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Regulating Constraint-Following Bound for Uncertain Mechanical Systems: An Indirect Control Approach

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ABSTRACT This paper proposes an indirect approach of constraint-following control for mechanical systems with (possibly fast) time-varying uncertainty. It proposes to design controller for the system to render bounded constraint-following error. First, it prescribes a desired hard bound for the constraint-following error. The system is required to lie within the bound at all time. Second, the constraint-following error can eventually be sufficiently small. To accomplish this, the original system is transformed into a constructed system. Then a robust control for the constructed system is designed, with renders uniform boundedness and uniform ultimate boundedness, regardless of the uncertainty. It is further proven that when the constructed system is uniformly bounded and uniformly ultimately bounded, the constraint-following error of the original system stays within the pre-determined bounded. In addition, it will become sufficiently small after a finite time. Therefore the desired bounded performance of the constraint-following error can be archived by the robust control. Since the control is not designed based on the original system, the approach is *indirect*.

INDEX TERMS Mechanical systems, constraint-following, uncertainty, coordinate transformation, robust control.

I. INTRODUCTION

In Lagrangian mechanics, constraints (holonomic or nonholonomic) are used to describe the performance of mechanical system. From the view of control, identification of the control force or torque to render the system to obey the constraints is the so-call constraint following. In the recently past, research efforts can be found in, e.g., [6]–[14] and their bibliographies. In practice, if the constraint-following error, despite being small ultimately, is too large during the transient period, the performance may not be acceptable. This issue, which we shall call the *bounded constraint-following error*, is of practical significance.

Robust control for uncertain systems has attracted many scholars' attention. Chang *et al.* proposed a novel two-step strategy of robust quantized feedback H_∞ control [1] and then addressed the feedback guaranteed cost control problem for discrete-time uncertain systems [2]. Cheng and Chen [3] designed a robust control for euler-lagrange mechanical systems with decentralized adaptive scheme. Yang *et al.* [4] proposed a robust control for under-actuated mechanical systems with model uncertainty. Rascón [5] explored a robust tracking control scheme for mechanical systems. However, for mechanical systems control, the past works usually apply a *control theory oriented* approach that is not solely intended for mechanical systems. This paper falls into a *physical system oriented* framework [6] that starts with the characteristics of the problem, which only exist for mechanical systems, while along the way, the knowledge of control theory will help to pave the way for a concrete control design. This makes our control action more aligned with how the Nature works (through the prescriptions of the physical laws such as d'Alembert's principle, Gauss's principle, etc.), and hopefully leading to better and more resilient performance and less cost.

In the past decades, the quest for the bounded performance has already attracted some scholars' attention. Leitmann and Skowronski [15] guaranteed the concerned system

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avoiding a given bounded set with an avoidance control. Prucz *et al.* [16]–[18] dealt with the structural control problem by applying bounded-state control. Li *et al.* [19] designed a nonlinear velocity controller for the road vehicles to stay on a desired velocity profile within a required error bound. Udwadia *et al.* [7]–[10] put forward several tracking control approaches to drive the controlled system to track a desired reference trajectory within given error bounds. The main focus of these efforts is on the control design for the *original* system. We call such approach the *direct* bounded control. In this approach, the overarching requirement is the initial condition needs to be sufficiently small.

To overcome this limitation, this paper explores a creative *indirect* approach for constraint following of uncertain mechanical systems. This paper specially considers the presence of modeling uncertainty as the engineer's knowledge of the system is often inadequate. In the past, many works have been done on uncertainty analysis and control such as [20]–[30]. By contrast, this paper deals with (relatively) more complicated uncertainty that is (possibly fast) time varying. The objective is to design control to render two layers of system performances, which are included in the bounded constraint-following error task. First, a prescribed constraint-following error is imposed by the designer. The system is to lie within the bound at all time, provided it starts within the bound. Second, the constraint-following error can become close to zero after some finite time and remains there thereafter.

To accomplish the bounded constraint-following error task, the concerned (original) system is transformed to a new (constructed) system via a one-to-one smooth diffeomorphism, and a robust control is designed based on Lyapunov stability theory such as applied in [20]–[27]. We show that if the constructed system exhibits uniform boundedness and uniform ultimate boundedness, the constraint-following error of the original system can be bounded and becomes close to zero after a finite time. This in turn means the desired performance in bounded constraint-following error can be achieved *indirectly*. We call this the *indirect* approach.

The main contributions of this article are fourfold. First, it extends the past constraint-following problem to a more general setting, allowing a desirable constraint-following error bound to be imposed. Second, a more practical *indirect* approach is proposed. A new *constructed* system is derived from the original system. In contrast to the past *direct* approach, which is based on the *original* system and a sufficiently small initial condition is needed, the *indirect* approach allows a rather broad range for the initial condition. Third, a robust control is designed for the constructed system. Fourth, it is shown that when the constructed system is uniformly bounded and uniformly ultimately bounded, the original system meets the desirable error bound. As a result, the proposed bounded constraint-following error problem is solved.

II. UNCERTAIN MECHANICAL SYSTEM AND CONSTRAINTS

Consider a mechanical system (i.e., the original system) described by an n dimensional coordinate [31], [32]:

$$M\left(q\left(t\right),\sigma\left(t\right),t\right)\ddot{q}\left(t\right)+C\left(q\left(t\right),\dot{q}\left(t\right),\sigma\left(t\right),t\right)\times\dot{q}\left(t\right)+G\left(q\left(t\right),\sigma\left(t\right),t\right)=u\left(t\right). \tag{1}$$

Here $q\in R^n$ is the position, $\dot{q}\in R^n$ is the velocity, $\ddot{q}\in R^n$ is the acceleration, $t\in R$ is the time, $\sigma\in\Sigma\subset R^p$ is the (possibly fast) time-varying uncertain parameter with Σ prescribed and compact, and $u\in R^n$ is the control. Furthermore, $M\left(q,\sigma,t\right)>0$ is the inertia matrix, $C\left(q,\dot{q},\sigma,t\right)\dot{q}\left(t\right)$ is the Coriolis/centrifugal force, and $G\left(q,\sigma,t\right)$ is the gravitational force. The matrices/vector $M\left(q,\sigma,t\right)$, $C\left(q,\dot{q},\sigma,t\right)$ and $G\left(q,\sigma,t\right)$ are of appropriate dimensions and the functions they make up are continuous.

The *first order* form constraints which the system needs to follow are proposed as

$$\sum_{i=1}^{n}A_{li}\left(q,t\right)\dot{q}_i=c_l\left(q,t\right),\quad l=1,\ldots,m, \tag{2}$$

where $\dot{q}_i$ is the $i-$th component of $\dot{q}$, $1\leq m\leq n$, $A_{li}(\cdot)$ and $c_l(\cdot)$ are both C^1 in q and t. The constraint may be holonomic and/or nonholonomic. They can be written in matrix form as

$$A\left(q,t\right)\dot{q}=c\left(q,t\right), \tag{3}$$

where $A=[A_{li}]_{m\times n}$, $c=[c_1\ c_2\ \ldots\ c_m]^T$.

Upon differentiation, these constraints can be converted into the *second order* and cast into matrix form:

$$A\left(q,t\right)\ddot{q}=b\left(\dot{q},q,t\right), \tag{4}$$

where $A=[A_{li}]_{m\times n}$, $b=[b_1\ b_2\ \ldots\ b_m]^T$.

For further analysis, a new notion of *constraint-following error* is defined as

$$\tilde{\beta}\left(q,\dot{q},t\right):=A\left(q,t\right)\dot{q}-c\left(q,t\right), \tag{5}$$

where $\tilde{\beta}_i$ is the $i-$th component of the vector $\tilde{\beta}=[\tilde{\beta}_1,\tilde{\beta}_2,\ldots,\tilde{\beta}_m]^T$. It will be used to describe the desired performance (i.e., the control objective) later. With (4), we have

$$\dot{\tilde{\beta}}\left(q,\dot{q},t\right)=A\left(q,t\right)\ddot{q}-b\left(\dot{q},q,t\right). \tag{6}$$

Using (1) in (6), we have

$$\begin{aligned}&\dot{\tilde{\beta}}\left(q(t),\dot{q}(t),t\right)\\&\quad=A\left(q(t),t\right)M^{-1}(q(t),\sigma(t),t)\\&\qquad\times[-C(q(t),\dot{q}(t),\sigma(t),t)\dot{q}(t)-G(q(t),\sigma(t),t)]\\&\qquad+A\left(q(t),t\right)M^{-1}(q(t),\sigma(t),t)u(t)\\&\qquad-b\left(\dot{q}(t),q(t),t\right).\end{aligned} \tag{7}$$

We then denote the constraint-following error with another function of $\beta(t):=\tilde{\beta}\left(q(t),\dot{q}(t),t\right)$, where β_i is the $i-$th component of the vector $\beta=[\beta_1,\beta_2,\ldots,\beta_m]^T$. Let $\beta_0:=\beta(t_0)$ to denote the initial condition, where $\beta_0=[\beta_{01},\beta_{02},\ldots,\beta_{0m}]^T$, β_{0i} is the $i-$th component of

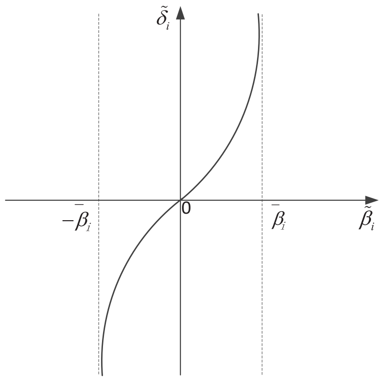

FIGURE 1. The relationship between $\tilde{\beta}_i$ and $\tilde{\delta}_i$.

the vector β_0. The objective of this paper proposed indirect approach of constraint-following control is to drive the constraint-following error to render following desired performances:

(i) Keep in a certain range of $\bar{\beta}$: for any given constraint with the initial condition $\|\beta_{0i}\| < \bar{\beta}_i$,

$$\|\beta_i(t)\| < \bar{\beta}_i \tag{8}$$

for all $t \geq t_0$, where $\bar{\beta}_i > 0$ is the $i-$th component of the vector $\bar{\beta} = [\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m]^T$.

(ii) Reach to an arbitrarily small range of $\bar{c}$: for any given trajectory, there exists a time $T < \infty$, such that

$$\|\beta_i(t)\| \leq \bar{c}_i, \tag{9}$$

for all $t \geq t_0 + T$, where $\bar{c}_i > 0$ is the $i-$th component of the vector $\bar{c} = [\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_m]^T$.

Remark 1: The first performance represents the overall bounded constraint-following error, while the second performance represents the ultimate constraint-following. These two together denotes the overall control objective of *bounded constraint-following*. This can be cast into the bounded control problem.

III. SYSTEM TRANSFORMATION

As the first step of the indirect approach for constraint-following control, based on the original system, a new system (called constructed system) is constructed by a coordinate transformation. Define

$$\tilde{\delta}_i(\tilde{\beta}_i) := \tan\left(\frac{\tilde{\beta}_i - \bar{\beta}_i}{2\bar{\beta}_i}\pi + \frac{\pi}{2}\right), \tag{10}$$

where $\tilde{\delta}_i$ is the $i-$th component of a vector $\tilde{\delta} = [\tilde{\delta}_1, \tilde{\delta}_2, \ldots, \tilde{\delta}_m]^T$. This is a one-to-one smooth diffeomorphism between $\tilde{\beta}_i$ and $\tilde{\delta}_i$, hence invertible. Figure 1 shows the relationship between $\tilde{\delta}_i$ and $\tilde{\beta}_i$. Differentiating (10) with respect to time, we have

$$\dot{\tilde{\delta}}_i(\tilde{\beta}_i) = \sec^2\left(\frac{\tilde{\beta}_i - \bar{\beta}_i}{2\bar{\beta}_i}\pi + \frac{\pi}{2}\right)\frac{\pi}{2\bar{\beta}_i}\dot{\tilde{\beta}}_i, \tag{11}$$

where $\dot{\tilde{\delta}} = [\dot{\tilde{\delta}}_1, \dot{\tilde{\delta}}_2, \ldots, \dot{\tilde{\delta}}_m]^T$, $\dot{\tilde{\delta}}_i$ is the $i-$th component of the vector $\dot{\tilde{\delta}}$. Recalling $\beta(t) = \tilde{\beta}(q(t), \dot{q}(t), t)$, we have $\tilde{\beta}_i = \beta_i$, and then (11) can be rewritten as

$$\dot{\tilde{\delta}}_i(\beta_i) = \sec^2\left(\frac{\beta_i - \bar{\beta}_i}{2\bar{\beta}_i}\pi + \frac{\pi}{2}\right)\frac{\pi}{2\bar{\beta}_i}\dot{\beta}_i, \tag{12}$$

Let

$$\varpi_i(\beta_i) := \sec^2\left(\frac{\beta_i - \bar{\beta}_i}{2\bar{\beta}_i}\pi + \frac{\pi}{2}\right) \tag{13}$$

as the $i-$th component of a vector $\varpi(\beta) = [\varpi_1(\beta_1), \varpi_2(\beta_2), \ldots, \varpi_m(\beta_m)]^T$, and then (12) can be rewritten as

$$\dot{\tilde{\delta}}_i(\beta_i) = \varpi_i(\beta_i)\frac{\pi}{2\bar{\beta}_i}\dot{\beta}_i. \tag{14}$$

Now we obtain its generalized vector form as

$$\dot{\tilde{\delta}}(\beta) = \varpi(\beta)\frac{\pi}{2\bar{\beta}}\dot{\beta}. \tag{15}$$

Let $\delta_i(t) := \tilde{\delta}_i(\tilde{\beta}(q_i(t), \dot{q}_i(t), t))$ as the $i-$ th component of the vector $\delta = [\delta_1, \delta_2, \ldots, \delta_m]^T$, then $\delta(t) = \tilde{\delta}(\tilde{\beta}(q(t), \dot{q}(t), t))$. Recalling $\beta(t) = \tilde{\beta}(q(t), \dot{q}(t), t)$ and using (7) in (15), we have,

$$\begin{aligned}\dot{\delta}(t) = \varpi(\beta(t))\frac{\pi}{2\bar{\beta}}\Big\{&A(q(t), t) M^{-1}(q(t), \sigma(t), t)\\ &\times[-C(q(t), \dot{q}(t), \sigma(t), t)\dot{q}(t) - G(q(t), \sigma(t), t)]\\ &-b(\dot{q}(t), q(t), t)\Big\} + \varpi(\beta(t))\frac{\pi}{2\bar{\beta}}A(q(t), t)\\ &\times M^{-1}(q(t), \sigma(t), t)u(t).\end{aligned} \tag{16}$$

Let

$$\begin{aligned}&f(\beta, q, \dot{q}, \sigma, t)\\ &\quad := \varpi(\beta)\frac{\pi}{2\bar{\beta}}\Big\{A(q, t) M^{-1}(q, \sigma, t)\\ &\qquad \times[-C(q, \dot{q}, \sigma, t)\dot{q} - G(q, \sigma, t)] - b(\dot{q}, q, t)\Big\}\end{aligned} \tag{17}$$

and

$$g(\beta, q, \dot{q}, \sigma, t) := \varpi(\beta)\frac{\pi}{2\bar{\beta}}A(q, t) M^{-1}(q, \sigma, t), \tag{18}$$

where $f \in R^m$ and $g \in R^{m\times n}$. Rearrange (11) with (17) and (18) as

$$\begin{aligned}\dot{\delta}(t) = &f(\beta(t), q(t), \dot{q}(t), \sigma(t), t)\\ &+g(\beta(t), q(t), \dot{q}(t), \sigma(t), t)u(t).\end{aligned} \tag{19}$$

It can be rewritten as

$$\begin{aligned}\dot{\delta}(t) = &\bar{f}(\delta(t), t) + [f(\beta(t), q(t), \dot{q}(t), \sigma(t), t)\\ &-\bar{f}(\delta(t), t)] + g(\beta(t), q(t), \dot{q}(t), \sigma(t), t)u(t),\end{aligned} \tag{20}$$

where $\bar{f} \in R^m$. Decompose $g(\cdot)$ as

$$\begin{aligned}g(\beta(t), q(t), \dot{q}(t), \sigma(t), t) = &\bar{g}(\beta(t), q(t), \dot{q}(t), t)\\ &+\Delta g(\beta(t), q(t), \dot{q}(t), \sigma(t), t).\end{aligned} \tag{21}$$

Here $\bar{g}(\cdot)$ denotes the "nominal" (known) portion, while $\Delta g(\cdot)$ denotes the uncertain (unknown) portion. With it, we rewrite (20) as

$$\begin{aligned}\dot{\delta}(t) = {} & \bar{f}(\delta(t), t) \\ & + \left[f(\beta(t), q(t), \dot{q}(t), \sigma(t), t) - \bar{f}(\delta(t), t)\right] \\ & + [\bar{g}(\beta(t), q(t), \dot{q}(t), t) \\ & + \Delta g(\beta(t), q(t), \dot{q}(t), \sigma(t), t)]\, u(t). \end{aligned} \tag{22}$$

such that we obtain the state equation of the constructed system as (22).

Remark 2: Through above analysis of system transformation, we shift the focus from state β (i.e., the original control object) to the other state δ (i.e., the constructed control object). This is the critical process of our proposed indirect approach of constraint-following control. From (10), as $\tilde{\beta}_i$ is the variable of $\tilde{\delta}_i$, $\tilde{\beta}_i$ should affect $\tilde{\delta}_i$, conversely, $\tilde{\delta}_i$ may determine $\tilde{\beta}_i$. In some sense, $\tilde{\beta}$ (i.e., β) can be seen as the bridge between the original system and the new constructed system. This is the essential reason that why we can achieve the desired performance for β indirectly by controlling δ in later study.

IV. INDIRECT ROBUST CONTROL DESIGN

For the desirable performance of the *original* system, we design a robust control for the *constructed* system. Consider the constructed system

$$\begin{aligned}\dot{\delta}(t) = {} & \bar{f}(\delta(t), t) \\ & + \left[f(\beta(t), q(t), \dot{q}(t), \sigma(t), t) - \bar{f}(\delta(t), t)\right] \\ & + [\bar{g}(\beta(t), q(t), \dot{q}(t), t) \\ & + \Delta g(\beta(t), q(t), \dot{q}(t), \sigma(t), t)]\, u(t), \quad \delta(t_0) = \delta_0. \end{aligned} \tag{23}$$

where $\delta(t_0) = \delta_0$ is the initial condition.

Assumption 1: There are mappings $h(\cdot) : R^m \times R^m \times R^n \times R^n \times R^p \times R \to R^n$ and $E(\cdot) : R^m \times R^n \times R^n \times R^p \times R \to R^{n\times n}$, such that

$$\begin{aligned} f(\beta, q, \dot{q}, \sigma, t) - \bar{f}(\delta, t) &= \bar{g}(\beta, q, \dot{q}, t) \\ & \quad \times h(\delta, \beta, q, \dot{q}, \sigma, t), \end{aligned} \tag{24}$$

$$\Delta g(\beta, q, \dot{q}, \sigma, t) = \bar{g}(\beta, q, \dot{q}, t) E(\beta, q, \dot{q}, \sigma, t), \tag{25}$$

for all $\delta, \beta \in R^m$, $q, \dot{q} \in R^n$, $\sigma \in \Sigma$ and $t \in R$.

Subject to Assumption 1, the constructed system can be represented as

$$\begin{aligned}\dot{\delta}(t) = {} & \bar{f}(\delta(t), t) \\ & + \bar{g}(\beta(t), q(t), \dot{q}(t), t) \\ & \times h(\delta(t), \beta(t), q(t), \dot{q}(t), \sigma(t), t) \\ & + [\bar{g}(\beta(t), q(t), \dot{q}(t), t) + \bar{g}(\beta(t), q(t), \dot{q}(t), t) \\ & \times E(\beta(t), q(t), \dot{q}(t), \sigma(t), t)]\, u(t), \qquad \delta(t_0) = \delta_0, \end{aligned} \tag{26}$$

in which $h(\cdot)$ and $E(\cdot)$ are the *uncertain* elements that represent the "lumped" uncertainties of this system.

Assumption 2: The functions $\bar{f}(\cdot) \in R^m$, $\bar{g}(\cdot) \in R^{m\times n}$, $h(\cdot) \in R^n$, and $E(\cdot) \in R^{n\times n}$ are continuous.

Based on the continuity of the functions and the compactness of the set Σ, there exists a continuous function $\rho(\cdot) : R^m \times R^m \times R^n \times R^n \times R \to R_+$, such that for all $(\delta, \beta, q, \dot{q}, \sigma, t) \in R^m \times R^m \times R^n \times R^n \times \Sigma \times R$, $\|h(\delta, \beta, q, \dot{q}, \sigma, t)\| \le \rho(\delta, \beta, q, \dot{q}, t)$.

Assumption 3: There exists a constant $\underline{\lambda} > -1$, such that for all $(\beta, q, \dot{q}, \sigma, t) \in R^m \times R^n \times R^n \times \Sigma \times R$,

$$\frac{1}{2} \min_{\sigma\in\Sigma} \lambda_{\min} \left(E(\beta, q, \dot{q}, \sigma, t) + E^T(\beta, q, \dot{q}, \sigma, t)\right) \ge \underline{\lambda}. \tag{27}$$

Remark 3: In the special case that $g = \bar{g}$ (i.e., no uncertainty), $E = 0$, and hence one can choose $\underline{\lambda} = 0$. Thus by continuity this assumption imposes the effect of uncertainty on the possible deviation of g from $\bar{g}$ to be within a certain threshold which is unidirectional (i.e., unbounded in one direction).

Assumption 4; There are a C^1 function $V(\cdot) : R^m \times R \to R_+$ and $\mathcal{K}_\infty$ functions $\gamma_i(\cdot) : R_+ \to R_+$, $i = 1, 2, 3$, such that for all $(\delta, t) \in R^m \times R$

$$\begin{aligned} \gamma_1 (\|\delta\|) \le V(\delta, t) \le \gamma_2 (\|\delta\|), \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \delta}\bar{f}(\delta, t) \le -\gamma_3 (\|\delta\|). \end{aligned} \tag{28}$$

This assumption simply implicates the choice of the nominal portion $\bar{f}(\delta, t)$. Notice that, $\|\cdot\|$ denotes the euclidean norm of the concerned matrix in this paper; hence, $\|\delta\|$ denotes the euclidean norm of δ here.

Next, subject to Assumptions 1-4, we are to design a robust control for the constructed system (23) or (26) to guarantee the every response of the system uniformly bounded and uniformly ultimately bounded. The control is proposed as

$$\begin{aligned} u(t) = -\lambda \bar{g}^T(\beta(t), q(t), \dot{q}(t), t) \frac{\partial V}{\partial \delta}(\delta(t), t) \\ \times \rho^2(\delta(t), \beta(t), q(t), \dot{q}(t), t), \end{aligned} \tag{29}$$

with a constant $\lambda > 0$.

Theorem 1: Consider the constructed system (23) or (26). Subject to Assumptions 1-4, the control (29) renders:

(i) Uniform boundedness: For any $r > 0$, there is a $d(r) < \infty$ such that if $\|\delta_0\| \le r$, then $\|\delta(t)\| \le d(r)$ for all $t \ge t_0$;

(ii) Uniform ultimate boundedness: For any $r > 0$ with $\|\delta_0\| \le r$, there exists a $\underline{d} > 0$ and a time $T\left(\bar{d}, r\right) < \infty$ such that $\|\delta(t)\| \le \bar{d}$ for any $\bar{d} > \underline{d}$ as $t \ge t_0 + T$.

Proof: Recalling Assumption 4, $V(\cdot) : R^m \times R \to R_+$; hence, $V(\cdot) \ge 0$. Moreover, $\gamma_1(\|\delta\|) \le V(\delta, t) \le \gamma_2(\|\delta\|)$, where $\gamma_{1,2}(\cdot) : R_+ \to R_+$ are $\mathcal{K}_\infty$ functions; hence, $V(\delta, t) = 0$ only as $\delta = 0$. This means $V(\delta, t)$ is a positive definite function. By this, we take it as the Lyapunov function candidate here. Its derivative along the trajectory of the controlled system is

$$\begin{aligned} \mathcal{L}(\delta, t) &:= \frac{\partial V(\delta, t)}{\partial t} + \frac{\partial^T V(\delta, t)}{\partial \delta}\dot{\delta} \\ &= \frac{\partial V(\delta, t)}{\partial t} + \frac{\partial^T V(\delta, t)}{\partial \delta}\left[\bar{f} + \bar{g}h + (\bar{g} + \bar{g}E)\, u\right] \\ &= \frac{\partial V(\delta, t)}{\partial t} + \frac{\partial^T V(\delta, t)}{\partial \delta}\bar{f} + \frac{\partial^T V(\delta, t)}{\partial \delta}\bar{g}h \\ & \quad + \frac{\partial^T V(\delta, t)}{\partial \delta}(\bar{g} + \bar{g}E)\, u. \end{aligned} \tag{30}$$

With Assumption 4, we can rewrite it as

$$\begin{aligned}\mathcal{L}(\delta,t) &\leq -\gamma_3(\|\delta\|) + \frac{\partial^T V(\delta,t)}{\partial\delta}\bar{g}h \\ &\quad + \frac{\partial^T V(\delta,t)}{\partial\delta}(\bar{g}+\bar{g}E)u \\ &\leq -\gamma_3(\|\delta\|) + \left\|\frac{\partial^T V(\delta,t)}{\partial\delta}\bar{g}\right\|\|h\| \\ &\quad + \frac{\partial^T V(\delta,t)}{\partial\delta}\bar{g}u + \frac{\partial^T V(\delta,t)}{\partial\delta}\bar{g}Eu. \end{aligned} \tag{31}$$

As a consequence of the control (29) with (31), we have

$$\begin{aligned}\mathcal{L}(\delta,t) &\leq -\gamma_3(\|\delta\|) + \left\|\frac{\partial^T V(\delta,t)}{\partial\delta}\bar{g}\right\|\|h\| \\ &\quad + \frac{\partial^T V(\delta,t)}{\partial\delta}\bar{g}\left[-\lambda\bar{g}^T\frac{\partial V}{\partial\delta}(\delta,t)\rho^2\right] \\ &\quad + \frac{\partial^T V(\delta,t)}{\partial\delta}\bar{g}E\left[-\lambda\bar{g}^T\frac{\partial V}{\partial\delta}(\delta,t)\rho^2\right] \\ &= -\gamma_3(\|\delta\|) + \left\|\frac{\partial^T V(\delta,t)}{\partial\delta}\bar{g}\right\|\|h\| \\ &\quad -\lambda\left\|\frac{\partial^T V(\delta,t)}{\partial\delta}\bar{g}\right\|^2\rho^2 \\ &\quad -\lambda\left[\frac{\partial^T V(\delta,t)}{\partial\delta}\bar{g}\right]E\left[\frac{\partial^T V}{\partial\delta}(\delta,t)\bar{g}\right]^T\rho^2. \end{aligned} \tag{32}$$

Define

$$\psi := \frac{\partial^T V(\delta,t)}{\partial\delta}\bar{g}. \tag{33}$$

Then by using (33) in (32), we have

$$\begin{aligned}\mathcal{L}(\delta,t) &\leq -\gamma_3(\|\delta\|) + \|\psi\|\rho \\ &\quad -\lambda\|\psi\|^2\rho^2 - \lambda\psi E\psi^T\rho^2 \\ &= -\gamma_3(\|\delta\|) + \|\psi\|\rho \\ &\quad -\lambda\|\psi\|^2\rho^2 - \frac{1}{2}\lambda\left(\psi E\psi^T + \psi E\psi^T\right)\rho^2 \\ &= -\gamma_3(\|\delta\|) + \|\psi\|\rho \\ &\quad -\lambda\|\psi\|^2\rho^2 - \frac{1}{2}\lambda\left(\psi E\psi^T + \psi E^T\psi^T\right)\rho^2. \end{aligned} \tag{34}$$

Rearranging it, we have

$$\begin{aligned}\mathcal{L}(\delta,t) &\leq -\gamma_3(\|\delta\|) + \|\psi\|\rho - \lambda\|\psi\|^2\rho^2 \\ &\quad -\frac{1}{2}\lambda\left[\psi\left(E+E^T\right)\psi^T\right]\rho^2. \end{aligned} \tag{35}$$

Recalling the Assumption 4 and the Rayleigh's principle, we have

$$\begin{aligned} 2\underline{\lambda}\|\psi\|^2 &\leq \lambda_{\min}(E+E^T)\|\psi\|^2 \\ &\leq \psi\left(E+E^T\right)\psi^T \leq \lambda_{\max}(E+E^T)\|\psi\|^2. \end{aligned} \tag{36}$$

Using it in (35), we have

$$\begin{aligned}\mathcal{L}(\delta,t) &\leq -\gamma_3(\|\delta\|) + \|\psi\|\rho \\ &\quad -\lambda\|\psi\|^2\rho^2 - \lambda\underline{\lambda}\|\psi\|^2\rho^2 \\ &= -\gamma_3(\|\delta\|) + \|\psi\|\rho - \lambda(1+\underline{\lambda})\|\psi\|^2\rho^2 \\ &\leq -\gamma_3(\|\delta\|) + \frac{1}{4\lambda\left(1+\underline{\lambda}\right)}, \end{aligned} \tag{37}$$

where $1/\left[4\lambda\left(1+\underline{\lambda}\right)\right] > 0$ as $\underline{\lambda} > -1$.

Recalling the standard arguments as in [33], we conclude the solution of the controlled system is uniform boundedness with

$$d(r) = \begin{cases}(\gamma_1^{-1}\circ\gamma_2)(R) & \text{if } r \leq R, \\ (\gamma_1^{-1}\circ\gamma_2)(r) & \text{if } r > R,\end{cases} \tag{38}$$

$$R = \gamma_3^{-1}\left(\frac{1}{4\lambda\left(1+\underline{\lambda}\right)}\right). \tag{39}$$

Furthermore, uniform ultimate boundedness follows with

$$\bar{R} = (\gamma_1^{-1}\circ\gamma_2)(\bar{d}), \tag{40}$$

$$T(\bar{d},r) = \begin{cases}0 & \text{if } r \leq \bar{R}, \\ \dfrac{\gamma_2(r)-\gamma_1(\bar{R})}{\gamma_3(\bar{R}) - \left[4\lambda\left(1+\underline{\lambda}\right)\right]^{-1}} & \text{if } r > \bar{R}.\end{cases} \tag{41}$$

□

Finally, we summarize the control design procedure as following:

(i) With desired constraint, obtain A and b, and then construct $\tilde{\beta}$ (i.e., β) with (5).

(ii) With A, b and β, obtain $f(\cdot)$ and $\bar{g}(\cdot)$ with (17) and (18).

(iii) Choose $\bar{f}(\cdot)$ and $V(\cdot)$ based on the Assumption 4, and then obtain $\partial V(\cdot)/\partial\delta$.

(iv) Obtain $h(\cdot)$ with (24) in Assumption 1. Choose a bounding function $\rho(\cdot)$ with $\|h(\cdot)\| \leq \rho(\cdot)$.

(v) With $\bar{g}(\cdot)$, $\rho(\cdot)$ and $\partial V(\cdot)/\partial\delta$, the control is obtained as in (29).

V. ANALYSIS OF THE ORIGINAL SYSTEM

We shall verify that the original system (1) renders the desired performance of bounded constraint-following error as the constructed system (26) renders uniform boundedness and uniform ultimate boundedness,.

Theorem 2: If there exists a controller $u(t) \in R^n$ such that $\delta(t)$ is rendered uniform boundedness, then the constraint-following error $\|\beta_i(t)\|$ can stay in a certain range of $\bar{\beta}_i$ with the initial condition of $\|\beta_{0i}\| < \bar{\beta}_i$, that is, $\|\beta_i(t)\| < \bar{\beta}_i$ for all time $t \geq t_0$.

Proof: By the definition of uniform boundedness, when state $\delta(t)$ renders uniform boundedness, we have

$$\|\delta(t)\| \leq d(r) \tag{42}$$

for all $t \geq t_0$. Such that

$$\|\delta_i(t)\| \leq d(r). \tag{43}$$

That is

$$-d(r) \leq \delta_i(t) \leq d(r). \tag{44}$$

Recalling the definition of δ_i as (10) and $\beta(t) = \tilde{\beta}(q(t), \dot{q}(t), t)$, we have

$$-d(r) \leq \tan\left(\frac{\beta_i - \bar{\beta}_i}{2\bar{\beta}_i}\pi + \frac{\pi}{2}\right) \leq d(r). \tag{45}$$

A lengthy but straightforward analysis can show that

$$\frac{2\bar{\beta}_i\left[-\arctan\left(d\left(r\right)\right)-\frac{\pi}{2}\right]}{\pi}+\bar{\beta}_i\leq\beta_i \leq\frac{2\bar{\beta}_i\left[\arctan\left(d\left(r\right)\right)-\frac{\pi}{2}\right]}{\pi}+\bar{\beta}_i. \quad (46)$$

Using

$$-\frac{\pi}{2}<\arctan\left(d\left(r\right)\right)<\frac{\pi}{2} \quad (47)$$

in (46), we have

$$\frac{2\bar{\beta}_i\left(-\frac{\pi}{2}-\frac{\pi}{2}\right)}{\pi}+\bar{\beta}_i<\beta_i<\frac{2\bar{\beta}_i\left(\frac{\pi}{2}-\frac{\pi}{2}\right)}{\pi}+\bar{\beta}_i. \quad (48)$$

Finally, we have

$$\|\beta_i\|<\bar{\beta}_i. \quad (49)$$

□

Remark 4: Under the robust control (29), the new constructed system (23) can render uniform boundedness, by the way the constraint-following error $\|\beta_i\|$ of the original system (1) can stay in a desired range $\bar{\beta}_i$. By this, the desired performance of bounded constraint-following error for the original system is indirectly achieved.

Remark 5: Note that, in the whole control process, the initial condition of the constraint-following error β_0 is not necessary to stay in any ideal range (except hereto desired range $\bar{\beta}$), such that it breaks the limitation of system initial state appearing in direct bounded control.

Theorem 3: If there exists a controller $u(t) \in R^n$ such that $\delta(t)$ is rendered uniform ultimate boundedness, then the constraint-following error $\|\beta_i(t)\|$ can reach to an arbitrary small range of $\bar{c}$, that is, there exists a time $T < \infty$ such that $\|\beta_i(t)\| \leq \bar{c}_i$ for all time $t \geq t_0+T$, where $\bar{c}=[\bar{c}_1,\bar{c}_2,\ldots,\bar{c}_m]^T$ and $\bar{c}_i>0$ is the $i-$th component of the vector $\bar{c}$.

Proof: By the definition of uniform ultimate boundedness, when the state $\delta(t)$ renders uniform ultimate boundedness, we have

$$\|\delta\left(t\right)\|\leq\bar{d} \quad (50)$$

for all $t\geq t_0+T$, such that

$$\|\delta_i\left(t\right)\|\leq\bar{d}. \quad (51)$$

When $t\geq t_0+T$, for all $\|\delta_i\left(t\right)\|\leq\bar{d}$, we have

$$\arctan\left(-\bar{d}\right)<\arctan\left(\delta_i\right)<\arctan\left(\bar{d}\right). \quad (52)$$

Recalling the definition of δ_i as (10), we have

$$\beta_i=\frac{2\bar{\beta}_i\left[\arctan\left(\delta_i\right)-\frac{\pi}{2}\right]}{\pi}+\bar{\beta}_i. \quad (53)$$

Finally, by using (52) in (53), we have

$$\frac{2\bar{\beta}_i\left[\arctan\left(-\bar{d}\right)-\frac{\pi}{2}\right]}{\pi}+\bar{\beta}_i<\beta_i(t) <\frac{2\bar{\beta}_i\left[\arctan\left(\bar{d}\right)-\frac{\pi}{2}\right]}{\pi}+\bar{\beta}_i. \quad (54)$$

We can obtain

$$\|\beta_i(t)\| \leq \max\left\{\left|\frac{2\bar{\beta}_i\left[\arctan\left(-\bar{d}\right)-\frac{\pi}{2}\right]}{\pi}+\bar{\beta}_i\right|, \left|\frac{2\bar{\beta}_i\left[\arctan\left(\bar{d}\right)-\frac{\pi}{2}\right]}{\pi}+\bar{\beta}_i\right|\right\}, =: \bar{c}_i(\bar{d}). \quad (55)$$

It can be seen when $\bar{d}\to 0$, $\bar{c}_i(\bar{d})\to 0$. Therefore, we can conclude when δ is uniform ultimate boundedness, $\|\beta_i\|$ can reach to an arbitrary small range of $\bar{c}_i$ with an arbitrary small $\bar{d}$. □

Remark 6: As the magnitude of $\bar{d}$ can get arbitrarily close to zero by adjusting the control parameters, $\|\beta_i\|$ can be controlled to be arbitrarily close to zero. In this sense, with the robust control (29), the constructed system (23) can render uniform ultimate boundedness, by the way the constraint-following error $\|\beta_i\|$ of the original system (1) can reach to an arbitrary small range of $\bar{c}_i$. By this, the desired performance of approximate constraint-following for the original system is indirectly realized.

Remark 7: Based on Theorems 2-3, the proposed problem of bounded constraint-following for uncertain mechanical systems can be indirectly solved by transforming the original system (1) into another new constructed system (23). When the new constructed system renders uniform boundedness and uniform ultimate boundedness, the constraint-following error of the original system renders bounded and arbitrarily close to zero, respectively. The constraint-following error rendering bounded reflects an *overall* bounded performance, meanwhile, the constraint-following error rendering arbitrarily close to zero reflects an *ultimately* constraint-following performance. These two together form the desired performance of bounded constraint-following, with which the controlled system not only can follow the excepted constraint approximatively, but also can keep the constraint-following error in a desired range to avoid any excessive fluctuation. Fig. 2 shows the design procedure.

VI. DIRECT APPROACH VERSUS INDIRECT APPROACH: WHY THE INDIRECT APPROACH IS SUPERIOR?

A. DIRECT APPROACH

For bounded constraint-following error, the past researches would take the *direct* approach such as [6], [12]–[14], which is briefly reviewed as below. One starts with designing a control which renders the uniform boundedness and uniform ultimate boundedness of $\beta(t)$. The uniform boundedness property means that for any $r>0$ with $\|\beta_0\|\leq r$, $\beta_0=\beta(t_0)$, there is a $d\left(r\right)<\infty$ such that if $\|\beta\left(t_0\right)\|\leq r$, then $\|\beta\left(t\right)\|\leq d\left(r\right)$ for all $t\geq t_0$.

Specifically,

$$d(r)=(\gamma_1^{-1}\circ\gamma_2)(R), \quad \textit{if } r\leq R \quad (56)$$

$$d(r)=(\gamma_1^{-1}\circ\gamma_2)(r), \quad \textit{if } r>R \quad (57)$$

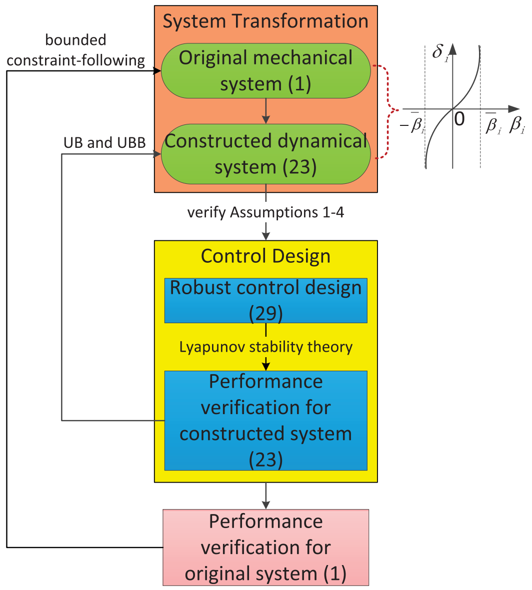

FIGURE 2. Design procedure of the proposed control scheme.

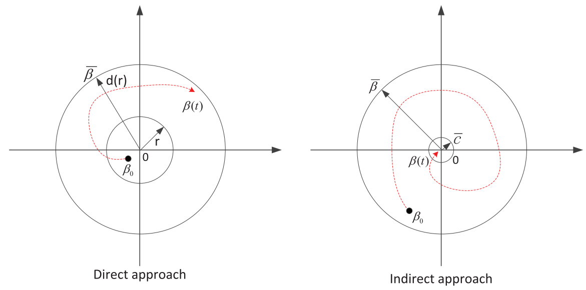

FIGURE 3. Comparison of the direct and indirect approaches of bounded control.

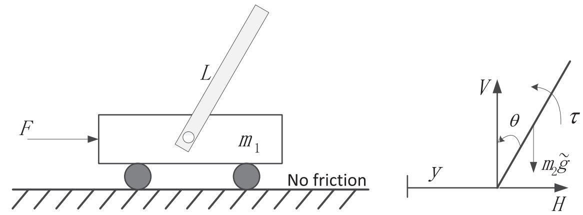

FIGURE 4. Vehicle with an inverted pendulum.

where $R > 0$ is such that the derivative of the Lyapunov function $V(\beta, t)$ along the trajectory of the *original* system is strictly negative when $\|\beta\| > R$, $\bar{\gamma}_{1,2}(\cdot)$ are such that they are $\mathcal{K}_\infty$ and $\bar{\gamma}_1(\|\beta\|) \leq V \leq \bar{\gamma}_2(\|\beta\|)$.

The desired bound of the constraint-following error corresponds $d\,(r)$. This in fact imposes a rather strict limitation on the initial condition β_0.

Suppose one needs $\|\beta_i(t)\| < \bar{\beta}_i$, $i = 1, 2, \cdots, m$, for all t, where $\bar{\beta}_{1,2,\cdots,m} > 0$. To meet all, one should choose $d(r) = \min\{\bar{\beta}_1, \bar{\beta}_2, \cdots, \bar{\beta}_m\}$. Notice that it would be incorrect to choose $d(r) = (\bar{\beta}_1^2 + \bar{\beta}_2^2 + \cdots + \bar{\beta}_m^2)^{1/2}$ since it is possible at one stage all but one components are zero. As a result, the choice of r, which bounds the initial constraint following error β_0, is also subject to the minimum of all bounds; therefore the minimum bound dictates the control design.

B. INDIRECT APPROACH

Based on sections 2 and 3, all $\bar{\beta}_i$'s, instead of only the minimum one, are taken into consideration in the transformation. This means the pre-imposed upper bound of each component of β plays a role in the control design, therefore a non-conservative setting. Furthermore, the ultimate boundedness region for β can be selected by an appropriate choice of λ.

C. COMPARISON

The comparison between direct approach and indirect approach is shown in Figure 3. In the direct approach, only the minimum bound $\min\{\beta_i,\ i = 1, 2, \cdots, m\}$ is considered for the design, while other bounds, no matter how large they are, make no influences. By contrast, in the indirect approach, all bounds $\bar{\beta}_i$'s are taken into consideration. By this, the direct approach is always subject to a rather conservative design of r, while the indirect approach allows a rather broad choice for the initial constraint-following error. This result is practically the limit any control theory can achieve.

VII. ILLUSTRATIVE EXAMPLE

Consider a vehicle with an inverted pendulum as shown in Figure 4. Assume no friction exists between the vehicle and the ground. With two generalized coordinates $q = [y\ \theta]^T$ (where y is the distance from the center of gravity to the pivot, θ is the angular rotation of the pendulum), the system can be described as in [34]

$$\begin{aligned} m_2\tilde{g}L\sin\theta - m_2L^2\ddot{\theta} - m_2L\cos\theta\ddot{y} + \tau &= 0, \\ F - m_1\ddot{y} - m_2\left(\ddot{y} + L\cos\theta\ddot{\theta} - L\sin\theta\dot{\theta}^2\right) &= 0. \end{aligned} \tag{58}$$

Here, m_1 and m_2 are the masses of the vehicle and the inverted pendulum, L is the distance from the center of gravity to the pivot, $\tilde{g}$ is the gravitational constant, F (the control) is an external force imposed on the vehicle, H is a horizontal reaction force imposed on the pendulum, and τ (the control) is an external torque imposed on the pendulum.

A. SYSTEM MODEL AND CONTROL OBJECTIVE

The system (58) can be rewritten in the form of (1) with

$$q = \begin{bmatrix} y \\ \theta \end{bmatrix}, \quad \dot{q} = \begin{bmatrix} \dot{y} \\ \dot{\theta} \end{bmatrix}, \quad \ddot{q} = \begin{bmatrix} \ddot{y} \\ \ddot{\theta} \end{bmatrix}, \quad u = \begin{bmatrix} F \\ \tau \end{bmatrix}, \tag{59}$$

$$M = \begin{bmatrix} m_1 + m_2 & m_2L\cos\theta \\ m_2L\cos\theta & m_2L^2 \end{bmatrix}, \tag{60}$$

$$C(q, \dot{q}, t)\dot{q} = \begin{bmatrix} -m_2L\sin\theta\dot{\theta}^2 \\ 0 \end{bmatrix}, \tag{61}$$

$$G(q) = \begin{bmatrix} 0 \\ -m_2\tilde{g}L\sin\theta \end{bmatrix}. \tag{62}$$

We consider the masses are uncertai (hence $\sigma = [m_1\ m_2]^T$): $m_1 = \bar{m}_1 + \Delta m_1(t)$, $m_2 = \bar{m}_2 + \Delta m_2(t)$. Here $\Delta m_{1,2}(t)$ are the uncertain parts with bounds $\Delta \underline{m}_{1,2} \leq \Delta m_{1,2} \leq \Delta \overline{m}_{1,2}$, where $\left|\Delta \underline{m}_{1,2}\right| \leq \left|\Delta \overline{m}_{1,2}\right|$.

As the *control objective* for a simple illustration, we desire to have the pendulum to be constrained by

$$\dot{y} + \dot{\theta} = 0. \tag{63}$$

Recalling the *second order* form of the constraint $A\ddot{q} = b$, we have

$$A = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad b = 0. \tag{64}$$

B. ASSUMPTIONS VERIFICATION

To illustrate the rationality of the Assumptions 1-4, we have following analysis, in which the argument of functions are largely omitted. First, by analysis of the Remark along with Assumption 3, we choose $\underline{\lambda} = 0$. Second, Assumption 4 can be met by choosing

$$\bar{f} = -k\delta \tag{65}$$

and

$$V(\delta, t) = \frac{1}{2}\delta^2, \tag{66}$$

with $k > 0$ a constant. Then, we have

$$\begin{aligned} \dot{V}(\delta, t) &= \frac{\partial V}{\partial t} + \frac{\partial^T V}{\partial \delta}\bar{f} \\ &= -k\delta^T \delta \\ &\leq -\frac{1}{2}k \left\|\delta\right\|^2, \end{aligned} \tag{67}$$

with

$$\frac{\partial^T V(\delta, t)}{\partial \delta} = \delta^T \tag{68}$$

and

$$\gamma_3(\|\delta\|) = \frac{1}{2}k \left\|\delta\right\|^2. \tag{69}$$

Third, for Assumptions 1-2, we have

$$\begin{aligned} &\varpi(\beta)\frac{\pi}{2\bar{\beta}}\left[AM^{-1}\left(-C\dot{q} - G\right) - b\right] + k\delta \\ &\qquad = \varpi(\beta)\frac{\pi}{2\bar{\beta}}A\bar{M}^{-1}h. \end{aligned} \tag{70}$$

Note that β is a scalar in this illustrative example, with (13), we have

$$\varpi(\beta) = \sec^2\left(\frac{\beta - \bar{\beta}}{2}\pi + \frac{\pi}{2}\right). \tag{71}$$

Taking it into (70) we have

$$\begin{aligned} &\sec^2\left(\frac{\beta - \bar{\beta}}{2}\pi + \frac{\pi}{2}\right)\frac{\pi}{2\bar{\beta}}\left[AM^{-1}\left(-C\dot{q} - G\right) - b\right] \\ &\quad + k\delta = \sec^2\left(\frac{\beta - \bar{\beta}}{2}\pi + \frac{\pi}{2}\right)\frac{\pi}{2\bar{\beta}}A\bar{M}^{-1}h. \end{aligned} \tag{72}$$

Dividing both sides by $\sec^2\left(\left(\beta - \bar{\beta}\right)\pi/2\bar{\beta} + \pi/2\right)\left(\pi/2\bar{\beta}\right)$, yields

$$\begin{aligned} &\left[AM^{-1}\left(-C\dot{q} - G\right) - b\right] \\ &\quad + \left[\sec^2\left(\frac{\beta - \bar{\beta}}{2\bar{\beta}}\pi + \frac{\pi}{2}\right)\frac{\pi}{2\bar{\beta}}\right]^{-1} k\delta = A\bar{M}^{-1}h. \end{aligned} \tag{73}$$

With $b = 0$, we have

$$\begin{aligned} &\left[AM^{-1}\left(-C\dot{q} - G\right)\right] \\ &\quad + \left[\sec^2\left(\frac{\beta - \bar{\beta}}{2\bar{\beta}}\pi + \frac{\pi}{2}\right)\frac{\pi}{2\bar{\beta}}\right]^{-1} k\delta = A\bar{M}^{-1}h. \end{aligned} \tag{74}$$

With (60), (61) and (62), we have

$$\begin{aligned} M^{-1} &= \frac{1}{m_1 m_2 L^2 + m_2^2 \sin^2\theta L^2} \\ &\quad \times \begin{bmatrix} m_2 L^2 & -m_2 L\cos\theta \\ -m_2 L\cos\theta & m_1 + m_2 \end{bmatrix}, \end{aligned} \tag{75}$$

$$\begin{aligned} \bar{M}^{-1} &= \frac{1}{\bar{m}_1 \bar{m}_2 L^2 + \bar{m}_2^2 \sin^2\theta L^2} \\ &\quad \times \begin{bmatrix} \bar{m}_2 L^2 & -\bar{m}_2 L\cos\theta \\ -\bar{m}_2 L\cos\theta & \bar{m}_1 + \bar{m}_2 \end{bmatrix}, \end{aligned} \tag{76}$$

and

$$-C\dot{q} - G = m_2 L\sin\theta \begin{bmatrix} \dot{\theta}^2 \\ \tilde{g} \end{bmatrix}. \tag{77}$$

After some calculations, we obtain

$$\begin{aligned} &AM^{-1}\left(-C\dot{q} - G\right) \\ &\quad = \frac{1}{m_1 m_2 L^2 + m_2^2 \sin^2\theta L^2} \\ &\quad \times m_2^2\Big(L^3\sin\theta\dot{\theta}^2 - L^2\sin\theta\cos\theta\dot{\theta}^2 \\ &\quad -L^2\sin\theta\cos\theta\tilde{g} + L\sin\theta\tilde{g}\Big) + m_1 m_2 L\sin\theta\tilde{g} \end{aligned} \tag{78}$$

and

$$\begin{aligned} A\bar{M}^{-1} &= \frac{1}{\bar{m}_1 \bar{m}_2 L^2 + \bar{m}_2^2 \sin^2\theta L^2} \\ &\quad \times \begin{bmatrix} \bar{m}_2 L^2 - \bar{m}_2 L\cos\theta & \bar{m}_1 + \bar{m}_2 - \bar{m}_2 L\cos\theta \end{bmatrix}. \end{aligned} \tag{79}$$

Using (78) and (79) in (74), and assuming $h = [\tilde{h}\; 0]^T$, we then have

$$\begin{aligned} A\bar{M}^{-1}h &= \frac{\bar{m}_2 L^2 - \bar{m}_2 L\cos\theta}{\bar{m}_1 \bar{m}_2 L^2 + \bar{m}_2^2 \sin^2\theta L^2}\tilde{h} \\ &= \frac{1}{m_1 m_2 L^2 + m_2^2 \sin^2\theta L^2} \\ &\quad \times m_2^2\Big(L^3\sin\theta\dot{\theta}^2 - L^2\sin\theta\cos\theta\dot{\theta}^2 \\ &\quad -L^2\sin\theta\cos\theta\tilde{g} + L\sin\theta\tilde{g}\Big) + m_1 m_2 L\sin\theta\tilde{g} \\ &\quad + \left[\sec^2\left(\frac{\beta - \bar{\beta}}{2\bar{\beta}}\pi + \frac{\pi}{2}\right)\frac{\pi}{2\bar{\beta}}\right]^{-1} k\delta. \end{aligned} \tag{80}$$

Let

$$\begin{aligned} \xi &:= \frac{1}{m_1 m_2 L^2 + m_2^2 \sin^2\theta L^2} \\ &\quad \times m_2^2\Big(L^3\sin\theta\dot{\theta}^2 - L^2\sin\theta\cos\theta\dot{\theta}^2 \\ &\quad -L^2\sin\theta\cos\theta\tilde{g} + L\sin\theta\tilde{g}\Big) + m_1 m_2 L\sin\theta\tilde{g} \end{aligned} \tag{81}$$

and

$$\eta := \frac{\bar{m}_1\bar{m}_2L^2 + \bar{m}_2^2\sin^2\theta L^2}{\bar{m}_2L^2 - \bar{m}_2L\cos\theta}. \tag{82}$$

With (80), aiming at $\tilde{h}$, we obtain

$$\begin{aligned}\tilde{h} &= \eta\xi + k\left[\sec^2\left(\frac{\beta-\bar{\beta}}{2\bar{\beta}}\pi + \frac{\pi}{2}\right)\frac{\pi}{2\bar{\beta}}\right]^{-1} \\ &\quad\times \tan\left(\frac{\beta-\bar{\beta}}{2\bar{\beta}}\pi + \frac{\pi}{2}\right)\eta \\ &= \eta\xi + \frac{k\bar{\beta}}{\pi}\sin\left(\frac{\beta-\bar{\beta}}{\bar{\beta}}\pi + \pi\right)\eta. \end{aligned}\tag{83}$$

We assume $L > 1$ here, such that $\bar{m}_2L^2 - \bar{m}_2L\cos\theta \neq 0$.

With (83), we first have

$$\begin{aligned}\|h\| &= \left|\tilde{h}\right| \\ &\leq |\xi|\,|\eta| + \left|\frac{k\bar{\beta}}{\pi}\sin\left(\frac{\beta-\bar{\beta}}{\bar{\beta}}\pi + \pi\right)\right|\,|\eta|. \end{aligned}\tag{84}$$

Aiming at $\|\xi\|$, with (81) and $\Delta\underline{m}_{1,2} \leq \Delta m_{1,2} \leq \Delta\bar{m}_{1,2}$, we obtain

$$\|\xi\| \leq \xi_1\xi_2 + \xi_1\xi_3 \tag{85}$$

with the definitions

$$\begin{aligned}\xi_1 &:= \Big[\left(\bar{m}_1 + \Delta\underline{m}_1\right)\left(\bar{m}_2 + \Delta\underline{m}_2\right)L^2 \\ &\quad + \left(\bar{m}_2 + \Delta\underline{m}_2\right)^2\sin^2\theta L^2\Big]^{-1} \end{aligned}\tag{86}$$

$$\begin{aligned}\xi_2 &:= (\bar{m}_2 + \Delta\bar{m}_2)^2\,\Big|L^3\sin\theta\dot{\theta}^2 - L^2\sin\theta\cos\theta\dot{\theta}^2 \\ &\quad - L^2\sin\theta\cos\theta\tilde{g} + L\sin\theta\tilde{g}\Big| \end{aligned}\tag{87}$$

and

$$\xi_3 := (\bar{m}_1 + \Delta\bar{m}_1)(\bar{m}_2 + \Delta\bar{m}_2)L\,|\sin\theta|\,\tilde{g}. \tag{88}$$

Then, aiming at $|\eta|$, with (82), we have

$$|\eta| = \frac{\bar{m}_1\bar{m}_2L^2 + \bar{m}_2^2\sin^2\theta L^2}{\left|\bar{m}_2L^2 - \bar{m}_2L\cos\theta\right|}. \tag{89}$$

Using (85) and (89) in (84), yields

$$\begin{aligned}\|h\| &\leq (\xi_1\xi_2 + \xi_1\xi_3)\,|\eta| \\ &\quad + \left|\frac{k\bar{\beta}}{\pi}\sin\left(\frac{\beta-\bar{\beta}}{\bar{\beta}}\pi + \pi\right)\right|\,|\eta| \\ &=: \rho. \end{aligned}\tag{90}$$

Note that, we choose ρ as in (90). Recalling the definition of $\bar{g}$, we have

$$\begin{aligned}\bar{g} &= \sec^2\left(\frac{\beta-\bar{\beta}}{2\bar{\beta}}\pi + \frac{\pi}{2}\right)\frac{\pi}{2\bar{\beta}}A\bar{M}^{-1} \\ &= \frac{\pi}{2\bar{\beta}}\frac{\sec^2\left(\frac{\beta-\bar{\beta}}{2\bar{\beta}}\pi + \frac{\pi}{2}\right)}{\bar{m}_1\bar{m}_2L^2 + \bar{m}_2^2\sin^2\theta L^2} \\ &\quad\times\begin{bmatrix}\bar{m}_2L^2 - \bar{m}_2L\cos\theta & \bar{m}_1 + \bar{m}_2 - \bar{m}_2L\cos\theta\end{bmatrix}. \end{aligned}\tag{91}$$

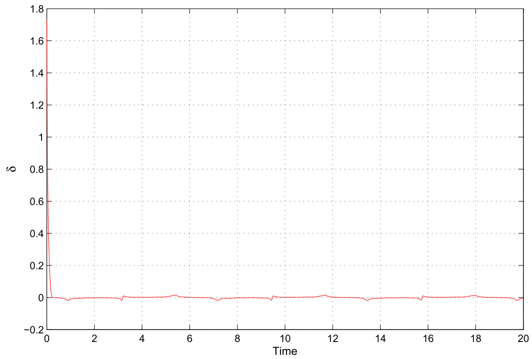

FIGURE 5. Performance of the constructed system (23).

Lastly, we obtain the control as (29) as

$$\begin{aligned}u(t) &= -\lambda\bar{g}^T\frac{\partial V}{\partial\delta}\rho^2 \\ &= -\frac{\lambda\rho^2\pi}{2\bar{\beta}}\frac{\sec^2\left(\frac{\beta-\bar{\beta}}{2\bar{\beta}}\pi + \frac{\pi}{2}\right)\tan\left(\frac{\beta-\bar{\beta}}{2\bar{\beta}}\pi + \frac{\pi}{2}\right)}{\bar{m}_1\bar{m}_2L^2 + \bar{m}_2^2\sin^2\theta L^2} \\ &\quad\times\begin{bmatrix}\bar{m}_2L^2 - \bar{m}_2L\cos\theta \\ \bar{m}_1 + \bar{m}_2 - \bar{m}_2L\cos\theta\end{bmatrix}. \end{aligned}\tag{92}$$

For comparison, we choose the nominal control as

$$\tilde{u}(t) = -\kappa\bar{g}^T\delta, \tag{93}$$

where $\tilde{u} = [\tilde{F}\ \tilde{\tau}]$, and κ is a constant.

C. SIMULATION RESULTS

For simulations, three different classes of uncertainties, namely constant, high frequency and random number, are adopted here. We choose a single uniformly distributed random number X in the interval (0, 1), and then select

$$\begin{aligned} m_1 &= 1 + 0.2\sin 10t + 0.2X, \\ m_2 &= 0.5 + 0.1\sin 10t + 0.1X, \\ \bar{m}_1 &= 1,\ \bar{m}_2 = 0.5, \quad \Delta\bar{m}_1 = 0.2, \\ \Delta\underline{m}_1 &= -0.2, \quad \Delta\bar{m}_2 = 0.1, \\ \Delta\underline{m}_2 &= -0.1, \quad L = 5,\ \tilde{g} = 9.8,\ k = 1, \\ \lambda &= 0.01, \quad \bar{\beta} = 1.5,\ \kappa = 1.6. \end{aligned}\tag{94}$$

Further, we take the initial conditions

$$y(0) = 0,\ \theta(0) = 0,\ \dot{y}(0) = 2,\ \dot{\theta}(0) = -1, \tag{95}$$

and define the control effort as

$$S = \int_0^T \|\beta\|\,dt \tag{96}$$

for performance comparison.

Figure 5 shows the performance of the constructed system (23) under the action of the robust control (29). It can be seen that state δ of the constructed system (23) approaches to a desirable neighborhood close to 0 before $t = 0.26$. Figure 6 shows the performance comparison of the original system (1) with the robust control (29) and the nominal control (93), while Figures 7-8 show the comparison of their

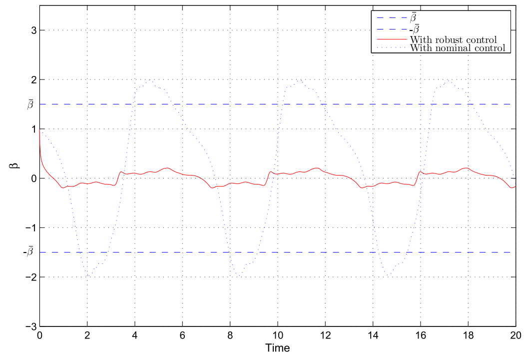

FIGURE 6. Comparison of the performance of the original system (1).

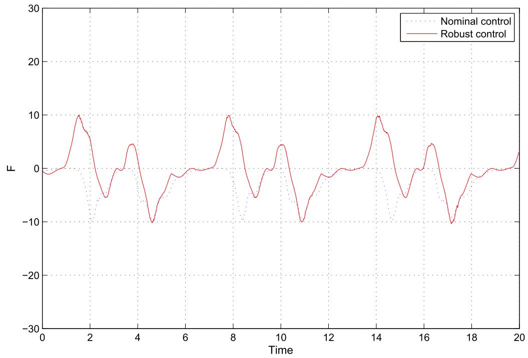

FIGURE 7. Comparison of the control F and $\hat{F}$.

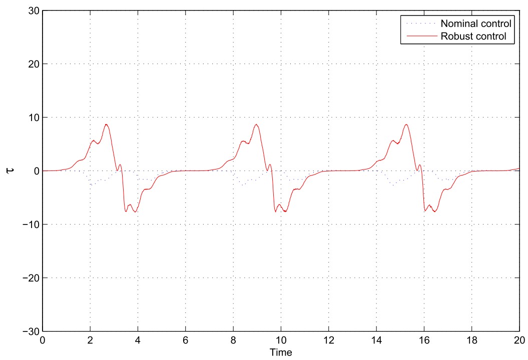

FIGURE 8. Comparison of the control τ and $\hat{\tau}$.

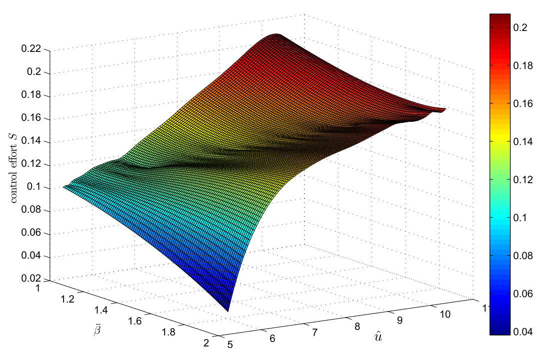

FIGURE 9. Relationship between $\bar{\beta}$, the control effort S and the maximum control $\hat{u}$.

control inputs F to $\tilde{F}$ and τ to $\tilde{\tau}$. By comparison, we find that although these two controls give out almost the same maximum control input of $F = 10$, the control effect is very different: with the robust control, the constraint-following error β always keeps in the previously given range of $\bar{\beta} = 1.5$, and approaches to a desirable neighborhood close to 0 at the same time as δ approaches to its desired range (i.e., before $t = 0.26$) eventually; whereas, with the nominal control, the constraint-following error β goes out of that given range. It demonstrates that the proposed robust control not only can realize approximate constraint-following, but also can realize bounded control. Furthermore, Figure 9 shows the relationship between $\bar{\beta}$, the control effort S and the maximum control $\hat{u}$. It presents that the maximum control output $\hat{u}$ and the control effort S increase as $\bar{\beta}$ increases.

VIII. CONCLUSION

A creative *indirect* approach of constraint-following control is proposed for uncertain mechanical systems. The system's uncertainty is (possibly fast) time-varying. The main objective is to design a control to guarantee that the constraint-following error is bounded in two ways: (i) the constraint-following error lies within a desirable bound for all time; (ii) the constraint-following error eventually enters a small region after a finite time and remains there thereafter. To accomplish this, the original system is creatively transformed into a constructed system by a one-to-one smooth diffeomorphism. A robust control is then designed to render the constructed system uniformly bounded and uniformly ultimately bounded. It is further proven that in such case the original system renders the desirable performance; hence, the task is *indirectly* realized. The major advantage of this *indirect* approach, comparing with the past *direct* approach, is that the magnitude of the initial constraint-following error can be large. On the other hand, the past *direct* approach requires the magnitude of the initial error to be sufficiently small. Our result in terms of the allowable bound for the initial error is practicably the limit any theory can achieve.

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