0, Car 6 and Car 13 are in the same chain of concept lattice, Car 13 contains all the attributes of Car 6; But when k increases to 1, Car 6 and Car 13 are not in the same chain. Thirdly, some objects contained in the lower concepts move upward, which shows that the increase of k makes the decision-maker reduce the evaluation of these objects' attributes. In other words, the higher the hesitation level of the attribute is, the lower the evaluation of the object is.

V. CONCLUSIONS AND FURTHER STUDY

The hesitant fuzzy concept lattice not only contains more uncertain information but also reflects the stability of the attributes. compared with other fuzzy concept lattices, when dealing with hesitant and irresolute evaluation problems, the hesitate fuzzy concept lattice is more practical and appropriate. In a way, it can reduce data errors and information loss. Based on these reasons, in this paper, the following innovative achievements have been achieved:

Firstly, we introduced the hesitation fuzzy set theory into the formal context, and built the hesitant fuzzy formal context, and then we studied the formal concept analysis theory with hesitant fuzzy information. Secondly, we provided two effective reduction methods of hesitant fuzzy formal context and compared the results of these two methods. Thirdly, we proposed an incremental construction algorithm for hesitant fuzzy concept lattice. Next, we proposed a method to calculate the similarity of hesitant fuzzy concepts, and also discussed the calculation of the weights. Finally, we provided a practical application to prove the correctness and validity of the theory and methods proposed in this paper, and we also discussed the influence of the dispersion of hesitant fuzzy sets on the result. We can see from the controlled experiment that the more discrete the hesitant fuzzy set is, the less reliable the attribute is. In other words, the hesitant fuzzy set can reflect the stability of the attribute.

However, in reality, the information is mainly in the form of natural language from various media instead of numerical values. In order to solve this problem, in the future, we can apply the linguistic hesitant fuzzy set to concept lattice theory, so as to solve the transformation of natural language and numerical value. Besides, for the reason that the weights of attributes depend on specific applications, so the calculation of weights can be further studied.

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