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New Conditions of Analysis and Synthesis for Periodic Piecewise Linear Systems With Matrix Polynomial Approach

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ABSTRACT In this paper, new conditions of the stability, stabilization and L_2-gain performance of periodic piecewise systems are proposed. Both the continuous and discontinuous Lyapunov functions with dwell-time related time-varying Lyapunov matrix polynomial are adopted, and methods guaranteeing the positive and negative definiteness of a matrix polynomial are introduced. Exponential stability conditions are derived based on continuous and discontinuous Lyapunov functions, respectively. A stabilizing controller with time-varying controller gain is designed with continuous Lyapunov function and the weighted L_2-gain performance based on the discontinuous Lyapunov matrix polynomial is studied as well. Numerical examples are used to verify the effectiveness of the proposed methods.

INDEX TERMS Periodic piecewise systems, time-varying Lyapunov matrix, matrix polynomial, analysis synthesis.

I. INTRODUCTION

The periodic system gives a framework for modeling systems with periodic features in engineering, ecological, and economic fields, such as rotor-blade systems [1], satellite attitude control systems [2], communication systems [3]. It has been widely investigated over the past decades [4]. Periodic piecewise system is a special kind of periodic system because it consists of several subsystems in one period and the switching signal between subsystems is fixed and periodic. Periodic piecewise systems can be found in mechanical and electric engineering [5]. Especially in power systems, kinds of power converters are typical periodic piecewise systems [6]. Apart from its broad applications in engineering, periodic piecewise system is used as the approximate system of continuous-time periodic system [7]. Since control problems of continuous-time periodic systems are more challenging than the discrete-time periodic systems [8], analysis and synthesis results of periodic piecewise systems can help study continuous-time periodic systems.

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Because of its value in application and continuous-time periodic systems study, the periodic piecewise system attracts much attention in recent years [9]–[15]. The stability, L_2-gain and generalized H_2 performance indices are investigated in [9], [10], respectively. The finite-time stability condition of the periodic piecewise system is proposed in [11]. An H_∞ controller is designed to control the system output under disturbance in that work. For periodic piecewise systems with time-delay, an H_∞ control problem is discussed in [12]. The disturbance attenuation performance of mechanical systems with periodic piecewise coefficients is investigated in [13], where a saturated controller is designed to attenuate the system vibration. Sufficient condition is established in [16] to study the global exponential stability of neural networks with periodic coefficients and piecewise constant arguments.

For a periodic piecewise system, its subsystems sequence and the dwell time of each subsystem are fixed. In other words, it has a known and inalterable switching law. Techniques used in switched systems [17]–[21] can be applied in the analysis and synthesis of periodic piecewise systems. In order to obtain possible less conservative results, efforts are put on improving the Lyapunov function and Lyapunov matrix formats. A continuous Lyapunov function with time-varying Lyapunov matrix is adopted in [9]–[11], [13]. On the contrary, multiple Lyapunov functions with time-varying Lyapunov matrices are applied in [12], [14]. No matter the continuous or the multiple Lyapunov functions, the time-varying Lyapunov matrices established in the results mentioned above are formulated in the linear interpolation form of time. To further improve the Lyapunov matrix, the matrix polynomial formulation is proposed in [15], where more free variables are introduced in the constructed Lyapunov function. In that work, the constraints on exponential order of each subsystems are relaxed as well.

The square matricial representation (SMR) and the sum of square (SOS) techniques [24], [25] are well-known techniques in dealing with the matrix polynomial. Nonconservative results are obtained for switched systems with guaranteed dwell time and arbitrary switching [26], [27]. These methods are also adopted for periodic piecewise systems in [15] to handle the time-varying Lyapunov matrix polynomial. Motivated by the above works, new conditions of the stability, stabilizing and L_2-gain performance are proposed in this paper based on the matrix polynomial approach. Different from [15], a dwell-time related time-varying Lyapunov matrix polynomial is adopted in this work, and a new method is introduced to handle the matrix polynomial issue. Compared with previous works, contributions of this work can be concluded as: 1. Both continuous Lyapunov function and discontinuous Lyapunov function with dwell-time related Lyapunov matrix polynomial are used. The discontinuous Lyapunov function could introduce more decision variables in the stability and performance analysis. The continuous Lyapunov function could result in a directly solvable periodic time-varying controller. 2. The weighted L_2-gain performance analysis is carried out based on the discontinuous Lyapunov function with time-varying Lyapunov matrix polynomial, which has not been reported in previous works. The paper is organized as follow. The problem is formulated and preliminaries are given in Section 2. In Section 3, stability, stabilization and L_2-gain performance analysis are provided. Numerical examples used to demonstrate the advantage of the proposed methods is given in Section 4. A brief conclusion is given in Section 5.

Notation: $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $\|\cdot\|$ stands for the Euclidean vector norm, the superscript $'$ refers to matrix transposition, $\mathbb{N}^+$ denotes the set of positive integers. $\mathcal{D}^+(\cdot)$ denotes the upper right Dini derivative and $P > 0$ means that P is a real symmetric and positive definite matrix. To facilitate the description, a non-negative integer ℓ is adopted to stand for the nominal number of fundamental periods, that is, $\ell = 0, 1, \ldots$.

II. PROBLEM FORMULATION AND PRELIMINARIES

Consider a continuous-time T-periodic piecewise linear system given by

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) + B_w(t)w(t), \\ z(t) &= C(t)x(t) + D(t)w(t) \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^r$, $u(t) \in \mathbb{R}^d$, $w(t) \in \mathbb{R}^s$ are the state vector, control input and disturbance vector, respectively. For $t \geq 0$, one has $A(t) = A(t+T)$, $B(t) = B(t+T)$, $B_w(t) = B_w(t+T)$, $C(t) = C(t+T)$, $D(t) = D(t+T)$. Suppose the interval $[0, T)$ is partitioned into S subintervals, each interval can be given as $[t_{i-1}, t_i)$, $i = 1, 2, \ldots, S$, where $t_0 = 0$, $t_S = T$ The dwell time for the ith subsystem is $T_i = t_i - t_{i-1}$ with $\sum_{i=1}^{S} T_i = T$. Under the ith subinterval (subsystems), $(A(t), B(t), B_w(t), C(t), D(t))$ is time-invariant and given by $(A_i, B_i, B_{wi}, C_i, D_i)$. Then, system (1) is equivalently represented by, for $t \in [\ell T_p + t_{i-1}, \ell T_p + t_i)$, $i = 1, 2, \ldots, S$,

$$\begin{aligned} \dot{x}(t) &= A_i x(t) + B_i u(t) + B_{wi} w(t), \\ z(t) &= C_i x(t) + D_i w(t). \end{aligned}$$

In this work, the exponential stability, stabilization and L_2-gain performance are studied for system (1) with Lyapunov matrix polynomial. Definitions of exponential stability of system (1) and matrix polynomial are given below for later development.

Definition 1 (Exponential Stability [9]): Periodic piecewise system (1) with $u(t) = 0, w(t) = 0$ is said to be λ^*-exponentially stable if the solution of the system from $x(0)$ satisfies $\|x(t)\| \leq \kappa e^{-\lambda^* t}\|x(0)\|$, $\forall t \geq 0$, for some constants $\kappa \geq 1$, $\lambda^* > 0$.

Definition 2 (Matrix Polynomial [24]): The function $P : \mathbb{R}^q \to \mathbb{R}^{l \times l}$ is a matrix polynomial if $P_{i,j}$, $i, j = 1, 2, \ldots, l$, is polynomial.

For system (1), consider a Lyapunov function given as $V(x, t) = x'P(t)x$ where $P(t) > 0$ is periodic with period T. For $t \in [\ell T + t_{i-1}, \ \ell T + t_i)$, $i = 1, 2, \ldots, S$, $V(x, t)$ can be rewritten as

$$V(x, t) = V_i(x, t) = x'P_i(t)x \tag{2}$$

where $P(t) = P_i(t)$.

Based on the Lyapunov function (2), both the continuous and discontinuous Lyapunov functions are formulated in this work. For the continuous Lyapunov function case, $P_i(t)$ is supposed to be continuous both during the ith subsystem and at the switching instants. It implies that at each switching instant, one has $\lim_{t \to \ell T + t_i} P(t) = P(\ell T + t_i)$. For the discontinuous Lyapunov function case, the switching of the subsystem from $i - 1$ to i causes a bounded mode-dependent increment. In other words, $P_i(t)$ is continuous in the ith subsystem, but discontinuous at each switching instant.

The following Lemma 1 is the general exponential stability result of periodic piecewise systems based on a continuous Lyapunov function.

Lemma 1: Consider periodic piecewise system (1) with $u(t) = 0, w(t) = 0$. Given $\lambda^* > 0$, if there exist λ_i, $i = 1, 2, \ldots, S$, two class $\mathcal{K}_\infty$ functions κ_1, κ_2 and continuous functions $V(x, t)$ defined on $t \in [0, \infty)$ such that

$$\begin{aligned} &\kappa_1(\|x\|) \leq V(x, t) \leq \kappa_2(\|x\|), \\ &\mathcal{D}^+ V(x, t) + \lambda_i V(x, t) < 0, \\ &2\lambda^* T - \sum_{i=1}^{S} \lambda_i T_i \leq 0, \end{aligned}$$

then system (1) is λ^*-exponentially stable.

Its proof could be easily concluded from [9] and [15].

Lemma 2 gives the general exponential stability condition with discontinuous Lyapunov function.

Lemma 2: Consider periodic piecewise system (1) with $u(t) = 0, w(t) = 0$*. Given* $\lambda^* > 0$*, if there exist* λ_i*,* $\mu_i > 1, i = 1, 2, \ldots, S$*, two class* $\mathcal{K}_\infty$ *functions* κ_1, κ_2 *and continuous functions* $V_i(x, t)$ *defined on* $t \in [\ell T_p + t_{i-1}, \ell T_p + t_i)$*,* $i = 1, 2, \ldots, S$ *satisfying*

$$\begin{aligned}
&\kappa_1(\|x\|) \leq V_i(x, t) \leq \kappa_2(\|x\|),\\
&\mathcal{D}^+ V_i(x, t) + \lambda_i V_i(x, t) < 0,\\
&V_i(x, \ell T + t_{i-1}) \leq \mu_i V_{i-1}(x, \ell T + t_{i-1}),\\
&V_1(x, \ell(T + 1)) \leq \mu_1 V_S(x, \ell(T + 1)),\\
&\sum_{i=1}^{S} \ln \mu_i + 2\lambda^* T \leq \sum_{i=1}^{S} \lambda_i T_i,
\end{aligned}$$

then system (1) is λ^**-exponentially stable.*

Its proof could be easily concluded from [14] and [15]. Based on the discontinuous Lyapunov function, the definition of weighted L_2-gain performance can be given as follow.

Definition 3 (Weighted L_2*-gain Performance): Given scalars* $\hat{\lambda} > 0, \sigma > 0$ *and* $\hat{\gamma} > 0$*, periodic piecewise system (1) with* $u(t) = 0$ *is said to be exponentially stable with a weighted* L_2*-gain* $\hat{\gamma}$*, if it is exponentially stable with* $w(t) = 0$*, and under initial condition* $x(0) = x_0$*, it holds for any nonzero disturbance* $w(t) \in L_2[0, +\infty)$ *that*

$$\int_0^\infty e^{-\hat{\lambda}\tau} z'(\tau) z(\tau) d\tau < \sigma V(x_0, 0) + \hat{\gamma}^2 \int_0^\infty w'(\tau) w(\tau) d\tau.$$

[14].

III. MAIN RESULTS

In this section, the periodic Lyapunov function with continuous and discontinuous time-varying Lyapunov matrix polynomials are constructed, respectively. Conditions concerning the negative and positive definiteness of a matrix polynomial are introduced. Based on these conditions, new results on the exponential stability of periodic piecewise system are proposed and a stabilizing controller is designed. In addition, weighted L_2-gain performance index of periodic piecewise system based on discontinuous Lyapunov matrix polynomial is also provided.

Construct a periodic Lyapunov matrix polynomial $P(t) = P(t + \ell T)$ such that, for $t \in [\ell T + t_{i-1}, \ell T + t_i)$, $i = 1, 2, \ldots, S$,

$$\begin{aligned}
P(t) = P_i(t) &= \sum_{j=0}^{n} \frac{(t - \ell T - t_{i-1})^j}{T_i^j} P_{i,j}\\
&= P_{i,0} + \frac{(t - \ell T - t_{i-1})}{T_i} P_{i,1} + \cdots\\
&\quad + \frac{(t - \ell T - t_{i-1})^n}{T_i^n} P_{i,n}
\end{aligned} \tag{3}$$

where $P_{i,j}, i = 1, 2, \ldots, S, j = 1, 2, \ldots, n$, are constant matrices and the degree of matrix polynomial $P(t)$ is given as n. One may observe that the Lyapunov matrix polynomial adopted in (3) is dwell-time related.

In the discontinuous Lyapunov function case, the Lyapunov matrix polynomial can be given as (3). That is, for the ith subsystems, one has an independent $P_i(t)$, which is continuous in the interval $[\ell T + t_{i-1}, \ell T + t_i)$. As to the continuous Lyapunov function case, the continuity is required at each switching instant as well. One has $P_{2,0} = \sum_{j=0}^{n} P_{1,j}$, $P_{3,0} = P_{1,0} + \sum_{i=1}^{2} \sum_{j=1}^{n} P_{i,j}, \cdots, P_{1,0} = P_{1,0} + \sum_{i=1}^{S} \sum_{j=1}^{n} P_{i,j}$. It indicates that $\sum_{i=1}^{S} \sum_{j=1}^{n} P_{i,j} = 0$. Then, for the continuous Lyapunov function case, $P(t)$ can be rewritten as

$$P(t) = \begin{cases}
\sum_{j=0}^{n} \frac{(t - \ell T - t_{i-1})^j}{T_{i_n}^j} P_{i,j}, \quad i = 1\\
P_{1,0} + \sum_{f=1}^{i-1} \sum_{j=1} P_{f,j}\\
+ \sum_{j=1}^{n} \frac{(t - \ell T - t_{i-1})^j}{T_i^j} P_{i,j}, \quad i = 2, \ldots, S - 1,\\
P_{1,0} + \sum_{f=1}^{i-1} \sum_{j=1}^{n} (1 - \frac{(t - \ell T - t_{S-1})^n}{T_S^n}) P_{f,j}\\
+ \sum_{j=1}^{n-1} \frac{(t - \ell T - t_{i-1})^j}{T_i^j} (1 - \frac{(t - \ell T - t_{S-1})^{n-j}}{T_S^{n-j}})\\
\times P_{S,j}, \quad i = S
\end{cases} \tag{4}$$

Remark 1: It should be noticed that different degrees of matrix polynomial could be allocated for different subsystems. In that case, n will be rewritten as n_i*, which can be chosen as any positive integers. In this work, a uniform degree denoted as n is allocated for all subsystems to facilitate the derivation.*

Before providing the theorems, the following Lemma concerning the negative definiteness of a matrix polynomial is introduced first.

Lemma 3 [28]: Consider a bounded matrix polynomial $f(\tau_1, \tau_2, \ldots, \tau_n)$ *given as*

$$f(\tau_1, \tau_2, \ldots, \tau_n) = \Upsilon_0 + \tau_1 \Upsilon_1 + \tau_1 \tau_2 \Upsilon_2 + \cdots + \left(\prod_{k=1}^{n} \tau_k\right) \Upsilon_n, \tag{5}$$

where $n \in \mathbb{N}^+, \Upsilon_j, j = 0, 1, \ldots, n$ *are real symmetric matrices,* $\tau_k, k = 1, 2, \ldots n$ *are variables and* $\tau_k \in [0, 1]$*. If*

$$\sum_{k=0}^{d} \Upsilon_k < 0, \quad d = 0, 1, \ldots, n,$$

then the matrix polynomial $f(\tau_1, \tau_2, \ldots, \tau_n) < 0$*.*

Similarly, one can obtain Lemma 4 concerning the positive definiteness of a matrix polynomial. The proof can be easily extended from the proof of Lemma 3.

Lemma 4: Consider a bounded matrix polynomial $f(\tau_1, \tau_2, \ldots, \tau_n)$ *given as in (5), where* $n \in \mathbb{N}^+$*,* $\Upsilon_j, j = 0, 1, \ldots, n$ *are real symmetric matrix,* $\tau_k, k = 1, 2, \ldots n$ *are variables and* $\tau_k \in [0, 1]$*. If*

$$\sum_{k=0}^{d} \Upsilon_k > 0, \quad d = 0, 1, \ldots, n, \tag{6}$$

then the matrix polynomial $f(\tau_1, \tau_2, \ldots, \tau_n) > 0$*.*

Then, by exploiting a Lyapunov function (2) with continuous Lyapunov matrix polynomial (4), a stability condition for periodic piecewise system (1) can be obtained in Theorem 1.

Theorem 1: Consider periodic piecewise system (1) with $u(t) = 0, w(t) = 0$. *Given* $\lambda^* > 0$, *if there exist* λ_i *and matrices* $P_{i,j}$, $i = 1, \ldots, S, j = 0, \ldots, n$ *satisfying*

$$P_{1,0} > 0, \tag{7}$$

$$P_{1,0}+\sum_{f=1}^{g}\sum_{j=1}^{k}P_{f,j}>0, \quad g=1,2,\ldots,S-1, k=1,\ldots,n, \tag{8}$$

$$P_{1,0} + \sum_{f=1}^{S-1}\sum_{j=1}^{n}P_{f,j} + \sum_{j=1}^{k}P_{S,j} > 0, \quad k = 1,\ldots,n-1 \tag{9}$$

$$\sum_{j=0}^{k}\Xi_{i,j} < 0, \quad i = 1,2,\ldots,S, k = 0,\ldots,n, \tag{10}$$

$$2\lambda^* T - \sum_{i=1}^{S}\lambda_i T_i \le 0, \tag{11}$$

where

$$\begin{aligned}
\Xi_{1,j} &= A_1'P_{1,j} + P_{1,j}A_1 + \lambda_1 P_{1,j} + \frac{j+1}{T_1}P_{1,j+1},\\
&\quad j = 0,1,\ldots,n-1,\\
\Xi_{1,n} &= A_1'P_{1,n} + P_{1,n}A_1 + \lambda_1 P_{1,n},\\
\Xi_{i,0} &= A_i'(P_{1,0} + \sum_{f=1}^{i-1}\sum_{g=1}^{n}P_{f,g}) + (P_{1,0}\\
&\quad + \sum_{f=1}^{i-1}\sum_{g=1}^{n}P_{f,g})A_i + \lambda_i(P_{1,0} + \sum_{f=1}^{i-1}\sum_{g=1}^{n}P_{f,g})\\
&\quad + \frac{1}{T_i}P_{i,1}, \quad i = 2,\ldots,S,\\
\Xi_{i,j} &= A_i'P_{i,j} + P_{i,j}A_i + \lambda_i P_{i,j} + \frac{j+1}{T_i}P_{i,j+1},\\
&\quad i = 2,\ldots,S-1, j = 1,2,\ldots,n-1,\\
\Xi_{i,n} &= A_i'P_{i,n} + P_{i,n}A_i, +\lambda_i P_{i,n}, \quad i = 2,\ldots,S-1,\\
\Xi_{S,j} &= A_S'P_{S,j} + P_{S,j}A_S + \lambda_S P_{S,j} + \frac{j+1}{T_S}P_{S,j+1},\\
&\quad j = 1,2,\ldots,n-2,\\
\Xi_{S,n-1} &= A_S'P_{S,n-1} + P_{S,n-1}A_S + \lambda_S P_{S,n-1}\\
&\quad - \frac{n}{T_S}(\sum_{f=1}^{S-1}\sum_{g=1}^{n}P_{f,g} + \sum_{g=1}^{n-1}P_{S,g}),\\
\Xi_{S,n} &= -A_S'(\sum_{f=1}^{S-1}\sum_{g=1}^{n}P_{f,g} + \sum_{g=1}^{n-1}P_{S,g}) - (\sum_{f=1}^{S-1}\sum_{g=1}^{n}P_{f,g}\\
&\quad + \sum_{g=1}^{n-1}P_{S,g})A_S - \lambda_S(\sum_{f=1}^{S-1}\sum_{g=1}^{n}P_{f,g} + \sum_{g=1}^{n-1}P_{S,g}),
\end{aligned} \tag{12}$$

then system (1) is λ^**-exponentially stable.*

Proof: Construct a Lyapunov function as in (2) with a continuous time-varying Lyapunov matrix polynomial given as in (4) for $t \in [\ell T + t_{i-1}, \ell T + t_i)$, $i = 1,2,\ldots,S$.

From (7)–(9), one can obtain that in the ith subsystem, one has $\sum_{j=0}^{k}P_{i,j} > 0, k = 0,1,\ldots,n$. Since $\frac{t-\ell T - t_{i-1}}{T_i} \in [0,1]$, according to Lemma 4, one has $P_i(t) > 0$. Because of the continuity of $P(t)$, one obtains $P(t) > 0$. It indicates that one could find constants $c_1 > 0, c_2 > 0$ such that $c_1\|x\|^2 \le V(x,t) \le c_2\|x\|^2$.

Moreover, with (10), one can obtain

$$\begin{aligned}
&\mathcal{D}^+V_i(x,t) + \lambda_i V_i(x,t)\\
&= \dot{x}'P_i(t)x + x'P_i(t)\dot{x} + x'\mathcal{D}^+P_i(t)x + \lambda_i x'P_i(t)x\\
&= x'\Bigg(A_i'\sum_{j=0}^{n}(\frac{(t-\ell T - t_{i-1})^j}{T_i^j}P_{i,j})\\
&\quad +(\sum_{j=0}^{n}\frac{(t-\ell T - t_{i-1})^j}{T_i^j}P_{i,j})A_i\\
&\quad + \sum_{j=1}^{n}(\frac{j(t-\ell T - t_{i-1})^{j-1}}{T_i^j}P_{i,j})\\
&\quad + \lambda_i\sum_{j=0}^{n}\frac{(t-\ell T - t_{i-1})^j}{T_i^j}P_{i,j}\Bigg)x\\
&= x'\Big(A_i'P_{i,0} + P_{i,0}A_i + \frac{1}{T_i}P_{i,1} + \lambda_i P_i\\
&\quad + \frac{t-\ell T - t_{i-1}}{T_i}(A_i'P_{i,1} + P_{i,1}A_i + \frac{2}{T_i}P_{i,2} + \lambda_i P_{i,1})\\
&\quad + \cdots + \frac{(t-\ell T - t_{i-1})^{(n-1)}}{T_i^{(n-1)}}(A_i'P_{i,n-1} + P_{i,n-1}A_i\\
&\quad + \frac{n}{T_i}P_{i,n} + \lambda_i P_{i,n-1}) + \frac{(t-\ell T - t_{i-1})^n}{T_i^n}(A_i'P_{i,n}\\
&\quad + P_{i,n}A_i + \lambda_i P_{i,n})\Big)x.
\end{aligned} \tag{13}$$

Since $P_{i,0} = P_{1,0} + \sum_{f=1}^{i-1}\sum_{j=1}^{n}P_{f,j}, i = 2,\ldots,S$ and $P_{S,n} = -\sum_{f=1}^{S-1}\sum_{j=1}^{n}P_{f,j} - \sum_{j=1}^{n-1}P_{S,j}$, (13) can be rewritten as

$$\begin{aligned}
&\mathcal{D}^+V_i(t)(t) + \lambda_i V_i(t)\\
&= x'(\Xi_{i,0} + \frac{(t-\ell T - t_{i-1})}{T_i}\Xi_{i,1} + \cdots\\
&\quad + \frac{(t-\ell T - t_{i-1})^j}{T_i^j}\Xi_{i,j} + \cdots\\
&\quad + \frac{(t-\ell T - t_{i-1})^n}{T_i^n}\Xi_{i,n})x
\end{aligned} \tag{14}$$

where $\Xi_{i,j}$ is given as in (12).

Since $\frac{t-\ell T - t_{i-1}}{T_i} \in [0,1]$, then according to Lemma 3, one has $\mathcal{D}^+V_i(x,t) + \lambda_i V_i(x,t) < 0$. Then combining with (11) and according to Lemma 1, one can conclude that system (1) is λ^*-exponentially stable. □

Apart from the continuous Lyapunov function, discontinuous Lyapunov function with piecewise Lyapunov matrix polynomial is employed as well to develop the following Theorem 2.

Theorem 2: Consider periodic piecewise system (1) with $u(t) = 0, w(t) = 0$. *Given* $\lambda^* > 0$, *if there exist* $\lambda_i, \mu_i > 1$ *and matrices* $P_{i,j},\ i = 1,\dots,S, j = 0, 1,\dots,n$ *satisfying*

$$\sum_{j=0}^{k} P_{i,j} > 0, \quad i = 1, 2, \dots, S, k = 0, 1, \dots, n, \tag{15}$$

$$\sum_{j=0}^{k} \Upsilon_{i,j} < 0, \quad i = 1, 2, \dots, S, k = 0, 1, \dots, n, \tag{16}$$

$$P_{i+1,0} \le \mu_{i+1} \sum_{j=0}^{n} P_{i,j}, \quad i = 1, 2, \dots, S-1, \tag{17}$$

$$P_{1,0} \le \mu_1 \sum_{j=0}^{n} P_{S,j}, \tag{18}$$

$$\sum_{i=1}^{S} \ln \mu_i + 2\lambda^* T \le \sum_{i=1}^{S} \lambda_i T_i \tag{19}$$

where

$$\begin{aligned}\Upsilon_{i,j} &= A_i' P_{i,j} + P_{i,j} A_i + \lambda_i P_{i,j} + \frac{j+1}{T_i} P_{i,j+1},\\ &\quad j = 0, \dots, n-1,\\ \Upsilon_{i,n} &= A_i' P_{i,n} + P_{i,n} A_i + \lambda_i P_{i,n},\end{aligned} \tag{20}$$

then system (1) is λ^**-exponentially stable.*

Proof: For $t \in [\ell T + t_{i-1}, \ell T + t_i), i = 1, 2, \dots, S$, construct a discontinuous Lyapunov function as in (2) with Lyapunov matrix given as in (3). With (15) and according to Lemma 4, one has $P_i(t) > 0$. Then one could find constants $c_1 > 0, c_2 > 0$, such that $c_1\|x\|^2 \le V_i(x, t) \le c_2\|x\|^2$.

Following similar arguments in the proof of Theorem 1, one would also have (13). With choosing $\tau_i = \frac{(t-\ell T - t_{i-1})}{T_i}$, one obtains

$$\begin{aligned}&\mathcal{D}^+ V_i(x, t) + \lambda_i V_i(x, t)\\ &\qquad = x'(\Upsilon_{i,0} + \dots + \tau_i^j \Upsilon_{i,j} + \dots + \tau_i^n \Upsilon_{i,n})x.\end{aligned}$$

Since $\tau_i \in [0, 1]$, with (16) and according to Lemma 3, one has $\mathcal{D}^+ V_i(x, t) + \lambda_i V_i(x, t) < 0$.

Moreover, with (17) and (18), one has $V_i(x, \ell T + t_{i-1}) \le \mu_i V_{i-1}(x, \ell T + t_{i-1}),\ i = 1, 2, \dots, S$, and $V_1(x, \ell(T + 1)) \le \mu_1 V_S(x, \ell(T + 1))$. Combining with (19) and according to Lemma 2, one can conclude that system (1) is λ^*-exponentially stable. □

Remark 2: One may find the results based on the Lyapunov matrix polynomial are also reported in [15]. Different from [15], a dwell-time related Lyapunov matrix polynomial is used in this work, and the techniques introduced in Lemma 3 is adopted to deal with the matrix polynomial issue. Moreover, the condition based on discontinuous Lyapunov function with Lyapunov matrix polynomial is also provided.

Remark 3: It can be seen that compared with Theorem 1, more variables have been introduced in Theorem 2. The constraint that the Lyapunov function should be continuous at each switching instant has been relaxed.

Remark 4: One may observe that $\Xi_{i,j}$ *is much more complicated than* $\Upsilon_{i,j}$. *It comes from the fact that* $\Xi_{i,j}$ *can be treated as a special case of* $\Upsilon_{i,j}$ *with* $P_{i,0} = P_{1,0} + \sum_{f=1}^{i-1}\sum_{j=1}^{n} P_{f,j}, i = 2, \dots, S, \sum_{i=1}^{S}\sum_{j=1}^{n} P_{i,j} = 0$.

In the following, a controller with time-varying controller gain based on continuous Lyapunov function is designed to stabilize the unstable periodic piecewise systems. Consider a periodic state-feedback control as, for $t \in [\ell T + t_{i-1}, \ell T + t_i),\quad i = 1, 2, \dots, S,\ u(t) = K_i(t)x(t)$, where $K_i(t)$ is continuous and $K_i(t + \ell T) = K_i(t)$. Then the closed-loop representation of system (1) can be given as

$$\begin{aligned}\dot{x}(t) &= A_{ci}(t)x(t) + B_{wi}w(t),\\ z(t) &= C_i x(t) + D_i w(t)\end{aligned} \tag{21}$$

where $A_{ci}(t) = A_i + B_i K_i(t)$. It can be seen that the closed-loop system with time-varying controller gain becomes a periodic piecewise time-varying system. The following Lemma 5 concerns the stability of periodic piecewise time-varying system based on a continuous Lyapunov function (2).

Lemma 5 [15]: Consider periodic piecewise time-varying system (21) with $w(t) = 0$, *let* $\lambda^* > 0$ *be a given constant. If there exist* $\lambda_i, i = 1, 2, \dots, S$, *and a real symmetric T−periodic, continuous and Dini-differentiable matrix function* $Z(t)$ *defined on* $t \in [0, \infty)$ *such that, for* $i = 1, 2, \dots, S, t \in [\ell T + t_{i-1}, \ell T + t_i), Z(t) = Z_i(t)$ *satisfies*

$$A_{ci}'(t)Z_i(t) + Z_i(t)A_{ci}(t) + \mathcal{D}^+ Z_i(t) + \lambda_i Z_i(t) < 0, \tag{22}$$

$$2\lambda^* T - \sum_{i=1}^{S} \lambda_i T_i \le 0, \tag{23}$$

then system (21) is λ^**-exponentially stable.*

Based on the above Lemma, a stabilizing controller can be given as in Theorem 3.

Theorem 3: Consider periodic piecewise system (1) with $w(t) = 0$, *let* $\lambda^* > 0$ *be a given constant. If there exist* λ_i *and matrices* $W_{i,j}, Q_{i,j}, i = 1, 2, \dots, S, j = 1, 2, \dots, n$ *satisfying*

$$\sum_{j=0}^{k} W_{i,j} > 0, i = 1, 2, \dots, S, k = 0, 1, \dots, n, \tag{24}$$

$$\sum_{j}^{k} \Delta_{i,j} < 0, i = 1, 2, \dots, S, k = 0, 1, \dots, n, \tag{25}$$

$$2\lambda^* T - \sum_{i=1}^{S} \lambda_i T_i \le 0 \tag{26}$$

where

$$W_{i,0} = \sum_{j=0}^{n} W_{i-1,j}, i = 2, \dots, S, \tag{27}$$

$$W_{S,n} = -\sum_{f=1}^{S-1}\sum_{j=1}^{n} W_{f,j} - \sum_{j=1}^{n-1} W_{S,j}, \tag{28}$$

$$\begin{aligned}\Delta_{i,j} &= A_i W_{i,j} + W_{i,j} A_i' + B_i Q_{i,j} + Q_{i,j}' B_i'\\ &\quad - \frac{j+1}{T_i} W_{i,j+1} + \lambda_i W_{i,j},\\ &\quad i = 1, 2, \dots, S, j = 0, \dots, n-1,\end{aligned} \tag{29}$$

$$\Delta_{i,n} = A_i W_{i,n} + W_{i,n} A_i' + B_i Q_{i,j} + Q_{i,j}' B_i' + \lambda_i W_{i,n}, \tag{30}$$

then the closed-loop system is λ^*-exponentially stable, the periodic state-feedback gain can be given as, for $t \in [\ell T + t_{i-1}, \ell T + t_i), i = 1, 2, \ldots, S$,

$$K(t) = K_i(t) = Q_i(t) W_i^{-1}(t) \tag{31}$$

with time-varying matrix function $Q_i(t)$ and continuous time-varying matrix function $W_i(t)$ given as

$$Q_i(t) = \sum_{j=0}^{n} \frac{(t - \ell T - t_{i-1})^j}{T_i^j} Q_{i,j}, \tag{32}$$

$$W_i(t) = \sum_{j=0}^{n} \frac{(t - \ell T - t_{i-1})^j}{T_i^j} W_{i,j}. \tag{33}$$

Proof: For $t \in [\ell T + t_{i-1}, \ell T + t_i), i = 1, 2, \ldots, S$, with (27), (28) and (33), one obtains $W(t)$ is continuous. Then with (24) and according to Lemma 4, one has $W(t) > 0$. Construct a Lyapunov function $V(x, t) = x' W^{-1}(t) x = x' Z(t) x$, then one has $Z(t) > 0$, and it is continuous for $x(t) \neq 0$.

Following the similar argument in the proof of Theorem 1, choosing $\tau_i = \frac{(t - \ell T - t_{i-1})}{T_i}$ one has

$$\begin{aligned} &W_i(t) A_i' + A_i W_i(t) + B_i Q_i(t) + Q_i'(t) B_i' \\ &\quad - \mathcal{D}^+ W_i(t) + \lambda_i W_i(t) \\ &= \Delta_{i,0} + \tau_i \Delta_{i,1} + \cdots + \tau_i^n \Delta_{i,n}. \end{aligned}$$

Since $\tau_i \in [0, 1]$, according to Lemma 3, then one has

$$W_i(t) A_i' + A_i W_i(t) + B_i Q_i(t) + Q_i'(t) B_i' - \mathcal{D}^+ W_i(t) + \lambda_i W_i(t) < 0. \tag{34}$$

Left-multiply and right-multiply $Z_i(t) = W_i^{-1}(t)$ of (34), one has

$$A_{ci}'(t) W_i^{-1}(t) + W_i^{-1}(t) A_{ci}(t) - W_i^{-1}(t) \mathcal{D}^+ W_i(t) W_i^{-1}(t) + \lambda_i W_i^{-1}(t) < 0. \tag{35}$$

Since $\mathcal{D}^+ W^{-1}(t) = -W^{-1}(t) \mathcal{D}^+ W(t) W^{-1}(t)$, (35) can be rewritten as

$$A_{ci}'(t) Z_i(t) + Z_i(t) A_{ci}(t) + \mathcal{D}^+ Z_i(t) + \lambda_i Z_i(t) < 0. \tag{36}$$

Then, combining with (26) and according to Lemma 5, the λ^*-exponential stability of the closed-loop system is established. □

Notice that substituting (27) and (28) into (24), (25), one could obtain the condition without the explicit expression of $W_{i,0}, i = 2, \ldots, S$ and $W_{S,n}$.

Remark 5: One may observe that the stabilizing controller with time-varying controller gain is established based on the continuous Lyapunov function. It should be noticed this method cannot be directly applied to the controller design based on the discontinuous Lyapunov function. Controller based on the discontinuous Lyapunov function can be formulated as a periodic piecewise controller with different constant gains for different subsystems, such as the one proposed in [14]. A corresponding iterative algorithm should be designed as well to obtain the controller gain.

In the following, a weighted L_2-gain performance of periodic piecewise system (1) is studied based on Lyapunov function (2) with discontinuous Lyapunov matrix polynomial (3).

Theorem 4: Consider periodic piecewise system (1) with $u(t) = 0$, given $\hat{\gamma} > 0, \lambda^* > 0$. If there exist λ_i, $\mu_i > 1, i = 1, 2, \ldots, S$, and matrices $P_{i,j}, i = 1, 2, \ldots, S$, $j = 0, 1, \ldots, n$, satisfying

$$\sum_{j=0}^{k} P_{i,j} > 0, \quad i = 1, 2, \ldots, S, k = 0, 1, \ldots, n, \tag{37}$$

$$\sum_{j=0}^{k} \begin{bmatrix} \Delta_{i,j} & \Lambda_{i,j}' & \mathcal{C}_{i,j}' \\ \Lambda_{i,j} & \mathcal{R}_{i,j} & \mathcal{D}_{i,j}' \\ \mathcal{C}_{i,j} & \mathcal{D}_{i,j} & \mathcal{I}_{i,j} \end{bmatrix} < 0, \quad i = 1, 2, \ldots, S, k = 0, 1, \ldots, n \tag{38}$$

$$P_{i+1,0} \leq \mu_{i+1} \sum_{j=0}^{n} P_{i,j}, \quad i = 1, 2, \ldots, S-1, \tag{39}$$

$$P_{1,0} \leq \mu_i \sum_{j=0}^{n} P_{S,j}, \tag{40}$$

$$\sum_{i=1}^{S} \lambda_i T_i \geq \sum_{i=1}^{S} \ln \mu_i + 2\lambda^* T \tag{41}$$

where

$$\begin{aligned} \Delta_{i,j} &= A_i' P_{i,j} + P_{i,j} A_i + \lambda_i P_{i,j} + \frac{j+1}{T_i} P_{i,j+1}, \\ &\quad j = 0, 1, \ldots, n-1, \\ \Delta_{i,n} &= A_i' P_{i,n} + P_{i,n} A_i + \lambda_i P_{i,n}, \\ \Lambda_{i,j}' &= P_{i,j} B_{wi}, \quad j = 0, 1, \ldots, n, \\ \mathcal{C}_{i,0}' &= C_i', \quad \mathcal{R}_{i,0} = -\gamma^2 I, \mathcal{D}_{i,0}' = D_i', \mathcal{I}_{i,0} = -I, \\ \mathcal{C}_{i,j} &= 0, \quad \mathcal{R}_{i,j} = 0, \ \mathcal{D}_{i,j} = 0, \\ \mathcal{I}_{i,j} &= 0, \quad j = 1, 2, \ldots, n, \end{aligned}$$

then system (1) is λ^*-exponentially stable with a disturbance attenuation level $\hat{\gamma}$ in the sense of Definition 3.

Proof: For $t \in [\ell T + t_{i-1}, \ell T + t_i), i = 1, 2, \ldots, S$, construct a Lyapunov function as in (2) with Lyapunov matrix given in (3). With (37) and according to Lemma 4, one has $V_i(x, t) > 0$. Define $\mathcal{F} = z'z - \gamma^2 w'w$ with a given scale $\gamma > 0$, one has

$$\begin{aligned} &\mathcal{D}^+ V_i(x, t) + \lambda_i V_i(x, t) + \mathcal{F} \\ &= (x' A_i' + w' B_{wi}') P_i(t) x + x' P_i(t)(A_i x + B_{wi} w) \\ &\quad + x' \mathcal{D}^+ P_i(t) x \\ &= x'(A_i' P_i(t) + P_i(t) A_i + \lambda_i P_i(t) + \mathcal{D}^+ P_i(t) \\ &\quad + C_i' C_i) x + w'(B_{wi}' P_i(t) + D_i' C_i) x \\ &\quad + x'(P_i(t) B_{wi} + C_i' D_i) w \\ &\quad + w'(-\gamma^2 I + D_i' D_i) w \\ &= \begin{bmatrix} x \\ w \end{bmatrix}' \left(\begin{bmatrix} \varrho(t) & P_i(t) B_{wi} \\ B_{wi}' P_i & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} C_i \\ D_i \end{bmatrix}' \begin{bmatrix} C_i & D_i \end{bmatrix} \right) \begin{bmatrix} x \\ w \end{bmatrix} \end{aligned} \tag{42}$$

where

$$\varrho(t) = A_i' P_i(t) + P_i(t) A_i + \mathcal{D}^+ P_i(t) + \lambda_i P_i(t).$$

Construct a matrix polynomial Ω given as

$$\begin{aligned}\Omega &= \begin{bmatrix} \varrho(t) & P_i(t)B_{wi} & C_i' \\ * & -\gamma^2 I & D_i' \\ * & * & -I \end{bmatrix} \\ &= \Omega_{i,0} + \frac{t-\ell T - t_{i-1}}{T_i}\Omega_{i,1} + \ldots + \frac{(t-\ell T-t_{i-1})^n}{T_i^n}\Omega_{i,n}\end{aligned} \tag{43}$$

where

$$\Omega_{i,0} = \begin{bmatrix} \Delta_{i,0} & \Lambda_{i,0}' & \mathcal{C}_{i,0}' \\ \Lambda_{i,0} & \mathcal{R}_{i,0} & \mathcal{D}_{i,0}' \\ \mathcal{C}_{i,0} & \mathcal{D}_{i,0} & \mathcal{I}_{i,0} \end{bmatrix},$$

$$\Omega_{i,1} = \begin{bmatrix} \Delta_{i,1} & \Lambda_{i,1}' & \mathcal{C}_{i,1}' \\ \Lambda_{i,1} & \mathcal{R}_{i,1} & \mathcal{D}_{i,1}' \\ \mathcal{C}_{i,1} & \mathcal{D}_{i,1} & \mathcal{I}_{i,1} \end{bmatrix},$$

$$\Omega_{i,n} = \begin{bmatrix} \Delta_{i,n} & \Lambda_{i,n}' & \mathcal{C}_{i,n}' \\ \Lambda_{i,n} & \mathcal{R}_{i,n} & \mathcal{D}_{i,n}' \\ \mathcal{C}_{i,n} & \mathcal{D}_{i,n} & \mathcal{I}_{i,n} \end{bmatrix}.$$

Since $0 \le \frac{t-\ell T - t_{i-1}}{T_i} \le 1$, with (38) and according to Lemma 3, one has $\Omega < 0$. Then applying Schur complement equivalence to (43), one has

$$\begin{bmatrix} \varrho(t) & P_i(t)B_{wi} \\ B_{wi}' P_i & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} C_i \\ D_i \end{bmatrix}' \begin{bmatrix} C_i & D_i \end{bmatrix} < 0. \tag{44}$$

It implies $\mathcal{D}^+ V_i(x,t) < -\lambda_i V_i(x,t) - \mathcal{F}$ if $x \neq 0$ or $w \neq 0$. Integrate it for $t \in [\ell T + t_{i-1}, \ell T + t_i)$, following the same steps in [14], with $V_i(t) \ge 0$, one has

$$\begin{aligned}&\sum_{k=1}^{\ell}\sum_{j=1}^{S}\int_{(k-1)T+t_{j-1}}^{(k-1)T+t_j} \exp(\Psi_1(j,k)) z'(\tau) z(\tau) d\tau \\ &+ \sum_{j=1}^{i-1}\int_{\ell T+t_{j-1}}^{\ell T+t_j} \exp(\Psi_2(j)) z'(\tau) z(\tau) d\tau \\ &+ \int_{\ell T+t_{i-1}}^{t} \exp(\Psi_3) z'(\tau) z(\tau) d\tau \\ &\le e^{\Xi_0} V(0) + \gamma^2 \Bigg\{ \sum_{k=1}^{\ell}\sum_{j=1}^{S}\int_{(k-1)T+t_{j-1}}^{(k-1)T+t_j} \exp(\Xi_1(j,k)) \\ &\quad \times w'(\tau) w(\tau) d\tau + \sum_{j=1}^{i-1}\int_{\ell T+t_{j-1}}^{\ell T+t_j} \exp(\Xi_2(j)) w'(\tau) w(\tau) d\tau \\ &\quad + \int_{\ell T+t_{i-1}}^{t} \exp(\Xi_3) w'(\tau) w(\tau) d\tau \Bigg\},\end{aligned} \tag{45}$$

where

$$\begin{aligned}\Psi_1(j,k) &= -(k-1)\sum_{q=1}^{S}\ln\mu_q - \sum_{q=2}^{j}\ln\mu_q - \lambda_j((k-1)T \\ &\quad + t_j - \tau) - \sum_{q=j+1}^{S}\lambda_q T_q - (\ell-k)\sum_{q=1}^{S}\lambda_q T_q \\ &\quad - \sum_{q=1}^{i-1}\lambda_q T_q - \lambda_i(t-(\ell T + t_{i-1})), \\ \Psi_2(j) &= -\ell\sum_{q=1}^{S}\ln\mu_q - \sum_{q=1}^{j}\ln\mu_q - \lambda_j(\ell T + t_j - \tau) \\ &\quad - \sum_{q=j+1}^{i-1}\lambda_q T_q - \lambda_i(t-(\ell T + t_{i-1})), \\ \Psi_3 &= -\ell\sum_{q=1}^{S}\ln\mu_q - \sum_{q=2}^{i}\ln\mu_q - \lambda_i(t-\tau), \\ \Xi_0 &= -\ell\sum_{q=1}^{S}\lambda_q T_q - \sum_{q=1}^{i-1}\lambda_q T_q - \lambda_i(t-(\ell T + t_{i-1})), \\ \Xi_1(j,k) &= -\lambda_j((k-1)T + t_j - \tau) \\ &\quad - \sum_{q=j+1}^{S}\lambda_q T_q - (\ell-k)\sum_{q=1}^{S}\lambda_q T_q \\ &\quad - \sum_{q=1}^{i-1}\lambda_q T_q - \lambda_i(t-(\ell T + t_{i-1})), \\ \Xi_2(j) &= -\lambda_j(\ell T + t_j - \tau) - \sum_{q=j+1}^{i-1}\lambda_q T_q - \lambda_i(t - (\ell T \\ &\quad + t_{i-1})), \\ \Xi_3 &= -\lambda_i(t-\tau).\end{aligned} \tag{46}$$

Since

$$\begin{aligned}\Psi_1(j,k) &\ge -k\sum_{q=1}^{S}\ln\mu_q - \lambda_{\max}((k-1)T + t_j - \tau) \\ &\quad - \lambda_{\max}(T - t_j) - (\ell-k)\lambda_{\max}T \\ &\quad - \lambda_{\max}t_{q-1} - \lambda_{\max}(t-(\ell T + t_{q-1})) \\ &\ge -k\sum_{q=1}^{S}\lambda_q T_q + 2\lambda^* kT - \lambda_{\max}(t-\tau) \\ &\ge -\lambda_{\max}t - \lambda_{\max}(kT - \tau) \\ &\ge -\lambda_{\max}t - \lambda_{\max}T,\end{aligned} \tag{47}$$

$$\begin{aligned}\Psi_2(j) &\ge -(\ell+1)\sum_{q=1}^{S}\ln\mu_q - \lambda_{\max}(\ell T + t_j - \tau) \\ &\quad - \lambda_{\max}(t_{i-1} - t_j) - \lambda_{\max}(t - \ell T - t_{i-1}) \\ &\ge -(\ell T + 1)\sum_{q=1}^{S} + 2(\ell+1)\lambda^* T - \lambda_{\max}(t-\tau) \\ &\ge -(\ell+1)\lambda_{\max}T - \lambda_{\max} + \lambda_{\max}\tau \\ &\ge -\lambda_{\max}t - \lambda_{\max}T,\end{aligned} \tag{48}$$

$$\begin{aligned}\Psi_3 &\ge -\ell\sum_{q=1}^{S}\lambda_q T_q + 2\ell T\lambda^* - \lambda_{\max}(t-\tau) \\ &\ge -\lambda_{\max}t - \lambda_{\max}T,\end{aligned} \tag{49}$$

and

$$
\begin{aligned}
\Xi_0 &\le -\ell 2\lambda^* T - \lambda_{\min} t_{m-1} \\
&\quad - \lambda_{\min}(t - (\ell T + t_{m-1})) \\
&= -2\lambda^* t + (2\lambda^* - \lambda_{\min})(t - \ell T) \\
&< -2\lambda^* t + \max(2\lambda^* - \lambda_{\min}, 0)T, \qquad (50) \\
\Xi_1(j,k) &\le -\lambda_{\min}(kT - \tau) - (\ell - k)2\lambda^* T \\
&\quad - \lambda_{\min}(t - \ell T) \\
&= (-\lambda_{\min} + 2\lambda^*)(t - \ell T) + 2\lambda^*(\tau - t) \\
&\quad + (-\lambda_{\min} + 2\lambda^*)(kT - \tau) \\
&\le -2\lambda^*(t - \tau) + \max(2\lambda^* \\
&\quad - \lambda_{\min}, 0)2T, \qquad (51) \\
\Xi_2(j) &\le -2\lambda^*(t - \tau) + 2\lambda^*(t - \tau) - \lambda_{\min}(t - \tau) \\
&\le -2\lambda^*(t - \tau) + \max(2\lambda^* \\
&\quad - \lambda_{\min}, 0)2T, \qquad (52) \\
\Xi_3 &= -2\lambda^*(t - \tau) + 2\lambda^*(t - \tau) - \lambda_i(t - \tau) \\
&\le -2\lambda^*(t - \tau) + 2\lambda^*(t - \tau) - \lambda_{\min}(t - \tau) \\
&\le -2\lambda^*(t - \tau) + \max(2\lambda^* \\
&\quad - \lambda_{\min}, 0)2T. \qquad (53)
\end{aligned}
$$

Combining (47)–(53), one has

$$
\begin{aligned}
&\int_0^t e^{-\lambda_{\max}(t+T)} z'(\tau) z(\tau) d\tau \\
&\le e^{T \max(2\lambda^* - \lambda_{\min}, 0) - 2\lambda^* t} V(x_0, 0) \\
&\quad + \gamma^2 \int_0^t e^{2T \max(2\lambda^* - \lambda_{\min}, 0) - 2\lambda^*(t-\tau)} w'(\tau) w(\tau) d\tau.
\end{aligned}
$$

Integrating t from 0 to ∞, we obtain

$$
\begin{aligned}
&\int_0^\infty e^{-\lambda_{\max}\tau} z'(\tau) z(\tau) d\tau \\
&\le \frac{\lambda_{\max}}{2\lambda^*} e^{2T \max(\lambda^* - \lambda_{\min}, 0)} V(x_0, 0) \\
&\quad + \frac{\lambda_{\max}}{2\lambda^*} e^{2T \max(2\lambda^* - \lambda_{\min}, 0)} \gamma^2 \int_0^\infty w'(\tau) w(\tau) d\tau.
\end{aligned}
$$

Finally, by denoting $\hat{\lambda} = \lambda_{\max}$, $\sigma = \frac{\lambda_{\max}}{2\lambda^*} e^{T \max(2\lambda^* - \lambda_{\min}, 0)}$, and $\hat{\gamma} = \gamma e^{T(\max(2\lambda^* - \lambda_{\min}, 0))} \sqrt{\frac{\lambda_{\max}}{2\lambda^*}}$, one can conclude that the system is exponentially stable with a weighted L_2-gain $\hat{\gamma}$ in the sense of Definition 3. □

Remark 6: One may notice that the above performance index is described in a weighted form. A similar description can be found in switched systems as well [29], [30]. The weighted index results from the adopted discontinuous Lyapunov function and the multiple Lyapunov functions [31].

IV. SIMULATION

In this section, numerical examples are used to demonstrate the effectiveness of the proposed approaches. The stabilizing controller design and L_2-gain performance index are studied in Example I and Example II, respectively.

Example I: Consider a periodic piecewise system with $T = 2$ s and $T_1 = 0.4$ s, $T_2 = 0.2$ s, $T_3 = 1$ s, the subsystems with $w(t) = 0$ are given as

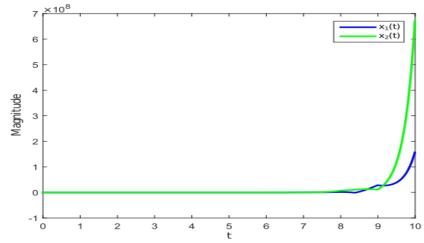

FIGURE 1. Trajectory of system state without control.

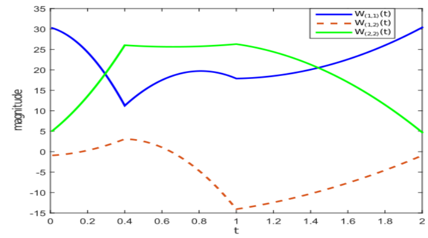

FIGURE 2. Time history of $W(t)$.

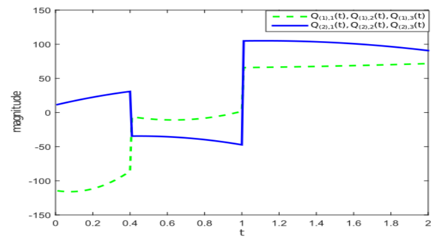

FIGURE 3. Time history of $Q_1(t)$, $Q_2(t)$, $Q_3(t)$.

$$
A_1 = \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix},
$$
$$
B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.
$$

One may observe that all subsystems are non-Hurwitz stable. Under initial condition $x_0 = [2, 1]'$, the trajectory of system state is shown in Figure 1. It can be seen that the periodic piecewise system is unstable.

Choosing $\lambda_1 = 0.2$, $\lambda_2 = -0.1$, $\lambda_3 = 0.5$, according to Theorem 3, a stabilizing controller with time-varying controller gain is designed with degree 2. The obtained $W(t)$ and $Q_1(t)$, $Q_2(t)$, $Q_3(t)$ are shown in Fig. 2 and Fig. 3, where $W(t)$ is continuous and $Q_1(t)$, $Q_2(t)$, $Q_3(t)$ switches at each switching instant, the legend provides the specific item location in the matrix and vector. Then, the obtained controller gain is shown in Fig. 4. One may observe that the controller gain is piecewise time-varying because of the discontinuous $Q_i(t)$. A continuous controller gain could be obtained with putting a continuity constraint on $Q_i(t)$. The trajectory of system state under the designed controller proposed by this work is shown in Fig.5. It can be seen that the unstable periodic piecewise system is stabilized with the designed controller.

Example II: Consider a periodic piecewise system with $T = 2$ and $T_1 = 0.7$, $T_2 = 0.8$, $T_3 = 0.5$, the subsystems are

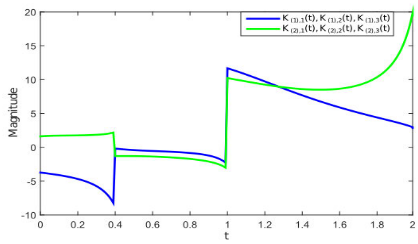

FIGURE 4. Time history of the controller gain.

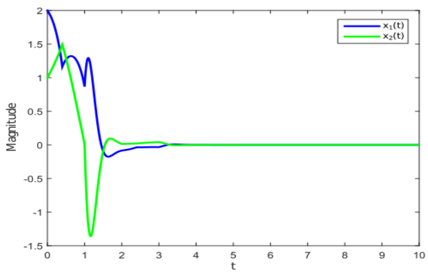

FIGURE 5. Trajectory of system state under control.

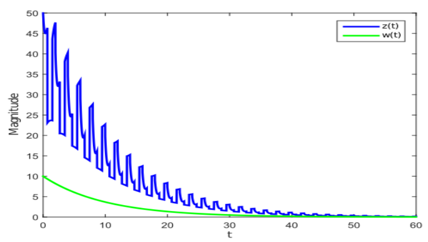

FIGURE 6. Disturbance and system response.

given as

$$A_1=\begin{bmatrix}-1 & 0\\ -5 & -3\end{bmatrix},\quad Bw_1=\begin{bmatrix}-1\\ -1\end{bmatrix},\ C_1=[1,2],D_1=5,$$
$$A_2=\begin{bmatrix}-1 & 0\\ -2 & -3\end{bmatrix},\quad B_{w2}=\begin{bmatrix}0\\ -1.25\end{bmatrix},\ C_2=[1,0],D_2=5,$$
$$A_3=\begin{bmatrix}1 & -1\\ 1 & -3\end{bmatrix},\quad B_{w3}=\begin{bmatrix}1\\ -0.83\end{bmatrix},\text{‘}C_3=[1,-2],\ D_3=5.$$

It can be seen that the third subsystem is non-Hurwitz stable. Let $\lambda_1 = 0.3$, $\lambda_2 = 0.25$, $\lambda_3 = 0.2$, $\mu_1 = \mu_2 = \mu_3 = 1.01$, $x(0) = [0,0]'$. Notice that, it does not require to allocate a negative λ_i for non-Hurwitz subsystem. For more details, one could refer to authors' previous work [15]. According to Theorem 4, one can obtain $\gamma = 5.6033$. Consider a disturbance $w(t) = 10e^{-0.1t}$, the disturbance and system output are shown in Fig. 6. And one has $\|z\|_2 = 935.39$, $\|w\|_2 = 223.7179$. It can be seen that the system output is within the given constraint.

V. CONCLUSION

In this paper, new conditions of the analysis and synthesis of periodic piecewise system are proposed based on continuous and discontinuous Lyapunov functions, respectively. Based on the dwell-time related Lyapunov matrix polynomial, new methods concerning the positive and negative definiteness of matrix polynomial are used to develop the condition. The exponential stability are studies based on both the continuous and discontinuous Lyapunov functions at first. Then, a stabilizing controller with time-varying controller gain is designed based on continuous Lyapunov function and the weighted L_2-gain performance analysis is also carried out with discontinuous Lyapunov function. Finally, the merits of the proposed methods are demonstrated with the numerical examples. The future work may consider the system with nonlinear property and using the neural networks [32], [33].

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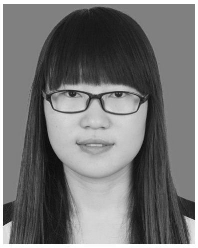

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