IEEE Access
Multidisciplinary | Rapid Review | Open Access Journal

Received February 18, 2020, accepted March 4, 2020, date of publication March 16, 2020, date of current version March 26, 2020.

Digital Object Identifier 10.1109/ACCESS.2020.2981044

Fractal Analysis of Overlapping Box Covering Algorithm for Complex Networks

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This work was supported in part by the National Natural Science Foundation of China under Grant 61867004, Grant 61762065, and Grant 61762067, and in part by the Science and Technology Project of the Education Department of Jiangxi Province under Grant GJJ180523.

ABSTRACT Due to extensive research on complex networks, fractal analysis with scale invariance is applied to measure the topological structure and self-similarity of complex networks. Fractal dimension can be used to quantify the fractal properties of the complex networks. However, in the existing box covering algorithms, accurately calculating the fractal dimension of complex networks is still an NP-hard problem. Therefore, in this paper, an improved overlapping box covering algorithm is proposed to explore a more accurate and effective method to calculate the fractal dimension of complex networks. Moreover, in order to verify the effectiveness of the algorithm, the improved algorithm is applied to six complex networks, and compared with other algorithms. Finally, the experimental results demonstrate that the improved overlapping box covering algorithm can cover the whole networks with fewer boxes. In addition, the improved overlapping box covering algorithm is a high accuracy and low time complexity method for calculating the fractal dimension of complex networks.

INDEX TERMS Overlapping box covering algorithm, fractal dimension, complex networks.

I. INTRODUCTION

Complex networks, because of the interdisciplinary and complex characteristics, involve the knowledge and theoretical basis of many disciplines and are widely applied in various fields, such as biology [1]–[3], physics [4]–[6] and social networks [7]–[9]. It's known that scale-free and small-world features are two fundamental properties of complex networks. The scale-free network has the characteristic of power law distribution [10]. Small-world feature means that the average path length of any two nodes is very small and the clustering coefficient is very large, which is also called six-degree separation theory in social network [11]. However, these two properties cannot fully depict the properties of complex networks, which in turn promotes scholars' continuous research on the nature of complex networks.

A fractal is a shape made of parts similar to the whole in some way, according to Mandelbrot who proposed the basic definition of fractal [12]. In simple terms, self-similarity is an important feature of fractal, and fractal dimension is the scale describing the phenomenon with self-similarity in terms of geometric properties. The research on the fractal features of complex networks is derived from the article published by Song *et al.*, ''Self-similarity of complex networks'', which first proposed two methods for calculating the fractal dimension of complex networks: box covering method and clustering-growing method [13]. The emergence of fractal features and self-similarity of complex networks not only provides a new perspective for people to better understand the internal structure and characteristics of networks, but also provides a new view for explaining the production mechanism, evolution process of networks and the coexistence of different characteristics [14]–[17]. Therefore, the self-similarity and fractal characteristics of complex networks are worthy of study.

Box covering method is the most commonly used method for calculating fractal dimension of complex networks. However, the key of box covering method is to cover the whole network with the minimum number of boxes, which is still an NP-hard problem [18]. Song *et al.* proposed three box covering methods, compact box burning method(CBB), maximum excluded mass burning method(MEMB) and greedy coloring method (GC) [19]. Later on, Kim *et al.* proposed a random sequential box covering algorithm to study the

The associate editor coordinating the review of this manuscript and approving it for publication was Yichuan Jiang.

skeleton and fractal scale in complex networks [20], [21]. Recently, Yao *et al.* constructed a kind of self-similar evolutionary network model by using the coding method in fractal geometry theory, replaced a node with an initial graph, and discussed the fractal property of the evolving self-similar network [22]. The traditional cluster growing method is improved to prevent the selection of seed nodes at the edge of the network, and the experimental results showed that the method could obtain the dimension of complex networks well, especially for heterogeneous networks [23]. A D-summable fractal dimension method is proposed, which modifies the definitions of box dimension and information dimension, and is first applied to the calculation of fractal dimension of complex networks [24].

Based on the fractal theory, the reliability model is constructed. The fractal unit and iterative process are used to simplify the reliability model and solve the problem of reliability model with complex systems [25], [26]. In addition, fractal dimension has been proved to directly measure the spatial filling capacity of graphs in complex networks, so it can be used as a parameter of the vulnerability model of complex network [27]. Fractals have been applied to the study of flow and transport in fracture networks and fractured porous media. Studies have shown that changes in fractures are related to the fractal dimension of fracture network [28]. By analyzing the data sets and models of bipartite networks, the self-similarity properties of bipartite networks are revealed in [29]. In wireless sensor network, the data with time series received by the sensor has fractal characteristics, thus providing a method to analyze the structure of wireless touch network [30].

A. MOTIVATION

However, the above mentioned algorithms all adopt separate boxes to cover the whole networks. Since the independent box does not take into account the relationship between the nodes in different boxes, the results obtained by the independent box covering algorithms tend to have large randomness. As we all know, in the complex network community structure, compared with the non-overlapping community structure, the overlapping community structure is closer to the real community organization structure of the network. The overlapping relationship between communities has well explored the application of networks of recommendation system, linking prediction and so on [31]–[33]. The same is true of the box covering algorithm in complex networks. The overlapping box covering algorithm is closer to the real network and can better explore the characteristics of the networks. Therefore, the fractal algorithm of overlapping boxes in complex networks is worth studying. Although the OBCA algorithm proposed by Sun *et al.* has demonstrated in detail the rationality of using overlapping boxes to cover the network [34], the method adopted by them still has a certain randomness and requires multiple experiments. In order to deal with the problems of randomness and high time complexity of box division in the OBCA algorithm, an improved overlapping box covering algorithm (IOB) is proposed in this paper.

B. OBJECTIVE AND CONTRIBUTIONS

The primary contributions of the improved method are outlined as follows:

(1) The improved overlapping box covering algorithm(IOB) proposed in this paper is a high accuracy and low time complexity method for calculating the fractal dimension of complex networks. In addition, the experimental results show that the IOB algorithm is better than the other two algorithms.

(2) Different from the traditional box covering algorithms(such as CBB, MEMB, etc.), the improved algorithm covers the whole network with overlapping boxes instead of separate boxes, which is of great significance for exploring the relationship between nodes in different boxes, and also more in line with the real networks.

(3) Although both the improved algorithm and the OBCA algorithm are overlapping box covering algorithms, compared with the OBCA algorithm, our improved overlapping box covering algorithm reduces the randomness and uncertainty of box division in complex networks. Moreover, the improved box is different from the OBCA algorithm in determining the effective box. The OBCA algorithm is to delete redundant boxes by comparing nodes after the box division, while the IOB algorithm is to define effective boxes and reduce the time complexity by determining effective box center node.

The remainder of this paper is organized as follows. Section II firstly introduces the relative algorithms of box covering method. Section III verifies the feasibility of the improved box covering algorithm and compares with other algorithms in several complex networks. Furthermore, the experimental results and analysis are discussed from two aspects: the number of boxes covering the complex networks and the time complexity of the algorithms. Finally, we discuss the conclusion and future work in Section IV.

II. PROPOSED METHOD

A. THE IMPROVED OVERLAPPING BOX COVERING ALGORITHM

The measurement of fractal dimension and self-similarity in complex networks is always an important part in the study of complex systems. Given a complex network G, the diameter of box is l_B and the distance between nodes is less than the box diameter. The whole networks should be covered by the minimum number of boxes, denoted by N_B:

$$N_B \sim l_B^{-D_B} \tag{1}$$

where D_B is the fractal dimension of the networks, it can be obtained by regression $\log(N_B)$ and $\log(l_B)$.

In order to apply the overlapping box covering algorithm to the calculation of complex networks fractal dimension, we improve the algorithm basing on the ratio of excluded mass to closeness centrality [35]. Different central nodes affect the effective division of the box of the complex networks, and the node with the smallest ratio of the excluded mass to closeness centrality is selected as the central node of the network. If the node with the largest network centrality ratio is selected to divide the overlapping box covering algorithm, the number of boxes in the entire network will not be reduced, because the nodes at the edge of the network still need separate boxes. Our idea is to reduce the number of boxes that individually cover the edge nodes as much as possible, thereby reducing the number of boxes. The ratio of the excluded mass to closeness centrality f_i is defined as follow:

$$f_i = \frac{m_i}{Ci} \tag{2}$$

where m_i is the excluded mass of complex networks, the distance is less than the radius r_B of the undiscovered node.Here $l_B = 2r_B + 1$. C_i is the closeness centrality of networks.

Closeness centrality measures the closeness of a vertex to all other vertices in the graph. In order to calculate the average of the distances between one node to the other nodes in the network, the smaller the distance, the closer the nodes are to other nodes.

$$C_i = \frac{N-1}{\sum_{j \in G} d_{ij}} \tag{3}$$

where d_{ij} is the minimum distance from node i to node j and G is the set of all nodes in the networks. In complex networks, Dijkstra algorithm is used to calculate the distance between nodes instead of Euclidean distance method.

Selecting the center node of box with the minimum ratio of excluded mass to closeness centrality algorithm, so that as many nodes can be divided into the same box as possible to reduce the number of boxes. Meanwhile, the improved overlapping box covering algorithm is adopted, Figure 1. shows the comparison between OBCA algorithm [34] and the improved overlapping box covering algorithm. The OBCA method needs three boxes and the IOB method needs two boxes. Obviously, within the same network structure, the number of boxes required for the IOB algorithm is less than the number of boxes required for OBCA algorithm in Figure 1. The calculation process for the center node selection of the two algorithms is shown in Table 1., the data in the table is calculated by eq.(2) and eq.(3). The OBCA algorithm selects the center node of the box according to the degree of node from small to large. In Figure 1(a), the degree of node 4 and 5 are both one as the center node of the box, dividing the nodes into boxes A and B, and box C is obtained on node 1 of degree two. While in Figure 1(b) the IOB algorithm divides boxes according to the minimum value of ratio f_i. The minimum value of node 2 divides boxes A, and then divides nodes 3 as the center of the box to get box B.

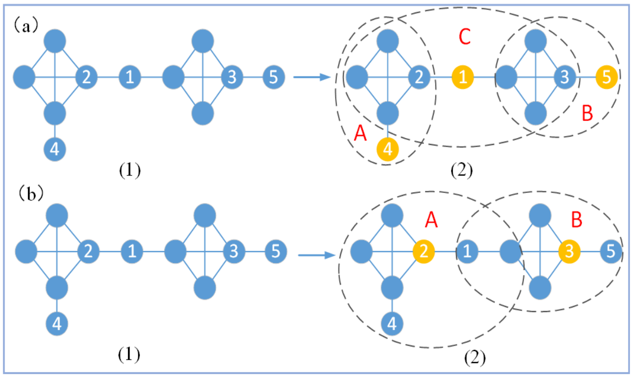


FIGURE 1. Comparison of box division methods between OBCA algorithm and IOB algorithm. (a) OBCA algorithm, the center node of the box is selected according to the order of node degree from small to large. The node degree of nodes 4 and 5 are both one, that is, the center node of the box, which according divided box A and B. Then the degree of node 1 is two to get box C. (b) IOB algorithm, the center node of the box is selected from small to large according to the ratio of the excluded mass to closeness centrality f_i. In addition, the central node of the box cannot be in the divided box. If it is in the divided box, the central node needs to be reselected. First, the node 2 with the smallest ratio f_i is selected as the central node of the box to get box A. Then, the node with the second smallest ratio f_i is selected. Node 3 is not in box A, so it can be used as the central node of the box to obtain box B. The number of boxes required by OBCA is three, but the number of boxes required by IOB is two. Within the same network structure and box radius, the number of boxes divided by IOB algorithm is less than that of OBCA. Here r_B is equal to 2, the yellow node is the center node of each box.

TABLE 1. The calculation process for the center node selection of the two algorithms.

Nodes	m_i	C_i	f_i	degree
1	9	0.50	18.00	2
2	7	0.48	14.58	4
3	6	0.39	15.38	4
4	5	0.29	17.24	1
5	5	0.29	17.24	1

B. ALGORITHM DESCRIPTION

The specific steps of the improved overlapping box covering algorithm are as shown in Algorithm 1.

In complex networks, the fractal interval is determined by the radius of the box, and the radius of the box required for each network is calculated basing on the diameter of the network. The radius of the box should be less than the radius of the network. Since the networks in this paper are unweighted complex networks, the fractal interval increases from one to the radius of the network. In the overlapping box covering algorithm, in order to ensure that the partitioned boxes are valid boxes, we adopt the following method: during box partitioning, the central node of the box cannot be in the divided box, ensuring that there is at least one uncovered node in each box, thus avoiding the completely overlapping boxes and reducing the comparison time of deleting redundant boxes.

In order to reduce the time complexity of calculation, the above algorithm is improved as follows. In the MEMB and a method based on the ratio of excluded mass to closeness centrality (REMCC) algorithms, the excluded mass needs to be recalculated each time, which takes a lot of time. The ratio of the excluded mass to closeness centrality f_i does not need to be updated every time in the improved method, but only the nodes that have been marked as the center nodes, which can

Algorithm 1 The IOB Algorithm

Input: Complex network data set;
Output: Fractal dimension D_B, The box numbers N_B;
Complex networks are represented by *Matrix*
function IOB($Matrix$)
  $T = sparse(Matrix)$
  $H = distance(T)$
  for $r_B = 1 \to k$ **do**
    $temp \leftarrow OVERLAP(T, r_B, H)$
    $N_B \leftarrow [N_B, temp]$
  end for
  $D_B \leftarrow N_B \sim l_B^{-D_B}$
  return N_B, D_B
end function

function OVERLAP(T, r_B, H)
  $m \leftarrow zeros(1, row)$; $C \leftarrow zeros(1, row)$
  $j \leftarrow 1 : T.row$
  for $i = 1 \to T.row$ **do**
    $num \leftarrow 0$
    if $H(i, j) <= r_B$ **then**
      $num + 1$
    end if
    $m(i) \leftarrow num$
    $C(i) = (row - 1)/sum(H(i, :))$
    $f(i) = m(i)/C(i)$
  end for
  $center = min(f)$
  $N_B \leftarrow uncover(i) = 0 \&\& H(center, i) <= r_B$
  Marks the covered node
  return N_B
end function

effectively reduce the calculation time and time complexity of the algorithm. Since the improved algorithm is an overlapping box covering algorithm, the boxes can be overlapped, but the completely overlapped boxes are invalid boxes. Therefore, the existence of invalid boxes is avoided in the steps of our improved method.

III. SIMULATION RESULTS AND DISCUSSIONS

A. COMPLEX NETWORKS DATA

In order to verify the effectiveness of the algorithm, in this section, we display the numbers of boxes covered and the execution time in six complex networks. The six networks are American college football network (Football, http://www-personal.umich.edu/mejn/netdata/), E.coli network (E.coli, http://www.ecmdb.ca), American western power network (Power, http://konect.uni-koblenz.de/networks/opsahl-powergrid), adolescent health network (Health, http://konect.uni-koblenz.de/networks/moreno_health), Yeast network (Yeast, http://www.ymdb.ca) and Facebook network (Facebook, http://snap.stanford.edu/data/ego-Facebook.html).

TABLE 2. Comparison of fractal dimension of different algorithms.

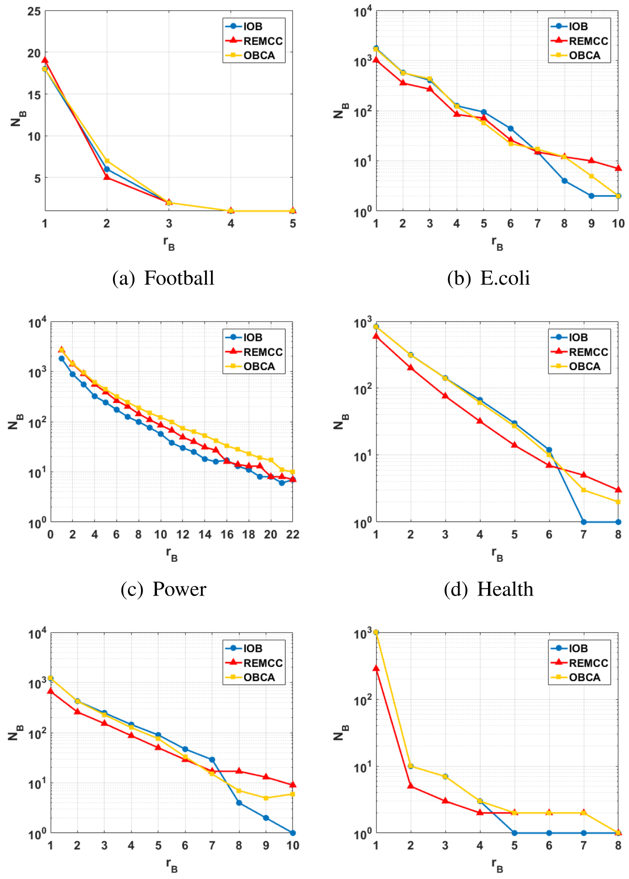
Networks	Nodes	Edges	REMCC	OBCA	IOB
Football	115	613	2.4949	2.1134	2.0912
E.coli	2859	6890	2.7719	3.1626	2.1899
Power	4941	6594	2.3364	2.4933	2.1899
Health	2539	12971	3.5993	3.1301	2.9623
Yeast	2375	11693	2.9615	3.1282	2.9561
Facebook	4039	88234	2.7443	3.1015	2.9420

B. METHOD VERIFICATION IN COMPLEX NETWORKS

First and foremost, the improved algorithm is applied to six complex networks. At the same time, the REMCC and OBCA algorithms are compared in the same network structure. The results of the fractal dimension calculated by eq.(1) are shown in Table 2. The REMCC method has the determined position of the center of each box, which has been proved better than the MEMB algorithm [35]. Nevertheless, networks are divided into separate boxes and the optimal number of boxes cannot be obtained, and the time complexity of the algorithm is relatively high. The OBCA adopts overlapping boxes, the construction process of the box is similar to the CBB algorithm and has some randomness. However, OBCA needs to compare nodes multiple times, which takes more time than the CBB method. The experimental results that IOB algorithm outperforms the OBCA algorithm and REMCC algorithm, in most cases, renders the more accurate fractal dimension. Since the IOB algorithm combines the determined center of the box with the overlapping box covering algorithm, the fractal dimension is more accurate, which is more in line with the reality.

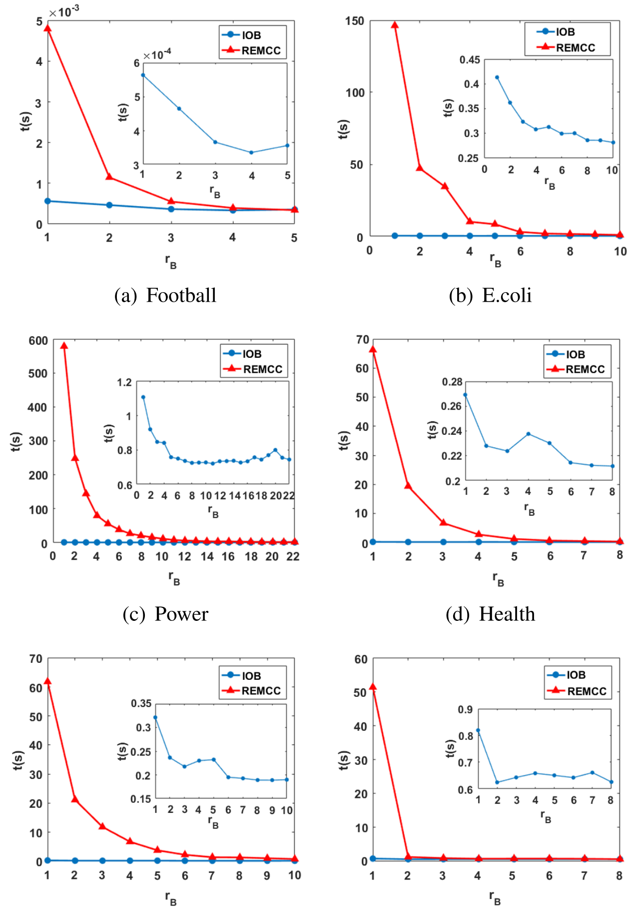
Secondly, it mainly elaborates from two aspects, one is the number of boxes, and the other is the execution time of algorithms. The main challenge of the box covering method is to cover the entire networks with fewer boxes. The box numbers of the IOB algorithm, REMCC algorithm and the OBCA algorithm at different scales in different complex networks are shown in Figure 2. Due to the randomness of the OBCA algorithm, the OBCA algorithm was tested for 20 times, and the experimental results were processed using the Monte Carlo method to obtain the number of boxes at different scales in Figure 2. The fractal scale (r_B) of the three algorithms is the same, but the number of boxes (N_B) is reduced, which leads to the reduction of fractal dimension. In other words, by reducing the number of boxes at the same scale, a more accurate number of boxes can be obtained.

In Figure 2, it is obvious that the IOB algorithm in Football, Power and Facebook networks requires fewer boxes than OBCA does. In particular, in the Power network, the number of boxes in each fractal scale for the IOB algorithm is less than or equal to the number of boxes required by the REMCC and OBCA algorithms. In the Football network, according to the experiment the number of boxes required by the IOB algorithm is less than that of OBCA when the box radius is two, and the number of boxes required by the two algorithms is the



(a) Football (b) E.coli (c) Power (d) Health (e) Yeast (f) Facebook

FIGURE 2. The relationship between the number of boxes N_B and box radius r_B.



(a) Football (b) E.coli (c) Power (d) Health (e) Yeast (f) Facebook

FIGURE 3. The IOB algorithm and REMCC algorithm execution time in different networks.

same at other scales. Although the E.coli, Health and Yeast networks do not accurately see the reduction in the number of boxes at each box scale, the IOB algorithm can cover the entire network faster, that is, the diameter of the box required by IOB is smaller than that of OBCA. This is because the method of determining the effective box in the overlapping box is different. Combined with the fractal dimension data in Table 2, the fractal dimension of IOB algorithm is smaller than that of OBCA, and has a more accurate one. The two algorithms are similar in that they are both box overlapping algorithms, and the main difference is the method of selecting the center node of each box. It can be seen that the improved algorithm is beneficial to reduce the number of boxes within a certain fractal scale and make the result more certain. The method of deleting the redundant box in the OBCA algorithm is not unique, and the box divided by the ratio method in the IOB algorithm is a valid box, thereby reducing unnecessary box division. On average, the IOB method has fewer boxes than the OBCA method in different network structures, due to the different methods of selecting the center node of the box. Accordingly, the IOB could yield fewer boxes on some scale and makes the results more deterministic. Although REMCC does not have fewer boxes than IOB at every box scale, REMCC requires significantly more boxes than IOB at a larger box scale. Therefore, when the box size is larger, the IOB algorithm can cover the entire network with fewer boxes, so the improved overlapping box covering algorithm proposed in this paper is effective.

Finally, time complexity refers to the computational effort required to execute the algorithm, which is qualitatively described as the running time of the algorithm. Time complexity measures the quality of algorithm. Figure 3. shows the calculation time of REMCC and the improved algorithm, we compare versus results of REMCC and IOB methods. It is obvious that the time consumption of the REMCC algorithm is larger than the IOB algorithm at the same box radius in most networks. Because the REMCC algorithm recalculates the ratio of excluded mass to closeness centrality after each partition of box, while IOB algorithm allows the overlap of boxes to be calculated only once, which greatly reduces the time complexity of the algorithm. In addition, Figure 3 depicts the time of our method costs significantly shorter than the REMCC method. Furthermore, the average time cost efficiency of the IOB algorithm increased by 42.12%, 89.27%, 84.06%, 80.39%, 90.82%, 31.32% respectively in six complex networks compared with REMCC. Moreover, the time consuming curves of the IOB method are different because

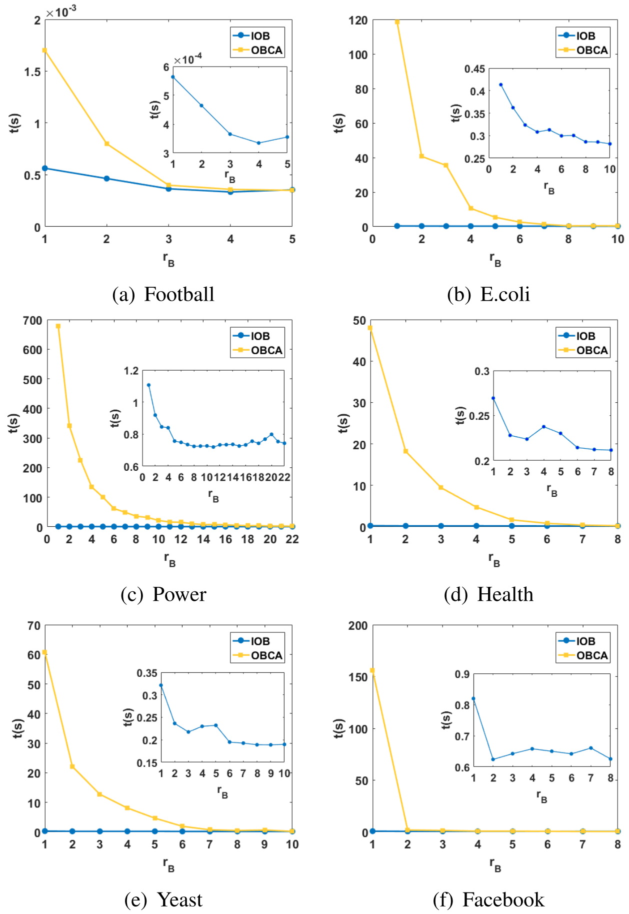


FIGURE 4. The IOB algorithm and OBCA algorithm execution time in different networks.

the different networks have different intrinsic feature. The time complexity of the OBCA algorithm to select the center node of the box is low, but it takes a lot of time to compare one node many times, and the redundant box needs to be deleted in the last step, leaving only the valid overlapping boxes. At the same time, IOB algorithm does not use the computation time required by multiple comparisons of nodes, so IOB algorithm requires less computation time. The randomness of OBCA makes it difficult to accurately measure its computation time, so multiple experiments have been performed to obtain its average calculation time. Figure 4 shows the calculation time of the IOB algorithm and the OBCA algorithm in different networks. Compared with the OBCA algorithm, the calculation time of the IOB algorithm is improved by 24.57%, 78.17%, 90.62%, 77.11%, 78.89%, 26.07% respectively. In summary, the IOB algorithm has less computational cost and the algorithm is more efficient than the other two algorithms.

IV. CONCLUSION

In this paper, an improved overlapping box covering algorithm (IOB) is proposed, which is different from the traditional box covering method. Instead of using individual boxes to cover the entire network, IOB algorithm uses overlapping boxes to cover it. Although Sun *et al.* also adopted the overlapping box, dividing the box from the node with small degree to the node with large degree, the selection of the center node of the box was quite random. The improved algorithm reduces the randomness of the center node of the selected box, and can determine the selection of the network center node, which increases the accuracy of the fractal dimension calculation. Experiments show that overlapping boxes are indeed a better way to reduce the number of boxes needed to cover the network. In addition, considering the quality of the algorithm from the aspect of time complexity, the improved algorithm calculation time is obviously less than the time required by the REMCC algorithm, and the comparison time required to repeatedly compare the overlapping boxes in the OBCA algorithm is also avoided.

In future work, the application of overlapping box covering algorithms can be considered in weighted networks to explore whether overlapping box covering algorithm is applicable to the weighted networks. In addition, the multifractal properties of complex networks are also worth of studying. Many networks cannot be represented by a single property, which requires multifractals. The fusion of multifractal analysis and overlapping box covering algorithm is also the future research direction.

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