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CR Iteration in Generation of Antifractals With s-Convexity

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ABSTRACT Complex graphics of nonlinear dynamical systems perform vital role in many fields, e.g., image compression or encryption, art, science, and so on. Antifractals are generated by applying a map recursively to an initial point in complex plane that have become a significant area of research these days. The purpose of this paper is to investigate the dynamics of antifractals like anti-Julia sets, tricornsand multicorns of antiholomorphic polynomials via CR iteration scheme with s-convexity. Many beautiful aesthetic patterns are visualized for antipolynomial $z \rightarrow \bar{z}^m + c$ of complex polynomial $z^m + c$, for $m \geq 2$ to explore the geometry of antifractals.

INDEX TERMS CR iteration, antifractals, s-convexity, escape criterion.

I. INTRODUCTION

Fractal art is generally established with the help of fractal-generating software, iterating in three steps: correct setting of parameters; executing desirable calculation and figure out the product. In some samples, graphics tools are used to further modify the produced patterns. It is called post-processing. The Julia and Mandelbrot sets can be recognized as symbol of fractal art [1]. It was considered that without computer fractal art could not have been organized because it provided extraordinary calculative capabilities [2]. Fractals are created by utilizing iterative methods to solve polynomial equations or non-linear equations. Generating fractals can be a mathematical model, an artistic endeavor, or just a soothing diversion. No doubt fractal art is distinguished from other digital activities [3]. The dynamics of antiholomorphic complex polynomials $z \rightarrow \bar{z}^m + c$ for $m \geq 2$ with a single critical point leads to fascinating tricorns and multicorns antifractals [4].

A sequence of iteratively obtained points is called an orbit and the orbit is said to be diverging when its points grow unbounded. The fractal dynamics of the orbits depends on the rule of iteration also. In the study of dynamics of quadratic holomorphic polynomials $Q_c(z) = z^2 + c$, the most important object is the Mandelbrot set [5]. Analogously, the connectedness locus for orbit of antiholomorphic polynomials $\bar{z}^2 + c$, coined "tricorn" by Milnor which plays an intermediate role between quadratic and cubic polynomials. Tricorn has many essential characteristics of the cubic connectedness locus which do not appear in the Mandelbrot set [6]. Although tricorn has many similarities with the Mandelbrot set due to a compact subset of $\mathbb{C}$. The three-cornered nature, the main feature of a tricorn, repetition with variations at various scales, follow the same sort of self-similarity as the Mandelbrot set. Crowe *et al.* [7] considered in formal analogy with Mandelbrot set and named it "Mandelbar sets" and demonstrates its components bifurcations along arcs instead at points. Milnor [6] and Branner [8] explained it in a real slice of the cubic connectedness locus. Winters interpreted that boundary of the tricorn consists of a smooth arc [9]. The symmetries of tricorn and multicorns are analyzed by Lau and Schleicher [10]. Nakane and Schleicher [4] explained various features of tricorn and multicorns along with beautiful figures and extracted that the multicorns are the generalized tricorns or the tricorns of higher order. Also explore that the Julia set of a polynomial of the form $A_c(z) = \bar{z}^m + c$ for $m \geq 2$ is either connected or totally disconnected.

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The set of parameters c such that the Julia set of A_c is connected, is called the multicorn M_c^*. These multicorns are simply higher order tricorns and these are used commercially such as tricorn mugs and tricorn dresses like tricorn shirts. Antifractals have been generalized in many different ways. One of these generalizations is the utilization of iteration procedures from the fixed point theory. Fixed point theory gives an appropriate framework to research different nonlinear phenomena emerging in the applied sciences including complex graphics, biology, geometry and material science [11], [12]. Complex graphical shapes such as fractals, were discovered as fixed points of certain set maps [11]. Many authors have used various iterative procedures to generate antifractals. Rani [13], [14] considered and investigated the dynamics of antiholomorphic complex polynomials $\bar{z}^m + c$ for $m \geq 2$ via Mann iteration. Mishra *et al.* displayed fixed point results for tricorn and multicorns by using Ishikawa iteratin with s-convexity [15]. The dynamics of the antifractals have been studied broadly by Chauhan *et al.* [16], Rani and Chugh [17], Kang *et al.* [18] and Partap *et al.* [19] for different fixed point iterative procedures. Kwun *et al.* [20] explored Julia sets, Mandelbrot sets and Tricorns and Multicorns by means of Jungck-CR iteration with s-convexity. Kumari *et al.* [21] exhibited antifractals in SP orbit. Recently, in [22], [23] authors used Noor orbit and modified S-iteration orbit to generate tricorns and multicorns with s-convextity. Chugh *et al.* [24] proved that CR iteration converges to a fixed point faster than Mann, Ishikawa, Noor and SP iteration schemes. We present an application of this faster fixed point iterative process in generation of antifractals.

In this paper we generate and visualize a new class of antifractals in modified CR orbit. The consequences of this paper are further extension of results presented in [25]. This paper is established as follows. Some basic definitions are presented in section II. Section III have the escape criterion for antifractals. In section IV we generate antifractals by using modified CR iteration scheme. At last, section V contains some concluding remarks.

II. PRELIMINARIES

Definition 1 (Multicorn [26]): The multicorns M_c^* for the function $A_c(z) = \bar{z}^m + c$ for $m \geq 2$ is defined as the collection of all $c \in \mathbb{C}$ for which the orbit of the point 0 is bounded, that is

$$M_c^* = \{c \in \mathbb{C} : A_c^n(0) \text{ does not tend to } \infty\},$$

where $\mathbb{C}$ is a complex plane, A_c^n is the nth iterate of the function $A_c(z)$.

It is seen that for $m = 2$, multicorns reduce to tricorn.

Definition 2 (Julia Set [27]): Let $f : \mathbb{C} \to \mathbb{C}$ symbolize a polynomial of degree ≥ 2. Let F_f be the set of points in $\mathbb{C}$ whose orbits do not converge to the point at infinity. That is, $F_f = \{x \in \mathbb{C} : \{|f^n(x)|, n \text{ varies from } 0 \text{ to } \infty\}$ is bounded$\}$. F_f is called as filled Julia set of the polynomial f. The boundary points of F_f are called as the points of Julia set of the polynomial f or simply the Julia set.

Definition 3 (Mandelbrot Set [26]): The Mandelbrot set M for the quadratic $Q_c(z) = z^2 + c$ is defined as the collection of all $c \in \mathbb{C}$ for which the orbit of the point 0 is bounded, that is

$$M = \{c \in \mathbb{C} : \{Q_c^n(0)\}; n = 0, 1, 2, \ldots \text{ is bounded}\}.$$

We choose the initial point 0, as 0 is the only critical point of Q_c.

Definition 4 (Noor Iteration [28]): Let $T : \mathbb{C} \to \mathbb{C}$ is a mapping. A sequence $\{z_n\}$ of iterates for initial point $z_0 \in \mathbb{C}$ such that

$$\begin{cases} z_{n+1} = (1-\alpha_n^1)z_n + \alpha_n^1 T u_n; \\ u_n = (1-\alpha_n^2)z_n + \alpha_n^2 T v_n; \\ v_n = (1-\alpha_n^3)z_n + \alpha_n^3 T z_n; \quad n \geq 0, \end{cases} \tag{1}$$

where $\alpha_n^1 \in (0, 1]$ and $\alpha_n^2, \alpha_n^3 \in [0, 1]$ is called Noor orbit, that is a function of five variables $(T, z_0, \alpha_n^1, \alpha_n^2, \alpha_n^3)$.

Definition 5 (SP Iteration [29]): Let $T : \mathbb{C} \to \mathbb{C}$ is a mapping. A sequence $\{z_n\}$ of iterates for initial point $z_0 \in \mathbb{C}$ such that

$$\begin{cases} z_{n+1} = (1-\alpha_n^1)u_n + \alpha_n^1 T u_n; \\ u_n = (1-\alpha_n^2)v_n + \alpha_n^2 T v_n; \\ v_n = (1-\alpha_n^3)z_n + \alpha_n^3 T z_n; \quad n \geq 0, \end{cases} \tag{2}$$

where $\alpha_n^1 \in (0, 1]$ and $\alpha_n^2, \alpha_n^3 \in [0, 1]$ is called SP orbit, that is a function of five variables $(T, z_0, \alpha_n^1, \alpha_n^2, \alpha_n^3)$.

Definition 6 (CR Iteration [24]): Let $T : \mathbb{C} \to \mathbb{C}$ be a mapping. Consider a sequence $\{z_n\}$ of iterates for initial point $z_0 \in \mathbb{C}$ such that

$$\begin{cases} z_{n+1} = (1-\alpha_n^{(1)})v_n + \alpha_k^{(1)} T v_n; \\ v_n = (1-\alpha_n^{(2)})T z_n + \alpha_n^{(2)} T w_n; \\ w_n = (1-\alpha_n^{(3)})z_n + \alpha_n^{(3)} T z_n; \quad n \geq 0, \end{cases} \tag{3}$$

where $\alpha_n^{(1)} \in (0, 1]$, $\alpha_n^{(2)}, \alpha_n^{(3)} \in [0, 1]$ and $\{\alpha_n^{(1)}\}, \{\alpha_n^{(2)}\}, \{\alpha_n^{(3)}\}$ are sequences of positive numbers is called as CR orbit, which is a function of five variables $(T, z_0, \alpha_n^{(1)}, \alpha_n^{(2)}, \alpha_n^{(3)})$.

Definition 7 (s-Convex Combination [30]): Let $z_1, z_2, \ldots, z_n \in C$ and $s \in (0, 1]$. The s-convex combination is defined in the following way:

$$\alpha_1^s z_1 + \alpha_2^s z_2 + \ldots + \alpha_n^s z_n, \tag{4}$$

where $\alpha_k \geq 0$ for $k \in \{1, 2, \ldots, n\}$ and $\sum_{k=1}^{n} \alpha_k = 1$.

It is seen that for $s = 1$ the s-convex combination arrange to the normal convex combination. We shall write the s-convex combination in the CR iteration. We take $z_o = z \in \mathbb{C}$, $\alpha_n^{(1)} = \alpha_1, \alpha_n^{(2)} = \alpha_2$ and $\alpha_n^{(3)} = \alpha_3$ (constant sequences are required to obtain desired consequences) then we can modify CR iteration scheme with s-convexity in the following way where $Q_c(z_n)$ be a quadratic, cubic or higher degree function.

$$\begin{cases} z_{n+1} = (1-\alpha_1)^s v_n + \alpha_1^s Q_c(\bar{v}_n) \\ v_n = (1-\alpha_2)^s Q_c(\bar{z}_n) + \alpha_2^s Q_c(\bar{w}_n) \\ w_n = (1-\alpha_3)^s z_n + \alpha_3^s Q_c(\bar{z}_n), \quad n \geq 0, \end{cases} \tag{5}$$

where $\alpha_1 \in (0, 1]$, $\alpha_2, \alpha_3 \in [0, 1]$ and $0 < s \leq 1$.

III. MAIN RESULTS

The escape criterion is important to generate the antifractals which is at the core of different applications in image encryption and computer graphics [31]. Now we establish a general escape criteria for antipolynomials of the form $Q_c(z) = \bar{z}^m + c, m \geq 2$ in CR orbit with s-convexity.

A. ESCAPE CRITERION

The escape criterion plays an important role to generate and analyze antifractals. Escape criterion for antipolynomials via modified CR iteration (5) presented in following result.

Theorem 1: Assume that $|z| \geq |c| > (\frac{2}{s\alpha_1})^{\frac{1}{m-1}}$, $|z| \geq |c| > (\frac{2}{s\alpha_2})^{\frac{1}{m-1}}$ and $|z| \geq |c| > (\frac{2}{s\alpha_3})^{\frac{1}{m-1}}$ here c be a complex number. Let $v_\circ = v, w_\circ = w$ and $z_\circ = z$ then sequence $\{z_n\}$ define as

$$\begin{cases} z_{n+1} = (1-\alpha_1)^s v_n + \alpha_1^s Q_c(\bar{v}_n) \\ v_n = (1-\alpha_2)^s Q_c(\bar{z}_n) + \alpha_2^s Q_c(\bar{w}_n) \\ w_n = (1-\alpha_3)^s z_n + \alpha_3^s Q_c(\bar{z}_n), \quad n \geq 0, \end{cases} \tag{6}$$

where $\alpha_1, s \in (0, 1]$, $\alpha_2, \alpha_3 \in [0, 1]$ and $Q_c(\bar{z}) = \bar{z}^m + c$, $m \geq 2$. Then $|z_n| \to \infty$ as $n \to \infty$.

Proof: Consider

$$|w| = \left|(1-\alpha_3)^s z + \alpha_3^s Q_c(\bar{z})\right|.$$

For $Q_c(\bar{z}) = \bar{z}^m + c, m \geq 2$,

$$|w| = \left|(1-\alpha_3)^s z + \alpha_3^s (\bar{z}^m + c)\right|.$$

Since $s \in (0, 1]$ and $\alpha_3 \in [0, 1]$, therefore $\alpha_3^s \geq s\alpha_3$

$$\begin{aligned} |w| &\geq \left|(1-\alpha_3)^s z + s\alpha_3(\bar{z}^m + c)\right| \\ &\geq \left|(1-\alpha_3)^s z + s\alpha_3 \bar{z}^m\right| - |s\alpha_3 c| \\ &\geq \left|s\alpha_3 \bar{z}^m + (1-\alpha_3)^s z\right| - |s\alpha_3 z| \because |z| \geq |c| \\ &\geq \left|s\alpha_3 \bar{z}^m\right| - \left|(1-\alpha_3)^s z\right| - |s\alpha_3 z|. \end{aligned}$$

By using binomial expansion upto linear terms of α_3, we get

$$\begin{aligned} |w| &\geq \left|s\alpha_3 \bar{z}^m\right| - |(1-s\alpha_3)z| - |s\alpha_3 z| \\ &\geq \left|s\alpha_3 \bar{z}^m\right| - |z| + |s\alpha_3 z| - |s\alpha_3 z| \\ &= \left|s\alpha_3 \bar{z}^{m-1}\right| |\bar{z}| - |z| \\ &= \left|s\alpha_3 \bar{z}^{m-1}\right| |z| - |z| \because |\bar{z}| = |z| \\ &= |z| (s\alpha_3 |\bar{z}|^{m-1} - 1). \end{aligned} \tag{7}$$

And

$$|v| = \left|(1-\alpha_2)^s Q_c(\bar{z}) + \alpha_2^s Q_c(\bar{w})\right|.$$

Since $s \in (0, 1]$ and $\alpha_2 \in [0, 1]$, therefore $\alpha_2^s \geq s\alpha_2$ so,

$$\begin{aligned} |v| &\geq \left|(1-\alpha_2)^s (\bar{z}^m + c) + s\alpha_2(\bar{w}^m + c)\right| \\ &\geq \left| \begin{matrix} (1-\alpha_2)^s (\bar{z}^m + c) + \\ s\alpha_2((|\bar{z}| (s\alpha_3 |\bar{z}|^{m-1} - 1))^m + c) \end{matrix} \right| \\ &= \left| \begin{matrix} (1-\alpha_2)^s (\bar{z}^m + c) + \\ s\alpha_2((|z| (s\alpha_3 |z|^{m-1} - 1))^m + c) \end{matrix} \right|. \end{aligned} \tag{8}$$

Since $|z| > (\frac{2}{s\alpha_3})^{\frac{1}{m-1}}$ implies $s\alpha_3 |z|^{m-1} - 1 > 1$ and $|z|^m (s\alpha_3 |z|^{m-1} - 1)^m > |z|^m$ using this in (8), we have

$$\begin{aligned} |v| &\geq \left|(1-\alpha_2)^s (\bar{z}^m + c) + s\alpha_2(|z|^m + c)\right| \\ &\geq \left|(1-\alpha_2)^s (\bar{z}^m + c) + s\alpha_2 |z|^m\right| - |s\alpha_2 c| \\ &= \left|s\alpha_2 |z|^m + (1-\alpha_2)^s (\bar{z}^m + c)\right| - |s\alpha_2 c| \\ &\geq \left|s\alpha_2 |z|^m\right| - \left|(1-\alpha_2)^s (\bar{z}^m + c)\right| - |s\alpha_2 c|. \end{aligned}$$

By using binomial expansion upto linear terms of α_2, we get

$$\begin{aligned} |v| &\geq \left|s\alpha_2 |z|^m\right| - \left|(1-s\alpha_2)(\bar{z}^m + c)\right| - |s\alpha_2 c| \\ &\geq \left|s\alpha_2 |z|^m\right| - \left|(1-s\alpha_2)\bar{z}^m\right| + |(1-s\alpha_2)c| - |s\alpha_2 c| \\ &\geq \left|s\alpha_2 |z|^m\right| - \left|s\alpha_2 \bar{z}^m\right| + \left|\bar{z}^m\right| + |s\alpha_2 c| - |c| - |s\alpha_2 c| \\ &\geq s\alpha_2 |z|^m - s\alpha_2 |z|^m + |z|^m - |z| \because |z| \geq |c| \\ &= |z| (|z|^{m-1} - 1). \end{aligned} \tag{9}$$

Also for

$$|z_1| = \left|(1-\alpha_1)^s v + \alpha_1^s Q_c(\bar{v})\right|.$$

Since $\alpha_1, s \in (0, 1]$ therefore $\alpha_1^s \geq s\alpha_1$, we get

$$\begin{aligned} |z_1| &\geq \left|(1-\alpha_1)^s v + s\alpha_1(\bar{v}^m + c)\right| \\ &\geq \left| \begin{matrix} (1-\alpha_1)^s |z| (|z|^{m-1} - 1) + \\ s\alpha_1((|\bar{z}| (|\bar{z}|^{m-1} - 1))^m + c) \end{matrix} \right| \\ &\geq \left| \begin{matrix} (1-\alpha_1)^s |z| (|z|^{m-1} - 1) + \\ s\alpha_1((|z| (|z|^{m-1} - 1))^m + c) \end{matrix} \right|. \end{aligned} \tag{10}$$

Since $|z| > (\frac{2}{s\alpha_1})^{\frac{1}{m-1}} \geq (2)^{\frac{1}{m-1}}$ because $\alpha_1, s \in (0, 1]$ implies $|z|^{m-1} - 1 > 1$, $|z|(|z|^{m-1} - 1) > |z|$ and $|z|^m (|z|^{m-1} - 1)^m > |z|^m$ using this in (10) and $|z| \geq |c|$, we have

$$\begin{aligned} |z_1| &\geq \left|(1-\alpha_1)^s |z| + s\alpha_1(|z|^m + c)\right| \\ &\geq \left|s\alpha_1 |z|^m\right| - \left|(1-\alpha_1)^s |z|\right| - |s\alpha_1 c| \\ &\geq \left|s\alpha_1 |z|^m\right| - \left|(1-\alpha_1)^s |z|\right| - |s\alpha_1 z|. \end{aligned}$$

By using binomial expansion upto linear terms of α_1, we get

$$\begin{aligned} |z_1| &\geq \left|s\alpha_1 |z|^m\right| - |(1-s\alpha_1) |z|| - |s\alpha_1 z| \\ &\geq \left|s\alpha_1 |z|^m\right| - |z| + |s\alpha_1 z| - |s\alpha_1 z| \\ &= |z| (s\alpha_1 |z|^{m-1} - 1). \end{aligned}$$

Since $|z| > (\frac{2}{s\alpha_1})^{\frac{1}{m-1}}$ implies $s\alpha_1 |z|^{m-1} - 1 > 1$, there exist a number $\beta > 0$, such that $s\alpha_1 |z|^{m-1} - 1 > 1 + \beta > 1$. Therefore

$$|z_1| > (1+\beta) |z|,$$

We can apply the similar argument repeatedly to obtain:

$$\begin{aligned} |z_2| &> (1+\beta)^2 |z|, \\ &\vdots \\ |z_n| &> (1+\beta)^n |z|. \end{aligned}$$

Hence $|z_n| \longrightarrow \infty$ as $n \to \infty$ and proved. □

Following corollary provides an algorithm for computing the antifractals for the functions of the form $Q_c(\bar{z}) = \bar{z}^m + c$, $m \geq 2$, via modified CR iterative procedure.

Corollary 1 (Escape Criterion): If $Q_c(\bar{z}) = \bar{z}^m + c$, $m \geq 2$, where c be a complex number. Suppose $|z| > \max\left\{|c|, (\frac{2}{s\alpha_1})^{\frac{1}{m-1}}, (\frac{2}{s\alpha_2})^{\frac{1}{m-1}}, (\frac{2}{s\alpha_3})^{\frac{1}{m-1}}\right\}$ then $|z_n| \to \infty$ as $n \to \infty$.

IV. GENERATION OF ANTIFRACTALS

In functional dynamics there exists two unique kinds of points. First sort of points contained in stable set of infinity that leaves the interval after a limited number of iterations are called the escape set (light color in figures) and second kind of points have bounded orbits that never leave the interval after any number of iterations are called a prisoner set (black color in figures). This section contains tricorns and multicorns for $z \to \bar{z}^m + c$ where $m \geq 2$ and anti-Julia sets for quadratic and cubic functions via CR iteration scheme with s-convexity. We used the escape time algorithm with the escape criterion to generate the images in the software Mathematica 9.0. The experiments were performed on a computer with the following specifications: Intel i5-2400 (@3.1 GHz) processor, 4 GB DDR3 RAM, and Microsoft Windows 10 (64-bit) operating system. Pseudocodes of antifractals generation algorithms are presented in Algorithm 1 and Algorithm 2.

Algorithm 1 Generation of Tricorn and Multicorn

Input: $Q_c(\bar{z}) = \bar{z}^m + c$, where $c \in \mathbb{C}$ and $m \geq 2$, $A \subset \mathbb{C}$ – area, K – iterations, $\alpha_1, \alpha_2, \alpha_3, s$ – parameters for the CR iteration with s-convexity, *colourmap*$[0..C-1]$ – with C colours.
Output: Tricorn or Multicorn for the area A.

for $c \in A$ **do**
  $R = \max\left\{|c|, (\frac{2}{s\alpha_1})^{\frac{1}{m-1}}, (\frac{2}{s\alpha_2})^{\frac{1}{m-1}}, (\frac{2}{s\alpha_3})^{\frac{1}{m-1}}\right\}$
  $n = 0$
  $z_0 = 0$
  while $n \leq K$ **do**
    $w_n = (1-\alpha_3)^s z_n + \alpha_3^s Q_c(\bar{z}_n)$,
    $v_n = (1-\alpha_2)^s Q_c(\bar{z}_n) + \alpha_2^s Q_c(\bar{w}_n)$
    $z_{n+1} = (1-\alpha_1)^s v_n + \alpha_1^s Q_c(\bar{v}_n)$
    if $|z_{n+1}| > R$ **then**
      break
    $n = n + 1$
  $i = \lfloor (C-1)\frac{n}{K} \rfloor$
  colour c with *colourmap*$[i]$

Algorithm 2 Generation of Anti-Julia Set

Input: $Q_c(\bar{z}) = \bar{z}^m + c$, where $c \in \mathbb{C}$ and $m \geq 2$, $A \subset \mathbb{C}$ – area, K – iterations, $\alpha_1, \alpha_2, \alpha_3, s$ – parameters for the CR iteration with s-convexity, *colourmap*$[0..C-1]$ – with C colours.
Output: Anti-Julia set for the area A.

$R = \max\left\{|c|, (\frac{2}{s\alpha_1})^{\frac{1}{m-1}}, (\frac{2}{s\alpha_2})^{\frac{1}{m-1}}, (\frac{2}{s\alpha_3})^{\frac{1}{m-1}}\right\}$
for $z_0 \in A$ **do**
  $n = 0$
  while $n \leq K$ **do**
    $w_n = (1-\alpha_3)^s z_n + \alpha_3^s Q_c(\bar{z}_n)$,
    $v_n = (1-\alpha_2)^s Q_c(\bar{z}_n) + \alpha_2^s Q_c(\bar{w}_n)$
    $z_{n+1} = (1-\alpha_1)^s v_n + \alpha_1^s Q_c(\bar{v}_n)$
    if $|z_{n+1}| > R$ **then**
      break
    $n = n + 1$
  $i = \lfloor (C-1)\frac{n}{K} \rfloor$
  colour z_0 with *colourmap*$[i]$

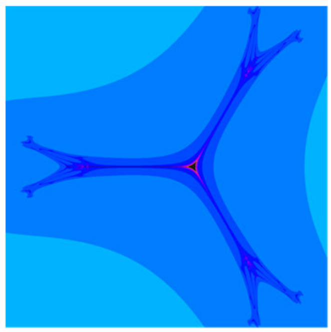

FIGURE 1. Tricorn generated with $s = 0.1$.

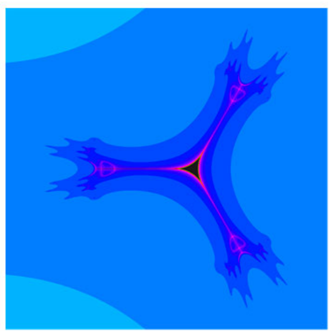

FIGURE 2. Tricorn generated with $s = 0.3$.

A. TRICORNS FOR QUADRATIC FUNCTION

In Figs. 1–6, tricorns are visualized for functions of the form $z \to \bar{z}^2 + c$ in CR orbit with s-convexity by choosing maximum number of iterations 30, $\alpha_1 = 0.04, \alpha_2 = 0.03, \alpha_3 = 0.02$, $A = [-3.5, 2.5] \times [-3.0, 3.0]$ and varying parameter s:

B. MULTICORNS FOR CUBIC FUNCTION

In Figs. 7–12, multicorns are presented for function of the form $z \to \bar{z}^3 + c$ in modified CR orbit by using number of iterations 30, $\alpha_1 = 0.05, \alpha_2 = 0.02, \alpha_3 = 0.03$, $A = [-1.7, 1.7]^2$ and shifting parameter s:

C. MULTICORNS FOR HIGHER DEGREE FUNCTIONS

Multicorns for higher degree functions $Q_c(\bar{z}) = \bar{z}^m + c$, $m \geq 4$ are visualized in modified CR orbit in Figs. 13–18.

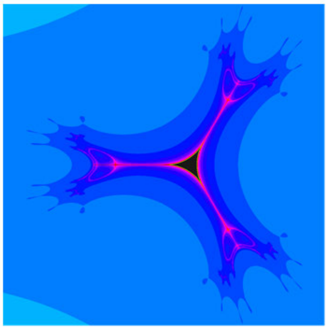

FIGURE 3. Tricorn generated with $s = 0.4$.

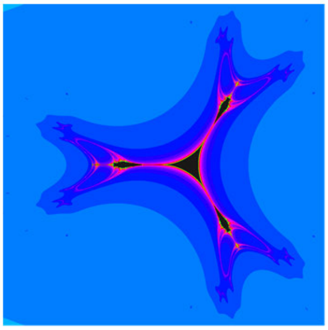

FIGURE 4. Tricorn generated with $s = 0.5$.

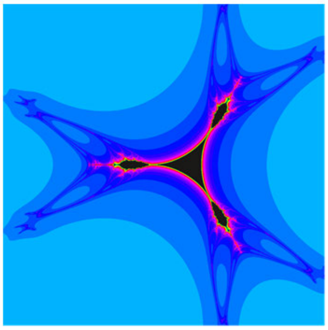

FIGURE 5. Tricorn generated with $s = 0.7$.

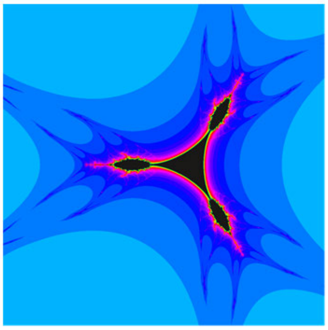

FIGURE 6. Tricorn generated with $s = 0.9$.

The number of iteration used to generate the images are 30 and $A = [-1.7, 1.7]^2$. Moreover, the shifting parameters are:

- Fig. 13: $\alpha_1 = 0.3, \alpha_2 = 0.5, \alpha_3 = 0.2$, $s = 0.6$ and $m = 4$,

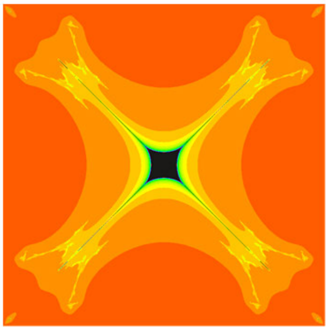

FIGURE 7. Multicorn generated with $s = 0.2$.

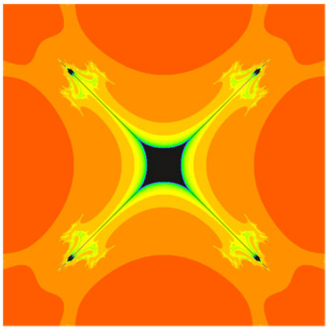

FIGURE 8. Multicorn generated with $s = 0.3$.

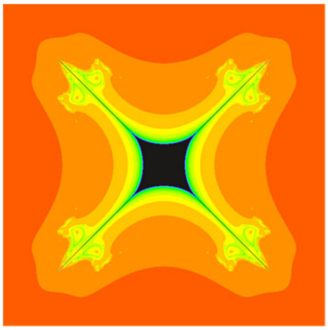

FIGURE 9. Multicorn generated with $s = 0.4$.

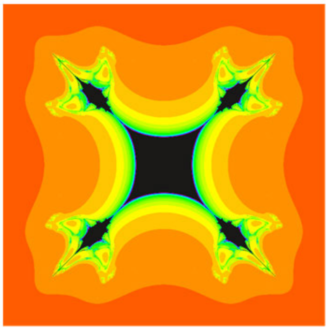

FIGURE 10. Multicorn generated with $s = 0.6$.

- Fig. 14: $\alpha_1 = 0.3, \alpha_2 = 0.6, \alpha_3 = 0.5$, $s = 0.8$ and $m = 4$,
- Fig. 15: $\alpha_1 = 0.03, \alpha_2 = 0.05, \alpha_3 = 0.06$, $s = 0.6$ and $m = 5$,

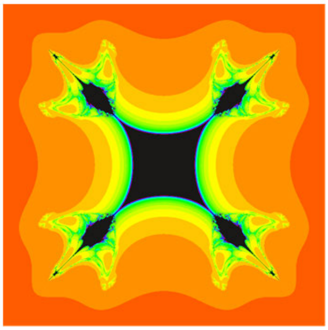

FIGURE 11. Multicorn generated with $s = 0.7$.

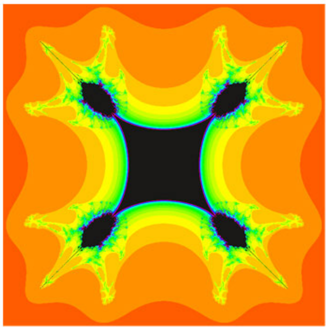

FIGURE 12. Multicorn generated with $s = 0.9$.

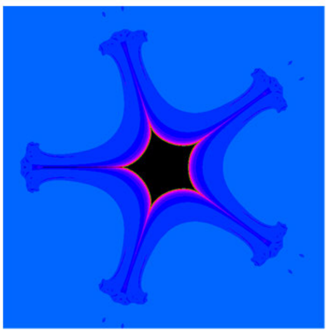

FIGURE 13. Multicorn generated for $m = 4$.

- Fig. 16: $\alpha_1 = 0.8, \alpha_2 = 0.5, \alpha_3 = 0.4$, $s = 0.6$ and $m = 5$,
- Fig. 17: $\alpha_1 = 0.7, \alpha_2 = 0.3, \alpha_3 = 0.6$, $s = 0.4$ and $m = 6$,
- Fig. 18: $\alpha_1 = 0.4, \alpha_2 = 0.6, \alpha_3 = 0.7$, $s = 0.4$ and $m = 6$.

The following observations are experienced after the generation of tricorns and multicorns in CR orbit with s-convexity.

- We observed a noticeable change in each shape of tricorns and multicorns for distinctive values of parameters.
- It is observed that when m is odd the symmetry of multicorn is around x-axis and y-axis and the symmetry is preserved only along x-axis when m is even.

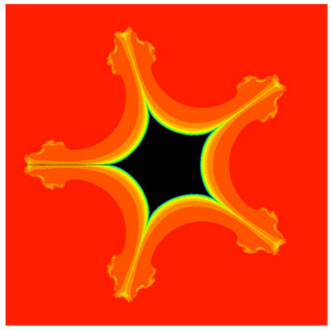

FIGURE 14. Multicorn generated for $m = 4$.

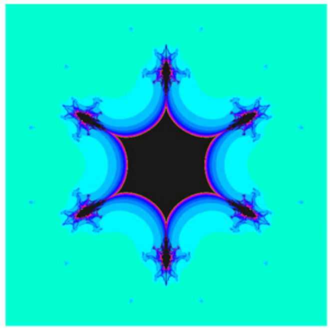

FIGURE 15. Multicorn generated for $m = 5$.

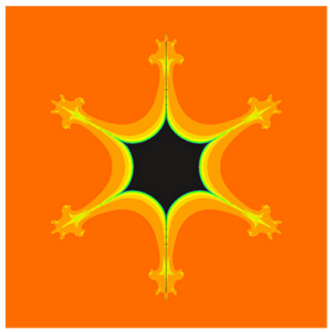

FIGURE 16. Multicorn generated for $m = 5$.

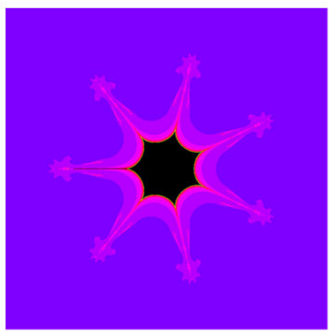

FIGURE 17. Multicorn generated for $m = 6$.

- It is observed that number of branches in the tricorns and multicorns are $m + 1$, where m is the power of $\bar{z}$. Also, some branches have some subbranches.

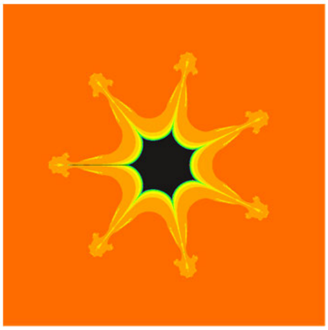

FIGURE 18. Multicorn generated for $m = 6$.

FIGURE 19. Quadratic anti-Julia set for $\alpha_1 = 0.1, \alpha_2 = 0.4$ and $\alpha_3 = 0.3$,.

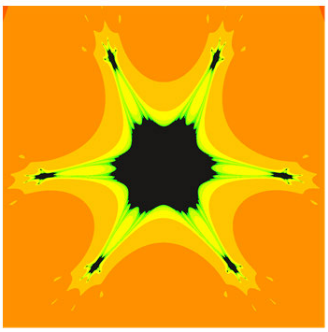

FIGURE 20. Quadratic anti-Julia set for $\alpha_1 = 0.03, \alpha_2 = 0.05$ and $\alpha_3 = 0.04$.

- The multicorns displayed $(m + 1)$-fold rotational symmetries.
- It is also observed that tricorns in Figs. 1–6 and and the multicorns in Figs. 7–12 became thicker if we increased the value of s.
- CPU time in seconds to generate Fig. 1 – 152.718750, Fig. 2 – 183.953125, Fig. 3 – 205.640625, Fig. 4 – 237.281250, Fig. 5 – 269.156250 and Fig. 6 – 304.562500 indicated that when s is increased generation time is increased.
- CPU time in seconds to generate Fig. 7 – 50.359375, Fig. 8 – 60.687500, Fig. 9 – 67.593750, Fig. 10 – 102.265625, Fig. 11 – 116.187500 and Fig. 12 – 138.609375 determined that when s is increased generation time is also increased.

FIGURE 21. Quadratic anti-Julia set for $\alpha_1 = 0.1, \alpha_2 = 0.6$ and $\alpha_3 = 0.5$.

FIGURE 22. Quadratic anti-Julia set for $\alpha_1 = 0.5, \alpha_2 = 0.4$ and $\alpha_3 = 0.2$.

FIGURE 23. Quadratic anti-Julia set for $\alpha_1 = 0.7, \alpha_2 = 0.5$ and $\alpha_3 = 0.3$.

FIGURE 24. Quadratic anti-Julia set for $\alpha_1 = 0.07, \alpha_2 = 0.06$ and $\alpha_3 = 0.05$.

D. ANTI-JULIA SETS FOR QUADRATIC FUNCTION

Anti-Julia sets for $Q_c(\bar{z}) = \bar{z}^2 + c$ are presented via modified CR iterative procedure in Figs. 19–24. The common

FIGURE 25. Cubic anti-Julia set for $\alpha_1 = 0.3, \alpha_2 = 0.5$ and $\alpha_3 = 0.4$.

FIGURE 26. Cubic anti-Julia set for $\alpha_1 = 0.7, \alpha_2 = 0.6$ and $\alpha_3 = 0.5$.

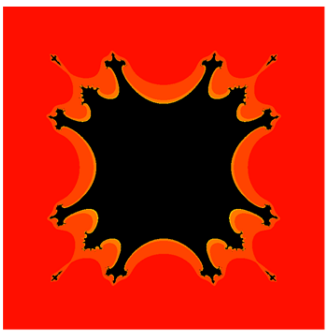

FIGURE 27. Cubic anti-Julia set for $\alpha_1 = 0.2, \alpha_2 = 0.3$ and $\alpha_3 = 0.4$.

FIGURE 28. Cubic anti-Julia set for $\alpha_1 = 0.5, \alpha_2 = 0.8$ and $\alpha_3 = 0.3$.

FIGURE 29. Cubic anti-Julia set for $\alpha_1 = 0.03, \alpha_2 = 0.2$ and $\alpha_3 = 0.2$.

FIGURE 30. Cubic anti-Julia set for $\alpha_1 = 0.05, \alpha_2 = 0.04$ and $\alpha_3 = 0.02$.

parameters used to visualize the images are the following: $K = 30$, $s = 0.5$, $A = [-3.0, 3.0]^2$ and $c = 0.01 - 0.01\mathbf{i}$. Moreover, the shifting parameters are α_1, α_2 and α_3:

E. ANTI-JULIA SETS FOR CUBIC FUNCTION

Anti-Julia sets for $Q_c(\bar{z}) = \bar{z}^3 + c$ are presented via modified CR iteration in Figs. 25–30. The common parameters used to generate the graphics are the following: $K = 30$, $s = 0.6$, $A = [-1.9, 1.9]^2$ and $c = -0.05$ and the varying parameters $\alpha_1, \alpha_2, \alpha_3$:

The following observations are considered after the generation of Anti-Julia sets in CR orbit with s-convexity.

- It is observed that in Figs. 19–24 quadratic Anti-Julia sets for fixed value of s and varying values of α_1, α_2 and α_3 maintain symmetry along y-axis.
- It is also observed that in Figs. 25–30 cubic Anti-Julia sets for fixed value of s and varying values of α_1, α_2 and α_3 maintain symmetry along both x-axis and y-axis.
- Drastic changes are observed in quadratic and cubic anti-Julia sets for different values of α_1, α_2 α_3 and fixed value of s.

V. CONCLUSIONS

Escape criterion to generate antifractals has been established via modified CR iteration scheme. We have visualized many examples of tricorns and multicorns for antipolynomials $z \to \bar{z}^m + c$ where $m \geq 2$. Tricorns in Figs. 1–6 and multicorns in Figs. 7–12 have been visualized for similar values of $\alpha_1, \alpha_2, \alpha_3$ and different values of s. It is seen that $m + 1$ branches, where m is the power of $\bar{z}$ joined to the main body of the tricorns and multicorns. Also, some branches have sub-branches. We have observed that when m is odd the symmetry of multicorn is around x-axis and y-axis and the symmetry is preserved only along x-axis when m is even. A few examples of connected anti-Julia sets have been visualized for quadratic and cubic functions with fix value of s and different values of α_1, α_2 and α_3. Attractive changes can be observed in antifractals visualized in CR orbit with s-convexity for distinctive values of $\alpha_1, \alpha_2, \alpha_3$ and s. We believe that consequences of this paper will be impress those who are interesting in creating aesthetic graphics.

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