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# Adaptive Robust Control for a Class of Stochastic Nonlinear Uncertain Systems

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**ABSTRACT** This paper is concerned with the control design for a class of stochastic nonlinear systems. Three uncertainties are considered; that is, nonlinear parameter uncertainty, matched uncertainty and stochastic disturbance. The nonlinear uncertainty contains some uncertain parameter and satisfies bound condition. Neither the exact value of the matched uncertainty nor its possible bound is known; its upper bound function satisfies certain concave condition. The stochastic disturbance is a standard Wiener process. Based on stochastic Lyapunov stability theory, the adaptive robust controller is designed, which renders the state variables of the closed-loop system bounded in probability, regardless of all uncertainties. The desired controller is constructed by the upper bound function and the saturation function, in which the upper bound function represents the magnitude of the control, while the saturation function indicates the control direction. The design of the adaptive robust controller is based on the minimum information of uncertainty, which is simple and can be easily realized in practical systems. Finally, a two-tank water level control example is used to demonstrate the effectiveness of our control design.

**INDEX TERMS** Stochastic system, nonlinear system, uncertainty, adaptive control, bounded in probability.

## I. INTRODUCTION

The stochastic disturbance often occurs in practical systems and causes instability. Therefore, it is necessary and challenging to investigate control problems for stochastic systems. In the middle of the 20th century, stochastic stability definitions were firstly formulated in [1] and [2]. And then, some classical stochastic stability theories have been put forward, such as stochastic Lyapunov stability theory [1], stochastic LaSalle invariance principle [3] and stochastic input-to-state stable (SISS) [4] etc. Meanwhile, some analysis and design tools for stochastic control systems were also developed from deterministic control systems. Florchinger extended the CLF [5] and Sontag stabilization formula [6] into stochastic systems, and introduced the passivity definition [7]. The stochastic nonlinear small-gain theorem was given in [8]. More recently, lots of outstanding results on stochastic nonlinear control have been reported in the literature; see, e.g. [9]–[14] and the references therein.

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In practical applications, mathematical models of systems always contain uncertainty. The earlier robust control technique can usually accommodate the uncertainty with upper bound. When the upper bound is unknown, which is related to some unknown parameters, the conservative estimation in the robust control will lead to large control laws. Therefore the adaptive algorithm of parameter is designed, instead of the robust controller design only depending on the upper bound. This is the core idea of adaptive robust control, which can provide less conservative results. In recent years, many important research results about advanced adaptive control of deterministic nonlinear systems have emerged. An adaptive neural tracking scheme was designed in [15] for a class of uncertain switched non-strict feedback nonlinear systems subject to both unknown backlash-like hysteresis and output dead-zone. By using output feedback, [16] investigated the problem of adaptive practical tracking for a class of uncertain nonlinear systems subject to non-symmetric dead-zone input nonlinearity. In [17], based on the Lyapunov stability theory and the backstepping design technique, the event-triggered adaptive fuzzy output feedback control was given for a class of multi-input and multi-output (MIMO) switched nonlinear systems with unknown nonlinear functions and unmeasured states. An improved backstepping-based fuzzy finite-time control scheme was proposed in [18] for a class of non-strict feedback nonlinear systems with unknown actuator faults. As far as stochastic nonlinear uncertain systems are concerned, there have been some representative results on the adaptive control problems. For example, a neural network (NN) was employed in [19] to compensate for the unknown upper function of the nonlinear interconnections, where a decentralized adaptive output-feedback stabilization controller for a class of large-scale stochastic nonlinear strict-feedback systems was designed. In the context of stochastic large-scale nonlinear systems with unknown dead-zone and unmodeled dynamics, a robust adaptive fuzzy decentralized controller was constructed in [20]. In [21], combining the backstepping design with the supply changing function technique, an adaptive fuzzy decentralized output feedback control approach was developed which was robust against unmodeled dynamics and unknown dead-zones, while in [22] an adaptive robust output feedback control scheme was developed for dynamically positioned ships with unavailable velocities and unknown dynamic parameters in an unknown time-variant disturbance environment. In [23], an adaptive NN state-feedback controller was presented for a class of nonlinear stochastic systems with unknown parameters, unknown nonlinear functions and stochastic disturbances. An adaptive control scheme for nonlinear stochastic systems with unknown parameters was provided in [24], where full state constraints in the considered systems were taken into consideration. In [25], an adaptive integral sliding mode controller was constructed for general Takagi-Sugeno (T-S) fuzzy systems with matched uncertainties. Reference [26] constructed an observer to estimate the unknown state variables, and solved the observer-based adaptive fuzzy control problem for nonstrict-feedback stochastic nonlinear systems with input saturation and prescribed performance. An observer-based fuzzy adaptive output feedback controller was proposed for a class of switched stochastic nonlinear uncertain systems with quantized input signals in [27]. It is noted that there are few results on the adaptive robust control design for stochastic nonlinear systems embracing nonlinear parameter uncertainty, matched uncertainty and stochastic disturbance.

In this paper we endeavor to explore the effect of those different uncertainties on stochastic systems performance analysis and controller design. Inspired by [28], [29] and [30], we address the state feedback stabilization problem for a class of stochastic nonlinear systems. The system is subject to three uncertainties including nonlinear parameter uncertainty, matched uncertainty and external disturbance. The zero dynamic which contains parameter uncertainty is bounded in probability, while the matched uncertainty satisfies certain concave condition. The stochastic disturbance is a standard Wiener process. Based on the stochastic Lyapunov stability theory, an adaptive robust controller is designed to guarantee that the state variables of the resulting closed-loop system are bounded in probability, regardless of all uncertainties. By properly choosing design parameters, all the responses of the closed-loop system can converge to a small neighbourhood of the origin. The controller is explicitly represented by the upper bound function and a saturation function. The control design can be based on the minimum information of the uncertainty, which is simple and can be easily realized in practical systems. We demonstrate the design procedure by using a two-tank water level control system, in which the water level height not only needs to be close to the desired height but also stays positive for all time.

Our control schemes are unprecedented, practical, and significant. In the past, the most advanced control design only embraced portion of uncertainties. And even for that, it is required to know the bound of uncertainty. Our control schemes, on the other hand, do not need to know the bound of uncertainty. They also capture the full spectrum of varieties of uncertainty. The proposed control scheme is applicable to many systems and requires only the minimum structural information of the uncertainty.

## II. PRELIMINARIES ON STABILITY IN PROBABILITY

The following symbols will be used in the sequel. $\mathbb{R}_+$ represents the set of non-negative real numbers. For a given vector or matrix $X$, $X^T$ denotes its transpose, and $Tr$ denotes the matrix trace. $\|\cdot\|$ represents the Euclidean vector norm. $C^i$ denotes the set of all functions with continuous $i$-th partial derivatives. $C^{i,j}$ denotes the family of all nonnegation functions which are $C^i$ in the first argument and $C^j$ in the second argument.

Consider a stochastic nonlinear system described by

$$dx(t) = \hat{f}(x(t), t)dt + \hat{g}(x(t), t)dw(t), \tag{1}$$

where $x(t) = [x_1(t), x_2(t), \cdots, x_n(t)]^T \in \mathbb{R}^n$ is the state, $w(t)$ is $r$-dimensional independent standard Wiener process. $\hat{f} : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ and $\hat{g} : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^{n\times r}$ are locally Lipschitz.

*Definition 1 ( [1]):* A continuous function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $\gamma \in \mathcal{K}$, and $\gamma(r) \to \infty$ as $r \to \infty$.

*Definition 2 ( [1]):* The solution process $\{x(t), t \geq 0\}$ of the stochastic system (1) is said to be bounded in probability, if $\lim_{\xi\to\infty} \sup_{0\leq t<\infty} P\{\|x(t)\| > \xi\} = 0$.

The following theorem provides a sufficient condition on the boundedness and stability properties for the system (1), which is based on stochastic Lyapunov stability theory.

*Lemma 1 ( [4]):* Consider the stochastic system (1) and assume that both $\hat{f}$ and $\hat{g}$ are locally bounded and Lipschitz continuous. If there exist a $C^{2,1}$ function $V : \mathbb{R}^n\times\mathbb{R}_+ \to \mathbb{R}_+$, $\mathcal{K}_\infty$ functions $\mu_1$, $\mu_2$, constants $c_1 > 0$, $c_2 \geq 0$, and a nonnegative continuous function $W(x, t)$, such that

$$\mu_1(\|x(t)\|) \leq V(x, t) \leq \mu_2(\|x(t)\|), \tag{2}$$

$$\mathcal{L}V(x, t) \leq -c_1 W(x, t) + c_2, \tag{3}$$

where

$$\mathcal{L}V(x,t) = \frac{\partial V(x,t)}{\partial t} + \nabla_x^T V(x,t)\hat{f}(x,t) + \frac{1}{2}Tr\{\hat{g}^T(x,t)\frac{\partial^2 V(x,t)}{\partial x^2}\hat{g}(x,t)\}. \quad (4)$$

then (a) for system (1) there exists an almost surely unique solution on $[0, \infty)$; (b) all the responses are bounded in probability, when $W(x,t) \geq \zeta V(x,t)$ for some constant $\zeta > 0$; (c) when $c_2 = 0$, $\hat{f}(0,t) \equiv 0$, $\hat{g}(0,t) \equiv 0$ and $W(x,t) = W(x)$ is continuous, the equilibrium $x = 0$ is globally stable in probability and the solution $x(t)$ satisfies $P\{\lim_{t\to\infty} W(x(t)) = 0\} = 1$.

## III. UNCERTAIN STOCHASTIC DYNAMICAL SYSTEMS

Consider the uncertain stochastic dynamical system described by the state equation

$$\begin{aligned} dx(t) = \{&f(x(t),t,\sigma(t)) + \Delta f(x(t),t,\sigma(t)) \\ &+ [B(x(t),t) + \Delta B(x(t),t,\sigma(t))]u(t)\}dt \\ &+ g(x(t),t)dw(t), \end{aligned} \quad (5)$$

where $x(t) = [x_1(t), x_2(t), \cdots, x_n(t)]^T \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control, $\sigma(t) \in \mathbb{R}^p$ is an uncertain (possibly time-varying) parameter, and $w(t)$ is an $r$-dimensional independent standard Wiener process. Furthermore, $f(x,t,\sigma)$, $\Delta f(x,t,\sigma)$, $B(x,t)$, $\Delta B(x,t,\sigma)$ and $g(x,t)$ are locally Lipschitz.

*Assumption 1:* The (unknown) parameter $\sigma : \mathbb{R} \to \Sigma \subset \mathbb{R}^p$ is Lebesgue measurable with $\Sigma$ compact and possibly unknown.

*Assumption 2 (i):* There are continuous functions $e : \mathbb{R}^n \times \mathbb{R} \times \Sigma \to \mathbb{R}^m$ and $E : \mathbb{R}^n \times \mathbb{R} \times \Sigma \to \mathbb{R}^{m\times m}$ such that

$$\begin{aligned} \Delta f(x,t,\sigma) &= B(x,t)e(x,t,\sigma), \\ \Delta B(x,t,\sigma) &= B(x,t)E(x,t,\sigma), \end{aligned} \quad (6)$$

and

$$\|e(x,t,\sigma)\| \leq \rho(x,t,\theta), \forall (x,t,\sigma) \in \mathbb{R}^n \times \mathbb{R} \times \Sigma. \quad (7)$$

for a known bound function $\rho(x,t,\theta)$ with an unknown constant $\theta \in \mathbb{R}^k$. Moreover, there exists a scalar constant $c > -1$ such that

$$\min_{\sigma\in\Sigma} \lambda_{\min}\frac{1}{2}\{E^T(x,t,\sigma) + E(x,t,\sigma)\} > c. \quad (8)$$

Here $\lambda_{\min}(\cdot)$ ($\lambda_{\max}(\cdot)$) denotes the minimum (maximum) eigenvalue of a symmetric matrix.

*Remark 1:* In reality, unknown parameters are always bounded, just like Assumption 1, unbounded uncertain parameters would require infinite energy to sustain. The inequality (8) implies that the direction of the control is not to be "reversed" nor "annihilated" by uncertainty.

Assumption 1 implies that the possible value of the uncertainty parameter $\sigma$ falls within a compact set, which may be unknown. Assumption 2 (i) states that $e$ must be bounded by a known function $\rho$, whose *value* at each $(x,t,\theta)$ may be unknown due to the fact that $\theta$ is unknown, in order to permit one to guarantee boundedness of the system (5).

*Assumption 2 (ii):* For each $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, the function $\rho(x,t,\cdot) : \mathbb{R}^k \to \mathbb{R}_+$ is $C^1$ and nondecreasing with respect to each component of its argument $\theta$, and $\rho(x,t,\cdot)$ is concave.

*Remark 2:* $\rho(x,t,\cdot)$ is concave means the effect of parameter on the upper bound function of uncertainty is nonlinear.

Thus, prior to controlling the effects of the uncertainty, it may be necessary to employ a portion of the control to render the following uncontrolled system bounded:

$$dx(t) = f(x(t),t,\sigma(t))dt + g(x(t),t)dw(t). \quad (9)$$

*Assumption 3:* There exist a $C^{2,1}$ function $V_0 : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$, $\mathcal{K}_\infty$ functions $\gamma_1$, $\gamma_2$, and parameters $l_1 > 0$, $l_2 > 0$ such that

$$\gamma_1(\|x\|) \leq V_0(x,t) \leq \gamma_2(\|x\|), \quad (10)$$

$$\mathcal{L}V_0(x,t) \leq -l_1 V_0(x,t) + 1/l_2. \quad (11)$$

where

$$\mathcal{L}V_0(x,t) = \frac{\partial V_0(x,t)}{\partial t} + \nabla_x^T V_0(x,t)f(x,t,\sigma) + \frac{1}{2}Tr\{g^T(x,t)\frac{\partial^2 V_0(x,t)}{\partial x^2}g(x,t)\}. \quad (12)$$

We denote $\alpha(x,t)$, for each $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, as follows

$$\alpha(x,t) = B^T(x,t)[\partial V_0(x,t)/\partial x]. \quad (13)$$

*Remark 3:* Both $f(x,t,\sigma)$ and $\Delta f(x,t,\sigma)$ depend on the uncertain parameter $\sigma$. The reason for distinguishing them in (5) is that $f(x,t,\sigma)$ provides a "bounded" portion for the system regardless of the appearance / effect of $\sigma$. The other portion $\Delta f(x,t,\sigma)$ satisfies the matched condition.

*Remark 4:* The matching condition, as delineated in Assumption 2 (i), is a structural condition, which preempts that $\Delta f(x,t,\sigma)$ and $\Delta B(x,t,\sigma)$ are within the range space of $B(x,t)$, as in (6). This can always be met if there is sufficient control to the system. That is, the satisfaction of the matching condition is simply the designer's discretion. It is satisfied in most physical systems, including mechanical manipulators, transportation systems, hydraulic systems, etc. This however does not mean the system needs to be completely matched. The system dynamics $f(x,t,\sigma)$ still contains uncertain parameter $\sigma$ and does not need to meet the matching condition. It only needs to meet certain bounded condition (See Assumption 3). So, the whole system is best described as partially matched. The matching condition can be relaxed in [31]. We adopt this condition in this paper for simplicity. We also note that for quadratic stabilizability, the matching condition is also proven to be a necessary condition in [32].

Our control objective is to design the adaptive robust controller $u(t)$ such that all the responses of the resulting closed-loop system are bounded in probability.

Next, we introduce one important Lemma, which is followed from Theorem 4.1 of [33].

*Lemma 2:* For some continuous function $h : \mathbb{R}^n \times \mathbb{R} \times \Sigma \times \mathbb{R}^m \to \mathbb{R}$, suppose there exist a function

$\rho_0 : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}_+$, a constant $\beta_0 > 0$ and a constant vector $\theta \in \mathbb{R}^k$, such that for all $(x, t, \sigma, u) \in \mathbb{R}^n \times \mathbb{R} \times \Sigma \times \mathbb{R}^m$,

$$u^T h(x, t, \sigma, u) \geq \beta_0 \|u\| [\|u\| - \rho_0(x, t, \theta)]. \tag{14}$$

Suppose, further, that for any vector $\alpha_0 \in \mathbb{R}^m$ with $u \|\alpha_0\| = -\|u\| \alpha_0$. Then the following inequality holds:

$$\alpha_0^T h(x, t, \sigma, u) \leq -\beta_0 \|\alpha_0\| [\|u\| - \rho_0(x, t, \theta)]. \tag{15}$$

where $\rho_0$ is a known function, and $\theta$ is a constant which may be unknown. And $h$ is some function which will be determined later. Lemma 2 is useful in the proof of our main results. See Appendix for proof.

## IV. CONTROLLER DESIGN AND PERFORMANCE ANALYSIS

The control $u(t)$ is proposed to be

$$u(t) = p(x(t), t, \hat{\theta}(t), \epsilon(t)), \tag{16}$$

with

$$\begin{aligned} &p(x(t), t, \hat{\theta}(t), \epsilon(t)) \\ &\quad = -\Pi(x(t), t, \hat{\theta}(t)) s(x(t), t, \hat{\theta}(t), \epsilon(t)), \end{aligned} \tag{17}$$

$$\dot{\hat{\theta}}(t) = L \|\alpha(x(t), t)\| \frac{\partial \Pi}{\partial \hat{\theta}}(x(t), t, \hat{\theta}(t)) - \sigma_1 L \hat{\theta}(t), \tag{18}$$

$$\dot{\epsilon}(t) = -(\sigma_2 + \tau)\epsilon(t), \hat{\theta}(t_0) \in (0, \infty)^k, \epsilon(t_0) \in (0, \infty), \tag{19}$$

where $L \in \mathbb{R}^{k \times k}$ is diagonal with positive elements, $\sigma_1, \sigma_2$ and $\tau$ are positive constants, $\Pi(x, t, \hat{\theta}) = \rho(x, t, \hat{\theta})/(1 + c)$, and $s : \mathbb{R}^n \times \mathbb{R} \times (0, \infty)^{k+1} \to \mathbb{R}^m$ is any continuous function which satisfies

$$s(x, t, \hat{\theta}, \epsilon) \|\alpha(x, t)\| = \left\|s(x, t, \hat{\theta}, \epsilon)\right\| \alpha(x, t), \tag{20}$$

i.e., the two vectors have the same direction, and

$$\left\|\mu(x, t, \hat{\theta})\right\| > \epsilon^2 \Rightarrow s(x, t, \hat{\theta}, \epsilon) = \frac{\alpha(x, t)}{\|\alpha(x, t)\|}. \tag{21}$$

where $\mu(x, t, \hat{\theta}) = \Pi(x, t, \hat{\theta})\alpha(x, t)$. A particular example of such a function $s$ is

$$s(x, t, \hat{\theta}, \epsilon) = sat[\mu(x, t, \hat{\theta})/\epsilon^2], \tag{22}$$

where

$$sat(\eta) = \begin{cases} \eta, & \|\eta\| \leq 1, \\ \eta/\|\eta\|, & \|\eta\| > 1. \end{cases} \tag{23}$$

Based on the aforementioned design, the adaptive parameter $\hat{\theta}(t) > 0$ for all $t \geq t_0$. The main result of this paper can be summarized by the following theorem.

*Theorem 1:* Consider the system (5). Suppose that Assumptions A1-A3 hold. Then, under the adaptive robust controller (16)-(19), the state variables of the closed-loop systems are bounded in probability. Moreover, if a prior bound on the unknown constant $\theta$ is available, all the responses of the closed-loop system can converge to a small neighborhood of the origin by appropriately choosing the parameters $l_1, l_2, L, \sigma_1, \sigma_2$ and $\tau$.

*Proof:* The Lyapunov function candidate is

$$V(x, t, \hat{\theta} - \theta, \epsilon) = V_0(x, t) + V_1(\hat{\theta} - \theta) + \frac{1 + c}{2\tau}\epsilon^2, \tag{24}$$

where

$$V_1(\hat{\theta} - \theta) = \frac{1 + c}{2}(\hat{\theta} - \theta)^T L^{-1}(\hat{\theta} - \theta). \tag{25}$$

Consider the infinitesimal generator of $V$ along the trajectory of the controlled system:

$$\mathcal{L}V(x, t, \hat{\theta} - \theta, \epsilon) = \mathcal{L}V_0(x, t) + \dot{V}_1(\hat{\theta} - \theta) + \tfrac{(1+c)}{\tau}\dot{\epsilon}\epsilon, \tag{26}$$

where

$$\begin{aligned} \mathcal{L}V_0(x, t) = &\frac{\partial V_0}{\partial t} + \nabla_x^T V_0(x, t) f(x, t, \sigma) \\ &+ \frac{1}{2} Tr\{g^T(x, t)\frac{\partial^2 V_0(x, t)}{\partial x^2} g(x, t)\} \\ &+ \nabla_x^T V_0(x, t) B\{[I + E(x, t, \sigma)] \\ &\times p(x, t, \hat{\theta}, \epsilon) + e(x, t, \sigma)\}. \end{aligned} \tag{27}$$

We denote the last term as $\Omega$:

$$\begin{aligned} \Omega :&= \nabla_x^T V_0(x, t) B\{[I + E(x, t, \sigma)] p(x, t, \hat{\theta}, \epsilon) \\ &\quad + e(x, t, \sigma)\} \\ &= \alpha^T\{[I + E(x, t, \sigma)] p(x, t, \hat{\theta}, \epsilon) + e(x, t, \sigma)\}. \end{aligned} \tag{28}$$

In view of (20), one has

$$\begin{aligned} &-\Pi(x, t, \hat{\theta}) s(x, t, \hat{\theta}, \epsilon) \|\alpha(x, t)\| \\ &\quad = -\Pi(x, t, \hat{\theta}) \left\|s(x, t, \hat{\theta}, \epsilon)\right\| \alpha(x, t), \end{aligned} \tag{29}$$

which, along with (17), implies that

$$p(x, t, \hat{\theta}, \epsilon) \|\alpha(x, t)\| = -\left\|p(x, t, \hat{\theta}, \epsilon)\right\| \alpha(x, t). \tag{30}$$

Letting

$$h(x, t, \sigma, u) := [I + E(x, t, \sigma)]u(t) + e(x, t, \sigma),$$

and combining with Assumption 2 (i), we have

$$\begin{aligned} u^T h(x, t, \sigma, u) &= u^T [I + E(x, t, \sigma)]u + u^T e(x, t, \sigma) \\ &\geq (1 + c)\|u\|^2 - \|u\| \rho(x, t, \theta) \\ &= (1 + c)\|u\| [\|u\| - \rho(x, t, \theta)/(1 + c)]. \end{aligned} \tag{31}$$

Using Lemma 2, (28) becomes

$$\begin{aligned} \Omega &\leq -(1 + c)\|\alpha(x, t)\| [\left\|p(x, t, \hat{\theta}, \epsilon)\right\| \\ &\quad - \rho(x, t, \theta)/(1 + c)] \\ &= (1 + c)\|\alpha(x, t)\| [\rho(x, t, \theta)/(1 + c) \\ &\quad - \Pi(x, t, \hat{\theta})] \\ &\quad + (1 + c)\|\alpha(x, t)\| [\Pi(x, t, \hat{\theta}) \\ &\quad - \left\|p(x, t, \hat{\theta}, \epsilon)\right\|]. \end{aligned} \tag{32}$$

Since $\Pi(x,t,\cdot)$ is concave for all $(x,t)\in\mathbb{R}^n\times\mathbb{R}$, we note

$$\Pi(x,t,\theta)-\Pi(x,t,\hat{\theta})\le-\frac{\partial\Pi^T}{\partial\hat{\theta}}(x,t,\hat{\theta})(\hat{\theta}-\theta), \quad (33)$$

For the first term on the right-hand side (RHS) of (32), we have

$$\begin{aligned}&(1+c)\|\alpha(x,t)\|\,[\rho(x,t,\theta)/(1+c)-\Pi(x,t,\hat{\theta})]\\&\quad=(1+c)\|\alpha(x,t)\|\,[\Pi(x,t,\theta)-\Pi(x,t,\hat{\theta})]\\&\quad\le-(1+c)\|\alpha(x,t)\|\frac{\partial\Pi^T}{\partial\hat{\theta}}(x,t,\hat{\theta})(\hat{\theta}-\theta). \end{aligned}\quad (34)$$

Now consider the second term on the RHS of (32). It follows from (21), (22) and (23) that, if $\|\mu\|>\epsilon^2$, $\left\|p(x,t,\hat{\theta},\epsilon)\right\|=\Pi(x,t,\hat{\theta})$ and

$$(1+c)\|\alpha(x,t)\|\,[\Pi(x,t,\hat{\theta})-\left\|p(x,t,\hat{\theta},\epsilon)\right\|]=0. \quad (35)$$

If $\|\mu\|\le\epsilon^2$, then

$$\begin{aligned}&(1+c)\|\alpha(x,t)\|\,[\Pi(x,t,\hat{\theta})-\left\|p(x,t,\hat{\theta},\epsilon)\right\|]\\&\quad\le(1+c)\|\alpha(x,t)\|\,\Pi(x,t,\hat{\theta})\\&\quad=(1+c)\left\|\mu(x,t,\hat{\theta})\right\|\\&\quad\le(1+c)\epsilon^2. \end{aligned}\quad (36)$$

Consequently,

$$(1+c)\|\alpha(x,t)\|\,[\Pi(x,t,\hat{\theta})-\left\|p(x,t,\hat{\theta},\epsilon)\right\|]\le(1+c)\epsilon^2. \quad (37)$$

Combining (34) and (37), one can obtain

$$\Omega\le-(1+c)\|\alpha(x,t)\|\,(\hat{\theta}-\theta)^T\frac{\partial\Pi}{\partial\hat{\theta}}(x,t,\hat{\theta})+(1+c)\epsilon^2. \quad (38)$$

Concerning $V_1$, it follows from (18) and (25) that

$$\begin{aligned}\dot{V}_1(\hat{\theta}-\theta)&=\frac{\partial V_1}{\partial\hat{\theta}}(\hat{\theta}-\theta)\dot{\hat{\theta}}\\&=(1+c)(\hat{\theta}-\theta)^TL^{-1}(L\|\alpha(x,t)\|\\&\quad\times\frac{\partial\Pi}{\partial\hat{\theta}}(x,t,\hat{\theta})-\sigma_1L\hat{\theta}). \end{aligned}\quad (39)$$

Substituting (11), (19), (38) and (39) into (26) yields

$$\begin{aligned}&\mathcal{L}V(x,t,\hat{\theta}-\theta,\epsilon)\\&\quad\le-l_1V_0(x,t)+1/l_2\\&\qquad-(1+c)\|\alpha(x,t)\|\,(\hat{\theta}-\theta)^T\frac{\partial\Pi}{\partial\hat{\theta}}(x,t,\hat{\theta})\\&\qquad+(1+c)\epsilon^2\\&\qquad+(1+c)(\hat{\theta}-\theta)^T\|\alpha(x,t)\|\frac{\partial\Pi}{\partial\hat{\theta}}(x,t,\hat{\theta})\\&\qquad-(1+c)(\hat{\theta}-\theta)^T\sigma_1\hat{\theta}\\&\qquad-(1+c)\epsilon^2-\frac{(1+c)\sigma_2}{\tau}\epsilon^2\\&\quad=-l_1V_0(x,t)+1/l_2-(1+c)\sigma_1(\hat{\theta}-\theta)^T\hat{\theta}\\&\qquad-\frac{(1+c)\sigma_2}{\tau}\epsilon^2. \end{aligned}\quad (40)$$

Denote $\tilde{\theta}=\hat{\theta}-\theta$, then it follows

$$-\bar{\sigma}_1\tilde{\theta}^T\hat{\theta}\le-\frac{\bar{\sigma}_1}{2}\left\|\tilde{\theta}\right\|^2+\frac{\bar{\sigma}_1}{2}\|\theta\|^2. \quad (41)$$

Finally, we have

$$\begin{aligned}\mathcal{L}V(x,t,\hat{\theta}-\theta,\epsilon)&\le-l_1V_0(x,t)-\frac{\bar{\sigma}_1}{2}\left\|\tilde{\theta}\right\|^2\\&\quad-\bar{\sigma}_2\epsilon^2+\frac{\bar{\sigma}_1}{2}\|\theta\|^2+1/l_2. \end{aligned}\quad (42)$$

where $\bar{\sigma}_1=(1+c)\sigma_1$, $\bar{\sigma}_2=\frac{(1+c)\sigma_2}{\tau}$ and $\kappa=\min\{l_1,\frac{\bar{\sigma}_1}{\lambda_{\max}(L)^{-1}(1+c)},\bar{\sigma}_2\}$. Therefore we can prove

$$\begin{aligned}\mathcal{L}V(x,t,\hat{\theta}-\theta,\epsilon)&\le-\kappa V(x,t,\hat{\theta}-\theta,\epsilon)\\&\quad+(\frac{\bar{\sigma}_1}{2}\|\theta\|^2+1/l_2). \end{aligned}\quad (43)$$

Based on Lemma 1, if a prior bound on the unknown constant $\theta$ is available, the state variables of the closed-loop system are bounded in probability. Moreover, all the responses of the closed-loop system can converge to a small neighborhood of the origin by appropriately choosing the parameters $l_1, l_2, L, \sigma_1, \sigma_2$ and $\tau$. Q.E.D.

*Remark 5:* In (16) (17), the upper bound function represents the magnitude of the controller, and the saturation function shows its direction. And neither the uncertain parameter nor the bound of uncertainty is known while constructing the control. The control design is based on the minimum information of uncertainty.

*Remark 6:* From (43), the bound of the closed-loop system trajectory can be tuned arbitrarily small when we choose $\sigma_1$ that is small enough and $l_2$ that is large enough at the same time.

Next we will take another control design into condition. The controller is given by

$$u(t)=p(x(t),t,\hat{\theta}(t)), \quad (44)$$

with

$$p(x(t),t,\hat{\theta}(t))=-\Pi(x(t),t,\hat{\theta}(t))s(x(t),t,\hat{\theta}(t)), \quad (45)$$

$$\begin{aligned}\dot{\hat{\theta}}(t)&=\bar{L}\,\|\alpha(x(t),t)\|\frac{\partial\Pi}{\partial\hat{\theta}}(x(t),t,\hat{\theta}(t))-\sigma\bar{L}\hat{\theta}(t),\\\hat{\theta}(t_0)&\in(0,\infty)^k, \end{aligned}\quad (46)$$

where $\bar{L}\in\mathbb{R}^{k\times k}$ is diagonal with positive elements, $\sigma$ is a positive constant, and $\Pi(x,t,\hat{\theta})=\rho(x,t,\hat{\theta})/(1+c)$,

$$s(x,t,\hat{\theta})=\begin{cases}\mu/\bar{\epsilon}^2, & \|\mu\|\le\bar{\epsilon}^2,\\ \mu/\|\mu\|, & \|\mu\|>\bar{\epsilon}^2.\end{cases} \quad (47)$$

with $\mu(x,t,\hat{\theta})=\Pi(x,t,\hat{\theta})\alpha(x,t)$. And $\bar{\epsilon}$ is a positive constant.

*Theorem 2:* Consider the system (5). Suppose Assumptions A1-A3 hold. Then, under the adaptive robust controller (44)-(46), the state variables of the closed-loop system are bounded in probability. Moreover, if a prior bound on the unknown constant $\theta$ is available, all the responses of the closed-loop system can be regulated into a small neighbourhood of the origin by appropriately choosing the parameters $l_1, l_2, \bar{L}, \sigma$ and $\bar{\epsilon}$.

*Proof:* Consider the Lyapunov function candidate

$$V(x, t, \hat{\theta} - \theta) = V_0(x, t) + V_1(\hat{\theta} - \theta), \tag{48}$$

where

$$V_1(\hat{\theta} - \theta) = \frac{1 + c}{2}(\hat{\theta} - \theta)^T \bar{L}^{-1}(\hat{\theta} - \theta). \tag{49}$$

Consider the infinitesimal generator of $V$, it follows from (48) that

$$\mathcal{L}V(x, t, \hat{\theta} - \theta) = \mathcal{L}V_0(x, t) + \dot{V}_1(\hat{\theta} - \theta). \tag{50}$$

Substituting (11), (44)-(46) and (49) into the expression of $\mathcal{L}V$, yields

$$\begin{aligned}
&\mathcal{L}V(x, t, \hat{\theta} - \theta) \\
&\quad \le -l_1 V_0(x, t) + 1/l_2 \\
&\qquad - (1 + c)\,\|\alpha(x, t)\|\,(\hat{\theta} - \theta)^T \frac{\partial \Pi}{\partial \hat{\theta}}(x, t, \hat{\theta}) \\
&\qquad + (1 + c)\bar{\epsilon}^2 \\
&\qquad + (1 + c)(\hat{\theta} - \theta)^T\,\|\alpha(x, t)\|\,\frac{\partial \Pi}{\partial \hat{\theta}}(x, t, \hat{\theta}) \\
&\qquad - (1 + c)(\hat{\theta} - \theta)^T \sigma \hat{\theta} \\
&\quad = -l_1 V_0(x, t) + 1/l_2 \\
&\qquad - (1 + c)\sigma(\hat{\theta} - \theta)^T \hat{\theta} + (1 + c)\bar{\epsilon}^2.
\end{aligned} \tag{51}$$

Applying the Young's inequality, we have

$$\begin{aligned}
\mathcal{L}V(x, t, \hat{\theta} - \theta) &\le -\pi V(x, t, \hat{\theta} - \theta) \\
&\quad + (\frac{\bar{\sigma}}{2}\,\|\theta\|^2 + 1/l_2 + (1 + c)\bar{\epsilon}^2).
\end{aligned} \tag{52}$$

where $\bar{\sigma} = (1 + c)\sigma$ and $\pi = \min\{l_1, \frac{\sigma}{\lambda_{\max}(\bar{L})^{-1}(1+c)}\}$. Q.E.D.

*Remark 7:* By comparing Theorem 1 with Theorem 2, it is concluded that when $\bar{\epsilon}$ is time-varying, the control effect is better than when it is constant, due to the smaller boundedness region.

*Remark 8:* When the low triangle system or higher-order system is considered, these results still hold and the boundedness in the mean square can be obtained.

## V. SPECIALIZATION TO LINEAR SYSTEMS

Consider the special case when the stochastic system is linear:

$$\begin{aligned}
dx(t) = \{&[A + \Delta A(\sigma(t)]x(t) \\
&+ [B + \Delta B(\sigma(t)]u(t)\}dt + Gx(t)dw(t),
\end{aligned} \tag{53}$$

where $A$ is Hurwitz, $\sigma$ is an unknown parameter, the matrices $A$, $\Delta A$, $B$, $\Delta B$, and $G$ are of appropriate dimensions. The counter part of Assumption 2 is that there are $e(\sigma)$ and $E(\sigma)$ such that

$$\begin{aligned}
\Delta A(\sigma) &= Be(\sigma), \\
\Delta B(\sigma) &= BE(\sigma),
\end{aligned} \tag{54}$$

and

$$\min_{\sigma \in \Sigma} \frac{1}{2}\lambda_{\min}\{E^T(\sigma) + E(\sigma)\} > \tilde{c} > -1, \tag{55}$$

$$\|e(\sigma)\| \le \rho(\theta). \tag{56}$$

where $\rho$ is concave on $\theta$.

Let $Q \in \mathbb{R}^{n \times n}$ be a positive-definite symmetric matrix. Let also $V_0(x) = x^T Qx$. By the Rayleigh's principle,

$$\lambda_{\min}(Q)\,\|x\|^2 \le x^T Qx \le \lambda_{\max}(Q)\,\|x\|^2. \tag{57}$$

Consider the matrix equation $H = QA + A^T Q + G^T QG$. We have

$$\begin{aligned}
\mathcal{L}V_0(x) &= x^T(QA + A^T Q + G^T QG)x \\
&\le -\varphi V_0(x).
\end{aligned} \tag{58}$$

where $\varphi = \lambda_{\max}(H)\lambda_{\max}(Q)$.

We propose the controller as

$$u(t) = p(x(t), \hat{\theta}(t), \tilde{\epsilon}(t)), \tag{59}$$

with

$$\begin{aligned}
&p(x(t), \hat{\theta}(t), \tilde{\epsilon}(t)) \\
&\quad = -\Pi(\hat{\theta}(t))s(x(t), \hat{\theta}(t), \tilde{\epsilon}(t)),
\end{aligned} \tag{60}$$

$$\dot{\hat{\theta}}(t) = 2\tilde{L}\frac{\partial \Pi^T}{\partial \hat{\theta}}(\hat{\theta}(t))\left\|B^T Qx(t)\right\| - \tilde{\sigma}_1 \tilde{L}\hat{\theta}(t), \tag{61}$$

$$\begin{aligned}
&\dot{\tilde{\epsilon}}(t) = -(\tilde{\tau} + \tilde{\sigma}_2)\tilde{\epsilon}(t), \\
&\hat{\theta}(t_0) \in (0, \infty)^k, \epsilon(t_0) \in (0, \infty),
\end{aligned} \tag{62}$$

where

$$\Pi(\hat{\theta}) = \rho(\hat{\theta})/(1 + \tilde{c}), \tag{63}$$

$$\mu(x, \hat{\theta}) = 2[\rho(\hat{\theta})/(1 + \tilde{c})]B^T Qx, \tag{64}$$

$$s(x, \hat{\theta}, \tilde{\epsilon}) = sat[\mu(x, \hat{\theta})/\tilde{\epsilon}^2]. \tag{65}$$

As in Theorem 1, all responses of the closed-loop system are bounded in probability. The control is more structured: $\mu$ is autonomous and linear in $x$. Moreover, the effects of $x$ and $\hat{\theta}$ in $\mu$ are decoupled. Furthermore, in comparing with the nonlinear case, the choice of $V_0(x, t)$ can be autonomous and $\mathcal{L}V_0(x, t)$ is obtained via a matrix equation.

In the past, the most advanced control design only embraced portion of them. And even for that, the control needed to know the bound of uncertainty. Our control schemes, on the other hand, do not need to know the bound of uncertainty. They also capture the full spectrum of varieties of uncertainty. The control scheme is applicable to many systems and requires only the minimum structural information of the uncertainty.

## VI. ILLUSTRATIVE EXAMPLE

Consider the two-tanks water level system. The model is given by

$$\begin{aligned}
&C_1(t)dh_1(t) \\
&\quad = (r_1(t) - \frac{h_1(t)}{R_1(t)})dt + C_1(t)\beta_1\frac{h_1^*}{h_1(t)}sin(2t)dw(t),
\end{aligned} \tag{66}$$

$$C_2(t)dh_2(t) = (r_2(t) + \frac{h_1(t)}{R_1(t)} - \frac{h_2(t)}{R_2(t)})dt + C_2(t)\beta_2 \frac{h_2^*}{h_2(t)} \times cos(5t)dw(t), \quad (67)$$

where $h_i(t),\ i = 1, 2$ is the water level height of tank $i$, $C_i(t)$ is the capacitance of tank $i$, $R_i(t)$ is the resistance of valve $i$, and $r_i(t)$ is the inflow rate to the tank $i$ ($r_i(t) > 0$ means to add water and $r_i(t) < 0$ means to drain the tank). We treat $r_{1,2}(t)$ as the control input. The control task is to design $r_i(t),\ i = 1,\ 2$, such that $h_i(t)$ is driven to be close to a desired value $h_i^* > 0$. $\beta_i,\ i = 1, 2$ is a positive constant. We consider that $C_i(t), R_i(t), i = 1, 2$ are unknown. However, their possible bounds are known *a priori*. Let $C_i^0,\ R_i^0,\ i = 1,\ 2$ be the nominal values of $C_i(t),\ R_i(t),\ i = 1, 2$. There are known strictly positive constants $\underline{C}_i,\ \bar{C}_i,\ \underline{R}_i$ and $\bar{R}_i$ with

$$0 < \underline{C}_i \le C_i(t) \le \overline{C}_i, \quad (68)$$

$$0 < \underline{R}_i \le R_i(t) \le \overline{R}_i, \quad i = 1,\ 2. \quad (69)$$

We note that the strict positiveness is assured by the physical implications of the capacitance and resistance. While designing the control, in addition to regulating $h_i$ to be close to $h_i^*$, it is also realistic that $h_i(t) > 0$ for all $t \ge t_0$ for otherwise (that is: $h_i(t) < 0$) we lose the physical meaning. This positive-constraint is very common in many practical systems, yet it has rarely been addressed.

In order to ensure $h_i(t) > 0$ for all time and $h_i(t)$ will be close to $h_i^*$, we creatively propose the following transformation:

Let

$$y_i = ln(\frac{h_i}{h_i^*}), \quad i = 1,\ 2. \quad (70)$$

The transformation is bijective (i.e., one-to-one) and smooth. The implication of this transformation is that $y_i = 0$ as $h_i = h_i^*$. Furthermore, the boundedness of $y_i$ means $h_i > 0$. The systems (66) and (67) can be represented in terms of $y_i$:

$$dy_1(t) = (-\frac{1}{C_1(t)R_1(t)} + \frac{1}{C_1(t)}\frac{1}{h_1^* e^{y_1(t)}} r_1(t))dt + \beta_1 sin(2t)e^{-y_1(t)}dw(t), \quad (71)$$

$$dy_2(t) = (\frac{1}{C_2(t)R_1(t)}\frac{h_1^* e^{y_1(t)}}{h_2^* e^{y_2(t)}} - \frac{1}{C_2(t)R_2(t)} + \frac{1}{C_2(t)}\frac{1}{h_2^* e^{y_2(t)}} r_2(t))dt + \beta_2 cos(5t)e^{-y_2(t)}dw(t). \quad (72)$$

The systems above in turn are represented by

$$dy_1(t) = [\frac{1}{C_1(t)R_1(t)}(e^{-y_1(t)} - 1) - \frac{1}{C_1(t)R_1(t)}e^{-y_1(t)} + \frac{1}{C_1(t)}\frac{1}{h_1^* e^{y_1(t)}} r_1(t)]dt + \beta_1 sin(2t)e^{-y_1(t)}dw(t), \quad (73)$$

$$dy_2(t) = [\frac{1}{C_2(t)R_2(t)}(e^{-y_2(t)} - 1) - \frac{1}{C_2(t)R_2(t)}e^{-y_2(t)} + \frac{1}{C_2(t)R_1(t)}\frac{h_1^* e^{y_1(t)}}{h_2^* e^{y_2(t)}} + \frac{1}{C_2(t)}\frac{1}{h_2^* e^{y_2(t)}} r_2(t)]dt + \beta_2 cos(5t)e^{-y_2(t)}dw(t). \quad (74)$$

The systems (73) and (74) is in the form of (5) by taking $y = [y_1, y_2]^T,\ u = [r_1, r_2]^T$,

$$f(y, t, \sigma) = \begin{bmatrix} \frac{1}{C_1R_1}(e^{-y_1} - 1) \\ \frac{1}{C_2R_2}(e^{-y_2} - 1) \end{bmatrix},$$

$$B(y, t) = \begin{bmatrix} \frac{1}{C_1^0 h_1^* e^{y_1}} & 0 \\ 0 & \frac{1}{C_2^0 h_2^* e^{y_2}} \end{bmatrix},$$

$$E(y, t, \sigma) = \begin{bmatrix} \frac{C_1^0 - C_1}{C_1} & 0 \\ 0 & \frac{C_2^0 - C_2}{C_2} \end{bmatrix},$$

$$g(y, t) = \begin{bmatrix} \beta_1 sin(2t)e^{-y_1(t)} \\ \beta_2 cos(5t)e^{-y_2(t)} \end{bmatrix},$$

$$\Delta f(y, t, \sigma) = \begin{bmatrix} -\frac{1}{C_1R_1}e^{-y_1} \\ -\frac{1}{C_2R_2}e^{-y_2} + \frac{1}{C_2R_1}\frac{h_1^* e^{y_1}}{h_2^*.e^{y_2}} \end{bmatrix},$$

$$\Delta B(y, t, \sigma) = \begin{bmatrix} \frac{C_1^0 - C_1}{C_1C_1^0}\frac{1}{h_1^* e^{y_1}} & 0 \\ 0 & \frac{C_2^0 - C_2}{C_2C_2^0}\frac{1}{h_2^* e^{y_2}} \end{bmatrix}. \quad (75)$$

Assumption 3 is met by taking

$$V_0(y) = \sum_{i=1}^{2} \frac{1}{2}(e^{y_i} - 1)^2. \quad (76)$$

This can be easily verified by first taking

$$\gamma_1(\eta) = \min_y\{\sum_{i=1}^{2} \frac{1}{2}(e^{y_i} - 1)^2 |\ \eta = \|y\|\}, \quad (77)$$

$$\gamma_2(\eta) = \max_y\{\sum_{i=1}^{2} \frac{1}{2}(e^{y_i} - 1)^2 |\ \eta = \|y\|\}. \quad (78)$$

Since $\frac{C_i^0}{C_i} > 0$, we have

$$\min\left\{\frac{C_1^0 - C_1}{C_1},\ \frac{C_2^0 - C_2}{C_2}\right\} =: c > -1. \quad (79)$$

After some simple calculations, one obtains

$$e(y, t, \sigma) = \begin{bmatrix} -\frac{C_1^0 h_1^*}{C_1R_1} \\ -\frac{C_2^0 h_2^*}{C_2R_2} + \frac{C_2^0 h_1^* e^{y_1}}{C_2R_1} \end{bmatrix}. \quad (80)$$

Letting $\theta = \max_{i,\, j=1,\, 2}\{(\frac{1}{\underline{C}_i\underline{R}_j})^2\}$, it yields

$$\|e\| \leq \rho(y, \theta), \tag{81}$$

where

$$\rho(y, \theta) = \sqrt{\theta} \cdot \sqrt{(C_1^0 h_1^*)^2 + (C_2^0 h_1^* e^{y_1} - C_2^0 h_2^*)^2}, \tag{82}$$

$$\Pi = \rho(y, \theta)/(1 + c). \tag{83}$$

$\rho(y, \cdot)$ is $C^1$ and nondecreasing, and $\rho(y, \cdot)$ is concave about $\theta$. $B$, $\Pi$ and $\partial\Pi/\partial\theta$ are continuous.
Furthermore, we have

$$\begin{aligned}\mathcal{L}V_0(y) &= \nabla_y^T V_0(y) f(y, t, \sigma) \\ &\quad + \frac{1}{2}Tr\{g^T(y, t)\frac{\partial^2 V_0(y)}{\partial y^2}g(y, t)\} \\ &= \sum_{i=1}^{2}(e^{y_i} - 1)\frac{1}{C_i R_i}e^{y_i}(e^{-y_i} - 1) \\ &\quad + \frac{1}{2}\sum_{i=1}^{2} g_i^2(2e^{2y_i} - e^{y_i}).\end{aligned} \tag{84}$$

Considering Assumption 3, we obtain

$$\mathcal{L}V_0(y) \leq -2V_0(y) + (\beta_1^2 + \beta_2^2). \tag{85}$$

The candidate Lyapunov function can be chosen as

$$V(y, \hat{\theta} - \theta, \epsilon) = V_0(y) + V_1(\hat{\theta} - \theta) + \frac{1 + c}{\tau}\epsilon^2, \tag{86}$$

where

$$V_1(\hat{\theta} - \theta) = \frac{1 + c}{2}(\hat{\theta} - \theta)^T L^{-1}(\hat{\theta} - \theta). \tag{87}$$

The adaptive robust controller can be given as

$$\begin{aligned}u(t) &= \begin{bmatrix} r_1(t) \\ r_2(t)\end{bmatrix} \\ &= -\Pi(y(t), \hat{\theta}(t))s(y(t), \hat{\theta}(t), \epsilon(t)),\end{aligned} \tag{88}$$

with

$$\Pi(y, \hat{\theta}) = \frac{\sqrt{\hat{\theta}} \cdot \sqrt{(C_1^0 h_1^*)^2 + (C_2^0 h_1^* e^{y_1} - C_2^0 h_2^*)^2}}{1 + c}, \tag{89}$$

$$\alpha(y) = \begin{bmatrix} \frac{1}{C_1^0 h_1^*}(e^{y_1} - 1) \\ \frac{1}{C_2^0 h_2^*}(e^{y_2} - 1)\end{bmatrix}, \tag{90}$$

$$\begin{aligned}\mu(y, \hat{\theta}) &= \Pi(y, \hat{\theta})\alpha(y) \\ &= \begin{bmatrix}\mu_1(y, \hat{\theta}) \\ \mu_2(y, \hat{\theta})\end{bmatrix} \\ &= \frac{\sqrt{\hat{\theta}} \cdot \sqrt{(C_1^0 h_1^*)^2 + (C_2^0 h_1^* e^{y_1} - C_2^0 h_2^*)^2}}{1 + c} \\ &\quad \times \begin{bmatrix} \frac{1}{C_1^0 h_1^*}(e^{y_1} - 1) \\ \frac{1}{C_2^0 h_2^*}(e^{y_2} - 1)\end{bmatrix},\end{aligned} \tag{91}$$

and

$$s(y, \hat{\theta}, \epsilon) = sat[\mu(y, \hat{\theta})/\epsilon^2]. \tag{92}$$

The controllers are established as

$$r_1(t) = \begin{cases} -\frac{\Pi^2}{\epsilon(t)^2} \cdot \frac{1}{C_1^0 h_1^*}(e^{y_1(t)} - 1), & \|\mu_1\| \leq \epsilon^2, \\ -sgn[\Pi \cdot \frac{1}{C_1^0 h_1^*}(e^{y_1(t)} - 1)]\Pi, & \|\mu_1\| > \epsilon^2. \end{cases} \tag{93}$$

$$r_2(t) = \begin{cases} -\frac{\Pi^2}{\epsilon(t)^2} \cdot \frac{1}{C_2^0 h_2^*}(e^{y_2(t)} - 1), & \|\mu_2\| \leq \epsilon^2, \\ -sgn[\Pi \cdot \frac{1}{C_2^0 h_2^*}(e^{y_2(t)} - 1)]\Pi, & \|\mu_2\| > \epsilon^2. \end{cases} \tag{94}$$

Additionally, we know

$$\frac{\partial\Pi}{\partial\hat{\theta}} = \frac{\sqrt{(C_1^0 h_1^*)^2 + (C_2^0 h_1^* e^{y_1} - C_2^0 h_2^*)^2}}{2(1 + c)\sqrt{\hat{\theta}}}, \tag{95}$$

$$\|\alpha\| = \sqrt{(\frac{e^{y_1} - 1}{C_1^0 h_1^*})^2 + (\frac{e^{y_2} - 1}{C_2^0 h_2^*})^2}. \tag{96}$$

So the adaptive laws of parameter are as follows

$$\begin{aligned}\dot{\hat{\theta}}(t) &= \frac{L\sqrt{(C_1^0 h_1^*)^2 + (C_2^0 h_1^* e^{y_1(t)} - C_2^0 h_2^*)^2}}{1 + c} \\ &\quad \times \sqrt{(\frac{e^{y_1(t)} - 1}{C_1^0 h_1^*})^2 + (\frac{e^{y_2(t)} - 1}{C_2^0 h_2^*})^2} \\ &\quad - \sigma_1 L\hat{\theta}(t), \\ \dot{\epsilon}(t) &= -(\tau + \sigma_2)\epsilon(t).\end{aligned} \tag{97}$$

For simulation purpose, we choose $C_1^0 = C_2^0 = 1$, $R_1^0 = 1$, $R_2^0 = 0.5$, $\Delta C_1 = 0.2\sin(7t)$, $\Delta C_2 = 0.2\sin(8t)$, $\Delta R_1 = 0.2\sin(5t)$, $\Delta R_2 = 0.2\cos(10t)$, $\underline{C}_1 = \underline{C}_2 = 0.8$, $\bar{C}_1 = \bar{C}_2 = 1.2$, $\underline{R}_1 = \underline{R}_2 = 0.8$, $\bar{R}_1 = \bar{R}_2 = 1.2$, $L = 1$, $\tau = 1$, $\sigma_1 = 60$, $\sigma_2 = 1$, $c = -0.17$, $\beta_1 = \beta_2 = 0.1$, and $y_1(0) = 2$, $y_2(0) = -2$, $\hat{\theta}(0) = 0.5$, $\epsilon(0) = 0.5$ are the initial conditions.

Hence, we obtain the controller

$$\mu_1(y, \hat{\theta}) = 1.2\sqrt{\hat{\theta}} \cdot \sqrt{1 + (e^{y_1} - 1)^2} \cdot (e^{y_1} - 1), \tag{98}$$

$$r_1(t) = \begin{cases} -\frac{\hat{\theta}(t)[1 + (e^{y_1(t)} - 1)^2](e^{y_1(t)} - 1)}{0.69\epsilon(t)^2}, \\ \qquad \|\mu_1\| \leq \epsilon^2, \\ -sgn[1.2\sqrt{\hat{\theta}(t)} \cdot \sqrt{1 + (e^{y_1(t)} - 1)^2} \\ \times(e^{y_1(t)} - 1)]\frac{\sqrt{\hat{\theta}(t)} \cdot \sqrt{1 + (e^{y_1} - 1)^2}}{0.83}, \\ \qquad \|\mu_1\| > \epsilon^2. \end{cases} \tag{99}$$

$$\mu_2(y, \hat{\theta}) = 1.2\sqrt{\hat{\theta}} \cdot \sqrt{1 + (e^{y_1} - 1)^2} \cdot (e^{y_2} - 1), \tag{100}$$

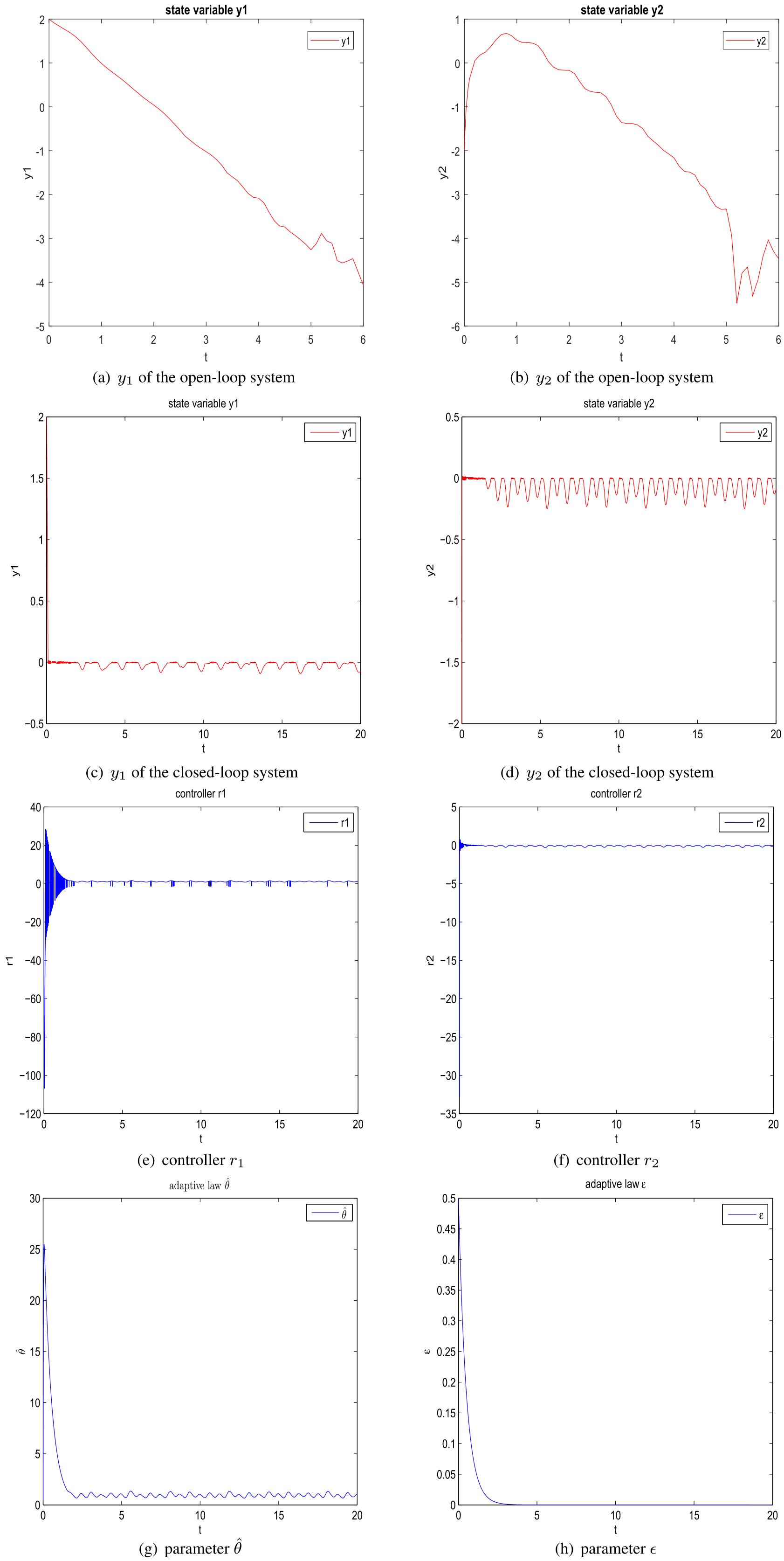


**FIGURE 1.** **(a) $y_1$ of the open-loop system. (b) $y_2$ of the open-loop system. (c) $y_1$ of the closed-loop system. (d) $y_2$ of the closed-loop system. (e) Controller $r_1$. (f) Controller $r_2$. (g) Parameter $\hat{\theta}$. (h) Parameter $\epsilon$.**

$$r_2(t)=\begin{cases}-\dfrac{\hat{\theta}(t)[1+(e^{y_1(t)}-1)^2](e^{y_2(t)}-1)}{0.69\epsilon(t)^2}, \\ \qquad \|\mu_2\| \le \epsilon^2, \\ -sgn[1.2\sqrt{\hat{\theta}(t)}\cdot\sqrt{1+(e^{y_1(t)}-1)^2} \\ \times(e^{y_2(t)}-1)]\dfrac{\sqrt{\hat{\theta}(t)}\cdot\sqrt{1+(e^{y_1(t)}-1)^2}}{0.83}, \\ \qquad \|\mu_2\| > \epsilon^2. \end{cases} \tag{101}$$

The adaptive laws are given as

$$\dot{\hat{\theta}}(t) = 1.2\sqrt{1+(e^{y_1(t)}-1)^2} \times\sqrt{(e^{y_1(t)}-1)^2+(e^{y_2(t)}-1)^2}-\hat{\theta}(t), \tag{102}$$

$$\dot{\epsilon}(t) = -2\epsilon(t). \tag{103}$$

The results of simulation are as follows. The magnitude of $y_i, i=1,2$ reflects the relation between $h_i$ and $h_i^*$. As time goes on, Fig. 1(a) and Fig. 1(b) show $y_i,\ i=1,2$ is divergent, which means $h_i \to h_i^*$ does not hold. From Fig. 1(c) and Fig. 1(d), we can see $y_i,\ i=1,2$ are bounded in probability. This in turns means $h_i$ will be close to $h_i^*$ and remain close thereafter. From Fig. 1(e) and Fig. 1(f), the controls $r_i, i=1,2$, remains bounded for all time. Furthermore, the control magnitude stays very small after the transient period. These demonstrate both efficiency and cost-effectiveness of the adaptive robust control. From Fig. 1(g) and Fig. 1(h), we can see that the other responses $\hat{\theta}(t)$ and $\epsilon(t)$ are also bounded in probability after about 3 seconds.

## VII. CONCLUSION

In this paper, we have considered the adaptive robust control problem for a class of stochastic nonlinear systems with three uncertainties. The nonlinear uncertainty is bounded in probability. Under the matched conditions, a known function $\rho$ with an unknown parameter $\theta$, is introduced to bound the uncertain $e$. The upper bound $\rho$ satisfies certain concave condition. And the external disturbance is standard Wiener Process. Based on stochastic Lyapunov stability theory, the adaptive robust controller has been designed, which renders the closed-loop system bounded in probability, regardless of all the uncertainties. As a result, neither $\sigma$ nor $\theta$ is known when constructing the control. The desired controller is constructed by the upper bound function and the saturation function, in which the upper bound function represents the magnitude of the control, while the saturation function indicates the control direction. The design of desired controllers is based on the minimum information of the uncertainty.

## APPENDIX

## THE PROOF OF LEMMA 2

*Proof:* Note that if $\alpha_0 \in R^m$ and $\gamma > 0$, then, utilizing the inequality (14),

$$\begin{aligned}&\alpha_0^T h(x,t,\sigma,-\gamma\alpha_0)\\ &= -\frac{1}{\gamma}(-\gamma\alpha_0)^T h(x,t,\sigma,-\gamma\alpha_0)\\ &\le -\frac{1}{\gamma}\beta_0\,|-\gamma\alpha_0|\,[|-\gamma\alpha_0|-\rho_0(x,t,\theta)]\\ &= -\beta_0\,|\alpha_0|\,[\gamma\,|\alpha_0|-\rho_0(x,t,\theta)],\end{aligned} \tag{104}$$

that is

$$\alpha_0^T h(x,t,\sigma,-\gamma\alpha_0) \le -\beta_0\,|\alpha_0|\,[\gamma\,|\alpha_0|-\rho_0(x,t,\theta)] \tag{105}$$

Since (106) is satisfied for all $\gamma > 0$ and each side is continuous in $\gamma$, it also holds for $\gamma=0$.
Now consider any two vectors $u, \alpha_0 \in R^m$ which satisfy

$$u\,|\alpha_0| = -\,|u|\,\alpha_0 \tag{106}$$

If $\alpha_0 \neq 0$, then $u=-\gamma\alpha_0$ where $\gamma = |u|\,/\,|\alpha_0|$, and using (106),

$$\begin{aligned}\alpha_0^T h(x,t,\sigma,u) &\le -\beta_0\,|\alpha_0|\,[(|u|\,/\,|\alpha_0|)\,|\alpha_0|-\rho_0(x,t,\theta)]\\ &= -\beta_0\,|\alpha_0|\,[|u|-\rho_0(x,t,\theta)].\end{aligned} \tag{107}$$

that is

$$\alpha_0^T h(x,t,\sigma,u) \le -\beta_0\,|\alpha_0|\,[|u|-\rho_0(x,t,\theta)] \tag{108}$$

Clearly, (109) also holds when $\alpha_0 = 0$.

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