IEEE Access
Multidisciplinary | Rapid Review | Open Access Journal

Received February 11, 2020, accepted March 1, 2020, date of publication March 10, 2020, date of current version March 20, 2020.

Digital Object Identifier 10.1109/ACCESS.2020.2979780

New q-Rung Orthopair Fuzzy Bonferroni Mean Dombi Operators and Their Application in Multiple Attribute Decision Making

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This work was supported in part by the National Natural Science Foundation of China under Grant 71971163, and in part by the Shaanxi Province Natural Science Fund of China under Grant 2019JM252.

ABSTRACT Some q-rung orthopair fuzzy Bonferroni mean Dombi aggregation operators have been developed based on the Bonferroni mean, Dombi T-norm and T-conorm in q-rung orthopair fuzzy environment. The q-rung orthopair fuzzy Bonferroni mean Dombi averaging (q-ROFBMDA) operator and the q-rung orthopair fuzzy geometric Bonferroni mean Dombi averaging (q-ROFGBMDA) operator are first developed. Then the q-rung orthopair fuzzy weighted Bonferroni mean Dombi averaging (q-ROFWBMDA) operator and the q-rung orthopair fuzzy weighted geometric Bonferroni mean Dombi averaging (q-ROFWGBMDA) operator have been developed. Based on the partitioned operation, the q-rung orthopair fuzzy partitioned Bonferroni mean Dombi averaging (q-ROFPBMDA) operator and the q-rung orthopair fuzzy partitioned geometric Bonferroni mean Dombi averaging (q-ROFPGBMDA) operator have been presented. Some desirable properties of the new aggregation operators have been studied. A new multiple attribute decision making method based on the q-ROFWBMDA (q-ROFWGBMDA) operator is proposed. Finally, a numerical example of new campus site selection has been presented to illustrate the new method.

INDEX TERMS q-rung orthopair fuzzy sets, Dombi, Bonferroni mean, aggregation operator.

I. INTRODUCTION

Q-rung orthopair fuzzy set is the extension of intuitionistic fuzzy set and Pythagorean fuzzy set [1], which was first developed by Yager [2]. In q-rung orthopair fuzzy set, the sum of the qth power of the membership and the qth power of the nonmembership is not more than 1. Hence, the q-rung fuzzy set is more flexible, which has more applications than intuitionistic fuzzy set and Pythagorean fuzzy set. The q-rung fuzzy set has attracted broad attentions. Some aggregation operators have been developed [3]–[14]. Considering correlation of the q-rung orthopair fuzzy values, Liang *et al.* [5] developed q-rung orthopair fuzzy Choquet integral operator. By using the power mean operator, some q-rung orthopair fuzzy power mean operators have been developed [6], [7]. Q-rung orthopair fuzzy Maclaurin symmetric mean operator has been presented by Wei *et al.* [8]. Some q-rung orthopair fuzzy Bonferroni mean operators have been presented in [9] and [10]. By using the Heronian mean operator, Wei *et al.* [11] presented some q-rung orthopair fuzzy Heronian mean operator. Based on Dombi aggregation, Jana *et al.* [14] developed q-rung orthopair fuzzy Dombi weighted averaging operator and q-rung orthopair fuzzy Dombi order weighted averaging operator. Some distance measures in q-rung orthopair fuzzy environment have been investigated [15]–[18]. Peng *et al.* [18] proposed q-rung orthopair fuzzy weighted distance-based approximation method. Q-rung fuzzy set has been further extended to accommodate uncertain linguistic arguments [19]–[21], interval values [22]. Q-rung orthopair fuzzy set has been applied in evaluation of renewable energy problem [23], teaching quality [24], etc. Since q-rung orthopair fuzzy set has advantages over existing fuzzy sets, q-rung orthopair fuzzy values are taken as evaluation values in this paper.

Aggregation operators are very important in decision making process [25]–[28]. The Bonferroni mean (BM) was first proposed by Bonferroni [29], which is interpreted as the product of each argument with the average of the other

The associate editor coordinating the review of this manuscript and approving it for publication was Ching-Ter Chang.

arguments by Yager [30]. BM operator has been further studied and applied extensively [31]–[37]. Geometric BM was developed by Xia *et al.* [31] and generalized BM was proposed by Xia *et al.* [32] by considering correlation of three aggregated arguments rather than two. Chen *et al.* [33] generalized extended BM by a composite aggregation function. Blanco-Mesa and Merigó [34] proposed Bonferroni-Hamming weighted distance operator. Different views of weighted BM operations have been studied by Mesiarova-Zemankova *et al.* [35]. Some intuitionistic fuzzy Dombi BM operators have been developed by Liu *et al.* [36]. Some intuitionistic fuzzy interaction partitioned BM operators have been proposed by Liu *et al.* [37]. The BM has been extended to accommodate interval type-2 fuzzy values [38], Pythagorean fuzzy values [39]–[41], 2-tuple intuitionistic fuzzy values [42], hesitant 2-tuple linguistic argument [43], neutrosophic fuzzy values [44], etc. Dombi [45] developed Dombi t-norm and Dombi t-conorm, which is more flexible by a parameter in aggregation process. Dombi aggregation has been studied and applied extensively [46]–[48]. Some aggregation operators based on the Dombi t-norm and Dombi t-conorm have been studied. Some picture fuzzy Dombi aggregation operators been developed including Picture fuzzy Dombi weighted average operator and picture fuzzy Dombi weighted geometric average operator [49], picture fuzzy Dombi Heronian mean operator [50]. Some intuitionistic fuzzy Dombi aggregation operators and interval-valued intuitionistic fuzzy Dombi aggregation operators have been presented [51]–[53]. Some bipolar fuzzy Dombi weighted averaging operator has been proposed by Jana *et al.* [54]. Dombi operation has been further extended to the neutrosophic fuzzy environment [55]–[57], 2-tuple linguistic neutrosophic fuzzy environment [58], [59], picture fuzzy environment [60], q-rung picture fuzzy environment [61], etc. Since q-rung orthopair fuzzy set is more flexible in evaluation process and Dombi is more flexible in aggregation, BM operator can consider the correlation of the arguments to be aggregated, then we develop new aggregation operators based on the Dombi and BM operator in q-rung orthopair fuzzy environments to give some more powerful and flexible aggregation operators. To the best of our knowledge, Dombi aggregation operation based on the BM in q-rung orthopair fuzzy environment is yet to be studied. Hence, the aim of this paper is to develop some q-rung orthopair fuzzy BM Dombi aggregation operators. We first developed q-rung orthopair fuzzy BM Dombi averaging (q-ROFBMDA) operator and q-rung orthopair fuzzy geometric BM Dombi averaging (q-ROFGBMDA) operator. Then we presented the q-rung orthopair fuzzy weighted BM Dombi averaging (q-ROFWBMDA) operator and q-rung orthopair fuzzy weighted geometric BM Dombi averaging (q-ROFWGBMDA) operator. Considering the partitioned aggregation operation, we developed the q-rung orthopair fuzzy partitioned BM Dombi averaging (q-ROFPBMDA) operator, the q-rung orthopair fuzzy partitioned weighted BM Dombi averaging (q-ROFPWBMDA) operator, the q-rung orthopair fuzzy partitioned geometric BM Dombi averaging (q-ROFPGBMDA) operator and the q-rung orthopair fuzzy partitioned weighted geometric BM Dombi averaging (q-ROFPWGBMDA) operator. The new aggregation operators can provide us a very useful means to deal with MADM problems in q-rung orthopair fuzzy environments.

The rest of the paper is organized as follows. Some basic concepts about q-rung orthopair fuzzy set, Dombi T-norm and Dombi T-conorm have been reviewed in Section 2. Some q-rung orthopair fuzzy BM Dombi aggregation operators have been developed including the q-ROFBMDA operator, the q-ROFGBMDA operator, the q-ROFWBMDA operator, the q-ROFWGBMDA operator, the q-ROFPBMDA operator, the q-ROFPWBMDA operator, the q-ROFPGBMDA operator and the q-ROFPWGBMDA operator in Section 3. Some properties have been studied. A new multiple attribute decision making method based on the q-ROFWBMDA (q-ROFWGBMDA) has been developed in Section 4. Numerical example is presented in Section 5 to illustrate the new method. Conclusions are given in the last Section.

II. PRELIMINARIES

Definition 1 [2]: Let X be a fixed set. A q-rung orthopair fuzzy set (q-ROFS) A on X can be represented as

$$P = \{< x, \mu_A(x), \nu_A(x) >| \; x \in X\}, \tag{1}$$

where $\mu_A(x) : X \to [0, 1]$ is the degree of membership and $\nu_A(x) : X \to [0, 1]$ is the degree of non-membership of $x \in X$ to the set A, respectively. For each $x \in X$, it satisfies the following condition $0 \leq (\mu_A(x))^q + (\nu_A(x))^q \leq 1, \; (q \geq 1)$. $\pi_A(x) = (1 - (\mu_A(x))^q - (\nu_A(x))^2)^{1/q}$ is the indeterminacy degree of x to X.

Definition 2 [4]: Let $\hat{\alpha} =< \mu_{\hat{\alpha}}, \nu_{\hat{\alpha}} >$ be a q-rung orthopair fuzzy number. The score function of $\hat{\alpha}$ can be defined as

$$S(\hat{\alpha}) = \mu_{\hat{\alpha}}^q - \nu_{\hat{\alpha}}^q. \tag{2}$$

The accuracy function of $\hat{\alpha}$ can be defined as

$$H(\hat{\alpha}) = \mu_{\hat{\alpha}}^q + \nu_{\hat{\alpha}}^q. \tag{3}$$

Let $\hat{\alpha} =< \mu_{\hat{\alpha}}, \nu_{\hat{\alpha}} >$ and $\hat{\beta} =< \mu_{\hat{\beta}}, \nu_{\hat{\beta}} >$ be two q-rung orthopair fuzzy numbers, then

If $S(\hat{\alpha}) > S(\hat{\beta})$, then $\hat{\alpha} > \hat{\beta}$,

If $S(\hat{\alpha}) = S(\hat{\beta})$, then

If $H(\hat{\alpha}) > H(\hat{\beta})$, then $\hat{\alpha} > \hat{\beta}$,

If $H(\hat{\alpha}) = H(\hat{\beta})$, then $\hat{\alpha} = \hat{\beta}$.

Definition 3 [45]: Let $(x, y) \in (0, 1) \times (0, 1)$ and $\gamma \geq 0$. The Dombi T-norm $T_{D,\gamma}$ and Dombi T-conorm $S_{D,\gamma}$ are defined as follows

$$T_{D,\gamma}(x, y) = \frac{1}{1 + ((\frac{1-x}{x})^\gamma + (\frac{1-y}{y})^\gamma)^{1/\gamma}}, \tag{4}$$

$$S_{D,\gamma}(x, y) = 1 - \frac{1}{1 + ((\frac{x}{1-x})^\gamma + (\frac{y}{1-y})^\gamma)^{1/\gamma}}. \tag{5}$$

Definition 4 [14]: Let $\hat{\alpha} =< \mu_{\hat{\alpha}}, \nu_{\hat{\alpha}} >$ and $\hat{\beta} =< \mu_{\hat{\beta}}, \nu_{\hat{\beta}} >$ be two q-rung orthopair fuzzy numbers.

The operational laws of q-rung orthopair fuzzy numbers based on the Dombi T-norm and Dombi T-conorm can be defined as

(1)

$$\hat{\alpha}\oplus\hat{\beta}=\Big\langle\Big(1-\frac{1}{1+\big((\frac{\mu_{\hat{\alpha}}^q}{1-\mu_{\hat{\alpha}}^q})^\gamma+(\frac{\mu_{\hat{\beta}}^q}{1-\mu_{\hat{\beta}}^q})^\gamma\big)^{1/\gamma}}\Big)^{1/q},\Big(\frac{1}{1+\big((\frac{1-\nu_{\hat{\alpha}}^q}{\nu_{\hat{\alpha}}^q})^\gamma+(\frac{1-\nu_{\hat{\beta}}^q}{\nu_{\hat{\beta}}^q})^\gamma\big)^{1/\gamma}}\Big)^{1/q}\Big\rangle,$$

(2)

$$\hat{\alpha}\otimes\hat{\beta}=\Big\langle\Big(\frac{1}{1+\big((\frac{1-\mu_{\hat{\alpha}}^q}{\mu_{\hat{\alpha}}^q})^\gamma+(\frac{1-\mu_{\hat{\beta}}^q}{\mu_{\hat{\beta}}^q})^\gamma\big)^{1/\gamma}}\Big)^{1/q},\Big(1-\frac{1}{1+\big((\frac{\nu_{\hat{\alpha}}^q}{1-\nu_{\hat{\alpha}}^q})^\gamma+(\frac{\nu_{\hat{\beta}}^q}{1-\nu_{\hat{\beta}}^q})^\gamma\big)^{1/\gamma}}\Big)^{1/q}\Big\rangle,$$

(3)

$$\lambda\hat{\alpha}=\Big\langle\Big(1-\frac{1}{1+\big(\lambda(\frac{\mu_{\hat{\alpha}}^q}{1-\mu_{\hat{\alpha}}^q})^\gamma\big)^{1/\gamma}}\Big)^{1/q},\Big(\frac{1}{1+\big(\lambda(\frac{1-\nu_{\hat{\alpha}}^q}{\nu_{\hat{\alpha}}^q})^\gamma\big)^{1/\gamma}}\Big)^{1/q}\Big\rangle,$$

(4)

$$\hat{\alpha}^\lambda=\Big\langle\Big(\frac{1}{1+\big(\lambda(\frac{1-\mu_{\hat{\alpha}}^q}{\mu_{\hat{\alpha}}^q})^\gamma\big)^{1/\gamma}}\Big)^{1/q},\Big(1-\frac{1}{1+\big(\lambda(\frac{\nu_{\hat{\alpha}}^q}{1-\nu_{\hat{\alpha}}^q})^\gamma\big)^{1/\gamma}}\Big)^{1/q}\Big\rangle.$$

III. SOME NEW q-RUNG ORTHOPAIR FUZZY DOMBI BONFERRONI MEAN OPERATOR

Definition 5 [29]: The BM aggregation operator of dimension n is a mapping $(R^+)^n\rightarrow R^+$:

$$\mathrm{BM}^{r,s}(\beta_1,\beta_2,\ldots,\beta_n)=\Big(\frac{1}{n(n-1)}\sum_{i,j=1,i\neq j}^{n}\beta_i^r\beta_j^s\Big)^{\frac{1}{r+s}},\quad(6)$$

where $r,s\geq 0$, β_j $(j=1,2,\ldots,n)$ is a collection of nonnegative real numbers.

Definition 6: Let $\hat{\alpha}_k$ $(k=1,2,\ldots,n)$ be a collection of q-rung orthopair fuzzy numbers. The q-rung orthopair fuzzy BM Dombi averaging (q-ROFBMDA) operator is defined as

$$\text{q-ROFBMDA}(\hat{\alpha}_1,\hat{\alpha}_2,\ldots,\hat{\alpha}_n)=\Big(\frac{1}{n(n-1)}\oplus_{i,j=1,i\neq j}^{n}(\hat{\alpha}_i^r\otimes\hat{\alpha}_j^s)\Big)^{\frac{1}{r+s}},\quad(7)$$

where $r,s>0$.

Theorem 1: Let $\hat{\alpha}_k=<\mu_{\hat{\alpha}_k},\nu_{\hat{\alpha}_k}>$ $(i=1,2,\ldots,n)$ be a collection of q-rung orthopair fuzzy numbers, $q,\gamma>0$. The aggregated result of q-ROFBMDA operator is still a q-rung orthopair fuzzy number and

$$\begin{aligned}&\text{q-ROFBMDA}(\hat{\alpha}_1,\hat{\alpha}_2,\ldots,\hat{\alpha}_n)\\&=\Big(\frac{1}{n(n-1)}\oplus_{i,j=1,i\neq j}^{n}(\hat{\alpha}_i^r\otimes\hat{\alpha}_j^s)\Big)^{\frac{1}{r+s}}\\&=\Big\langle\Big(\frac{1}{1+\big(\frac{1}{r+s}(\frac{1}{\frac{1}{n(n-1)}\sum_{i,j=1,i\neq j}^{n}u_{\hat{\alpha}_{ij}}})\big)^{1/\gamma}}\Big)^{1/q},\\&\Big(1-\frac{1}{1+\big(\frac{1}{r+s}(\frac{1}{\frac{1}{n(n-1)}\sum_{i,j=1,i\neq j}^{n}v_{\hat{\alpha}_{ij}}})\big)^{1/\gamma}}\Big)^{1/q}\Big\rangle,\end{aligned}\quad(8)$$

where

$$u_{\hat{\alpha}_{ij}}=\frac{1}{r(\frac{1-\mu_{\hat{\alpha}_i}^q}{\mu_{\hat{\alpha}_i}^q})^\gamma+s(\frac{1-\mu_{\hat{\alpha}_j}^q}{\mu_{\hat{\alpha}_j}^q})^\gamma},$$

$$v_{\hat{\alpha}_{ij}}=\frac{1}{r(\frac{\nu_{\hat{\alpha}_i}^q}{1-\nu_{\hat{\alpha}_i}^q})^\gamma+s(\frac{\nu_{\hat{\alpha}_j}^q}{1-\nu_{\hat{\alpha}_j}^q})^\gamma}.$$

Proof:

$$\hat{\alpha}_i^r=\Big\langle\Big(\frac{1}{1+\big(r(\frac{1-\mu_{\hat{\alpha}_i}^q}{\mu_{\hat{\alpha}_i}^q})^\gamma\big)^{1/\gamma}}\Big)^{1/q},\Big(1-\frac{1}{1+\big(r(\frac{\nu_{\hat{\alpha}_i}^q}{1-\nu_{\hat{\alpha}_i}^q})^\gamma\big)^{1/\gamma}}\Big)^{1/q}\Big\rangle.$$

$$\hat{\alpha}_j^s=\Big\langle\Big(\frac{1}{1+\big(s(\frac{1-\mu_{\hat{\alpha}_j}^q}{\mu_{\hat{\alpha}_j}^q})^\gamma\big)^{1/\gamma}}\Big)^{1/q},\Big(1-\frac{1}{1+\big(s(\frac{\nu_{\hat{\alpha}_j}^q}{1-\nu_{\hat{\alpha}_j}^q})^\gamma\big)^{1/\gamma}}\Big)^{1/q}\Big\rangle.$$

$$\begin{aligned}&\hat{\alpha}_i^r\otimes\hat{\alpha}_j^s\\&=\Big\langle\Big(\frac{1}{1+\big(r(\frac{1-\mu_{\hat{\alpha}_i}^q}{\mu_{\hat{\alpha}_i}^q})^\gamma+s(\frac{1-\mu_{\hat{\alpha}_j}^q}{\mu_{\hat{\alpha}_j}^q})^\gamma\big)^{1/\gamma}}\Big)^{1/q},\\&\Big(1-\frac{1}{1+\big(r(\frac{\nu_{\hat{\alpha}_i}^q}{1-\nu_{\hat{\alpha}_i}^q})^\gamma+s(\frac{\nu_{\hat{\alpha}_j}^q}{1-\nu_{\hat{\alpha}_j}^q})^\gamma\big)^{1/\gamma}}\Big)^{1/q}\Big\rangle.\end{aligned}$$

$$\begin{aligned}&\oplus_{i,j=1,i\neq j}^{n}(\hat{\alpha}_i^r\otimes\hat{\alpha}_j^s)\\&=\Big\langle\Big(1-\frac{1}{1+\big(\sum_{i,j=1,i\neq j}^{n}\frac{1}{r(\frac{1-\mu_{\hat{\alpha}_i}^q}{\mu_{\hat{\alpha}_i}^q})^\gamma+s(\frac{1-\mu_{\hat{\alpha}_j}^q}{\mu_{\hat{\alpha}_j}^q})^\gamma}\big)^{1/\gamma}}\Big)^{1/q},\\&\Big(\frac{1}{1+\big(\sum_{i,j=1,i\neq j}^{n}\frac{1}{r(\frac{\nu_{\hat{\alpha}_i}^q}{1-\nu_{\hat{\alpha}_i}^q})^\gamma+s(\frac{\nu_{\hat{\alpha}_j}^q}{1-\nu_{\hat{\alpha}_j}^q})^\gamma}\big)^{1/\gamma}}\Big)^{1/q}\Big\rangle.\end{aligned}$$

Let

$$u_{\hat{\alpha}_{ij}} = \frac{1}{r\left(\frac{1-\mu^q_{\hat{\alpha}_i}}{\mu^q_{\hat{\alpha}_i}}\right)^\gamma + s\left(\frac{1-\mu^q_{\hat{\alpha}_j}}{\mu^q_{\hat{\alpha}_j}}\right)^\gamma},$$

$$v_{\hat{\alpha}_{ij}} = \frac{1}{r\left(\frac{v^q_{\hat{\alpha}_i}}{1-v^q_{\hat{\alpha}_i}}\right)^\gamma + s\left(\frac{v^q_{\hat{\alpha}_j}}{1-v^q_{\hat{\alpha}_j}}\right)^\gamma}.$$

$$\frac{1}{n(n-1)} \oplus^n_{i,j=1,i\neq j} (\hat{\alpha}_i^r \otimes \hat{\alpha}_j^s)$$
$$= \left\langle \left(1 - \frac{1}{1+\left(\frac{1}{n(n-1)}\sum^n_{i,j=1,i\neq j} u_{\hat{\alpha}_{ij}}\right)^{1/\gamma}}\right)^{1/q}, \left(\frac{1}{1+\left(\frac{1}{n(n-1)}\sum^n_{i,j=1,i\neq j} v_{\hat{\alpha}_{ij}}\right)^{1/\gamma}}\right)^{1/q}\right\rangle,$$

$$\left(\frac{1}{n(n-1)} \oplus^n_{i,j=1,i\neq j} (\hat{\alpha}_i^r \otimes \hat{\alpha}_j^s)\right)^{\frac{1}{r+s}}$$
$$= \left\langle \left(\frac{1}{1+\left(\frac{1}{r+s}\left(\frac{1}{\frac{1}{n(n-1)}\sum^n_{i,j=1,i\neq j} u_{\hat{\alpha}_{ij}}}\right)\right)^{1/\gamma}}\right)^{1/q}, \left(1 - \frac{1}{1+\left(\frac{1}{r+s}\left(\frac{1}{\frac{1}{n(n-1)}\sum^n_{i,j=1,i\neq j} v_{\hat{\alpha}_{ij}}}\right)\right)^{1/\gamma}}\right)^{1/q}\right\rangle.$$

Moreover, $0 \leq \mu^q_{\hat{\alpha}_j} + v^q_{\hat{\alpha}_j} \leq 1$, $\mu^q_{\hat{\alpha}_j} \leq 1 - v^q_{\hat{\alpha}_j}$, $v^q_{\hat{\alpha}_j} \leq 1 - \mu^q_{\hat{\alpha}_j}$, $\frac{v^q_{\hat{\alpha}_j}}{1-v^q_{\hat{\alpha}_j}} \leq \frac{1-\mu^q_{\hat{\alpha}_j}}{\mu^q_{\hat{\alpha}_j}}$, $\frac{v^q_{\hat{\alpha}_i}}{1-v^q_{\hat{\alpha}_i}} \leq \frac{1-\mu^q_{\hat{\alpha}_i}}{\mu^q_{\hat{\alpha}_i}}$, $r\left(\frac{v^q_{\hat{\alpha}_i}}{1-v^q_{\hat{\alpha}_i}}\right)^\gamma + s\left(\frac{v^q_{\hat{\alpha}_j}}{1-v^q_{\hat{\alpha}_j}}\right)^\gamma \leq r\left(\frac{1-\mu^q_{\hat{\alpha}_i}}{\mu^q_{\hat{\alpha}_i}}\right)^\gamma + s\left(\frac{1-\mu^q_{\hat{\alpha}_j}}{\mu^q_{\hat{\alpha}_j}}\right)^\gamma$,

$$\frac{1}{n(n-1)} \sum^n_{i,j=1,i\neq j} v_{\hat{\alpha}_{ij}} \geq \frac{1}{n(n-1)} \sum^n_{i,j=1,i\neq j} u_{\hat{\alpha}_{ij}},$$

$$\frac{1}{r+s}\left(\frac{1}{\frac{1}{n(n-1)}\sum^n_{i,j=1,i\neq j} v_{\hat{\alpha}_{ij}}}\right) \leq \frac{1}{r+s}\left(\frac{1}{\frac{1}{n(n-1)}\sum^n_{i,j=1,i\neq j} u_{\hat{\alpha}_{ij}}}\right),$$

$$1 \geq \frac{1}{1+\left(\frac{1}{r+s}\left(\frac{1}{\frac{1}{n(n-1)}\sum^n_{i,j=1,i\neq j} v_{\hat{\alpha}_{ij}}}\right)\right)} \geq \frac{1}{1+\left(\frac{1}{r+s}\left(\frac{1}{\frac{1}{n(n-1)}\sum^n_{i,j=1,i\neq j} u_{\hat{\alpha}_{ij}}}\right)\right)} \geq 0,$$

$$0 \leq \frac{1}{1+\left(\frac{1}{r+s}\left(\frac{1}{\frac{1}{n(n-1)}\sum^n_{i,j=1,i\neq j} u_{\hat{\alpha}_{ij}}}\right)\right)^{1/\gamma}} + 1 - \frac{1}{1+\left(\frac{1}{r+s}\left(\frac{1}{\frac{1}{n(n-1)}\sum^n_{i,j=1,i\neq j} v_{\hat{\alpha}_{ij}}}\right)\right)^{1/\gamma}} \leq 1.$$

Hence, the aggregated result of the q-ROFBMDA operator is still a q-rung orthopair fuzzy number.

Theorem 2 (Idempotency): Let $\hat{\alpha}_k = <\mu_{\hat{\alpha}_k}, v_{\hat{\alpha}_k}>$ $(i = 1, 2, \ldots, n)$ be a collection of q-rung orthopair fuzzy numbers. If $\hat{\alpha}_k = \hat{\alpha}$, that is $<\mu_{\hat{\alpha}_k}, v_{\hat{\alpha}_k}> = <\mu_{\hat{\alpha}}, v_{\hat{\alpha}}>$ $(k = 1, 2, \ldots, n)$, $q, \gamma > 0$. Then

$$\text{q-ROFBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) = \hat{\alpha}.$$

Proof: Since $\mu_{\hat{\alpha}_i} = \mu_{\hat{\alpha}_j} = \mu_{\hat{\alpha}}$, then $u_{\hat{\alpha}_{ij}} = \frac{1}{r\left(\frac{1-\mu^q_{\hat{\alpha}_i}}{\mu^q_{\hat{\alpha}_i}}\right)^\gamma + s\left(\frac{1-\mu^q_{\hat{\alpha}_j}}{\mu^q_{\hat{\alpha}_j}}\right)^\gamma} = \frac{1}{r\left(\frac{1-\mu^q_{\hat{\alpha}}}{\mu^q_{\hat{\alpha}}}\right)^\gamma + s\left(\frac{1-\mu^q_{\hat{\alpha}}}{\mu^q_{\hat{\alpha}}}\right)^\gamma} = \frac{1}{(r+s)\left(\frac{1-\mu^q_{\hat{\alpha}}}{\mu^q_{\hat{\alpha}}}\right)^\gamma}$, then

$$\left(\frac{1}{1+\left(\frac{1}{r+s}\left(\frac{1}{\frac{1}{n(n-1)}\sum^n_{i,j=1,i\neq j} u_{\hat{\alpha}_{ij}}}\right)\right)^{1/\gamma}}\right)^{1/q}$$
$$= \left(\frac{1}{1+\left(\frac{1}{r+s}\left(\frac{1}{\frac{1}{n(n-1)}\sum^n_{i,j=1,i\neq j} \frac{1}{(r+s)\left(\frac{1-\mu^q_{\hat{\alpha}}}{\mu^q_{\hat{\alpha}}}\right)^\gamma}}\right)\right)^{1/\gamma}}\right)^{1/q}$$
$$= \left(\frac{1}{1+\left(\frac{1}{r+s}\frac{1}{\frac{1}{\left((r+s)\left(\frac{1-\mu^q_{\hat{\alpha}}}{\mu^q_{\hat{\alpha}}}\right)^\gamma\right)}}\right)^{1/\gamma}}\right)^{1/q}$$
$$= \left(\frac{1}{1+\left(\frac{1}{r+s}\left((r+s)\left(\frac{1-\mu^q_{\hat{\alpha}}}{\mu^q_{\hat{\alpha}}}\right)^\gamma\right)\right)^{1/\gamma}}\right)^{1/q}$$
$$= \left(\frac{1}{1+\left(\left(\frac{1-\mu^q_{\hat{\alpha}}}{\mu^q_{\hat{\alpha}}}\right)^\gamma\right)^{1/\gamma}}\right)^{1/q}$$
$$= \left(\frac{1}{1+\frac{1-\mu^q_{\hat{\alpha}}}{\mu^q_{\hat{\alpha}}}}\right)^{1/q} = \mu_{\hat{\alpha}}.$$

$v_{\hat{\alpha}_i} = v_{\hat{\alpha}_j} = v_{\hat{\alpha}}$, $v_{\hat{\alpha}_{ij}} = \frac{1}{r\left(\frac{v^q_{\hat{\alpha}_i}}{1-v^q_{\hat{\alpha}_i}}\right)^\gamma + s\left(\frac{v^q_{\hat{\alpha}_j}}{1-v^q_{\hat{\alpha}_j}}\right)^\gamma} = \frac{1}{r\left(\frac{v^q_{\hat{\alpha}}}{1-v^q_{\hat{\alpha}}}\right)^\gamma + s\left(\frac{v^q_{\hat{\alpha}}}{1-v^q_{\hat{\alpha}}}\right)^\gamma} = \frac{1}{(r+s)\left(\frac{v^q_{\hat{\alpha}}}{1-v^q_{\hat{\alpha}}}\right)^\gamma}$,

$$\left(1 - \frac{1}{1+\left(\frac{1}{r+s}\left(\frac{1}{\frac{1}{n(n-1)}\sum^n_{i,j=1,i\neq j} v_{\hat{\alpha}_{ij}}}\right)\right)^{1/\gamma}}\right)^{1/q}$$
$$= \left(1 - \frac{1}{1+\left(\frac{1}{r+s}\left(\frac{1}{\frac{1}{n(n-1)}\sum^n_{i,j=1,i\neq j} \frac{1}{(r+s)\left(\frac{v^q_{\hat{\alpha}}}{1-v^q_{\hat{\alpha}}}\right)^\gamma}}\right)\right)^{1/\gamma}}\right)^{1/q}$$
$$= \left(1 - \frac{1}{1+\left(\frac{1}{r+s}\left(\frac{1}{\frac{1}{\left((r+s)\left(\frac{v^q_{\hat{\alpha}}}{1-v^q_{\hat{\alpha}}}\right)^\gamma\right)}}\right)\right)^{1/\gamma}}\right)^{1/q}$$
$$= \left(1 - \frac{1}{1+\left(\frac{1}{r+s}\left((r+s)\left(\frac{v^q_{\hat{\alpha}}}{1-v^q_{\hat{\alpha}}}\right)^\gamma\right)\right)^{1/\gamma}}\right)^{1/q}$$
$$= \left(1 - \frac{1}{1+\left(\left(\frac{v^q_{\hat{\alpha}}}{1-v^q_{\hat{\alpha}}}\right)^\gamma\right)^{1/\gamma}}\right)^{1/q}$$
$$= \left(1 - \frac{1}{1+\frac{v^q_{\hat{\alpha}}}{1-v^q_{\hat{\alpha}}}}\right)^{1/q} = v_{\hat{\alpha}}.$$

Hence, q-ROFBMDA$(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) = \hat{\alpha}$.

Theorem 3: (Monotonicity) Let $(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n)$ and $(\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_n)$ be two collections of q-rung orthopair fuzzy numbers. If $\hat{\alpha}_k =< \mu_{\hat{\alpha}_k}, \nu_{\hat{\alpha}_k} >, \hat{\beta}_k =< \mu_{\hat{\beta}_k}, \nu_{\hat{\beta}_k} >$ $(k = 1, 2, \ldots, n)$ and $\mu_{\hat{\alpha}_k} \leq \mu_{\hat{\beta}_k}, \nu_{\hat{\alpha}_k} \geq \nu_{\hat{\beta}_k}$, then

$$\text{q-ROFBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) \leq \text{q-ROFBMDA}(\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_n).$$

Proof: Since $\mu_{\hat{\alpha}_i} \leq \mu_{\hat{\beta}_i}, \mu_{\hat{\alpha}_j} \leq \mu_{\hat{\beta}_j}, \mu^p_{\hat{\alpha}_i} \leq \mu^p_{\hat{\beta}_i}, \mu^p_{\hat{\alpha}_j} \leq \mu^p_{\hat{\beta}_j}, (\frac{1-x}{x})' = \frac{1}{x^2} < 0$. $\frac{1-\mu^p_{\hat{\alpha}_i}}{\mu^p_{\hat{\alpha}_i}} \geq \frac{1-\mu^p_{\hat{\beta}_i}}{\mu^p_{\hat{\beta}_i}}$ and $\frac{1-\mu^p_{\hat{\alpha}_j}}{\mu^p_{\hat{\alpha}_j}} \geq \frac{1-\mu^p_{\hat{\beta}_j}}{\mu^p_{\hat{\beta}_j}}$. $r(\frac{1-\mu^p_{\hat{\alpha}_i}}{\mu^p_{\hat{\alpha}_i}})^\gamma + s(\frac{1-\mu^p_{\hat{\alpha}_j}}{\mu^p_{\hat{\alpha}_j}})^\gamma \geq r(\frac{1-\mu^p_{\hat{\beta}_i}}{\mu^p_{\hat{\beta}_i}})^\gamma + s(\frac{1-\mu^p_{\hat{\beta}_j}}{\mu^p_{\hat{\beta}_j}})^\gamma$. Let $u_{\hat{\alpha}_{ij}} = r(\frac{1-\mu^p_{\hat{\alpha}_i}}{\mu^p_{\hat{\alpha}_i}})^\gamma + s(\frac{1-\mu^p_{\hat{\alpha}_j}}{\mu^p_{\hat{\alpha}_j}})^\gamma, u_{\hat{\beta}_{ij}} = r(\frac{1-\mu^p_{\hat{\beta}_i}}{\mu^p_{\hat{\beta}_i}})^\gamma + s(\frac{1-\mu^p_{\hat{\beta}_j}}{\mu^p_{\hat{\beta}_j}})^\gamma$,

$$\left(\frac{1}{n(n-1)} \sum_{i,j=1,i\neq j}^n \frac{1}{u_{\hat{\alpha}_{ij}}}\right) \leq \left(\frac{1}{n(n-1)} \sum_{i,j=1,i\neq j}^n \frac{1}{u_{\hat{\beta}_{ij}}}\right),$$

$$\frac{1}{r+s}\left(\frac{1}{\left(\frac{1}{n(n-1)} \sum_{i,j=1,i\neq j}^n \frac{1}{u_{\hat{\alpha}_{ij}}}\right)}\right) \geq \frac{1}{r+s}\left(\frac{1}{\left(\frac{1}{n(n-1)} \sum_{i,j=1,i\neq j}^n \frac{1}{u_{\hat{\beta}_{ij}}}\right)}\right),$$

$$\left(\frac{1}{1+\left(\frac{1}{r+s}\left(\frac{1}{\left(\frac{1}{n(n-1)} \sum_{i,j=1,i\neq j}^n \frac{1}{u_{\hat{\alpha}_{ij}}}\right)}\right)\right)^{1/\gamma}}\right)^{1/q} \leq \left(\frac{1}{1+\left(\frac{1}{r+s}\left(\frac{1}{\left(\frac{1}{n(n-1)} \sum_{i,j=1,i\neq j}^n \frac{1}{u_{\hat{\beta}_{ij}}}\right)}\right)\right)^{1/\gamma}}\right)^{1/q}.$$

$\nu_{\hat{\alpha}_i} \geq \nu_{\hat{\beta}_i}, \nu_{\hat{\alpha}_j} \geq \nu_{\hat{\beta}_j}, \nu^p_{\hat{\alpha}_i} \geq \nu^p_{\hat{\beta}_i}, \nu^p_{\hat{\alpha}_j} \geq \nu^p_{\hat{\beta}_j}, (\frac{x}{1-x})' = \frac{1}{(1-x)^2} > 0$. $\frac{\nu^p_{\hat{\alpha}_i}}{1-\nu^p_{\hat{\alpha}_i}} \geq \frac{\nu^p_{\hat{\beta}_i}}{1-\nu^p_{\hat{\beta}_i}}, \frac{\nu^p_{\hat{\alpha}_j}}{1-\nu^p_{\hat{\alpha}_j}} \geq \frac{\nu^p_{\hat{\beta}_j}}{1-\nu^p_{\hat{\beta}_j}}$, $r(\frac{\nu^p_{\hat{\alpha}_i}}{1-\nu^p_{\hat{\alpha}_i}})^\gamma + s(\frac{\nu^p_{\hat{\alpha}_j}}{1-\nu^p_{\hat{\alpha}_j}})^\gamma \geq r(\frac{\nu^p_{\hat{\beta}_i}}{1-\nu^p_{\hat{\beta}_i}})^\gamma + s(\frac{\nu^p_{\hat{\beta}_j}}{1-\nu^p_{\hat{\beta}_j}})^\gamma$. Let $v_{\hat{\alpha}_{ij}} = r(\frac{\nu^p_{\hat{\alpha}_i}}{1-\nu^p_{\hat{\alpha}_i}})^\gamma + s(\frac{\nu^p_{\hat{\alpha}_j}}{1-\nu^p_{\hat{\alpha}_j}})^\gamma, v_{\hat{\beta}_{ij}} = r(\frac{\nu^p_{\hat{\beta}_i}}{1-\nu^p_{\hat{\beta}_i}})^\gamma + s(\frac{\nu^p_{\hat{\beta}_j}}{1-\nu^p_{\hat{\beta}_j}})^\gamma$.

$$\frac{1}{n(n-1)} \sum_{i,j=1,i\neq j}^n \frac{1}{v_{\hat{\alpha}_{ij}}} \leq \frac{1}{n(n-1)} \sum_{i,j=1,i\neq j}^n \frac{1}{v_{\hat{\beta}_{ij}}},$$

$$\left(\frac{1}{r+s}\left(\frac{1}{\frac{1}{n(n-1)} \sum_{i,j=1,i\neq j}^n \frac{1}{v_{\hat{\alpha}_{ij}}}}\right)\right)^{1/\gamma} \geq \left(\frac{1}{r+s}\left(\frac{1}{\frac{1}{n(n-1)} \sum_{i,j=1,i\neq j}^n \frac{1}{v_{\hat{\beta}_{ij}}}}\right)\right)^{1/\gamma},$$

$$1-\frac{1}{1+\left(\frac{1}{r+s}\left(\frac{1}{\frac{1}{n(n-1)} \sum_{i,j=1,i\neq j}^n \frac{1}{v_{\hat{\alpha}_{ij}}}}\right)\right)^{1/\gamma}} \geq 1-\frac{1}{1+\left(\frac{1}{r+s}\left(\frac{1}{\frac{1}{n(n-1)} \sum_{i,j=1,i\neq j}^n \frac{1}{v_{\hat{\beta}_{ij}}}}\right)\right)^{1/\gamma}}.$$

By using the score function, we can get

$$\text{q-ROFBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) \leq \text{q-ROFBMDA}(\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_n).$$

Theorem 4 (Boundedness): Let $\hat{\alpha}_k =< \mu_{\hat{\alpha}_k}, \nu_{\hat{\alpha}_k} >$ $(k = 1, 2, \ldots, n)$ be a collection of q-rung orthopair fuzzy numbers. $\hat{\alpha}^- =< \mu^-, \nu^+ >=< \min_k \mu_{\hat{\alpha}_k}, \max_k \nu_{\hat{\alpha}_k} >, \hat{\alpha}^+ =< \mu^+, \nu^- >=< \max \mu_{\hat{\alpha}_k}, \min \nu_{\hat{\alpha}_k} >$, then

$$\hat{\alpha}^- \leq \text{q-ROFBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) \leq \hat{\alpha}^+.$$

Proof: The property of boundedness can be proved easily by using the property of monotonicity.

Theorem 5: (Commutativity) Let $\hat{\alpha}_k =< \mu_{\hat{\alpha}_k}, \nu_{\hat{\alpha}_k} >$ and $\hat{\alpha}'_k =< \mu'_{\hat{\alpha}_k}, \nu'_{\hat{\alpha}_k} >$ $(k = 1, 2, \ldots, n)$ be two collection of q-rung orthopair fuzzy numbers. If $\hat{\alpha}'_k =< \mu'_{\hat{\alpha}_k}, \nu'_{\hat{\alpha}_k} >$ is any permutation of $\hat{\alpha}_k =< \mu_{\hat{\alpha}_k}, \nu_{\hat{\alpha}_k} >$, then

$$\text{q-ROFBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) = \text{q-ROFBMDA}(\hat{\alpha}'_1, \hat{\alpha}'_2, \ldots, \hat{\alpha}'_n).$$

Proof:

$$\begin{aligned}\text{q-ROFBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) &= \left(\frac{1}{n(n-1)} \oplus_{i,j=1,i\neq j}^n (\hat{\alpha}_i^r \otimes \hat{\alpha}_j^s)\right)^{\frac{1}{r+s}} \\ &= \left(\frac{1}{n(n-1)} \oplus_{i,j=1,i\neq j}^n ((\hat{\alpha}'_i)^r \otimes (\hat{\alpha}'_j)^s)\right)^{\frac{1}{r+s}} \\ &= \text{q-ROFBMDA}(\hat{\alpha}'_1, \hat{\alpha}'_2, \ldots, \hat{\alpha}'_n).\end{aligned}$$

Definition 7: Let $\hat{\alpha}_k$ $(k = 1, 2, \ldots, n)$ be a collection of q-rung orthopair fuzzy numbers. The q-rung orthopair fuzzy geometric BM Dombi averaging (q-ROFGBMDA) operator is defined as

$$\text{q-ROFGBMDA}^{r,s}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) = \frac{1}{r+s}\left(\otimes_{i,j=1,i\neq j}^n (r\hat{\alpha}_i \oplus s\hat{\alpha}_j)^{\frac{1}{n(n-1)}}\right), \quad (9)$$

where $r, s > 0$.

Theorem 6: Let $\hat{\alpha}_k =< \mu_{\hat{\alpha}_k}, \nu_{\hat{\alpha}_k} >$ $(i = 1, 2, \ldots, n)$ be a collection of q-rung orthopair fuzzy numbers, $q, \gamma > 0$. The aggregated result of q-ROFGBMDA operator is still q-rung orthopair fuzzy number and

$$\begin{aligned}\text{q-ROFGBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) = \Bigg\langle &\left(1-\frac{1}{1+\left(\frac{1}{r+s}\left(\frac{1}{\left(\frac{1}{n(n-1)} \sum_{i,j=1,i\neq j}^n u_{\hat{\alpha}_{ij}}\right)}\right)\right)^{1/\gamma}}\right)^{1/q}, \\ &\left(\frac{1}{1+\left(\frac{1}{r+s}\left(\frac{1}{\left(\frac{1}{n(n-1)} \sum_{i,j=1,i\neq j}^n v_{\hat{\alpha}_{ij}}\right)}\right)\right)^{1/\gamma}}\right)^{1/q}\Bigg\rangle, \quad (10)\end{aligned}$$

where

$$u_{\hat{\alpha}_{ij}} = \left(\frac{1}{r\left(\frac{\mu^q_{\hat{\alpha}_i}}{1-\mu^q_{\hat{\alpha}_i}}\right)^\gamma + s\left(\frac{\mu^q_{\hat{\alpha}_j}}{1-\mu^q_{\hat{\alpha}_j}}\right)^\gamma}\right),$$

$$v_{\hat{\alpha}_{ij}} = \left(\frac{1}{r\left(\frac{1-\nu^q_{\hat{\alpha}_i}}{\nu^q_{\hat{\alpha}_i}}\right)^\gamma + s\left(\frac{1-\nu^q_{\hat{\alpha}_j}}{\nu^q_{\hat{\alpha}_j}}\right)^\gamma}\right).$$

Theorem 7 (Idempotency): Let $\hat{\alpha}_k =< \mu_{\hat{\alpha}_k}, \nu_{\hat{\alpha}_k} >$ $(i = 1, 2, \ldots, n)$ be a collection of q-rung orthopair fuzzy numbers. If $\hat{\alpha}_k = \hat{\alpha}$, that is $< \mu_{\hat{\alpha}_k}, \nu_{\hat{\alpha}_k} >=< \mu_{\hat{\alpha}}, \nu_{\hat{\alpha}} >$ $(k = 1, 2, \ldots, n)$, $q, \gamma > 0$. Then

$$\text{q-ROFGBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) = \hat{\alpha}.$$

Theorem 8 (Monotonicity): Let $(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n)$ and $(\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_n)$ be two collections of q-rung orthopair fuzzy numbers. If $\hat{\alpha}_k =< \mu_{\hat{\alpha}_k}, \nu_{\hat{\alpha}_k} >, \hat{\beta}_k =< \mu_{\hat{\beta}_k}, \nu_{\hat{\beta}_k} >$ $(k = 1, 2, \ldots, n)$ and $\mu_{\hat{\alpha}_k} \leq \mu_{\hat{\beta}_k}, \nu_{\hat{\alpha}_k} \geq \nu_{\hat{\beta}_k}$, then

$$\text{q-ROFGBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) \leq \text{q-ROFGBMDA}(\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_n).$$

Theorem 9 (Boundedness): Let $\hat{\alpha}_k =< \mu_{\hat{\alpha}_k}, \nu_{\hat{\alpha}_k} >$ $(k = 1, 2, \ldots, n)$ be a collection of q-rung orthopair fuzzy numbers. $\hat{\alpha}^- =< \mu^-, \nu^+ >=< \min_k \mu_{\hat{\alpha}_k}, \max_k \nu_{\hat{\alpha}_k} >, \hat{\alpha}^+ =< \mu^+, \nu^- >=< \max \mu_{\hat{\alpha}_k}, \min \nu_{\hat{\alpha}_k} >$, then

$$\hat{\alpha}^- \leq \text{q-ROFGBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) \leq \hat{\alpha}^+.$$

Theorem 10 (Commutativity): Let $\hat{\alpha}_k =< \mu_{\hat{\alpha}_k}, \nu_{\hat{\alpha}_k} >$ $(k = 1, 2, \ldots, n)$ and $\hat{\alpha}'_k =< \mu'_{\hat{\alpha}_k}, \nu'_{\hat{\alpha}_k} >$ $(k = 1, 2, \ldots, n)$ be two collection of q-rung orthopair fuzzy numbers. If $\hat{\alpha}'_k =< \mu'_{\hat{\alpha}_k}, \nu'_{\hat{\alpha}_k} >$ is any permutation of $\hat{\alpha}_k =< \mu_{\hat{\alpha}_k}, \nu_{\hat{\alpha}_k} >$, then

$$\text{q-ROFGBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) = \text{q-ROFGBMDA}(\hat{\alpha}'_1, \hat{\alpha}'_2, \ldots, \hat{\alpha}'_n).$$

Definition 8: Let $\hat{\alpha}_k$ $(k = 1, 2, \ldots, n)$ be a collection of q-rung orthopair fuzzy numbers and $(w_1, w_2, \ldots, w_n)$ be the weight vector of $\hat{\alpha}_k$. The q-rung orthopair fuzzy weighted BM Dombi averaging (q-ROFWBMDA) operator is defined as

$$\text{q-ROFWBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) = \left(\frac{1}{n(n-1)} \oplus^n_{i,j=1,i\neq j} \left((w_i\hat{\alpha}_i)^r \otimes (w_j\hat{\alpha}_j)^s\right)\right)^{\frac{1}{r+s}}, \quad (11)$$

where $r, s > 0$.

Theorem 11: Let $\hat{\alpha}_k$ $(k = 1, 2, \ldots, n)$ be a collection of q-rung orthopair fuzzy numbers. The aggregated result of q-ROFWBMDA operator is still q-rung orthopair fuzzy number and

$$\text{q-ROFWBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) = \left\langle \left(\frac{1}{1 + \left(\frac{1}{r+s}\left(\frac{1}{\frac{1}{n(n-1)}\sum^n_{i,j=1,i\neq j} u_{\hat{\alpha}_{ij}}}\right)\right)^{1/\gamma}}\right)^{1/q}, \left(1 - \frac{1}{1 + \left(\frac{1}{r+s}\left(\frac{1}{\frac{1}{n(n-1)}\sum^n_{i,j=1,i\neq j} v_{\hat{\alpha}_{ij}}}\right)\right)^{1/\gamma}}\right)^{1/q} \right\rangle, \quad (12)$$

where

$$u_{\hat{\alpha}_{ij}} = \frac{1}{r\frac{1}{w_i\left(\frac{\mu^q_{\hat{\alpha}_i}}{1-\mu^q_{\hat{\alpha}_i}}\right)^\gamma} + s\frac{1}{w_j\left(\frac{\mu^q_{\hat{\alpha}_j}}{1-\mu^q_{\hat{\alpha}_j}}\right)^\gamma}},$$

$$v_{\hat{\alpha}_{ij}} = \frac{1}{r\frac{1}{w_i\left(\frac{1-\nu^q_{\hat{\alpha}_i}}{\nu^q_{\hat{\alpha}_i}}\right)^\gamma} + s\frac{1}{w_j\left(\frac{1-\nu^q_{\hat{\alpha}_j}}{\nu^q_{\hat{\alpha}_j}}\right)^\gamma}}.$$

Theorem 12 (Commutativity): Let $\hat{\alpha}_k$ $(k = 1, 2, \ldots, n)$ be a collection of q-rung orthopair fuzzy numbers. If $(\hat{\alpha}'_1, \hat{\alpha}'_2, \ldots, \hat{\alpha}'_n)$ is any permutation of $(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n)$, then

$$\text{q-ROFWBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) = \text{q-ROFWBMDA}(\hat{\alpha}'_1, \hat{\alpha}'_2, \ldots, \hat{\alpha}'_n).$$

Definition 9: Let $\hat{\alpha}_k$ $(k = 1, 2, \ldots, n)$ be a collection of q-rung orthopair fuzzy numbers and $(w_1, w_2, \ldots, w_n)$ be the weight vector of $\hat{\alpha}_k$. The q-rung orthopair fuzzy weighted geometric BM Dombi averaging (q-ROFWGBMDA) operator is defined as

$$\text{q-ROFWGBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) = \frac{1}{r+s}\left(\otimes^n_{i,j=1,i\neq j} (r\hat{\alpha}_i^{w_i} \oplus s\hat{\alpha}_j^{w_j})^{\frac{1}{n(n-1)}}\right), \quad (13)$$

where $r, s > 0$.

Theorem 13: Let $\hat{\alpha}_k$ $(k = 1, 2, \ldots, n)$ be a collection of q-rung orthopair fuzzy numbers. The aggregated result of q-ROFWGBMDA operator is still q-rung orthopair fuzzy number and

$$\text{q-ROFWGBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) = \left\langle \left(1 - \frac{1}{1 + \left(\frac{1}{r+s}\left(\frac{1}{\left(\sum^n_{i,j=1,i\neq j} \frac{1}{n(n-1)} u_{\hat{\alpha}_{ij}}\right)}\right)\right)^{1/\gamma}}\right)^{1/q}, \left(\frac{1}{1 + \left(\frac{1}{r+s}\left(\frac{1}{\left(\sum^n_{i,j=1,i\neq j} \frac{1}{n(n-1)} v_{\hat{\alpha}_{ij}}\right)}\right)\right)^{1/\gamma}}\right)^{1/q} \right\rangle, \quad (14)$$

where $u_{\hat{\alpha}_{ij}} = \frac{1}{r\left(\frac{1}{\left(w_i\frac{1-\mu^q_{\hat{\alpha}_i}}{\mu^q_{\hat{\alpha}_i}}\right)^\gamma}\right) + s\left(\frac{1}{\left(w_j\frac{1-\mu^q_{\hat{\alpha}_j}}{\mu^q_{\hat{\alpha}_j}}\right)^\gamma}\right)}$, $v_{\hat{\alpha}_{ij}} = \frac{1}{r\left(\frac{1}{\left(w_i\frac{\nu^q_{\hat{\alpha}_i}}{1-\nu^q_{\hat{\alpha}_i}}\right)^\gamma}\right) + s\left(\frac{1}{\left(w_j\frac{\nu^q_{\hat{\alpha}_j}}{1-\nu^q_{\hat{\alpha}_j}}\right)^\gamma}\right)}$.

Theorem 14 (Commutativity): Let $\hat{\alpha}_k$ $(k = 1, 2, \ldots, n)$ be a collection of q-rung orthopair fuzzy numbers. If $(\hat{\alpha}'_1, \hat{\alpha}'_2, \ldots, \hat{\alpha}'_n)$ is any permutation of $(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n)$, then

$$\text{q-ROFWGBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) = \text{q-ROFWGBMDA}(\hat{\alpha}'_1, \hat{\alpha}'_2, \ldots, \hat{\alpha}'_n).$$

Definition 10: Let $\hat{\alpha}_k$ $(k = 1, 2, \ldots, n)$ be a collection of q-rung orthopair fuzzy numbers, which is partitioned into

m distinct sorts $P_1, P_2, \ldots, P_m$. The q-rung orthopair fuzzy partitioned BM Dombi averaging (q-ROFPBMDA) operator is defined as

$$\text{q-ROFPBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) = \frac{1}{m}\Big(\oplus_{i=1}^{m}\Big(\frac{1}{|P_h|}\oplus_{i\in P_h}\Big(\hat{\alpha}_i^r \otimes \Big(\frac{1}{|P_h|-1} \oplus_{j\in P_h, j\neq i}\hat{\alpha}_j^s\Big)\Big)\Big)^{\frac{1}{r+s}}\Big), \tag{15}$$

where $r, s \geq 0$ and $r + s > 0$. $|P_h|$ is the cardinality of P_h and m is the number of partitioned sorts and $\sum_{h=1}^{m}|P_h| = n$.

Theorem 15: Let $\hat{\alpha}_k$ $(k = 1, 2, \ldots, n)$ be a collection of q-rung orthopair fuzzy numbers. The aggregated result of the q-ROFPBMDA operator is still of a q-rung orthopair fuzzy number and we have

$$\text{q-ROFPBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) = \left\langle\left(1 - \frac{1}{1+\left(\frac{1}{m}\left(\sum_{i=1}^{m}\frac{1}{\left(\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|}\sum_{i\in P_h}\frac{1}{u_{\hat{\alpha}_{ij}}}}\right)}\right)\right)^{1/\gamma}}\right)^{1/q}, \left(\frac{1}{1+\left(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\left(\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|}\sum_{i\in P_h}\frac{1}{v_{\hat{\alpha}_{ij}}}}\right)}\right)^{1/\gamma}}\right)^{1/q}\right\rangle. \tag{16}$$

where $u_{\hat{\alpha}_{ij}} = r\left(\frac{1-\mu_{\hat{\alpha}_i}^q}{\mu_{\hat{\alpha}_i}^q}\right)^\gamma + \frac{1}{\frac{1}{|P_h|-1}\sum_{j\in P_h, j\neq i}\left(\frac{1}{s\left(\frac{1-\mu_{\hat{\alpha}_j}^q}{\mu_{\hat{\alpha}_j}^q}\right)^\gamma}\right)}$, $v_{\hat{\alpha}_{ij}} = r\left(\frac{v_{\hat{\alpha}_i}^q}{1-v_{\hat{\alpha}_i}^q}\right)^\gamma + \frac{1}{\frac{1}{|P_h|-1}\sum_{j\in P_h, j\neq i}\left(\frac{1}{s\left(\frac{v_{\hat{\alpha}_j}^q}{1-v_{\hat{\alpha}_j}^q}\right)^\gamma}\right)}$, $r, s \geq 0$ and $r + s > 0$. $|P_h|$ is the cardinality of P_h and m is the number of partitioned sorts and $\sum_{h=1}^{m}|P_h| = n$.

Proof:

$$\hat{\alpha}_j^s = \left\langle\left(\frac{1}{1+\left(s\left(\frac{1-\mu_{\hat{\alpha}_j}^q}{\mu_{\hat{\alpha}_j}^q}\right)^\gamma\right)^{1/\gamma}}\right)^{1/q}, \left(1 - \frac{1}{1+\left(s\left(\frac{v_{\hat{\alpha}_j}^q}{1-v_{\hat{\alpha}_j}^q}\right)^\gamma\right)^{1/\gamma}}\right)^{1/q}\right\rangle,$$

$$\oplus_{j\in P_h, j\neq i}\hat{\alpha}_j^s = \left\langle\left(1 - \frac{1}{1+\left(\sum_{j\in P_h, j\neq i}\left(\frac{1}{\left(s\left(\frac{1-\mu_{\hat{\alpha}_j}^q}{\mu_{\hat{\alpha}_j}^q}\right)^\gamma\right)}\right)\right)^{1/\gamma}}\right)^{1/q}, \left(\frac{1}{1+\left(\sum_{j\in P_h, j\neq i}\left(\frac{1}{\left(s\left(\frac{v_{\hat{\alpha}_j}^q}{1-v_{\hat{\alpha}_j}^q}\right)^\gamma\right)}\right)\right)^{1/\gamma}}\right)^{1/q}\right\rangle,$$

$$\frac{1}{|P_h|-1}\oplus_{j\in P_h, j\neq i}\hat{\alpha}_j^s = \left\langle\left(1 - \frac{1}{1+\left(\frac{1}{|P_h|-1}\sum_{j\in P_h, j\neq i}\left(\frac{1}{\left(s\left(\frac{1-\mu_{\hat{\alpha}_j}^q}{\mu_{\hat{\alpha}_j}^q}\right)^\gamma\right)}\right)\right)^{1/\gamma}}\right)^{1/q}, \left(\frac{1}{1+\left(\frac{1}{|P_h|-1}\sum_{j\in P_h, j\neq i}\left(\frac{1}{\left(s\left(\frac{v_{\hat{\alpha}_j}^q}{1-v_{\hat{\alpha}_j}^q}\right)^\gamma\right)}\right)\right)^{1/\gamma}}\right)^{1/q}\right\rangle,$$

$$\hat{\alpha}_i^r = \left\langle\left(\frac{1}{1+\left(r\left(\frac{1-\mu_{\hat{\alpha}_i}^q}{\mu_{\hat{\alpha}_i}^q}\right)^\gamma\right)^{1/\gamma}}\right)^{1/q}, \left(1 - \frac{1}{1+\left(r\left(\frac{v_{\hat{\alpha}_i}^q}{1-v_{\hat{\alpha}_i}^q}\right)^\gamma\right)^{1/\gamma}}\right)^{1/q}\right\rangle.$$

Let

$$u_{\hat{\alpha}_{ij}} = r\left(\frac{1-\mu_{\hat{\alpha}_i}^q}{\mu_{\hat{\alpha}_i}^q}\right)^\gamma + \frac{1}{\frac{1}{|P_h|-1}\sum_{j\in P_h, j\neq i}\left(\frac{1}{s\left(\frac{1-\mu_{\hat{\alpha}_j}^q}{\mu_{\hat{\alpha}_j}^q}\right)^\gamma}\right)},$$

$$v_{\hat{\alpha}_{ij}} = r\left(\frac{v_{\hat{\alpha}_i}^q}{1-v_{\hat{\alpha}_i}^q}\right)^\gamma + \frac{1}{\frac{1}{|P_h|-1}\sum_{j\in P_h, j\neq i}\left(\frac{1}{s\left(\frac{v_{\hat{\alpha}_j}^q}{1-v_{\hat{\alpha}_j}^q}\right)^\gamma}\right)},$$

$$\hat{\alpha}_i^r \otimes \left(\frac{1}{|P_h|-1}\oplus_{j\in P_h, j\neq i}\hat{\alpha}_j^s\right) = \left\langle\left(\frac{1}{1+(u_{\hat{\alpha}_{ij}})^{1/\gamma}}\right)^{1/q}, \left(1 - \frac{1}{1+\left(r(v_{\hat{\alpha}_{ij}}\right)^{1/\gamma}}\right)^{1/q}\right\rangle,$$

$$\oplus_{i\in P_h}\left(\hat{\alpha}_i^r \otimes \left(\frac{1}{|P_h|-1}\oplus_{j\in P_h, j\neq i}\hat{\alpha}_j^s\right)\right) = \left\langle\left(1 - \frac{1}{1+\left(\sum_{i\in P_h}\frac{1}{u_{\hat{\alpha}_{ij}}}\right)^{1/\gamma}}\right)^{1/q}, \left(\frac{1}{1+\left(\sum_{i\in P_h}\frac{1}{v_{\hat{\alpha}_{ij}}}\right)^{1/\gamma}}\right)^{1/q}\right\rangle,$$

$$\frac{1}{|P_h|}\oplus_{i\in P_h}\left(\hat{\alpha}_i^r \otimes \left(\frac{1}{|P_h|-1}\oplus_{j\in P_h, j\neq i}\hat{\alpha}_j^s\right)\right) = \left\langle\left(1 - \frac{1}{1+\left(\frac{1}{|P_h|}\sum_{i\in P_h}\frac{1}{u_{\hat{\alpha}_{ij}}}\right)^{1/\gamma}}\right)^{1/q}, \left(\frac{1}{1+\left(\frac{1}{|P_h|}\sum_{i\in P_h}\frac{1}{v_{\hat{\alpha}_{ij}}}\right)^{1/\gamma}}\right)^{1/q}\right\rangle,$$

$$\left(\frac{1}{|P_h|}\oplus_{i\in P_h}\left(\hat{\alpha}_i^r \otimes \left(\frac{1}{|P_h|-1}\oplus_{j\in P_h, j\neq i}\hat{\alpha}_j^s\right)\right)\right)^{\frac{1}{r+s}} = \left\langle\left(\frac{1}{1+\left(\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|}\sum_{i\in P_h}\frac{1}{u_{\hat{\alpha}_{ij}}}}\right)^{1/\gamma}}\right)^{1/q},$$

$$\big(1-\frac{1}{1+\big(\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|}\sum_{i\in P_h}\frac{1}{v_{\hat{\alpha}_{ij}}}}\big)^{1/\gamma}}\big)^{1/q}\big\rangle,$$

$$\oplus_{i=1}^{m}\big(\frac{1}{|P_h|}\oplus_{i\in P_h}\big(\hat{\alpha}_i^r\otimes\big(\frac{1}{|P_h|-1}\oplus_{j\in P_h,j\neq i}\hat{\alpha}_j^s\big)\big)\big)^{\frac{1}{r+s}}$$
$$=\big\langle\big(1-\frac{1}{1+\big(\sum_{i=1}^{m}\frac{1}{\big(\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|}\sum_{i\in P_h}\frac{1}{u_{\hat{\alpha}_{ij}}}}\big)}\big)^{1/\gamma}}\big)^{1/q},$$
$$\big(\frac{1}{1+\big(\sum_{i=1}^{m}\frac{1}{\big(\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|}\sum_{i\in P_h}\frac{1}{v_{\hat{\alpha}_{ij}}}}\big)}\big)^{1/\gamma}}\big)^{1/q}\big\rangle,$$

$$\frac{1}{m}\big(\oplus_{i=1}^{m}\big(\frac{1}{|P_h|}\oplus_{i\in P_h}\big(\hat{\alpha}_i^r\otimes\big(\frac{1}{|P_h|-1}\oplus_{j\in P_h,j\neq i}\hat{\alpha}_j^s\big)\big)\big)^{\frac{1}{r+s}}\big)$$
$$=\big\langle\big(1-\frac{1}{1+\big(\frac{1}{m}\big(\sum_{i=1}^{m}\frac{1}{\big(\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|}\sum_{i\in P_h}\frac{1}{u_{\hat{\alpha}_{ij}}}}\big)}\big)\big)^{1/\gamma}}\big)^{1/q},$$
$$\big(\frac{1}{1+\big(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\big(\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|}\sum_{i\in P_h}\frac{1}{v_{\hat{\alpha}_{ij}}}}\big)}\big)^{1/\gamma}}\big)^{1/q}\big\rangle.$$

Moreover, $0\leq\mu_{\hat{\alpha}_j}^q+v_{\hat{\alpha}_j}^q\leq 1$, $\mu_{\hat{\alpha}_j}^q\leq 1-v_{\hat{\alpha}_j}^q$, $v_{\hat{\alpha}_j}^q\leq 1-\mu_{\hat{\alpha}_j}^q$, $\frac{v_{\hat{\alpha}_j}^q}{1-v_{\hat{\alpha}_j}^q}\leq\frac{1-\mu_{\hat{\alpha}_j}^q}{\mu_{\hat{\alpha}_j}^q}$, $s\big(\frac{v_{\hat{\alpha}_j}^q}{1-v_{\hat{\alpha}_j}^q}\big)^\gamma\leq s\big(\frac{1-\mu_{\hat{\alpha}_j}^q}{\mu_{\hat{\alpha}_j}^q}\big)^\gamma$, $\frac{v_{\hat{\alpha}_i}^q}{1-v_{\hat{\alpha}_i}^q}\leq\frac{1-\mu_{\hat{\alpha}_i}^q}{\mu_{\hat{\alpha}_i}^q}$, $r\big(\frac{v_{\hat{\alpha}_i}^q}{1-v_{\hat{\alpha}_i}^q}\big)^\gamma\leq r\big(\frac{1-\mu_{\hat{\alpha}_i}^q}{\mu_{\hat{\alpha}_i}^q}\big)^\gamma$.

$$\frac{1}{|P_h|-1}\sum_{j\in P_h,j\neq i}\big(\frac{1}{s\big(\frac{v_{\hat{\alpha}_j}^q}{1-v_{\hat{\alpha}_j}^q}\big)^\gamma}\big)$$
$$\geq\frac{1}{|P_h|-1}\sum_{j\in P_h,j\neq i}\big(\frac{1}{s\big(\frac{1-\mu_{\hat{\alpha}_j}^q}{\mu_{\hat{\alpha}_j}^q}\big)^\gamma}\big),$$
$$r\big(\frac{v_{\hat{\alpha}_i}^q}{1-v_{\hat{\alpha}_i}^q}\big)^\gamma+\frac{1}{\frac{1}{|P_h|-1}\sum_{j\in P_h,j\neq i}\big(\frac{1}{s\big(\frac{v_{\hat{\alpha}_j}^q}{1-v_{\hat{\alpha}_j}^q}\big)^\gamma}\big)}$$
$$\leq r\big(\frac{1-\mu_{\hat{\alpha}_i}^q}{\mu_{\hat{\alpha}_i}^q}\big)^\gamma+\frac{1}{\frac{1}{|P_h|-1}\sum_{j\in P_h,j\neq i}\big(\frac{1}{s\big(\frac{1-\mu_{\hat{\alpha}_j}^q}{\mu_{\hat{\alpha}_j}^q}\big)^\gamma}\big)},$$

that is $v_{\hat{\alpha}_{ij}}\leq u_{\hat{\alpha}_{ij}}$, then

$$\frac{1}{|P_h|}\sum_{j\in P_h,j\neq i}\frac{1}{v_{\hat{\alpha}_{ij}}}$$
$$\geq\frac{1}{|P_h|}\sum_{j\in P_h,j\neq i}\frac{1}{u_{\hat{\alpha}_{ij}}},$$
$$\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|}\sum_{j\in P_h,j\neq i}\frac{1}{v_{\hat{\alpha}_{ij}}}}$$
$$\leq\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|}\sum_{j\in P_h,j\neq i}\frac{1}{u_{\hat{\alpha}_{ij}}}},$$
$$\big(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|-1}\sum_{j\in P_h,j\neq i}\frac{1}{v_{\hat{\alpha}_{ij}}}}}\big)^{1/\gamma}$$
$$\geq\big(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|-1}\sum_{j\in P_h,j\neq i}\frac{1}{u_{\hat{\alpha}_{ij}}}}}\big)^{1/\gamma},$$
$$\frac{1}{1+\big(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|-1}\sum_{j\in P_h,j\neq i}\frac{1}{v_{\hat{\alpha}_{ij}}}}}\big)^{1/\gamma}}$$
$$\leq\frac{1}{1+\big(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|-1}\sum_{j\in P_h,j\neq i}\frac{1}{u_{\hat{\alpha}_{ij}}}}}\big)^{1/\gamma}},$$
$$0\leq 1-\frac{1}{1+\big(\frac{1}{m}\big(\sum_{i=1}^{m}\frac{1}{\big(\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|}\sum_{i\in P_h}\frac{1}{u_{\hat{\alpha}_{ij}}}}\big)}\big)\big)^{1/\gamma}}$$
$$+\frac{1}{1+\big(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\big(\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|}\sum_{i\in P_h}\frac{1}{v_{\hat{\alpha}_{ij}}}}\big)}\big)^{1/\gamma}}\leq 1.$$

Hence, by using the score function, we can get the aggregated result of the q-ROFPBMDA operator is still a q-rung orthopair fuzzy number.

Theorem 16 (Idempotency): Let $\hat{\alpha}_k=<\mu_{\hat{\alpha}_k},v_{\hat{\alpha}_k}>(k=1,2,\ldots,n)$ be a collection of q-rung orthopair fuzzy numbers. If $\hat{\alpha}_k=\hat{\alpha}$, that is $<\mu_{\hat{\alpha}_k},v_{\hat{\alpha}_k}>=<\mu_{\hat{\alpha}},v_{\hat{\alpha}}>(k=1,2,\ldots,n)$. Then

$$\text{q-ROFPBMDA}(\hat{\alpha}_1,\hat{\alpha}_2,\ldots,\hat{\alpha}_n)=\hat{\alpha}.$$

Proof: Since $\mu_{\hat{\alpha}_i}=\mu_{\hat{\alpha}_j}=\mu_{\hat{\alpha}}$, $v_{\hat{\alpha}_i}=v_{\hat{\alpha}_j}=v_{\hat{\alpha}}$, we have

$$u_{\hat{\alpha}_{ij}}=r\big(\frac{1-\mu_{\hat{\alpha}_i}^q}{\mu_{\hat{\alpha}_i}^q}\big)^\gamma+\frac{1}{\frac{1}{|P_h|-1}\sum_{j\in P_h,j\neq i}\big(\frac{1}{s\big(\frac{1-\mu_{\hat{\alpha}_j}^q}{\mu_{\hat{\alpha}_j}^q}\big)^\gamma}\big)}$$
$$=r\big(\frac{1-\mu_{\hat{\alpha}}^q}{\mu_{\hat{\alpha}}^q}\big)^\gamma+\frac{1}{\frac{1}{|P_h|-1}\sum_{j\in P_h,j\neq i}\big(\frac{1}{s\big(\frac{1-\mu_{\hat{\alpha}}^q}{\mu_{\hat{\alpha}}^q}\big)^\gamma}\big)}$$
$$=r\big(\frac{1-\mu_{\hat{\alpha}}^q}{\mu_{\hat{\alpha}}^q}\big)^\gamma+s\big(\frac{1-\mu_{\hat{\alpha}}^q}{\mu_{\hat{\alpha}}^q}\big)^\gamma=(r+s)\big(\frac{1-\mu_{\hat{\alpha}}^q}{\mu_{\hat{\alpha}}^q}\big)^\gamma,$$
$$\big(1-\frac{1}{1+\big(\frac{1}{m}\big(\sum_{i=1}^{m}\frac{1}{\big(\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|}\sum_{i\in P_h}\frac{1}{u_{\hat{\alpha}_{ij}}}}\big)}\big)\big)^{1/\gamma}}\big)^{1/q}$$
$$=\big(1-\frac{1}{1+\big(\frac{1}{m}\big(\sum_{i=1}^{m}\frac{1}{\big(\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|}\sum_{i\in P_h}\frac{1}{(r+s)\big(\frac{1-\mu_{\hat{\alpha}}^q}{\mu_{\hat{\alpha}}^q}\big)^\gamma}}\big)}\big)\big)^{1/\gamma}}\big)^{1/q}$$

$$= \left(1-\frac{1}{1+\left(\frac{1}{m}\left(\sum_{i=1}^{m}\frac{1}{\left(\frac{1}{r+s}\frac{1}{\frac{1}{(r+s)\left(\frac{1-\mu_{\hat{\alpha}}^{q}}{\mu_{\hat{\alpha}}^{q}}\right)^{\gamma}}}\right)}\right)\right)^{1/\gamma}}\right)^{1/q}$$

$$= \left(1-\frac{1}{1+\left(\frac{1}{m}\left(\sum_{i=1}^{m}\frac{1}{\left(\frac{1}{r+s}(r+s)\left(\frac{1-\mu_{\hat{\alpha}}^{q}}{\mu_{\hat{\alpha}}^{q}}\right)^{\gamma}\right)}\right)\right)^{1/\gamma}}\right)^{1/q}$$

$$= \left(1-\frac{1}{1+\left(\frac{1}{m}\left(\sum_{i=1}^{m}\frac{1}{\left(\left(\frac{1-\mu_{\hat{\alpha}}^{q}}{\mu_{\hat{\alpha}}^{q}}\right)^{\gamma}\right)}\right)\right)^{1/\gamma}}\right)^{1/q}$$

$$= \left(1-\frac{1}{1+\left(\frac{1}{\left(\frac{1-\mu_{\hat{\alpha}}^{q}}{\mu_{\hat{\alpha}}^{q}}\right)^{\gamma}}\right)^{1/\gamma}}\right)^{1/q}$$

$$= \left(1-\frac{1}{1+\frac{1}{\frac{1-\mu_{\hat{\alpha}}^{q}}{\mu_{\hat{\alpha}}^{q}}}}\right)^{1/q} = \left(1-\frac{1}{1+\frac{\mu_{\hat{\alpha}}^{q}}{1-\mu_{\hat{\alpha}}^{q}}}\right)^{1/q} = \mu_{\hat{\alpha}},$$

$$v_{\hat{\alpha}_{ij}} = r\left(\frac{v_{\hat{\alpha}_i}^{q}}{1-v_{\hat{\alpha}_i}^{q}}\right)^{\gamma} + \frac{1}{\frac{1}{|P_h|-1}\sum_{j\in P_h, j\neq i}\left(\frac{1}{s\left(\frac{v_{\hat{\alpha}_j}^{q}}{1-v_{\hat{\alpha}_j}^{q}}\right)^{\gamma}}\right)}$$

$$= r\left(\frac{v_{\hat{\alpha}}^{q}}{1-v_{\hat{\alpha}}^{q}}\right)^{\gamma} + \frac{1}{\frac{1}{|P_h|-1}\sum_{j\in P_h, j\neq i}\left(\frac{1}{s\left(\frac{v_{\hat{\alpha}}^{q}}{1-v_{\hat{\alpha}}^{q}}\right)^{\gamma}}\right)}$$

$$= r\left(\frac{v_{\hat{\alpha}}^{q}}{1-v_{\hat{\alpha}}^{q}}\right)^{\gamma} + s\left(\frac{v_{\hat{\alpha}}^{q}}{1-v_{\hat{\alpha}}^{q}}\right)^{\gamma} = (r+s)\left(\frac{v_{\hat{\alpha}}^{q}}{1-v_{\hat{\alpha}}^{q}}\right)^{\gamma}.$$

Since $v_{\hat{\alpha}_i} = v_{\hat{\alpha}_j} = v_{\hat{\alpha}}$,

$$\left(\frac{1}{1+\left(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\left(\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|}\sum_{i\in P_h}\frac{1}{v_{\hat{\alpha}_{ij}}}}\right)}\right)^{1/\gamma}}\right)^{1/q}$$

$$= \left(\frac{1}{1+\left(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\left(\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|}\sum_{i\in P_h}\frac{1}{(r+s)\left(\frac{v_{\hat{\alpha}}^{q}}{1-v_{\hat{\alpha}}^{q}}\right)^{\gamma}}}\right)}\right)^{1/\gamma}}\right)^{1/q}$$

$$= \left(\frac{1}{1+\left(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\left(\frac{1}{r+s}\frac{1}{\frac{1}{(r+s)\left(\frac{v_{\hat{\alpha}}^{q}}{1-v_{\hat{\alpha}}^{q}}\right)^{\gamma}}}\right)}\right)^{1/\gamma}}\right)^{1/q}$$

$$= \left(\frac{1}{1+\left(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\left(\frac{1}{r+s}(r+s)\left(\frac{v_{\hat{\alpha}}^{q}}{1-v_{\hat{\alpha}}^{q}}\right)^{\gamma}\right)}\right)^{1/\gamma}}\right)^{1/q}$$

$$= \left(\frac{1}{1+\left(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\left(\left(\frac{v_{\hat{\alpha}}^{q}}{1-v_{\hat{\alpha}}^{q}}\right)^{\gamma}\right)}\right)^{1/\gamma}}\right)^{1/q}$$

$$= \left(\frac{1}{1+\left(\frac{1}{\left(\left(\frac{v_{\hat{\alpha}}^{q}}{1-v_{\hat{\alpha}}^{q}}\right)^{\gamma}\right)}\right)^{1/\gamma}}\right)^{1/q} = \left(\frac{1}{1+\frac{1}{\frac{v_{\hat{\alpha}}^{q}}{1-v_{\hat{\alpha}}^{q}}}}\right)^{1/q} = v_{\hat{\alpha}}.$$

Hence, q-ROFPBMDA$(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) = \hat{\alpha}$.

Theorem 17 (Commutativity): Let $\hat{\alpha}_k = < \mu_{\hat{\alpha}_k}, v_{\hat{\alpha}_k} >$ and $\hat{\alpha}'_k = < \mu'_{\hat{\alpha}_k}, v'_{\hat{\alpha}_k} >$ $(k = 1, 2, \ldots, n)$ be two collections of q-rung orthopair fuzzy numbers. If $(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n)$ is any permutation of $(\hat{\alpha}'_1, \hat{\alpha}'_2, \ldots, \hat{\alpha}'_n)$ and they have the same partitioned sorts, then

$$\text{q-ROFPBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) = \text{q-ROFPBMDA}(\hat{\alpha}'_1, \hat{\alpha}'_2, \ldots, \hat{\alpha}'_n).$$

Proof:

$$\begin{aligned}&\text{q-ROFPBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n)\\ &= \frac{1}{m}\left(\oplus_{i=1}^{m}\left(\frac{1}{|P_h|}\oplus_{i\in P_h}\left(\hat{\alpha}_i^{r}\otimes\left(\frac{1}{|P_h|-1}\oplus_{j\in P_h, j\neq i}\hat{\alpha}_j^{s}\right)\right)\right)^{\frac{1}{r+s}}\right)\\ &= \frac{1}{m}\left(\oplus_{i=1}^{m}\left(\frac{1}{|P_h|}\oplus_{i\in P_h}\left((\hat{\alpha}'_i)^{r}\otimes\left(\frac{1}{|P_h|-1}\oplus_{j\in P_h, j\neq i}(\hat{\alpha}'_j)^{s}\right)\right)\right)^{\frac{1}{r+s}}\right)\\ &= \text{q-ROFPBMDA}(\hat{\alpha}'_1, \hat{\alpha}'_2, \ldots, \hat{\alpha}'_n).\end{aligned}$$

Theorem 18 (Monotonicity): Let $\hat{\alpha}_k = < \mu_{\hat{\alpha}_k}, v_{\hat{\alpha}_k} >$ and $\hat{\beta}_k = < \mu_{\hat{\beta}_k}, v_{\hat{\beta}_k} >$ be two collection of q-rung orthopair fuzzy numbers. If $\mu_{\hat{\alpha}_k} \geq \mu_{\hat{\beta}_k}$ and $v_{\hat{\alpha}_k} \leq v_{\hat{\beta}_k}$ for all $k = 1, 2, \ldots, n$, then

$$\text{q-ROFPBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) \geq \text{q-ROFPBMDA}(\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_n).$$

Proof: Since $\mu_{\hat{\alpha}_k} \geq \mu_{\hat{\beta}_k}$, $\mu_{\hat{\alpha}_k}^{q} \geq \mu_{\hat{\beta}_k}^{q}$, $(\frac{1-x}{x})' = -\frac{1}{x^2}$, $\frac{1-\mu_{\hat{\alpha}_k}^{q}}{\mu_{\hat{\alpha}_k}^{q}} \leq \frac{1-\mu_{\hat{\alpha}_k}^{q}}{\mu_{\hat{\alpha}_k}^{q}}$, $r\left(\frac{1-\mu_{\hat{\alpha}_i}^{q}}{\mu_{\hat{\alpha}_i}^{q}}\right)^{\gamma} \leq r\left(\frac{1-\mu_{\hat{\beta}_i}^{q}}{\mu_{\hat{\beta}_i}^{q}}\right)^{\gamma}$,

$$\frac{1}{|P_h|-1}\sum_{j\in P_h, j\neq i}\frac{1}{s\left(\frac{1-\mu_{\hat{\alpha}_j}^{q}}{\mu_{\hat{\alpha}_j}^{q}}\right)^{\gamma}} \geq \frac{1}{|P_h|-1}\sum_{j\in P_h, j\neq i}\frac{1}{s\left(\frac{1-\mu_{\hat{\beta}_j}^{q}}{\mu_{\hat{\beta}_j}^{q}}\right)^{\gamma}},$$

$$\frac{1}{\frac{1}{|P_h|-1}\sum_{j\in P_h, j\neq i}\frac{1}{s\left(\frac{1-\mu_{\hat{\alpha}_j}^{q}}{\mu_{\hat{\alpha}_j}^{q}}\right)^{\gamma}}} \leq \frac{1}{\frac{1}{|P_h|-1}\sum_{j\in P_h, j\neq i}\frac{1}{s\left(\frac{1-\mu_{\hat{\beta}_j}^{q}}{\mu_{\hat{\beta}_j}^{q}}\right)^{\gamma}}},$$

$$r\left(\frac{1-\mu_{\hat{\alpha}_i}^{q}}{\mu_{\hat{\alpha}_i}^{q}}\right)^{\gamma} + \frac{1}{\frac{1}{|P_h|-1}\sum_{j\in P_h, j\neq i}\frac{1}{1}s\left(\frac{1-\mu_{\hat{\alpha}_j}^{q}}{\mu_{\hat{\alpha}_j}^{q}}\right)^{\gamma}} \leq r\left(\frac{1-\mu_{\hat{\beta}_i}^{q}}{\mu_{\hat{\beta}_i}^{q}}\right)^{\gamma} + \frac{1}{\frac{1}{|P_h|-1}\sum_{j\in P_h, j\neq i}\frac{1}{s\left(\frac{1-\mu_{\hat{\beta}_j}^{q}}{\mu_{\hat{\beta}_j}^{q}}\right)^{\gamma}}}.$$

Let

$$u_{\hat{\alpha}_{ij}} = r\left(\frac{1-\mu^q_{\hat{\alpha}_i}}{\mu^q_{\hat{\alpha}_i}}\right)^\gamma + \frac{1}{\frac{1}{|P_h|-1}\sum_{j\in P_h, j\neq i}\frac{1}{1}s\left(\frac{1-\mu^q_{\hat{\alpha}_j}}{\mu^q_{\hat{\alpha}_j}}\right)^\gamma},$$

$$u_{\hat{\beta}_{ij}} = r\left(\frac{1-\mu^q_{\hat{\beta}_i}}{\mu^q_{\hat{\beta}_i}}\right)^\gamma + \frac{1}{\frac{1}{|P_h|-1}\sum_{j\in P_h, j\neq i}\frac{1}{s\left(\frac{1-\mu^q_{\hat{\beta}_j}}{\mu^q_{\hat{\beta}_j}}\right)^\gamma}}.$$

Then $\frac{1}{|P_h|}\sum_{i\in P_h}\frac{1}{u_{\alpha_{\hat{\alpha}_{ij}}}} \geq \frac{1}{|P_h|}\sum_{i\in P_h}\frac{1}{u_{\beta_{\hat{\alpha}_{ij}}}}$.

$$\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|}\sum_{i\in P_h}\frac{1}{u_{\hat{\alpha}_{ij}}}} \leq \frac{1}{r+s}\frac{1}{\frac{1}{|P_h|}\sum_{i\in P_h}\frac{1}{u_{\hat{\beta}_{ij}}}},$$

$$\frac{1}{m}\sum_{i=1}^m \frac{1}{\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|}\sum_{i\in P_h}\frac{1}{u_{\hat{\alpha}_{ij}}}}} \geq \frac{1}{m}\sum_{i=1}^m \frac{1}{\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|}\sum_{i\in P_h}\frac{1}{u_{\hat{\beta}_{ij}}}}},$$

$$\frac{1}{1+\frac{1}{m}\sum_{i=1}^m \frac{1}{\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|}\sum_{i\in P_h}\frac{1}{u_{\hat{\alpha}_{ij}}}}}} \leq \frac{1}{1+\frac{1}{m}\sum_{i=1}^m \frac{1}{\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|}\sum_{i\in P_h}\frac{1}{u_{\hat{\beta}_{ij}}}}}},$$

$$\left(1-\frac{1}{1+\left(\frac{1}{m}\left(\sum_{i=1}^m\frac{1}{\left(\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|}\sum_{i\in P_h}\frac{1}{u_{\hat{\alpha}_{ij}}}}\right)}\right)\right)^{1/\gamma}}\right)^{1/q} \geq \left(1-\frac{1}{1+\left(\frac{1}{m}\left(\sum_{i=1}^m\frac{1}{\left(\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|}\sum_{i\in P_h}\frac{1}{u_{\hat{\beta}_{ij}}}}\right)}\right)\right)^{1/\gamma}}\right)^{1/q}.$$

$\nu_{\hat{\alpha}_k} \leq \nu_{\hat{\beta}_k}$, $\nu^q_{\hat{\alpha}_k} \leq \nu^q_{\hat{\beta}_k}$, $(\frac{x}{1-x})' = \frac{1}{(1-x)^2} > 0$, $\frac{\nu^q_{\hat{\alpha}_k}}{1-\nu^q_{\hat{\alpha}_k}} \leq \frac{\nu^q_{\hat{\beta}_k}}{1-\nu^q_{\hat{\beta}_k}}$, $s(\frac{\nu^q_{\hat{\alpha}_k}}{1-\nu^q_{\hat{\alpha}_k}})^\gamma \leq s(\frac{\nu^q_{\hat{\beta}_k}}{1-\nu^q_{\hat{\beta}_k}})^\gamma$, $r(\frac{\nu^q_{\hat{\alpha}_k}}{1-\nu^q_{\hat{\alpha}_k}})^\gamma \leq r(\frac{\nu^q_{\hat{\beta}_k}}{1-\nu^q_{\hat{\beta}_k}})^\gamma$,

$$\frac{1}{|P_h|-1}\sum_{j\in P_h, j\neq i}\left(\frac{1}{s\left(\frac{\nu^q_{\hat{\alpha}_j}}{1-\nu^q_{\hat{\alpha}_j}}\right)^\gamma}\right) \geq \frac{1}{|P_h|-1}\sum_{j\in P_h, j\neq i}\left(\frac{1}{s\left(\frac{\nu^q_{\hat{\beta}_j}}{1-\nu^q_{\hat{\beta}_j}}\right)^\gamma}\right),$$

$$r\left(\frac{\nu^q_{\hat{\alpha}_i}}{1-\nu^q_{\hat{\alpha}_i}}\right)^\gamma + \frac{1}{\frac{1}{|P_h|-1}\sum_{j\in P_h, j\neq i}\left(\frac{1}{s\left(\frac{\nu^q_{\hat{\alpha}_j}}{1-\nu^q_{\hat{\alpha}_j}}\right)^\gamma}\right)} \leq r\left(\frac{\nu^q_{\hat{\beta}_i}}{1-\nu^q_{\hat{\beta}_i}}\right)^\gamma + \frac{1}{\frac{1}{|P_h|-1}\sum_{j\in P_h, j\neq i}\left(\frac{1}{s\left(\frac{\nu^q_{\hat{\beta}_j}}{1-\nu^q_{\hat{\beta}_j}}\right)^\gamma}\right)}.$$

Let

$$v_{\hat{\alpha}_{ij}} = r\left(\frac{\nu^q_{\hat{\alpha}_i}}{1-\nu^q_{\hat{\alpha}_i}}\right)^\gamma + \frac{1}{\frac{1}{|P_h|-1}\sum_{j\in P_h, j\neq i}\left(\frac{1}{s\left(\frac{\nu^q_{\hat{\alpha}_j}}{1-\nu^q_{\hat{\alpha}_j}}\right)^\gamma}\right)},$$

$$v_{\hat{\beta}_{ij}} = r\left(\frac{\nu^q_{\hat{\beta}_i}}{1-\nu^q_{\hat{\beta}_i}}\right)^\gamma + \frac{1}{\frac{1}{|P_h|-1}\sum_{j\in P_h, j\neq i}\left(\frac{1}{s\left(\frac{\nu^q_{\hat{\beta}_j}}{1-\nu^q_{\hat{\beta}_j}}\right)^\gamma}\right)}.$$

$$\frac{1}{|P_h|}\sum_{i\in P_h}\frac{1}{v_{\hat{\alpha}_{ij}}} \geq \frac{1}{|P_h|}\sum_{i\in P_h}\frac{1}{v_{\hat{\beta}_{ij}}},$$

$$\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|}\sum_{i\in P_h}\frac{1}{v_{\hat{\alpha}_{ij}}}} \geq \frac{1}{r+s}\frac{1}{\frac{1}{|P_h|}\sum_{i\in P_h}\frac{1}{v_{\hat{\beta}_{ij}}}},$$

$$\frac{1}{m}\sum_{i=1}^m \frac{1}{\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|}\sum_{i\in P_h}\frac{1}{v_{\hat{\alpha}_{ij}}}}} \leq \frac{1}{m}\sum_{i=1}^m \frac{1}{\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|}\sum_{i\in P_h}\frac{1}{v_{\hat{\beta}_{ij}}}}},$$

$$\left(\frac{1}{1+\frac{1}{m}\sum_{i=1}^m \frac{1}{\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|}\sum_{i\in P_h}\frac{1}{v_{\hat{\alpha}_{ij}}}}}}\right)^{1/q} \leq \left(\frac{1}{1+\frac{1}{m}\sum_{i=1}^m \frac{1}{\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|}\sum_{i\in P_h}\frac{1}{v_{\hat{\beta}_{ij}}}}}}\right)^{1/q}.$$

By using the score function, we can get

$$\text{q-ROFPBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) \geq \text{q-ROFPBMDA}(\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_n).$$

Theorem 19 (Boundedness): Let $\hat{\alpha}_k =< \mu_{\hat{\alpha}_k}, \nu_{\hat{\alpha}_k} >$ $(k = 1, 2, \ldots, n)$ be a collection of q-rung orthopair fuzzy numbers. $\hat{\alpha}^- =< \mu^-, \nu^+ >=< \min_k \mu_{\hat{\alpha}_k}, \max_k \nu_{\hat{\alpha}_k} >$, $\hat{\alpha}^+ =< \mu^+, \nu^- >=< \max \mu_{\hat{\alpha}_k}, \min \nu_{\hat{\alpha}_k} >$, then

$$\hat{\alpha}^- \leq \text{q-ROFPBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) \leq \hat{\alpha}^+.$$

Proof: The property of boundedness can be proved by using the property of monotonicity.

Definition 11: Let $\hat{\alpha}_k =< \mu_{\hat{\alpha}_k}, \nu_{\hat{\alpha}_k} >$ $(k = 1, 2, \ldots, n)$ be a collection of q-rung orthopair fuzzy numbers, which are partitioned into m distinct sorts $P_1, P_2, \ldots, P_m$, $\sum_{j=1}^m |P_j| = n$. The q-rung orthopair fuzzy partitioned weighted BM Dombi averaging (q-ROFPWBMDA) operator is defined as

$$\text{q-ROFPWBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) = \frac{1}{m}\left(\oplus_{i=1}^m\left(\oplus_{i,j\in P_h}\left(w_i w_j\left(\hat{\alpha}_i^r \otimes \hat{\alpha}_j^s\right)\right)\right)^{\frac{1}{r+s}}\right), \quad (17)$$

where $(w_1, w_2, \ldots, w_n)$ is the weight vector of $(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n)$ satisfying $w_j \geq 0$ $(j = 1, 2, \ldots, n)$ and $\sum_{j=1}^n w_j = 1$.

Theorem 20: Let $\hat{\alpha}_k =< \mu_{\hat{\alpha}_k}, \nu_{\hat{\alpha}_k} >$ $(k = 1, 2, \ldots, n)$ be a collection of q-rung orthopair fuzzy numbers, which are partitioned into m distinct sorts $P_1, P_2, \ldots, P_m$, $\sum_{j=1}^{m} |P_j| = n$. The aggregated result of the q-ROFPWBMDA operator is still a q-rung orthopair fuzzy number, which has the following form

$$\begin{aligned}&\text{q-ROFPWBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n)\\ &= \Big\langle\Big(1 - \frac{1}{1 + \Big(\frac{1}{m}\sum_{i=1}^{m} \frac{1}{\frac{1}{r+s}\frac{1}{\sum_{i,j\in P_h} w_i w_j\left(\frac{1}{u_{\hat{\alpha}_{ij}}}\right)}}\Big)^{1/\gamma}}\Big)^{1/q},\\ &\quad\Big(\frac{1}{1 + \Big(\frac{1}{m}\sum_{i=1}^{m} \frac{1}{\frac{1}{r+s}\frac{1}{\sum_{i,j\in P_h} w_i w_j\left(\frac{1}{v_{\hat{\alpha}_{ij}}}\right)}}\Big)^{1/\gamma}}\Big)^{1/q}\Big\rangle, \end{aligned} \tag{18}$$

where $u_{\hat{\alpha}_{ij}} = r\left(\frac{1-\mu_{\hat{\alpha}_i}^q}{\mu_{\hat{\alpha}_i}^q}\right)^\gamma + s\left(\frac{1-\mu_{\hat{\alpha}_j}^q}{\mu_{\hat{\alpha}_j}^q}\right)^\gamma$, $v_{\hat{\alpha}_{ij}} = r\left(\frac{\nu_{\hat{\alpha}_i}^q}{1-\nu_{\hat{\alpha}_i}^q}\right)^\gamma + s\left(\frac{\nu_{\hat{\alpha}_j}^q}{1-\nu_{\hat{\alpha}_j}^q}\right)^\gamma$, $r, s > 0$ and $r + s > 0$. $(w_1, w_2, \ldots, w_n)$ is the weight vector of $(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n)$ and $w_j \geq 0$ $(j = 1, 2, \ldots, n)$, $\sum_{j=1}^{n} w_j = 1$.

Theorem 21 (Idempotency): Let $\hat{\alpha}_k = \hat{\alpha}$, $< \mu_{\hat{\alpha}_k}, \nu_{\hat{\alpha}_k} >=< \mu_{\hat{\alpha}}, \nu_{\hat{\alpha}} >$ $(i = 1, 2, \ldots, n)$. Then

$$\text{q-ROFPWBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) = \hat{\alpha}.$$

Theorem 22 (Monotonicity): Let $\hat{\alpha}_k =< \mu_{\hat{\alpha}_k}, \nu_{\hat{\alpha}_k} >$ and $\hat{\beta}_k =< \mu_{\hat{\beta}_k}, \nu_{\hat{\beta}_k} >$ $(k = 1, 2, \ldots, n)$ be two collections of q-rung orthopair fuzzy numbers, which have the same partitioned sorts $P_1, P_2, \ldots, P_m$, $\sum_{j=1}^{m} |P_j| = n$. $(w_1, w_2, \ldots, w_n)$ is the weight vector of $(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n)$ satisfying $w_j \geq 0$, $j = 1, 2, \ldots, n$, and $\sum_{j=1}^{n} w_j = 1$. If $\mu_{\hat{\alpha}_k} \leq \mu_{\hat{\beta}_k}$ and $\nu_{\hat{\alpha}_k} \geq \nu_{\hat{\beta}_k}$, then

$$\begin{aligned}&\text{q-ROFPWBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n)\\ &\qquad\leq \text{q-ROFPWBMDA}(\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_n).\end{aligned}$$

Theorem 23 (Boundedness): Let $\hat{\alpha}_k =< \mu_{\hat{\alpha}_k}, \nu_{\hat{\alpha}_k} >$ $(k = 1, 2, \ldots, n)$ be a collection of q-rung orthopair fuzzy numbers. $\hat{\alpha}^- =< \mu^-, \nu^+ >=< \min_k \mu_{\hat{\alpha}_k}, \max_k \nu_{\hat{\alpha}_k} >$, $\hat{\alpha}^+ =< \mu^+, \nu^- >=< \max \mu_{\hat{\alpha}_k}, \min \nu_{\hat{\alpha}_k} >$, then

$$\hat{\alpha}^- \leq \text{q-ROFPWBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) \leq \hat{\alpha}^+.$$

The Boundedness property of q-ROFPWBMDA operator can be proved easily by using the property of monotonicity.

Definition 12: Let $\hat{\alpha}_k =< \mu_{\hat{\alpha}_k}, \nu_{\hat{\alpha}_k} >$ $(k = 1, 2, \ldots, n)$ be a collection of q-rung orthopair fuzzy numbers, which are partitioned into m distinct sorts $P_1, P_2, \ldots, P_m$. The q-rung orthopair fuzzy partitioned geometric BM Dombi averaging (q-ROFPGBMDA) operator is defined as

$$\begin{aligned}&\text{q-ROFPGBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n)\\ &= \Big(\otimes_{i=1}^{m}\Big(\frac{1}{r+s}\Big(\otimes_{i,j\in P_h, i\neq j}((r\hat{\alpha}_i) \oplus (s\hat{\alpha}_j))\Big)^{\frac{1}{|P_h|(|P_h|-1)}}\Big)\Big)^{\frac{1}{m}},\end{aligned} \tag{19}$$

where $r, s \geq 0$ and $r + s > 0$. $| P_h |$ is the cardinality of P_h and m is the number of partitioned sorts and $\sum_{h=1}^{m} |P_h| = n$.

Theorem 24: Let $\hat{\alpha}_k =< \mu_{\hat{\alpha}_k}, \nu_{\hat{\alpha}_k} >$ $(k = 1, 2, \ldots, n)$ be a collection of q-rung orthopair fuzzy numbers. The aggregated result of the q-ROFPGBMDA operator is still a q-rung orthopair fuzzy number and

$$\begin{aligned}&\text{q-ROFPGBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n)\\ &= \Big(\otimes_{i=1}^{m}\Big(\frac{1}{r+s}\Big(\otimes_{i,j\in P_h, i\neq j}((r\hat{\alpha}_i) \oplus\\ &\quad (s\hat{\alpha}_j))\Big)^{\frac{1}{|P_h|(|P_h|-1)}}\Big)\Big)^{\frac{1}{m}}\\ &= \Big\langle\Big(\frac{1}{1 + \Big(\frac{1}{m}\sum_{i=1}^{m} \frac{1}{\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|(|P_h|-1)}\left(\sum_{i,j\in P_h, i\neq j}\frac{1}{u_{\hat{\alpha}_{ij}}}\right)}}\Big)^{1/\gamma}}\Big)^{1/q},\\ &\quad\Big(1 - \frac{1}{1 + \Big(\frac{1}{m}\sum_{i=1}^{m} \frac{1}{\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|(|P_h|-1)}\left(\sum_{i,j\in P_h, i\neq j}\frac{1}{v_{\hat{\alpha}_{ij}}}\right)}}\Big)^{1/\gamma}}\Big)^{1/q}\Big\rangle, \end{aligned} \tag{20}$$

where $u_{\hat{\alpha}_{ij}} = r\left(\frac{\mu_{\hat{\alpha}_i}^q}{1-\mu_{\hat{\alpha}_i}^q}\right)^\gamma + s\left(\frac{\mu_{\hat{\alpha}_j}^q}{1-\mu_{\hat{\alpha}_j}^q}\right)^\gamma$, $v_{\hat{\alpha}_{ij}} = r\left(\frac{1-\nu_{\hat{\alpha}_i}^q}{\nu_{\hat{\alpha}_i}^q}\right)^\gamma + s\left(\frac{1-\nu_{\hat{\alpha}_j}^q}{\nu_{\hat{\alpha}_j}^q}\right)^\gamma$.

Theorem 25 (Idempotency): Let $\hat{\alpha}_k$ $(k = 1, 2, \ldots, n)$ be a collection of q-rung orthopair fuzzy numbers. If $\hat{\alpha}_k = \hat{\alpha}$, that is $< \mu_{\hat{\alpha}_k}, \nu_{\hat{\alpha}_k} >=< \mu_{\hat{\alpha}}, \nu_{\hat{\alpha}} >$ $(i = 1, 2, \ldots, n)$. Then

$$\text{q-ROFPGBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) = \hat{\alpha}.$$

Theorem 26: (Commutativity) Let $\hat{\alpha}_k$ $(k = 1, 2, \ldots, n)$ be a collection of q-rung orthopair fuzzy numbers. If $(\hat{\alpha}'_1, \hat{\alpha}'_2, \ldots, \hat{\alpha}'_n)$ is any permutation of $(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n)$, then

$$\begin{aligned}&\text{q-ROFPGBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n)\\ &\qquad= \text{q-ROFPGBMDA}(\hat{\alpha}'_1, \hat{\alpha}'_2, \ldots, \hat{\alpha}'_n).\end{aligned}$$

Theorem 27 (Monotonicity): Let $\hat{\alpha}_k =< \mu_{\hat{\alpha}_k}, \nu_{\hat{\alpha}_k} >$ and $\hat{\beta}_k =< \mu_{\hat{\beta}_k}, \nu_{\hat{\beta}_k} >$ $(i = 1, 2, \ldots, n)$ be two collections of q-rung orthopair fuzzy numbers, which have the same partitioned sorts $P_1, P_2, \ldots, P_m$, $\sum_{j=1}^{m} |P_j| = n$. If $\mu_{\hat{\alpha}_k} \leq \mu_{\hat{\beta}_k}$ and $\nu_{\hat{\alpha}_k} \geq \nu_{\hat{\beta}_k}$, then

$$\begin{aligned}&\text{q-ROFPGBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n)\\ &\qquad\leq \text{q-ROFPGBMDA}(\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_n).\end{aligned}$$

Theorem 28 (Boundedness): Let $\hat{\alpha}_k =< \mu_{\hat{\alpha}_k}, \nu_{\hat{\alpha}_k} >$ $(k = 1, 2, \ldots, n)$ be a collection of q-rung orthopair fuzzy numbers. $\hat{\alpha}^- =< \mu^-, \nu^+ >=< \min_k \mu_{\hat{\alpha}_k}, \max_k \nu_{\hat{\alpha}_k} >$, $\hat{\alpha}^+ =< \mu^+, \nu^- >=< \max \mu_{\hat{\alpha}_k}, \min \nu_{\hat{\alpha}_k} >$, then

$$\hat{\alpha}^- \leq \text{q-ROFPGBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) \leq \hat{\alpha}^+.$$

Some special cases of the q-ROFPGBMDA operator are considered by considering some special r, s.

(1) If $s \to 0$, the the q-ROFPGBMDA operator reduces to the following operator

$$\text{q-ROFPGBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) = \left(\otimes_{i=1}^{m} \left(\frac{1}{r} \left(\otimes_{i \in P_h} (r\hat{\alpha}_i) \right)^{\frac{1}{|P_h|}} \right) \right)^{\frac{1}{m}}$$
$$= \left\langle \left(\frac{1}{1 + \left(\frac{1}{m} \sum_{i=1}^{m} \frac{1}{\frac{1}{r} \frac{1}{\frac{1}{|P_h|} \left(\sum_{i \in P_h} \frac{1}{r \left(\frac{\mu_{\hat{\alpha}_i}^q}{1-\mu_{\hat{\alpha}_i}^q} \right)^{\gamma}} \right)}} \right)^{1/\gamma}} \right)^{1/q}, \left(1 - \frac{1}{1 + \left(\frac{1}{m} \sum_{i=1}^{m} \frac{1}{\frac{1}{r} \frac{1}{\frac{1}{|P_h|} \left(\sum_{i \in P_h} \frac{1}{r \left(\frac{1-\nu_{\hat{\alpha}_i}^q}{\nu_{\hat{\alpha}_i}^q} \right)^{\gamma}} \right)}} \right)^{1/\gamma}} \right)^{1/q} \right\rangle. \tag{21}$$

(2) If $s \to 0$ and all the q-rung orthopair fuzzy numbers are partitioned into one sort, then the q-ROFPGBMDA operator reduces to the following operator

$$\text{q-ROFPGBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) = \frac{1}{r} \left(\otimes_{i=1}^{n} (r\hat{\alpha}_i) \right)^{\frac{1}{n}}$$
$$= \left\langle \left(1 - \frac{1}{1 + \left(\frac{1}{\frac{1}{n} \sum_{i=1}^{n} \frac{1}{\left(\frac{\mu_{\hat{\alpha}_i}^q}{1-\mu_{\hat{\alpha}_i}^q} \right)^{\gamma}}} \right)^{1/\gamma}} \right)^{1/q}, \left(\frac{1}{1 + \left(\frac{1}{\frac{1}{n} \sum_{i=1}^{n} \frac{1}{\left(\frac{1-\nu_{\hat{\alpha}_i}^q}{\nu_{\hat{\alpha}_i}^q} \right)^{\gamma}}} \right)^{1/\gamma}} \right)^{1/q} \right\rangle. \tag{22}$$

(3) If $s \to 0$ and $r = 1$, the q-ROFPGBMDA operator reduces to the following operator

$$\text{q-ROFPGBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) = \left(\otimes_{i=1}^{m} \left(\otimes_{i \in P_h} \hat{\alpha}_i \right)^{\frac{1}{|P_h|}} \right)^{\frac{1}{m}}$$
$$= \left\langle \left(\frac{1}{1 + \left(\frac{1}{m} \sum_{i=1}^{m} \frac{1}{|P_h|} \sum_{i \in P_h} \left(\frac{1-\mu_{\hat{\alpha}_i}^q}{\mu_{\hat{\alpha}_i}^q} \right)^{\gamma} \right)^{1/\gamma}} \right)^{1/q}, \left(1 - \frac{1}{1 + \left(\frac{1}{m} \sum_{i=1}^{m} \frac{1}{|P_h|} \sum_{i \in P_h} \left(\frac{\nu_{\hat{\alpha}_i}^q}{1-\nu_{\hat{\alpha}_i}^q} \right)^{\gamma} \right)^{1/\gamma}} \right)^{1/q} \right\rangle. \tag{23}$$

(4) If $s \to 0$, $r = 1$ and all the q-rung orthopair fuzzy numbers are partitioned into one sort, then the q-ROFPGBMDA operator reduces to the following operator

$$\left(\otimes_{i=1}^{n} \hat{\alpha}_i \right)^{\frac{1}{n}} = \left\langle \left(\frac{1}{1 + \sum_{i=1}^{n} \left(\frac{1-\mu_{\hat{\alpha}_i}^q}{\mu_{\hat{\alpha}_i}^q} \right)^{\gamma} \right)^{1/\gamma}} \right)^{1/q}, \left(1 - \frac{1}{1 + \left(\frac{1}{n} \sum_{i=1}^{n} \left(\frac{\nu_{\hat{\alpha}_i}^q}{1-\nu_{\hat{\alpha}_i}^q} \right)^{\gamma} \right)^{1/\gamma}} \right)^{1/q} \right\rangle. \tag{24}$$

(5) If $r = 1$ and $s = 1$, the q-ROFPGBMDA operator reduces to the following operator

$$\text{q-ROFPGBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) = \left(\otimes_{i=1}^{m} \left(\frac{1}{2} \left(\otimes_{i,j \in P_h, i \neq j} (\hat{\alpha}_i \oplus \hat{\alpha}_j) \right)^{\frac{1}{|P_h|(|P_h|-1)}} \right) \right)^{\frac{1}{m}}$$
$$= \left\langle \left(\frac{1}{1 + \left(\frac{1}{m} \sum_{i=1}^{m} \frac{1}{\frac{1}{2} \frac{1}{\frac{1}{|P_h|(|P_h|-1)} \sum_{i,j \in P_h, i \neq j} \frac{1}{u_{\hat{\alpha}_{ij}}}}} \right)^{1/\gamma}} \right)^{1/q}, \left(1 - \frac{1}{1 + \left(\frac{1}{m} \sum_{i=1}^{m} \frac{1}{\frac{1}{2} \frac{1}{\frac{1}{|P_h|(|P_h|-1)} \sum_{i,j \in P_h, i \neq j} \frac{1}{v_{\hat{\alpha}_{ij}}}}} \right)^{1/\gamma}} \right)^{1/q} \right\rangle, \tag{25}$$

where $u_{\hat{\alpha}_{ij}} = \left(\frac{\mu_{\hat{\alpha}_i}^q}{1-\mu_{\hat{\alpha}_i}^q} \right)^{\gamma} + \left(\frac{\mu_{\hat{\beta}_j}^q}{1-\mu_{\hat{\beta}_j}^q} \right)^{\gamma}$, $v_{\hat{\alpha}_{ij}} = \left(\frac{\mu_{\hat{\alpha}_i}^q}{1-\mu_{\hat{\alpha}_i}^q} \right)^{\gamma} + \left(\frac{\mu_{\hat{\beta}_j}^q}{1-\mu_{\hat{\beta}_j}^q} \right)^{\gamma}$.

Definition 13: Let $\hat{\alpha}_k = < \mu_{\hat{\alpha}_k}, \nu_{\hat{\alpha}_k} >$ $(k = 1, 2, \ldots, n)$ be a collection of q-rung orthopair fuzzy numbers, which are partitioned into m distinct sorts $P_1, P_2, \ldots, P_m$, $\sum_{j=1}^{m} |P_j| = n$. The q-rung orthopair fuzzy partitioned geometric weighted BM Dombi averaging (q-ROFPGWBMDA) operator is defined as follows:

$$\text{q-ROFPGWBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) = \left(\otimes_{i=1}^{m} \left(\frac{1}{r+s} \left(\otimes_{i,j \in P_h, i \neq j} (r\hat{\alpha}_i^{w_i} \oplus s\hat{\alpha}_j^{w_j}) \right)^{\frac{1}{|P_h|(|P_h|-1)}} \right) \right)^{\frac{1}{m}}. \tag{26}$$

Theorem 29: Let $\hat{\alpha}_k = < \mu_{\hat{\alpha}_k}, \nu_{\hat{\alpha}_k} >$ $(k = 1, 2, \ldots, n)$ be a collection of q-rung orthopair fuzzy numbers. The aggregated result of the q-ROFPGWBMDA operator is still a q-rung orthopair fuzzy number, which has the following form

$$\text{q-ROFPGWBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) = \left\langle \left(\frac{1}{1 + \left(\frac{1}{m} \sum_{i=1}^{m} \frac{1}{\frac{1}{r+s} \frac{1}{\frac{1}{|P_h|(|P_h|-1)} \sum_{i,j \in P_h, i \neq j} \frac{1}{u_{\hat{\alpha}_{ij}}}}} \right)^{1/\gamma}} \right)^{1/q}, \left(1 - \frac{1}{1 + \left(\frac{1}{m} \sum_{i=1}^{m} \frac{1}{\frac{1}{r+s} \frac{1}{\frac{1}{|P_h|(|P_h|-1)} \sum_{i,j \in P_h, i \neq j} \frac{1}{v_{\hat{\alpha}_{ij}}}}} \right)^{1/\gamma}} \right)^{1/q} \right\rangle, \tag{27}$$

where $u_{\hat{\alpha}_{ij}} = r \frac{1}{w_i \left(\frac{1-\mu_{\hat{\alpha}_i}^q}{\mu_{\hat{\alpha}_i}^q} \right)^{\gamma}} + s \frac{1}{w_j \left(\frac{1-\mu_{\hat{\alpha}_j}^q}{\mu_{\hat{\alpha}_j}^q} \right)^{\gamma}}$, $v_{\hat{\alpha}_{ij}} = r \frac{1}{w_i \left(\frac{\nu_{\hat{\alpha}_i}^q}{1-\nu_{\hat{\alpha}_i}^q} \right)^{\gamma}} + s \frac{1}{w_j \left(\frac{\nu_{\hat{\alpha}_j}^q}{1-\nu_{\hat{\alpha}_j}^q} \right)^{\gamma}}$.

Theorem 30: (Commutativity) Let $(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n)$ be a collection of q-rung orthopair fuzzy numbers. If $(\hat{\alpha}'_1, \hat{\alpha}'_2, \ldots, \hat{\alpha}'_n)$ is any permutation of $(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n)$. Then

$$\text{q-ROFPGWBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) = \text{q-ROFPGWBMDA}(\hat{\alpha}'_1, \hat{\alpha}'_2, \ldots, \hat{\alpha}'_n).$$

Theorem 31 (Monotonicity): Let $\hat{\alpha} =< \mu_{\hat{\alpha}_k}, \nu_{\hat{\alpha}_k} >$ and $\hat{\beta} =< \mu_{\hat{\beta}_k}, \nu_{\hat{\beta}_k} >$ $(k = 1, 2, \ldots, n)$ be two collection of q-rung orthopair fuzzy numbers. If $\mu_{\hat{\alpha}_k} \geq \mu_{\hat{\beta}_k}$ and $\nu_{\hat{\alpha}_k} \leq \nu_{\hat{\beta}_k}$ for $i = 1, 2, \ldots, n$, then

$$\text{q-ROFPGWBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) \geq \text{q-ROFPGWBMDA}(\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_n).$$

Theorem 32 (Boundedness): Let $\hat{\alpha}_k =< \mu_{\hat{\alpha}_k}, \nu_{\hat{\alpha}_k} >$ $(k = 1, 2, \ldots, n)$ be a collection of q-rung orthopair fuzzy numbers. $\hat{\alpha}^- =< \mu^-, \nu^+ > =< \min_k \mu_{\hat{\alpha}_k}, \max_k \nu_{\hat{\alpha}_k} >$, $\hat{\alpha}^+ =< \mu^+, \nu^- > =< \max \mu_{\hat{\alpha}_k}, \min \nu_{\hat{\alpha}_k} >$, then

$$\hat{\alpha}^- \leq \text{q-ROFPGWBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) \leq \hat{\alpha}^+.$$

Some special cases of the q-ROFPGWBMDA operator are discussed as follows.

(1) If $s \to 0$, the q-ROFPGWBMDA operator reduces to the following operator

$$\begin{aligned} &\text{q-ROFPGWBMDA}^{r,0}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) \\ &= \Big(\otimes_{i=1}^{m} \Big(\frac{1}{r} \Big(\otimes_{i \in P_h} (r\hat{\alpha}_i^{w_i}) \Big)^{\frac{1}{|P_h|}} \Big) \Big)^{\frac{1}{m}} \\ &= \langle \Big(\frac{1}{1 + \Big(\frac{1}{m} \sum_{i=1}^{m} \frac{1}{\frac{1}{r} \frac{1}{\frac{1}{|P_h|} \sum_{i \in P_h} \frac{1}{r \frac{1}{u_i}}}} \Big)^{1/\gamma}} \Big)^{1/q}, \Big(1 - \\ &\quad \frac{1}{1 + \Big(\frac{1}{m} \sum_{i=1}^{m} \frac{1}{\frac{1}{r} \frac{1}{\frac{1}{|P_h|(|P_h|-1)} \sum_{i,j \in P_h, i \neq j} \frac{1}{r \frac{1}{v_i}}}} \Big)^{1/\gamma}} \Big)^{1/q} \rangle, \end{aligned} \tag{28}$$

where $u_i = w_i \Big(\frac{1-\mu_{\hat{\alpha}_i}^q}{\mu_{\hat{\alpha}_i}^q} \Big)^{\gamma}$, $v_i = w_i \Big(\frac{\nu_{\hat{\alpha}_i}^q}{1-\nu_{\hat{\alpha}_i}^q} \Big)^{\gamma}$.

(2) If $s \to 0$ and all the q-rung orthopair fuzzy numbers are partitioned into one sort, the q-ROFPGWBMDA operator reduces to the following operator

$$\begin{aligned} &\text{q-ROFPGWBMDA}^{r,0}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) \\ &= \frac{1}{r} \Big(\otimes_{i=1}^{n} (r\hat{\alpha}_i^{w_i}) \Big)^{\frac{1}{n}} \\ &= \langle \Big(1 - \frac{1}{1 + \Big(\frac{1}{r} \frac{1}{\frac{1}{n} \sum_{i=1}^{n} \frac{1}{r \frac{1}{w_i \big(\frac{1-\mu_{\hat{\alpha}_i}^q}{\mu_{\hat{\alpha}_i}^q} \big)^{\gamma}}}} \Big)^{1/\gamma}} \Big)^{1/q}, \\ &\quad \Big(\frac{1}{1 + \Big(\frac{1}{r} \frac{1}{\frac{1}{n} \sum_{i=1}^{n} \frac{1}{r \frac{1}{w_i \big(\frac{\nu_{\hat{\alpha}_i}^q}{1-\nu_{\hat{\alpha}_i}^q} \big)^{\gamma}}}} \Big)^{1/\gamma}} \Big)^{1/q} \rangle. \end{aligned} \tag{29}$$

(3) If $s \to 0$ and $r = 1$, the q-ROFPGWBMDA operator reduces to the following operator

$$\begin{aligned} &\text{q-ROFPGWBMDA}^{1,0}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) \\ &= \Big(\otimes_{i=1}^{m} \Big(\otimes_{i \in P_h} \hat{\alpha}_i^{w_i} \Big)^{\frac{1}{|P_h|}} \Big)^{\frac{1}{m}} \\ &= \langle \Big(\frac{1}{1 + \Big(\frac{1}{m} \sum_{i=1}^{m} \frac{1}{|P_h|} \sum_{i \in P_h} w_i \big(\frac{1-\mu_{\hat{\alpha}_i}^q}{\mu_{\hat{\alpha}_i}^q} \big)^{\gamma} \Big)^{1/\gamma}} \Big)^{1/q}, \\ &\quad \Big(1 - \frac{1}{1 + \Big(\frac{1}{m} \sum_{i=1}^{m} \frac{1}{|P_h|} \sum_{i \in P_h} w_i \big(\frac{\nu_{\hat{\alpha}_i}^q}{1-\nu_{\hat{\alpha}_i}^q} \big)^{\gamma} \Big)^{1/\gamma}} \Big)^{1/q} \rangle. \end{aligned} \tag{30}$$

(4) If $s \to 0$ and $r = 1$, and all the q-rung orthopair fuzzy numbers are partitioned into one sort, the q-ROFPGWBMDA operator reduces to the following operator

$$\begin{aligned} &\text{q-ROFPGWBMDA}^{1,0}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) \\ &= \Big(\otimes_{i=1}^{n} \hat{\alpha}_i^{w_i} \Big)^{\frac{1}{n}} \\ &= \langle \Big(\frac{1}{1 + \Big(\frac{1}{n} \sum_{i=1}^{n} w_i \big(\frac{1-\mu_{\hat{\alpha}_i}^q}{\mu_{\hat{\alpha}_i}^q} \big)^{\gamma} \Big)^{1/\gamma}} \Big)^{1/q}, \\ &\quad \Big(1 - \frac{1}{1 + \Big(\frac{1}{n} \sum_{i=1}^{n} w_i \big(\frac{\nu_{\hat{\alpha}_i}^q}{1-\nu_{\hat{\alpha}_i}^q} \big)^{\gamma} \Big)^{1/\gamma}} \Big)^{1/q} \rangle. \end{aligned} \tag{31}$$

(5) If $r = 1$ and $s = 1$, the q-ROFPGWBMDA operator reduces to the following operator

$$\begin{aligned} &\text{q-ROFPGWBMDA}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) \\ &= \Big(\otimes_{i=1}^{m} \Big(\frac{1}{2} \Big(\otimes_{i,j \in P_h, i \neq j} (\hat{\alpha}_i^{w_i} \oplus \hat{\alpha}_j^{w_j}) \Big)^{\frac{1}{|P_h|(|P_h|-1)}} \Big) \Big)^{\frac{1}{m}} \\ &= \langle \Big(\frac{1}{1 + \Big(\frac{1}{m} \sum_{i=1}^{m} \frac{1}{\frac{1}{2} \frac{1}{\frac{1}{|P_h|(|P_h|-1)} \sum_{i,j \in P_h, i \neq j} \frac{1}{u_{\hat{\alpha}_{ij}}}}} \Big)^{1/\gamma}} \Big)^{1/q}, \\ &\quad \Big(1 - \frac{1}{1 + \Big(\frac{1}{m} \sum_{i=1}^{m} \frac{1}{\frac{1}{2} \frac{1}{\frac{1}{|P_h|(|P_h|-1)} \sum_{i,j \in P_h, i \neq j} \frac{1}{\nu_{\hat{\alpha}_{ij}}}}} \Big)^{1/\gamma}} \Big)^{1/q} \rangle, \end{aligned} \tag{32}$$

where $u_{\hat{\alpha}_{ij}} = \frac{1}{w_i \big(\frac{1-\mu_{\hat{\alpha}_i}^q}{\mu_{\hat{\alpha}_i}^q} \big)^{\gamma}} + \frac{1}{w_i \big(\frac{1-\mu_{\hat{\alpha}_j}^q}{\mu_{\hat{\alpha}_j}^q} \big)^{\gamma}}$, $\nu_{\hat{\alpha}_{ij}} = \frac{1}{w_i \big(\frac{\nu_{\hat{\alpha}_i}^q}{1-\nu_{\hat{\alpha}_i}^q} \big)^{\gamma}} + \frac{1}{w_j \big(\frac{\nu_{\hat{\alpha}_j}^q}{1-\nu_{\hat{\alpha}_j}^q} \big)^{\gamma}}$.

IV. NEW MULTIPLE ATTRIBUTE DECISION MAKING METHOD BASED ON THE NEW q-RUNG ORTHOPAIR FUZZY DOMBI BONFERRONI MEAN OPERATORS

In this section, we propose a new multiple attribute decision making method based on the new q-rung orthopair fuzzy Bonferroni mean Dombi aggregation operators.

Consider a multiple attribute decision making problem, which is composed of m alternatives $\{A_1, A_2, \ldots, A_m\}$ and n attributes $\{C_1, C_2, \ldots, C_n\}$. The weight vector of attributes is $(w_1, w_2, \ldots, w_n)$ with $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$. The alternatives are evaluated using the q-rung orthopair fuzzy numbers $\hat{\alpha}_{ij} =< \mu_{\hat{\alpha}_{ij}}, \nu_{\hat{\alpha}_{ij}} >$ and the decision matrix is formed as $\hat{D} = (\hat{\alpha}_{ij})_{m \times n}$. The new method is as follows.

Step 1. The q-rung orthopair fuzzy evaluation value $\hat{\alpha}_{ij} =< \mu_{\hat{\alpha}_{ij}}, \nu_{\hat{\alpha}_{ij}} >$ is given by decision maker when evaluating alternative A_i with respect to the attribute C_j and decision matrix is formed as $\hat{D} = (\hat{\alpha}_{ij})_{m \times n}$.

Step 2. The collective evaluation values $\hat{\alpha}_i$ $(i = 1, 2, \ldots., m)$ of alternatives A_i $(i = 1, 2, \ldots., m)$ are aggregated by using the q-ROFWBMDA operator or the q-ROFWGBMDA operator.

$$\begin{aligned}\hat{\alpha}_i &= \text{q-ROFWBMDA}(\hat{\alpha}_{i1}, \hat{\alpha}_{i2}, \ldots, \hat{\alpha}_{in})\\ &= \Big\langle \Big(\frac{1}{1+\big(\frac{1}{r+s}\big(\frac{1}{\frac{1}{n(n-1)}\sum_{k,l=1,k\neq l}^{n}\left(\frac{1}{u_{ikl}}\right)}\big)\big)^{1/\gamma}} \Big)^{1/q},\\ &\quad \Big(1-\frac{1}{1+\big(\frac{1}{r+s}\big(\frac{1}{\frac{1}{n(n-1)}\sum_{k,l=1,k\neq l}^{n}\left(\frac{1}{v_{ikl}}\right)}\big)\big)^{1/\gamma}}\Big)^{1/q}\Big\rangle,\end{aligned} \tag{33}$$

$$u_{\hat{\alpha}_{ikl}} = r\frac{1}{w_k\Big(\frac{\mu^q_{\hat{\alpha}_{ik}}}{1-\mu^q_{\hat{\alpha}_{ik}}}\Big)^{\gamma}} + s\frac{1}{w_l\Big(\frac{\mu^q_{\hat{\alpha}_{il}}}{1-\mu^q_{\hat{\alpha}_{il}}}\Big)^{\gamma}} \quad v_{\hat{\alpha}_{ikl}} = r\frac{1}{w_k\Big(\frac{1-v^q_{\hat{\alpha}_{ik}}}{v^q_{\hat{\alpha}_{ik}}}\Big)^{\gamma}} + s\frac{1}{w_l\Big(\frac{1-v^q_{\hat{\alpha}_{il}}}{v^q_{\hat{\alpha}_{il}}}\Big)^{\gamma}}.$$

$$\begin{aligned}\hat{\alpha}_i &= \text{q-ROFWGBMDA}(\hat{\alpha}_{i1}, \hat{\alpha}_{i2}, \ldots, \hat{\alpha}_{in})\\ &= \Big\langle \Big(1-\frac{1}{1+\big(\frac{1}{r+s}\big(\frac{1}{\left(\sum_{k,l=1,k\neq l}^{n}\left(\frac{1}{n(n-1)}\left(\frac{1}{u'_{ikl}}\right)\right)\right)}\big)\big)^{1/\gamma}}\Big)^{1/q},\\ &\quad \Big(\frac{1}{1+\big(\frac{1}{r+s}\big(\frac{1}{\left(\sum_{k,l=1,k\neq l}^{n}\frac{1}{n(n-1)}\left(\frac{1}{v'_{ikl}}\right)\right)}\big)\big)^{1/\gamma}}\Big)^{1/q}\Big\rangle,\end{aligned} \tag{34}$$

$$u'_{\hat{\alpha}_{ikl}} = r\Big(\frac{1}{\Big(w_k\frac{1-\mu^q_{\hat{\alpha}_{ik}}}{\mu^q_{\hat{\alpha}_{ik}}}\Big)^{\gamma}}\Big) + s\Big(\frac{1}{\Big(w_l\frac{1-\mu^q_{\hat{\alpha}_{il}}}{\mu^q_{\hat{\alpha}_{il}}}\Big)^{\gamma}}\Big), \quad v'_{\hat{\alpha}_{ikl}} = r\Big(\frac{1}{\Big(w_k\frac{v^q_{\hat{\alpha}_{ik}}}{1-v^q_{\hat{\alpha}_{ik}}}\Big)^{\gamma}}\Big) + s\Big(\frac{1}{\Big(w_l\frac{v^q_{\hat{\alpha}_{il}}}{1-v^q_{\hat{\alpha}_{il}}}\Big)^{\gamma}}\Big).$$

Step 3. Calculate the score degree and accuracy degree of $\hat{\alpha}_i$ $(i = 1, 2, \ldots., m)$ by using the Eq.(2)-(3). Rank $\hat{\alpha}_i$ $(i = 1, 2, \ldots., m)$ by using Definition 2.

Step 4. Rank alternatives according to the ranking of the $\hat{\alpha}_i$ $(i = 1, 2, \ldots., m)$ and select the optimal alternative.

V. NUMERICAL EXAMPLE AND COMPARATIVE ANALYSIS

A. NUMERICAL EXAMPLE

In this section, we give a numerical example to illustrate the feasibility and practical advantages of the new method. With the development of Chinese higher education, many universities choose to construct new campuses in suburbs of cities. Suppose there is a university in Xi'an, which want to construct a new campus in the suburb of Xi'an. There are five different locations for further evaluation A_1-Chanba, A_2-Caotangsi, A_3-Chuangxingang, A_4-Lintong, A_5−Yanliang. Four attributes are considered including: C_1-development of city, C_2-price of land, C_3-environment, C_4−transportation. The proposed method is used to rank alternatives.

Step 1. Decision makers give the evaluation values in the form of q-rung orthopair fuzzy numbers, which are shown in Table 1.

Step 2. If the q-ROFWBMDA operator is used as Eq.(33), the aggregated results are shown in Table 2. Here $q = 2, r = s = 2$ and $\gamma = 1, 2, 3, 4, 5, 6, 8, 10$, respectively. The weight vector of the attributes are assumed to known as $(0.20, 0.30, 0.15, 0.35)$.

Step 3. Calculate the scores $S(\hat{\alpha}_i)$ $(i = 1, 2, \ldots, 5)$ of collective evaluation values $\hat{\alpha}_i$ $(i = 1, 2, \ldots, 5)$ by using the Eq.(2). The results are shown in Table 3.

Step 4. Rank $\hat{\alpha}_i$ $(i = 1, 2, \ldots, 5)$ according to $S(\hat{\alpha}_i)$ $(i = 1, 2, \ldots, 5)$ and rank alternatives A_i $(i = 1, 2, \ldots, 5)$ according to $\hat{\alpha}_i$ $(i = 1, 2, \ldots, 5)$. The results are shown in Table 4.

TABLE 1. Q-rung orthopair fuzzy decision matrix $\hat{D}$.

Alternative	C_1	C_2	C_3	C_4
A_1	$< 0.6, 0.8 >$	$< 0.6, 0.3 >$	$< 0.7, 0.4 >$	$< 0.4, 0.3 >$
A_2	$< 0.7, 0.2 >$	$< 0.7, 0.4 >$	$< 0.6, 0.2 >$	$< 0.3, 0.6 >$
A_3	$< 0.5, 0.7 >$	$< 0.6, 0.2 >$	$< 0.9, 0.2 >$	$< 0.6, 0.3 >$
A_4	$< 0.8, 0.1 >$	$< 0.6, 0.4 >$	$< 0.7, 0.3 >$	$< 0.3, 0.5 >$
A_5	$< 0.4, 0.5 >$	$< 0.8, 0.1 >$	$< 0.5, 0.6 >$	$< 0.7, 0.2 >$

TABLE 2. The aggregated results by using the q-ROFWBMDA operator for $q = 2, r = s = 2$.

	$\gamma = 1$	$\gamma = 2$	$\gamma = 3$	$\gamma = 4$
A_1	$< 0.5687, 0.4259 >$	$< 0.5747, 0.3880 >$	$< 0.5824, 0.3686 >$	$< 0.5878, 0.3567 >$
A_2	$< 0.5713, 0.3259 >$	$< 0.6013, 0.2911 >$	$< 0.6094, 0.2635 >$	$< 0.6335, 0.2472 >$
A_3	$< 0.6327, 0.3054 >$	$< 0.6101, 0.2727 >$	$< 0.5904, 0.2554 >$	$< 0.5974, 0.2441 >$
A_4	$< 0.5984, 0.3288 >$	$< 0.6276, 0.3405 >$	$< 0.6345, 0.3386 >$	$< 0.6559, 0.3337 >$
A_5	$< 0.5903, 0.3042 >$	$< 0.6068, 0.2600 >$	$< 0.6193, 0.2384 >$	$< 0.6474, 0.2281 >$
	$\gamma = 5$	$\gamma = 6$	$\gamma = 8$	$\gamma = 10$
A_1	$< 0.5912, 0.3567 >$	$< 0.5932, 0.3411 >$	$< 0.5977, 0.3444 >$	$< 0.5922, 0.3252 >$
A_2	$< 0.6335, 0.2472 >$	$< 0.6496, 0.2306 >$	$< 0.6806, 0.2136 >$	$< 0.6802, 0.2137 >$
A_3	$< 0.5974, 0.2441 >$	$< 0.5957, 0.2302 >$	$< 0.5801, 0.2319 >$	$< 0.5869, 0.2252 >$
A_4	$< 0.6559, 0.3337 >$	$< 0.6683, 0.3249 >$	$< 0.7664, 0.1117 >$	$< 0.7624, 0.1131 >$
A_5	$< 0.6474, 0.2281 >$	$< 0.6660, 0.2183 >$	$< 0.6589, 0.2226 >$	$< 0.7623, 0.1131 >$

TABLE 3. The scores by using the q-ROFWBMDA operator for $q = 2, r = s = 2$.

	$\gamma=1$	$\gamma=2$	$\gamma=3$	$\gamma=4$	$\gamma=5$	$\gamma=6$	$\gamma=8$	$\gamma=10$
A_1	0.1421	0.1797	0.2033	0.2183	0.2284	0.2355	0.2387	0.2449
A_2	0.2202	0.2767	0.3019	0.3402	0.3565	0.3688	0.4176	0.4170
A_3	0.3071	0.2979	0.2834	0.2973	0.2996	0.3019	0.2827	0.2938
A_4	0.3288	0.2779	0.2879	0.3188	0.3314	0.3411	0.5749	0.5684
A_5	0.3042	0.3006	0.3267	0.3671	0.3844	0.3959	0.3846	0.5683

TABLE 4. Ranking results of the q-ROFWBMDA operator.

	Ordering	Optimal alternative
$\gamma=1$	$A_4 > A_3 > A_5 > A_2 > A_1$	A_4
$\gamma=2$	$A_5 > A_3 > A_4 > A_2 > A_1$	A_5
$\gamma=3$	$A_5 > A_4 > A_3 > A_2 > A_1$	A_5
$\gamma=4$	$A_5 > A_2 > A_4 > A_3 > A_1$	A_5
$\gamma=5$	$A_5 > A_2 > A_4 > A_3 > A_1$	A_5
$\gamma=6$	$A_5 > A_2 > A_4 > A_3 > A_1$	A_5
$\gamma=8$	$A_4 > A_2 > A_5 > A_3 > A_1$	A_4
$\gamma=10$	$A_4 > A_5 > A_2 > A_3 > A_1$	A_4

TABLE 5. The aggregated results by using the q-ROFWGBMDA operator for $q = 2, r = s = 2$.

	$\gamma=1$	$\gamma=2$	$\gamma=3$	$\gamma=4$
A_1	$<0.8355, 0.2050>$	$<0.7305, 0.2666>$	$<0.6919, 0.2896>$	$<0.6711, 0.3047>$
A_2	$<0.8536, 0.1560>$	$<0.7655, 0.2411>$	$<0.7318, 0.2848>$	$<0.7115, 0.3109>$
A_3	$<0.8688, 0.1465>$	$<0.7487, 0.1990>$	$<0.6956, 0.2212>$	$<0.6686, 0.2361>$
A_4	$<0.8659, 0.1500>$	$<0.7699, 0.2408>$	$<0.7278, 0.2849>$	$<0.7016, 0.3106>$
A_5	$<0.8482, 0.1510>$	$<0.7414, 0.2627>$	$<0.6834, 0.3333>$	$<0.6430, 0.3745>$
	$\gamma=5$	$\gamma=6$	$\gamma=8$	$\gamma=10$
A_1	$<0.6581, 0.3136>$	$<0.6491, 0.3261>$	$<0.6374, 0.3416>$	$<0.6301, 0.3519>$
A_2	$<0.6972, 0.3275>$	$<0.6858, 0.3389>$	$<0.6685, 0.3535>$	$<0.6562, 0.3625>$
A_3	$<0.6531, 0.2468>$	$<0.6433, 0.2546>$	$<0.6317, 0.2652>$	$<0.6251, 0.2718>$
A_4	$<0.6839, 0.3272>$	$<0.6712, 0.3387>$	$<0.6544, 0.3534>$	$<0.6438, 0.3625>$
A_5	$<0.6153, 0.3999>$	$<0.5959, 0.4168>$	$<0.5716, 0.4377>$	$<0.5570, 0.4501>$

TABLE 6. The scores by using the q-ROFWGBMDA operator for $q = 2, r = s = 2$.

	$\gamma=1$	$\gamma=2$	$\gamma=3$	$\gamma=4$	$\gamma=5$	$\gamma=6$	$\gamma=8$	$\gamma=10$
A_1	0.6560	0.4626	0.3948	0.3576	0.3331	0.3150	0.2898	0.2731
A_2	0.7043	0.5279	0.4544	0.4096	0.3788	0.3555	0.3220	0.2992
A_3	0.7334	0.5210	0.4350	0.3913	0.3656	0.3489	0.3287	0.3169
A_4	0.7273	0.5347	0.4485	0.3958	0.3607	0.3359	0.3033	0.2831
A_5	0.6966	0.4807	0.3560	0.2733	0.2186	0.1814	0.1351	0.1077

From results we can see that, the ranking of alternatives is $A_3 > A_5 > A_4 > A_2 > A_1$ and the optimal alternative is A_3 for $\lambda = 1$. The optimal alternative is A_5 for $\lambda = 2, 3, 4, 5, 6$. If $\lambda = 2$, the suboptimal alternative is A_3, which is the same as that for $\lambda = 1$. If $\lambda = 3$, the suboptimal alternative is A_4 and A_3 is ranked third. The ranking of alternatives are the same as $A_5 > A_2 > A_4 > A_3 > A_1$ for $\lambda = 4, 5, 6$. In this case, A_2 is the suboptimal alternative, A_4 is ranked third and A_3 is ranked the second from last. The optimal alternative becomes A_4 for $\lambda = 8, 10$. The suboptimal alternative is A_2 and A_5 is ranked third for $\lambda = 8$ and the suboptimal alternative is A_5 and A_2 is ranked third. The aggregated results of A_4 and A_5 are nearly the same for $\lambda = 10$. The λ can be seen as the risk attitude of decision maker. Decision maker is more risk-seeking with the increasing of λ.

If the q-ROFWGBMDA operator is used in step 3 in the aggregation process, results are shown in Table 5, where $q = 2, r = s = 2$. The scores are calculated by using the Eq.(2) and results are shown in Table 6. The ranking results are shown in Table 7. If $\gamma = 1$, the optimal alternative is A_3 and suboptimal alternative is A_4. If $\gamma = 2$, the optimal alternative is A_4 and suboptimal alternative is A_3. For $\gamma = 3, 4, 5, 6$, the optimal alternative is A_2. The suboptimal alternative is A_4 if $\gamma = 3, 4$ and suboptimal alternative is A_2 if $\gamma = 5, 6$. When $\gamma \geq 7$, the optimal alternative becomes A_2.

In order to consider influence of parameter q, we consider $q = 2, 3, 4, 5$ for $\gamma = 2$ and $\gamma = 3$ in q-ROFWBMDA operator, respectively. If $\gamma = 2$, A_5 is the optimal alternative for $q = 3$ and $q = 5$ and A_4 is the optimal alternative for $q = 8$ and $q = 10$. If $\gamma = 3$, A_5 is the optimal alternative

TABLE 7. Ranking results of the q-ROFWGBMDA operator for $q = 2, r = s = 2$.

	Ordering	Optimal alternative
$\gamma = 1$	$A_3 > A_4 > A_2 > A_5 > A_1$	A_3
$\gamma = 2$	$A_4 > A_3 > A_2 > A_5 > A_1$	A_4
$\gamma = 3$	$A_2 > A_4 > A_3 > A_1 > A_5$	A_2
$\gamma = 4$	$A_2 > A_4 > A_3 > A_1 > A_5$	A_2
$\gamma = 5$	$A_2 > A_3 > A_4 > A_1 > A_5$	A_2
$\gamma = 6$	$A_2 > A_3 > A_4 > A_1 > A_5$	A_2
$\gamma = 8$	$A_3 > A_2 > A_4 > A_1 > A_5$	A_3
$\gamma = 10$	$A_3 > A_2 > A_4 > A_1 > A_5$	A_3

TABLE 8. The aggregated results by using the q-ROFWBMDA operator for different q, γ.

	$q = 3, \gamma = 2$	$q = 5, \gamma = 2$	$q = 8, \gamma = 2$	$q = 10, \gamma = 2$
A_1	$< 0.5772, 0.3720 >$	$< 0.5842, 0.3519 >$	$< 0.5912, 0.3347 >$	$< 0.5936, 0.3279 >$
A_2	$< 0.6096, 0.2660 >$	$< 0.6236, 0.2391 >$	$< 0.6384, 0.2237 >$	$< 0.6460, 0.2187 >$
A_3	$< 0.6044, 0.2569 >$	$< 0.5980, 0.2377 >$	$< 0.5955, 0.2236 >$	$< 0.5955, 0.2187 >$
A_4	$< 0.6341, 0.3389 >$	$< 0.6455, 0.3308 >$	$< 0.6581, 0.3209 >$	$< 0.6644, 0.3168 >$
A_5	$< 0.6135, 0.2400 >$	$< 0.6312, 0.2232 >$	$< 0.6525, 0.2142 >$	$< 0.6616, 0.2113 >$
	$q = 3, \gamma = 3$	$q = 5, \gamma = 3$	$q = 8, \gamma = 3$	$q = 10, \gamma = 3$
A_1	$< 0.5859, 0.3549 >$	$< 0.5913, 0.3368 >$	$< 0.5951, 0.3232 >$	$< 0.5962, 0.3185 >$
A_2	$< 0.6283, 0.2433 >$	$< 0.6401, 0.2254 >$	$< 0.6539, 0.2155 >$	$< 0.6610, 0.2123 >$
A_3	$< 0.5979, 0.2411 >$	$< 0.5956, 0.2252 >$	$< 0.5959, 0.2155 >$	$< 0.5965, 0.2123 >$
A_4	$< 0.6509, 0.3326 >$	$< 0.6602, 0.3222 >$	$< 0.6706, 0.3140 >$	$< 0.6755, 0.3112 >$
A_5	$< 0.6395, 0.2257 >$	$< 0.6552, 0.2152 >$	$< 0.6694, 0.2094 >$	$< 0.6151, 0.2027 >$

TABLE 9. The scores by using the q-ROFWBMDA operator for different q, γ.

	$q = 3$	$q = 5$	$q = 8$	$q = 10$	$q = 3$	$q = 5$	$q = 8$	$q = 10$
			$\gamma = 2$				$\gamma = 3$	
A_1	0.1409	0.0626	0.0148	0.0054	0.1564	0.1350	0.0156	0.0057
A_2	0.2077	0.0935	0.0276	0.0127	0.2336	0.1985	0.0334	0.0159
A_3	0.2038	0.0757	0.0158	0.0056	0.1997	0.1950	0.0159	0.0057
A_4	0.2160	0.1081	0.0351	0.0167	0.2389	0.2123	0.0408	0.0198
A_5	0.2171	0.0996	0.0328	0.0161	0.2501	0.2197	0.0403	0.0197

TABLE 10. Ranking results of the q-ROFWBMDA operator for different q, γ.

	Ordering	Optimal alternative
$q = 3, \gamma = 2$	$A_5 > A_4 > A_2 > A_3 > A_1$	A_5
$q = 5, \gamma = 2$	$A_5 > A_4 > A_2 > A_3 > A_1$	A_5
$q = 8, \gamma = 2$	$A_4 > A_5 > A_2 > A_3 > A_1$	A_4
$q = 10, \gamma = 2$	$A_4 > A_5 > A_2 > A_3 > A_1$	A_4
$q = 3, \gamma = 3$	$A_5 > A_4 > A_2 > A_3 > A_1$	A_5
$q = 5, \gamma = 3$	$A_4 > A_5 > A_2 > A_3 > A_1$	A_4
$q = 8, \gamma = 3$	$A_4 > A_5 > A_2 > A_3 > A_1$	A_4
$q = 10, \gamma = 3$	$A_4 > A_5 > A_2 > A_1 > A_3$	A_4

for $q = 3$ and A_4 becomes the optimal alternative for $q = 5, 8, 10$. Moreover, the aggregated results and the scores are more close with the increasing of q.

In order to consider influence of parameter r, s, we consider $r = 1, \ldots, 4, s = 1, \ldots, 4$ for $q = \gamma = 2$ in q-ROFWBMDA operator. The ranking of alternatives is as $A_5 > A_3 > A_2 > A_4 > A_1$ for $r = s = 1, r = 1, s = 2$ and $r = 1, s = 3$. The optimal alternative is A_5 and suboptimal alternative is A_3 in other cases. The ranking of alternatives is $A_5 > A_3 > A_4 > A_2 > A_1$. The optimal alternative and suboptimal alternative are the same as above. But the rankings of A_2 and A_4 have changed.

B. COMPARATIVE ANALYSIS

If we aggregate the alternative evaluation values by using the q-rung orthopair fuzzy weighted averaging (q-ROFWA) operator as q-ROFWA$(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) = \sum_{j=1}^{n} w_j \hat{\alpha}_j = \langle (1 - \prod_{j=1}^{n}(1 - \mu_j^q)^{w_j})^{1/q}, \prod_{j=1}^{n} \nu_j^{w_j} \rangle$, we can get $\hat{\alpha}_1 =< 0.5654, 0.3811 >, \hat{\alpha}_2 =< 0.5948, 0.3617 >$, $\hat{\alpha}_3 =< 0.6704, 0.2961 >, \hat{\alpha}_4 =< 0.6135, 0.3139 >, \hat{\alpha}_5 =< 0.6798, 0.2301 >$, where $q = 2$. The scores of $\hat{\alpha}_i (i = 1, 2, \ldots, 5)$ can be calculated as $S(\hat{\alpha}_1) = 0.1744, S(\hat{\alpha}_2) = 0.2230, S(\hat{\alpha}_3) = 0.3617, S(\hat{\alpha}_4) = 0.2778, S(\hat{\alpha}_5) = 0.4092$. The alternatives can be ranked as $A_5 > A_3 > A_4 > A_2 > A_1$. The optimal alternative is A_5. If the q-rung orthopair fuzzy

TABLE 11. The aggregated results by using the operator for different *r*, *s*.

	$r=s=1$	$r=1,s=2$	$r=1,s=3$	$r=2,s=3$
A_1	$<0.4391,0.5109>$	$<0.5048,0.4418>$	$<0.5175,0.4338>$	$<0.5757,0.3875>$
A_2	$<0.4525,0.4169>$	$<0.5285,0.3533>$	$<0.5396,0.3422>$	$<0.6018,0.2907>$
A_3	$<0.4926,0.3742>$	$<0.5576,0.3186>$	$<0.5649,0.3100>$	$<0.6128,0.2722>$
A_4	$<0.4778,0.4711>$	$<0.5543,0.3922>$	$<0.5649,0.3888>$	$<0.6291,0.3379>$
A_5	$<0.4878,0.3327>$	$<0.5529,0.2901>$	$<0.5608,0.2850>$	$<0.6085,0.2581>$
	$r=3,s=3$	$r=3,s=4$	$r=4,s=4$	$r=4,s=5$
A_1	$<0.5747,0.3880>$	$<0.5752,0.3878>$	$<0.5747,0.3880>$	$<0.5750,0.3879>$
A_2	$<0.6013,0.2911>$	$<0.6015,0.2909>$	$<0.6013,0.2911>$	$<0.6014,0.2910>$
A_3	$<0.6101,0.2727>$	$<0.6115,0.2724>$	$<0.6101,0.2727>$	$<0.6110,0.2725>$
A_4	$<0.6276,0.3405>$	$<0.6283,0.3392>$	$<0.6276,0.3405>$	$<0.6280,0.3397>$
A_5	$<0.6068,0.2620>$	$<0.6077,0.2590>$	$<0.6068,0.2600>$	$<0.6073,0.2594>$

TABLE 12. The scores by using the q-ROFWBMDA operator for different *r*, *s*.

	$r=1$	$r=1$	$r=1$	$r=2$	$r=3$	$r=3$	$r=4$	$r=4$
	$s=1$	$s=2$	$s=3$	$s=3$	$s=3$	$s=4$	$s=4$	$s=5$
A_1	-0.0682	0.0596	0.0796	0.1812	0.1797	0.1805	0.1797	0.1802
A_2	0.0310	0.1545	0.1741	0.2777	0.2767	0.2772	0.2767	0.2770
A_3	0.1026	0.2096	0.2230	0.3015	0.2979	0.2997	0.2979	0.2990
A_4	0.0063	0.1534	0.1680	0.2815	0.2779	0.2798	0.2779	0.2790
A_5	0.1273	0.2216	0.2333	0.3037	0.3006	0.3022	0.3006	0.3015

TABLE 13. Ranking results of the q-ROFWBMDA operator for different *r*, *s*.

	Ordering	Optimal alternative
$r=1,s=1$	$A_5>A_3>A_2>A_4>A_1$	A_5
$r=1,s=2$	$A_5>A_3>A_2>A_4>A_1$	A_5
$r=1,s=3$	$A_5>A_3>A_2>A_4>A_1$	A_5
$r=2,s=3$	$A_5>A_3>A_4>A_2>A_1$	A_5
$r=3,s=3$	$A_5>A_3>A_4>A_2>A_1$	A_5
$r=3,s=4$	$A_5>A_3>A_4>A_2>A_1$	A_5
$r=4,s=4$	$A_5>A_3>A_4>A_2>A_1$	A_5
$r=4,s=5$	$A_5>A_3>A_4>A_2>A_1$	A_5

TABLE 14. The scores by using the q-ROFWA operator and the q-ROFWGA operator.

	$S(\hat{\alpha}_1)$	$S(\hat{\alpha}_2)$	$S(\hat{\alpha}_3)$	$S(\hat{\alpha}_4)$	$S(\hat{\alpha}_5)$	Ranking of the alternatives
						the q-ROFWA operator
$q=3$	0.1323	0.1786	0.2907	0.2201	0.3145	$A_5>A_3>A_4>A_2>A_1$
$q=5$	0.0611	0.0935	0.1731	0.1207	0.1727	$A_3>A_5>A_4>A_2>A_1$
$q=8$	0.0170	0.0314	0.0917	0.0494	0.0737	$A_3>A_5>A_4>A_2>A_1$
$q=10$	0.0073	0.0151	0.0662	0.0284	0.0433	$A_3>A_5>A_4>A_2>A_1$
						the q-ROFWGA operator
$q=3$	-0.0061	0.0292	0.1397	0.0645	0.1735	$A_5>A_3>A_4>A_2>A_1$
$q=5$	-0.0363	0.0029	0.0507	0.0201	0.0727	$A_5>A_3>A_4>A_2>A_1$
$q=8$	-0.0297	-0.0016	0.0086	0.0030	0.0183	$A_5>A_3>A_4>A_2>A_1$
$q=10$	-0.0206	-0.0010	0.0020	0.0008	0.0072	$A_5>A_3>A_4>A_2>A_1$

weighted geometric averaging (q-ROFWGA) operator is used in aggregation process as q-ROFWGA$(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) = \prod_{j=1}^{n} \hat{\alpha}_j^{w_j} = \langle \prod_{j=1}^{n} \mu_j^{w_j}, (1-\prod_{j=1}^{n}(1-\nu_j^q)^{w_j})^{1/q}\rangle$, we can get $\hat{\alpha}_1 =< 0.5328, 0.5031 >$, $\hat{\alpha}_2 =< 0.5085, 0.4469 >$, $\hat{\alpha}_3 =< 0.6148, 0.4120 >$, $\hat{\alpha}_4 =< 0.5103, 0.3945 >$, $\hat{\alpha}_5 = < 0.6194, 0.3636 >$ and $S(\hat{\alpha}_1) = 0.0308$, $S(\hat{\alpha}_2) = 0.0588$, $S(\hat{\alpha}_3) = 0.2082$, $S(\hat{\alpha}_4) = 0.1048$, $S(\hat{\alpha}_5) = 0.2515$. The ranking of alternatives is $A_5 > A_3 > A_2 > A_4 > A_1$, which is similar to that of the q-ROFWA operator. We also consider $q = 3, 5, 8, 10$ in the q-ROFWA operator and the q-ROFWGA operator. The results are shown in Table 14.

If the TOPSIS method is used to rank alternatives, we first determine the q-rung orthopair fuzzy positive ideal solution $\hat{\alpha}^+$ and q-rung orthopair fuzzy negative ideal solution $\hat{\alpha}^-$ as $\hat{\alpha}^+ = (\hat{\alpha}_1^+, \hat{\alpha}_2^+, \hat{\alpha}_3^+, \hat{\alpha}_4^+) = (< 0.8, 0.1 >, < 0.8, 0.1 >, < 0.9, 0.2 >, < 0.7, 0.2 >)$, $\hat{\alpha}^- = (\hat{\alpha}_1^-, \hat{\alpha}_2^-, \hat{\alpha}_3^-, \hat{\alpha}_4^-) = (< 0.4, 0.5 >, < 0.6, 0.4 >, < 0.5, 0.6 >, < 0.3, 0.6 >)$. Calculate the distance of each alternative evaluation values to $\hat{\alpha}^+$ and $\hat{\alpha}^-$ by using the distance measure $d(\hat{\alpha}_i, \hat{\alpha}_j) = \sqrt{(|\mu_i^2-\mu_j^2| + |\nu_i^2-\nu_j^2|)/2}$. The weighted distances can be calculated by using $d(\hat{\alpha}_i, \hat{\alpha}^+) = \sum_{j=1}^{4} w_j d(\hat{\alpha}_{ij}, \hat{\alpha}_j^+)$, $d(\hat{\alpha}_i, \hat{\alpha}^-) = \sum_{j=1}^{4} w_j d(\hat{\alpha}_{ij}, \hat{\alpha}_j^-)$ to get $d(\hat{\alpha}_1, \hat{\alpha}^+) = 0.4851$, $d(\hat{\alpha}_2, \hat{\alpha}^+) = 0.4573$, $d(\hat{\alpha}_3, \hat{\alpha}^+) = 0.3550$, $d(\hat{\alpha}_4, \hat{\alpha}^+) = 0.3969$, $d(\hat{\alpha}_5, \hat{\alpha}^+) = 0.2195$, $d(\hat{\alpha}_1, \hat{\alpha}^-) = 0.3936$, $d(\hat{\alpha}_2, \hat{\alpha}^-) = 0.3634$, $d(\hat{\alpha}_3, \hat{\alpha}^-) = 0.3964$, $d(\hat{\alpha}_4, \hat{\alpha}^-) = 0.3357$, $d(\hat{\alpha}_5, \hat{\alpha}^-) = 0.3158$. The relative closeness coefficients can be calculated by $CC_i = \frac{d(\hat{\alpha}_i,\hat{\alpha}^+)}{d(\hat{\alpha}_i,\hat{\alpha}^+)+d(\hat{\alpha}_i,\hat{\alpha}^+)}$ to get

TABLE 15. The characteristic comparisons of different methods.

Methods	information by q-rung fuzzy number	whether consider the interrelationships between any two aggregating arguments	whether a parameter vector exists to manipulate the ranking results
Xu and Yager [25]	No	No	No
Xu and Yager [26]	No	Yes	No
Liu and Wang [9]	Yes	Yes	No
Wei et al. [11]	Yes	No	No
Liu et al. [37]	No	Yes	Yes
Our proposed method	Yes	Yes	Yes

$CC_1 = 0.4479, CC_2 = 0.4428, CC_3 = 0.5275, CC_4 = 0.4583, CC_5 = 0.5900$. The alternatives can be ranked as $A_5 > A_3 > A_4 > A_1 > A_2$.

The main differences of the proposed method from the existing methods have been summarized in Table 15. The evaluation values of decision maker are given in the form of q-rung fuzzy numbers, which are more accurate and flexible in modeling fuzzy and uncertain information. The Bonferroni mean have been used to model interrelationship between any two aggregating arguments and Dombi mean has been used to make aggregation process more flexible by using a parameter. The decision makers' risk attitudes can be reflected by using the parameters in the proposed method. The existing methods don't have all these characteristics.

VI. CONCLUSION

In this paper, we develop some q-rung orthopair fuzzy Bonferroni mean Dombi aggregation operators based on the Bonferroni mean, Dombi t-norm and Dombi t-conorm. We have developed the q-ROFBMDA operator, the q-ROFGBMDA operator based on the arithmetic averaging and geometric averaging operation. Then we have developed the q-ROFWBMDA operator and the q-ROFWGBMDA operator based on the weighted arithmetic averaging and weighted geometric averaging operation. Considering partitioned operation, we have developed q-ROFPBMDA operator and q-ROFPWBMDA operator. The new aggregation operators are more flexible comparing with the existing aggregation operators. We have developed a new multiple attribute decision making method based on the proposed operators and presented a realistic example to illustrate the new method. We also have conducted some comparisons of the new methods with some existing methods to demonstrate its applicability and advantages. In the future, we will apply our new methods to solve some other large-scale complicated decision problems including the evaluation of sharing economy, environment, energy, logistics, etc.

APPENDIX

Proof of Theorem 6:

$$r\hat{\alpha}_i = \left\langle \left(1 - \frac{1}{1+\left(r\left(\frac{\mu^q_{\hat{\alpha}_i}}{1-\mu^q_{\hat{\alpha}_i}}\right)^\gamma\right)^{1/\gamma}}\right)^{1/q}, \left(\frac{1}{1+\left(r\left(\frac{1-\nu^q_{\hat{\alpha}_i}}{\nu^q_{\hat{\alpha}_i}}\right)^\gamma\right)^{1/\gamma}}\right)^{1/q} \right\rangle.$$

$$s\hat{\alpha}_i = \left\langle \left(1 - \frac{1}{1+\left(s\left(\frac{\mu^q_{\hat{\alpha}_j}}{1-\mu^q_{\hat{\alpha}_j}}\right)^\gamma\right)^{1/\gamma}}\right)^{1/q}, \left(\frac{1}{1+\left(s\left(\frac{1-\nu^q_{\hat{\alpha}_j}}{\nu^q_{\hat{\alpha}_j}}\right)^\gamma\right)^{1/\gamma}}\right)^{1/q} \right\rangle.$$

$$r\hat{\alpha}_i \oplus s\hat{\alpha}_i = \left\langle \left(1 - \frac{1}{1+\left(r\left(\frac{\mu^q_{\hat{\alpha}_i}}{1-\mu^q_{\hat{\alpha}_i}}\right)^\gamma + \left(s\left(\frac{\mu^q_{\hat{\alpha}_j}}{1-\mu^q_{\hat{\alpha}_j}}\right)^\gamma\right)\right)^{1/\gamma}}\right)^{1/q}, \left(\frac{1}{1+\left(r\left(\frac{1-\nu^q_{\hat{\alpha}_i}}{\nu^q_{\hat{\alpha}_i}}\right)^\gamma + s\left(\frac{1-\nu^q_{\hat{\alpha}_j}}{\nu^q_{\hat{\alpha}_j}}\right)^\gamma\right)^{1/\gamma}}\right)^{1/q} \right\rangle.$$

Let

$$u_{\hat{\alpha}_{ij}} = \frac{1}{r\left(\frac{\mu^q_{\hat{\alpha}_i}}{1-\mu^q_{\hat{\alpha}_i}}\right)^\gamma + s\left(\frac{\mu^q_{\hat{\alpha}_j}}{1-\mu^q_{\hat{\alpha}_j}}\right)^\gamma},$$

$$\nu_{\hat{\alpha}_{ij}} = \frac{1}{\left(r\left(\frac{1-\nu^q_{\hat{\alpha}_i}}{\nu^q_{\hat{\alpha}_i}}\right)^\gamma + s\left(\frac{1-\nu^q_{\hat{\alpha}_j}}{\nu^q_{\hat{\alpha}_j}}\right)^\gamma\right)},$$

$$\left(r\hat{\alpha}_i \oplus s\hat{\alpha}_i\right)^{\frac{1}{n(n-1)}} = \left\langle \left(\frac{1}{1+\left(\frac{1}{n(n-1)}u_{\hat{\alpha}_{ij}}\right)^{1/\gamma}}\right)^{1/q}, \left(1 - \frac{1}{1+\left(\frac{1}{n(n-1)}\nu_{\hat{\alpha}_{ij}}\right)^{1/\gamma}}\right)^{1/q} \right\rangle,$$

$$\otimes^n_{i,j=1,i\neq j}\left(r\hat{\alpha}_i \oplus s\hat{\alpha}_i\right)^{\frac{1}{n(n-1)}} = \left\langle \left(\frac{1}{1+\left(\frac{1}{n(n-1)}\sum^n_{i,j=1,i\neq j}u_{\hat{\alpha}_{ij}}\right)^{1/\gamma}}\right)^{1/q}, \left(1 - \frac{1}{1+\left(\frac{1}{n(n-1)}\sum^n_{i,j=1,i\neq j}\nu_{\hat{\alpha}_{ij}}\right)^{1/\gamma}}\right)^{1/q} \right\rangle,$$

$$\frac{1}{r+s}\left(\otimes^n_{i,j=1,i\neq j}\left(r\hat{\alpha}_i \oplus s\hat{\alpha}_j\right)^{\frac{1}{n(n-1)}}\right) = \left\langle \left(1 - \frac{1}{1+\left(\frac{1}{r+s}\frac{1}{\frac{1}{n(n-1)}\sum^n_{i,j=1,i\neq j}u_{\hat{\alpha}_{ij}}}\right)^{1/\gamma}}\right)^{1/q}, \left(\frac{1}{1+\left(\frac{1}{r+s}\frac{1}{\frac{1}{n(n-1)}\sum^n_{i,j=1,i\neq j}\nu_{\hat{\alpha}_{ij}}}\right)^{1/\gamma}}\right)^{1/q} \right\rangle.$$

Since $0 \leq \mu_{\hat{\alpha}_j}^q + \nu_{\hat{\alpha}_j}^q \leq 1$, $\mu_{\hat{\alpha}_j}^q \leq 1-\nu_{\hat{\alpha}_j}^q$, $\nu_{\hat{\alpha}_j}^q \leq 1-\mu_{\hat{\alpha}_j}^q$, $\frac{\mu_{\hat{\alpha}_j}^q}{1-\mu_{\hat{\alpha}_j}^q} \leq \frac{1-\nu_{\hat{\alpha}_j}^q}{\nu_{\hat{\alpha}_j}^q}$. Similarly, we have $\frac{\mu_{\hat{\alpha}_i}^q}{1-\mu_{\hat{\alpha}_i}^q} \leq \frac{1-\nu_{\hat{\alpha}_i}^q}{\nu_{\hat{\alpha}_i}^q}$. Then $(\frac{\mu_{\hat{\alpha}_i}^q}{1-\mu_{\hat{\alpha}_i}^q})^\gamma \leq (\frac{1-\nu_{\hat{\alpha}_i}^q}{\nu_{\hat{\alpha}_i}^q})^\gamma$ and $(\frac{\mu_{\hat{\alpha}_j}^q}{1-\mu_{\hat{\alpha}_j}^q})^\gamma \leq (\frac{1-\nu_{\hat{\alpha}_j}^q}{\nu_{\hat{\alpha}_j}^q})^\gamma$. $r(\frac{\mu_{\hat{\alpha}_i}^q}{1-\mu_{\hat{\alpha}_i}^q})^\gamma + s(\frac{\mu_{\hat{\alpha}_j}^q}{1-\mu_{\hat{\alpha}_j}^q})^\gamma \leq r(\frac{1-\nu_{\hat{\alpha}_i}^q}{\nu_{\hat{\alpha}_i}^q})^\gamma + s(\frac{1-\nu_{\hat{\alpha}_j}^q}{\nu_{\hat{\alpha}_j}^q})^\gamma$,

$$\frac{1}{n(n-1)}\sum_{i,j=1,i\neq j}^{n}\frac{1}{r(\frac{\mu_{\hat{\alpha}_i}^q}{1-\mu_{\hat{\alpha}_i}^q})^\gamma + s(\frac{\mu_{\hat{\alpha}_j}^q}{1-\mu_{\hat{\alpha}_j}^q})^\gamma} \geq \frac{1}{n(n-1)}\sum_{i,j=1,i\neq j}^{n}\frac{1}{r(\frac{1-\nu_{\hat{\alpha}_i}^q}{\nu_{\hat{\alpha}_i}^q})^\gamma + s(\frac{1-\nu_{\hat{\alpha}_j}^q}{\nu_{\hat{\alpha}_j}^q})^\gamma},$$

By using $u_{\hat{\alpha}_{ij}}$ and $\nu_{\hat{\alpha}_{ij}}$, we can get

$$\frac{1}{n(n-1)}\sum\nolimits_{i,j=1,i\neq j}^{n} u_{\hat{\alpha}_{ij}} \geq \frac{1}{n(n-1)}\sum\nolimits_{i,j=1,i\neq j}^{n} \nu_{\hat{\alpha}_{ij}},$$

$$\left(\frac{1}{r+s}\frac{1}{\frac{1}{n(n-1)}\sum_{i,j=1,i\neq j}^{n} u_{\hat{\alpha}_{ij}}}\right)^{1/\gamma} \leq \left(\frac{1}{r+s}\frac{1}{\frac{1}{n(n-1)}\sum_{i,j=1,i\neq j}^{n} \nu_{\hat{\alpha}_{ij}}}\right)^{1/\gamma},$$

$$\frac{1}{1+\left(\frac{1}{r+s}\frac{1}{\frac{1}{n(n-1)}\sum_{i,j=1,i\neq j}^{n} u_{\hat{\alpha}_{ij}}}\right)^{1/\gamma}} \leq \frac{1}{1+\left(\frac{1}{r+s}\frac{1}{\frac{1}{n(n-1)}\sum_{i,j=1,i\neq j}^{n} \nu_{\hat{\alpha}_{ij}}}\right)^{1/\gamma}},$$

$$0 \leq 1-\frac{1}{1+\left(\frac{1}{r+s}\frac{1}{\frac{1}{n(n-1)}\sum_{i,j=1,i\neq j}^{n} u_{\hat{\alpha}_{ij}}}\right)^{1/\gamma}} + \frac{1}{1+\left(\frac{1}{r+s}\frac{1}{\frac{1}{n(n-1)}\sum_{i,j=1,i\neq j}^{n} \nu_{\hat{\alpha}_{ij}}}\right)^{1/\gamma}} \leq 1.$$

Hence, the aggregate result of the q-ROFGBMDA operator is still a q-rung orthopair fuzzy number.

Proof of Theorem 7:

Proof: Since $\mu_{\hat{\alpha}_k} = \mu_{\hat{\alpha}}$, $(i = 1, 2, \ldots, n)$, $u_{\hat{\alpha}_{ij}} = \frac{1}{r(\frac{\mu_{\hat{\alpha}_i}^q}{1-\mu_{\hat{\alpha}_i}^q})^\gamma + s(\frac{\mu_{\hat{\alpha}_j}^q}{1-\mu_{\hat{\alpha}_j}^q})^\gamma} = \frac{1}{r(\frac{\mu_{\hat{\alpha}}^q}{1-\mu_{\hat{\alpha}}^q})^\gamma + s(\frac{\mu_{\hat{\alpha}}^q}{1-\mu_{\hat{\alpha}}^q})^\gamma} = \frac{1}{(r+s)(\frac{\mu_{\hat{\alpha}}^q}{1-\mu_{\hat{\alpha}}^q})^\gamma}$,

$$\left(1-\frac{1}{1+\left(\frac{1}{r+s}\left(\frac{1}{\left(\frac{1}{n(n-1)}\sum_{i,j=1,i\neq j}^{n} u_{\hat{\alpha}_{ij}}\right)}\right)\right)^{1/\gamma}}\right)^{1/q}$$

$$= \left(1-\frac{1}{1+\left(\frac{1}{r+s}\left(\frac{1}{\left(\frac{1}{n(n-1)}\sum_{i,j=1,i\neq j}^{n}\left(\frac{1}{(r+s)(\frac{\mu_{\hat{\alpha}}^q}{1-\mu_{\hat{\alpha}}^q})^\gamma}\right)\right)}\right)\right)^{1/\gamma}}\right)^{1/q}$$

$$= \left(1-\frac{1}{1+\left(\frac{1}{r+s}\left(\frac{1}{\frac{1}{(r+s)(\frac{\mu_{\hat{\alpha}}^q}{1-\mu_{\hat{\alpha}}^q})^\gamma}}\right)\right)^{1/\gamma}}\right)^{1/q}$$

$$= \left(1-\frac{1}{1+\left(\frac{1}{r+s}\left((r+s)(\frac{\mu_{\hat{\alpha}}^q}{1-\mu_{\hat{\alpha}}^q})^\gamma\right)\right)^{1/\gamma}}\right)^{1/q}$$

$$= \left(1-\frac{1}{1+\left((\frac{\mu_{\hat{\alpha}}^q}{1-\mu_{\hat{\alpha}}^q})^\gamma\right)^{1/\gamma}}\right)^{1/q}$$

$$= \left(1-\frac{1}{1+\frac{\mu_{\hat{\alpha}}^q}{1-\mu_{\hat{\alpha}}^q}}\right)^{1/q} = \left(1-(1-\mu_{\hat{\alpha}}^q)\right)^{1/q} = \mu_{\hat{\alpha}}.$$

Since $\nu_{\hat{\alpha}_k} = \nu_{\hat{\alpha}}$ $(k = 1, 2, \ldots, n)$, $\nu_{\hat{\alpha}_{ij}} = \frac{1}{r(\frac{1-\nu_{\hat{\alpha}}^q}{\nu_{\hat{\alpha}}^q})^\gamma + s(\frac{1-\nu_{\hat{\alpha}}^q}{\nu_{\hat{\alpha}}^q})^\gamma} = \frac{1}{(r+s)(\frac{1-\nu_{\hat{\alpha}}^q}{\nu_{\hat{\alpha}}^q})^\gamma}$, then

$$\left(\frac{1}{1+\left(\frac{1}{r+s}\left(\frac{1}{\left(\frac{1}{n(n-1)}\sum_{i,j=1,i\neq j}^{n} \nu_{\hat{\alpha}_{ij}}\right)}\right)\right)^{1/\gamma}}\right)^{1/q}$$

$$= \left(\frac{1}{1+\left(\frac{1}{r+s}\left(\frac{1}{\left(\frac{1}{n(n-1)}\sum_{i,j=1,i\neq j}^{n}\frac{1}{(r+s)(\frac{1-\nu_{\hat{\alpha}}^q}{\nu_{\hat{\alpha}}^q})^\gamma}\right)}\right)\right)^{1/\gamma}}\right)^{1/q}$$

$$= \left(\frac{1}{1+\left(\frac{1}{r+s}\left(\frac{1}{\frac{1}{(r+s)(\frac{1-\nu_{\hat{\alpha}}^q}{\nu_{\hat{\alpha}}^q})^\gamma}}\right)\right)^{1/\gamma}}\right)^{1/q}$$

$$= \left(\frac{1}{1+\left(\frac{1}{r+s}\left((r+s)(\frac{1-\nu_{\hat{\alpha}}^q}{\nu_{\hat{\alpha}}^q})^\gamma\right)\right)^{1/\gamma}}\right)^{1/q}$$

$$= \left(\frac{1}{1+\left((\frac{1-\nu_{\hat{\alpha}}^q}{\nu_{\hat{\alpha}}^q})^\gamma\right)^{1/\gamma}}\right)^{1/q} = \left(\frac{1}{1+\frac{1-\nu_{\hat{\alpha}}^q}{\nu_{\hat{\alpha}}^q}}\right)^{1/q} = \nu_{\hat{\alpha}}.$$

Hence q-ROFGBMDA$(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) = \hat{\alpha}$.

Proof of Theorem 8:

Proof: $\mu_{\hat{\alpha}_i} \leq \mu_{\hat{\beta}_i}$, $\mu_{\hat{\alpha}_j} \leq \mu_{\hat{\beta}_j}$, $\mu_{\hat{\alpha}_i}^p \leq \mu_{\hat{\beta}_i}^p$, $\mu_{\hat{\alpha}_j}^p \leq \mu_{\hat{\beta}_j}^p$, $(\frac{x}{1-x})' = \frac{1}{(1-x)^2} > 0$, $\frac{\mu_{\hat{\alpha}_i}^p}{1-\mu_{\hat{\alpha}_i}^p} \leq \frac{\mu_{\hat{\beta}_i}^p}{1-\mu_{\hat{\beta}_i}^p}$, $\frac{\mu_{\hat{\alpha}_j}^p}{1-\mu_{\hat{\alpha}_j}^p} \leq \frac{\mu_{\hat{\beta}_j}^p}{1-\mu_{\hat{\beta}_j}^p}$, $r(\frac{\mu_{\hat{\alpha}_i}^p}{1-\mu_{\hat{\alpha}_i}^p})^\gamma + s(\frac{\mu_{\hat{\alpha}_j}^p}{1-\mu_{\hat{\alpha}_j}^p})^\gamma \leq r(\frac{\mu_{\hat{\beta}_i}^p}{1-\mu_{\hat{\beta}_i}^p})^\gamma + s(\frac{\mu_{\hat{\beta}_j}^p}{1-\mu_{\hat{\beta}_j}^p})^\gamma$. $u_{\hat{\alpha}_{ij}} = \frac{1}{r(\frac{\mu_{\hat{\alpha}_i}^p}{1-\mu_{\hat{\alpha}_i}^p})^\gamma + s(\frac{\mu_{\hat{\alpha}_j}^p}{1-\mu_{\hat{\alpha}_j}^p})^\gamma}$, $u_{\hat{\beta}_{ij}} = \frac{1}{r(\frac{\mu_{\hat{\beta}_i}^p}{1-\mu_{\hat{\beta}_i}^p})^\gamma + s(\frac{\mu_{\hat{\beta}_j}^p}{1-\mu_{\hat{\beta}_j}^p})^\gamma}$,

$$\frac{1}{n(n-1)}\sum\nolimits_{i,j=1,i\neq j}^{n} u_{\hat{\alpha}_{ij}} \geq \frac{1}{n(n-1)}\sum\nolimits_{i,j=1,i\neq j}^{n} u_{\hat{\beta}_{ij}},$$

$$\left(\frac{1}{r+s}\left(\frac{1}{\left(\frac{1}{n(n-1)}\sum_{i,j=1,i\neq j}^{n} u_{\hat{\alpha}_{ij}}\right)}\right)\right)^{1/\gamma} \leq \left(\frac{1}{r+s}\left(\frac{1}{\left(\frac{1}{n(n-1)}\sum_{i,j=1,i\neq j}^{n} u_{\hat{\beta}_{ij}}\right)}\right)\right)^{1/\gamma},$$

$$\left(1-\frac{1}{1+\left(\frac{1}{r+s}\left(\frac{1}{\left(\frac{1}{n(n-1)}\sum_{i,j=1,i\neq j}^{n}u_{\hat{\alpha}_{ij}}\right)}\right)\right)^{1/\gamma}}\right)^{1/q}$$
$$\leq\left(1-\frac{1}{1+\left(\frac{1}{r+s}\left(\frac{1}{\left(\frac{1}{n(n-1)}\sum_{i,j=1,i\neq j}^{n}u_{\hat{\alpha}_{ij}}\right)}\right)\right)^{1/\gamma}}\right)^{1/q},$$

$\nu_{\hat{\alpha}_i}\geq\nu_{\hat{\beta}_i}$, $\nu_{\hat{\alpha}_j}\geq\nu_{\hat{\beta}_j}$, $\nu^p_{\hat{\alpha}_i}\geq\nu^p_{\hat{\beta}_i}$, $\nu^p_{\hat{\alpha}_j}\geq\nu^p_{\hat{\beta}_j}$, $\left(\frac{1-x}{x}\right)'=-\frac{1}{x^2}<0$, $\frac{1-\nu^p_{\hat{\alpha}_i}}{\nu^p_{\hat{\alpha}_i}}\leq\frac{1-\nu^p_{\hat{\beta}_i}}{\nu^p_{\hat{\beta}_i}}$, $\frac{1-\nu^p_{\hat{\alpha}_j}}{\nu^p_{\hat{\alpha}_j}}\leq\frac{1-\nu^p_{\hat{\beta}_j}}{\nu^p_{\hat{\beta}_j}}$, $r\left(\frac{1-\nu^p_{\hat{\alpha}_i}}{\nu^p_{\hat{\alpha}_i}}\right)^\gamma+s\left(\frac{1-\nu^p_{\hat{\alpha}_j}}{\nu^p_{\hat{\alpha}_j}}\right)^\gamma\leq r\left(\frac{1-\nu^p_{\hat{\beta}_i}}{\nu^p_{\hat{\beta}_i}}\right)^\gamma+s\left(\frac{1-\nu^p_{\hat{\beta}_j}}{\nu^p_{\hat{\beta}_j}}\right)^\gamma$, $\nu_{\hat{\alpha}_{ij}}=\frac{1}{r\left(\frac{1-\nu^p_{\hat{\alpha}_i}}{\nu^p_{\hat{\alpha}_i}}\right)^\gamma+s\left(\frac{1-\nu^p_{\hat{\alpha}_j}}{\nu^p_{\hat{\alpha}_j}}\right)^\gamma}$, $\nu_{\hat{\beta}_{ij}}=\frac{1}{r\left(\frac{1-\nu^p_{\hat{\beta}_i}}{\nu^p_{\hat{\beta}_i}}\right)^\gamma+s\left(\frac{1-\nu^p_{\hat{\beta}_j}}{\nu^p_{\hat{\beta}_j}}\right)^\gamma}$,

$$\frac{1}{n(n-1)}\sum_{i,j=1,i\neq j}^{n}\nu_{\hat{\alpha}_{ij}}\geq\frac{1}{n(n-1)}\sum_{i,j=1,i\neq j}^{n}\nu_{\hat{\beta}_{ij}},$$
$$\left(\frac{1}{r+s}\frac{1}{\left(\frac{1}{n(n-1)}\sum_{i,j=1,i\neq j}^{n}\nu_{\hat{\alpha}_{ij}}\right)}\right)^{1/\gamma}\leq\left(\frac{1}{r+s}\frac{1}{\left(\frac{1}{n(n-1)}\sum_{i,j=1,i\neq j}^{n}\nu_{\hat{\beta}_{ij}}\right)}\right)^{1/\gamma},$$
$$\frac{1}{1+\left(\frac{1}{r+s}\frac{1}{\left(\frac{1}{n(n-1)}\sum_{i,j=1,i\neq j}^{n}\nu_{\hat{\alpha}_{ij}}\right)}\right)^{1/\gamma}}\geq\frac{1}{1+\left(\frac{1}{r+s}\frac{1}{\left(\frac{1}{n(n-1)}\sum_{i,j=1,i\neq j}^{n}\nu_{\hat{\beta}_{ij}}\right)}\right)^{1/\gamma}}.$$

By using the score function, we can get

$$\text{q-ROFGBMDA}(\hat{\alpha}_1,\hat{\alpha}_2,\ldots,\hat{\alpha}_n)\leq\text{q-ROFGBMDA}(\hat{\beta}_1,\hat{\beta}_2,\ldots,\hat{\beta}_n).$$

Proof of Theorem 9:

Proof: The property of boundedness can be proved easily by using the property of monotonicity.

Proof of Theorem 24:

Proof:

$$r\hat{\alpha}_i=\left\langle\left(1-\frac{1}{1+\left(r\left(\frac{\mu^q_{\hat{\alpha}_i}}{1-\mu^q_{\hat{\alpha}_i}}\right)^\gamma\right)^{1/\gamma}}\right)^{1/q},\left(\frac{1}{1+\left(r\left(\frac{1-\nu^q_{\hat{\alpha}_i}}{\nu^q_{\hat{\alpha}_i}}\right)^\gamma\right)^{1/\gamma}}\right)^{1/q}\right\rangle,$$

$$s\hat{\alpha}_j=\left\langle\left(1-\frac{1}{1+\left(s\left(\frac{\mu^q_{\hat{\alpha}_j}}{1-\mu^q_{\hat{\alpha}_j}}\right)^\gamma\right)^{1/\gamma}}\right)^{1/q},\left(\frac{1}{1+\left(s\left(\frac{1-\nu^q_{\hat{\alpha}_j}}{\nu^q_{\hat{\alpha}_j}}\right)^\gamma\right)^{1/\gamma}}\right)^{1/q}\right\rangle,$$

$$(r\hat{\alpha}_i)\oplus(s\hat{\alpha}_i)=\left\langle\left(1-\frac{1}{1+\left(r\left(\frac{\mu^q_{\hat{\alpha}_i}}{1-\mu^q_{\hat{\alpha}_i}}\right)^\gamma+s\left(\frac{\mu^q_{\hat{\alpha}_j}}{1-\mu^q_{\hat{\alpha}_j}}\right)^\gamma\right)^{1/\gamma}}\right)^{1/q},\left(\frac{1}{1+\left(r\left(\frac{1-\nu^q_{\hat{\alpha}_i}}{\nu^q_{\hat{\alpha}_i}}\right)^\gamma+s\left(\frac{1-\nu^q_{\hat{\alpha}_j}}{\nu^q_{\hat{\alpha}_j}}\right)^\gamma\right)^{1/\gamma}}\right)^{1/q}\right\rangle.$$

Let $u_{\hat{\alpha}_{ij}}=r\left(\frac{\mu^q_{\hat{\alpha}_i}}{1-\mu^q_{\hat{\alpha}_i}}\right)^\gamma+s\left(\frac{\mu^q_{\hat{\alpha}_j}}{1-\mu^q_{\hat{\alpha}_j}}\right)^\gamma$, $\nu_{\hat{\alpha}_{ij}}=r\left(\frac{1-\nu^q_{\hat{\alpha}_i}}{\nu^q_{\hat{\alpha}_i}}\right)^\gamma+s\left(\frac{1-\nu^q_{\hat{\alpha}_j}}{\nu^q_{\hat{\alpha}_j}}\right)^\gamma$. Then

$$(r\hat{\alpha}_i)\oplus(s\hat{\alpha}_i)=\left\langle\left(1-\frac{1}{1+(u_{\hat{\alpha}_{ij}})^{1/\gamma}}\right)^{1/q},\left(\frac{1}{1+(\nu_{\hat{\alpha}_{ij}})^{1/\gamma}}\right)^{1/q}\right\rangle.$$

$$\otimes_{i,j\in P_h,i\neq j}((r\hat{\alpha}_i)\oplus(s\hat{\alpha}_j))=\left\langle\left(\frac{1}{1+\left(\sum_{i,j\in P_h,i\neq j}\frac{1}{u_{\hat{\alpha}_{ij}}}\right)^{1/\gamma}}\right)^{1/q},\left(1-\frac{1}{1+\left(\sum_{i,j\in P_h,i\neq j}\frac{1}{\nu_{\hat{\alpha}_{ij}}}\right)^{1/\gamma}}\right)^{1/q}\right\rangle,$$

$$\left(\otimes_{i,j\in P_h,i\neq j}((r\hat{\alpha}_i)\oplus(s\hat{\alpha}_j))\right)^{\frac{1}{|P_h|(|P_h|-1)}}=\left\langle\left(\frac{1}{1+\left(\frac{1}{|P_h|(|P_h|-1)}\sum_{i,j\in P_h,i\neq j}\frac{1}{u_{\hat{\alpha}_{ij}}}\right)^{1/\gamma}}\right)^{1/q},\left(1-\frac{1}{1+\left(\frac{1}{|P_h|(|P_h|-1)}\sum_{i,j\in P_h,i\neq j}\frac{1}{\nu_{\hat{\alpha}_{ij}}}\right)^{1/\gamma}}\right)^{1/q}\right\rangle,$$

$$\frac{1}{r+s}\left(\otimes_{i,j\in P_h,i\neq j}((r\hat{\alpha}_i)\oplus(s\hat{\alpha}_j))\right)^{\frac{1}{|P_h|(|P_h|-1)}}=\left\langle\left(1-\frac{1}{1+\left(\frac{1}{r+s}\frac{1}{\left(\frac{1}{|P_h|(|P_h|-1)}\sum_{i,j\in P_h,i\neq j}\frac{1}{u_{\hat{\alpha}_{ij}}}\right)}\right)^{1/\gamma}}\right)^{1/q},\left(\frac{1}{1+\left(\frac{1}{r+s}\frac{1}{\left(\frac{1}{|P_h|(|P_h|-1)}\sum_{i,j\in P_h,i\neq j}\frac{1}{\nu_{\hat{\alpha}_{ij}}}\right)}\right)^{1/\gamma}}\right)^{1/q}\right\rangle,$$

$$\otimes_{i=1}^{m}\left(\frac{1}{r+s}\left(\otimes_{i,j\in P_h,i\neq j}((r\hat{\alpha}_i)\oplus(s\hat{\alpha}_j))\right)^{\frac{1}{|P_h|(|P_h|-1)}}\right)=\left\langle\left(\frac{1}{1+\left(\sum_{i=1}^{m}\frac{1}{\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|(|P_h|-1)}\left(\sum_{i,j\in P_h,i\neq j}\frac{1}{u_{\hat{\alpha}_{ij}}}\right)}}\right)^{1/\gamma}}\right)^{1/q},\left(1-\frac{1}{1+\left(\sum_{i=1}^{m}\frac{1}{\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|(|P_h|-1)}\left(\sum_{i,j\in P_h,i\neq j}\frac{1}{\nu_{\hat{\alpha}_{ij}}}\right)}}\right)^{1/\gamma}}\right)^{1/q}\right\rangle,$$

$$\left(\otimes_{i=1}^{m}\left(\frac{1}{r+s}\left(\otimes_{i,j\in P_h,i\neq j}((r\hat{\alpha}_i)\oplus(s\hat{\alpha}_j))\right)^{\frac{1}{|P_h|(|P_h|-1)}}\right)\right)^{\frac{1}{m}}$$

$$= \langle \Big(\frac{1}{1+\big(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\frac{1}{r+s}\frac{1}{\frac{1}{\frac{1}{|P_h|(|P_h|-1)}\left(\sum_{i,j\in P_h,i\neq j}\frac{1}{u_{\hat{\alpha}_{ij}}}\right)}}}\big)^{1/\gamma}} \Big)^{1/q},$$

$$\Big(1-\frac{1}{1+\big(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\frac{1}{r+s}\frac{1}{\frac{1}{\frac{1}{|P_h|(|P_h|-1)}\left(\sum_{i,j\in P_h,i\neq j}\frac{1}{v_{\hat{\alpha}_{ij}}}\right)}}}\big)^{1/\gamma}} \Big)^{1/q}\rangle.$$

$0 \le \mu_{\hat{\alpha}_i}^q+\nu_{\hat{\alpha}_i}^q \le 1, 0 \le \mu_{\hat{\alpha}_j}^q+\nu_{\hat{\alpha}_j}^q \le 1, \mu_{\hat{\alpha}_i}^q \le 1-\nu_{\hat{\alpha}_i}^q, \mu_{\hat{\alpha}_j}^q \le 1-\nu_{\hat{\alpha}_j}^q, \nu_{\hat{\alpha}_i}^q \le 1-\mu_{\hat{\alpha}_i}^q, \nu_{\hat{\alpha}_j}^q \le 1-\mu_{\hat{\alpha}_j}^q, \frac{\mu_{\hat{\alpha}_i}^q}{1-\mu_{\hat{\alpha}_i}^q} \le \frac{1-\nu_{\hat{\alpha}_i}^q}{\nu_{\hat{\alpha}_i}^q}, \frac{\mu_{\hat{\alpha}_j}^q}{1-\mu_{\hat{\alpha}_j}^q} \le \frac{1-\nu_{\hat{\alpha}_j}^q}{\nu_{\hat{\alpha}_j}^q}, r(\frac{\mu_{\hat{\alpha}_i}^q}{1-\mu_{\hat{\alpha}_i}^q})^\gamma + s(\frac{\mu_{\hat{\alpha}_j}^q}{1-\mu_{\hat{\alpha}_j}^q})^\gamma \le r(\frac{1-\nu_{\hat{\alpha}_i}^q}{\nu_{\hat{\alpha}_i}^q})^\gamma + s(\frac{1-\nu_{\hat{\alpha}_j}^q}{\nu_{\hat{\alpha}_j}^q})^\gamma,$

$$\frac{1}{|P_h|(|P_h|-1)}\Big(\sum_{i,j\in P_h,i\neq j}\frac{1}{r(\frac{1-\nu_{\hat{\alpha}_i}^q}{\nu_{\hat{\alpha}_i}^q})^\gamma + s(\frac{1-\nu_{\hat{\alpha}_j}^q}{\nu_{\hat{\alpha}_j}^q})^\gamma}\Big)$$
$$\le \frac{1}{|P_h|(|P_h|-1)}\Big(\sum_{i,j\in P_h,i\neq j}\frac{1}{r(\frac{\mu_{\hat{\alpha}_i}^q}{1-\mu_{\hat{\alpha}_i}^q})^\gamma + s(\frac{\mu_{\hat{\alpha}_j}^q}{1-\mu_{\hat{\alpha}_j}^q})^\gamma}\Big),$$

that is

$$\frac{1}{|P_h|(|P_h|-1)}\Big(\sum_{i,j\in P_h,i\neq j}\frac{1}{v_{\hat{\alpha}_{ij}}}\Big) \le \frac{1}{|P_h|(|P_h|-1)}\Big(\sum_{i,j\in P_h,i\neq j}\frac{1}{u_{\hat{\alpha}_{ij}}}\Big),$$

$$\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|(|P_h|-1)}\left(\sum_{i,j\in P_h,i\neq j}\frac{1}{v_{\hat{\alpha}_{ij}}}\right)} \ge \frac{1}{r+s}\frac{1}{\frac{1}{|P_h|(|P_h|-1)}\left(\sum_{i,j\in P_h,i\neq j}\frac{1}{u_{\hat{\alpha}_{ij}}}\right)},$$

$$\Big(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|(|P_h|-1)}\left(\sum_{i,j\in P_h,i\neq j}\frac{1}{v_{\hat{\alpha}_{ij}}}\right)}}\Big)^{1/\gamma} \le \Big(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|(|P_h|-1)}\left(\sum_{i,j\in P_h,i\neq j}\frac{1}{u_{\hat{\alpha}_{ij}}}\right)}}\Big)^{1/\gamma},$$

$$1 \ge \frac{1}{1+\Big(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|(|P_h|-1)}\left(\sum_{i,j\in P_h,i\neq j}\frac{1}{v_{\hat{\alpha}_{ij}}}\right)}}\Big)^{1/\gamma}} \ge \frac{1}{1+\Big(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|(|P_h|-1)}\left(\sum_{i,j\in P_h,i\neq j}\frac{1}{u_{\hat{\alpha}_{ij}}}\right)}}\Big)^{1/\gamma}} \ge 0,$$

$$0 \le \frac{1}{1+\big(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|(|P_h|-1)}\left(\sum_{i,j\in P_h,i\neq j}\frac{1}{u_{\hat{\alpha}_{ij}}}\right)}}\big)^{1/\gamma}} + 1 - \frac{1}{1+\big(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|(|P_h|-1)}\left(\sum_{i,j\in P_h,i\neq j}\frac{1}{v_{\hat{\alpha}_{ij}}}\right)}}\big)^{1/\gamma}} \le 1.$$

By using the score function, we can get the aggregated result of the q-ROFPGBMDA operator is still a q-rung orthopair fuzzy number.

Proof of Theorem 25:

Proof: Since $\mu_{\hat{\alpha}_k} = \mu_{\hat{\alpha}}$, $u_{\hat{\alpha}_{ij}} = r(\frac{\mu_{\hat{\alpha}_i}^q}{1-\mu_{\hat{\alpha}_i}^q})^\gamma + s(\frac{\mu_{\hat{\alpha}_j}^q}{1-\mu_{\hat{\alpha}_j}^q})^\gamma = r(\frac{\mu_{\hat{\alpha}}^q}{1-\mu_{\hat{\alpha}}^q})^\gamma + s(\frac{\mu_{\hat{\alpha}}^q}{1-\mu_{\hat{\alpha}}^q})^\gamma = (r+s)(\frac{\mu_{\hat{\alpha}}^q}{1-\mu_{\hat{\alpha}}^q})^\gamma$, then

$$\frac{1}{|P_h|(|P_h|-1)}\Big(\sum_{i,j\in P_h,i\neq j}\frac{1}{u_{\hat{\alpha}_{ij}}}\Big) = \frac{1}{|P_h|(|P_h|-1)}\Big(\sum_{i,j\in P_h,i\neq j}\frac{1}{(r+s)(\frac{\mu_{\hat{\alpha}}^q}{1-\mu_{\hat{\alpha}}^q})^\gamma}\Big) = \frac{1}{(r+s)(\frac{\mu_{\hat{\alpha}}^q}{1-\mu_{\hat{\alpha}}^q})^\gamma}.$$

$$\Big(\frac{1}{1+\big(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|(|P_h|-1)}\left(\sum_{i,j\in P_h,i\neq j}\frac{1}{u_{\hat{\alpha}_{ij}}}\right)}}\big)^{1/\gamma}}\Big)^{1/q} = \Big(\frac{1}{1+\big(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\frac{1}{r+s}\frac{1}{\left(\frac{1}{(r+s)(\frac{\mu_{\hat{\alpha}}^q}{1-\mu_{\hat{\alpha}}^q})^\gamma}\right)}}\big)^{1/\gamma}}\Big)^{1/q}$$
$$= \Big(\frac{1}{1+\big(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\frac{1}{r+s}(\frac{\mu_{\hat{\alpha}}^q}{1-\mu_{\hat{\alpha}}^q})^\gamma}\big)^{1/\gamma}}\Big)^{1/q} = \Big(\frac{1}{1+\big(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{(\frac{\mu_{\hat{\alpha}}^q}{1-\mu_{\hat{\alpha}}^q})^\gamma}\big)^{1/\gamma}}\Big)^{1/q}$$
$$= \Big(\frac{1}{1+\big(\frac{1}{(\frac{\mu_{\hat{\alpha}}^q}{1-\mu_{\hat{\alpha}}^q})^\gamma}\big)^{1/\gamma}}\Big)^{1/q} = \Big(\frac{1}{1+\big(\frac{1}{(\frac{\mu_{\hat{\alpha}}^q}{1-\mu_{\hat{\alpha}}^q})}\big)}\Big)^{1/q} = \mu_{\hat{\alpha}}.$$

$v_{\hat{\alpha}_{ij}} = r(\frac{1-\nu_{\hat{\alpha}}^q}{\nu_{\hat{\alpha}}^q})^\gamma + s(\frac{1-\nu_{\hat{\alpha}}^q}{\nu_{\hat{\alpha}}^q})^\gamma = r(\frac{1-\nu_{\hat{\alpha}}^q}{\nu_{\hat{\alpha}}^q})^\gamma + s(\frac{1-\nu_{\hat{\alpha}}^q}{\nu_{\hat{\alpha}}^q})^\gamma = (r+s)(\frac{1-\nu_{\hat{\alpha}}^q}{\nu_{\hat{\alpha}}^q})^\gamma.$

Then

$$\frac{1}{|P_h|(|P_h|-1)}\Big(\sum_{i,j\in P_h,i\neq j}\frac{1}{v_{\hat{\alpha}_{ij}}}\Big) = \frac{1}{|P_h|(|P_h|-1)}\Big(\sum_{i,j\in P_h,i\neq j}\frac{1}{(r+s)(\frac{1-\nu_{\hat{\alpha}}^q}{\nu_{\hat{\alpha}}^q})^\gamma}\Big) = \frac{1}{(r+s)(\frac{1-\nu_{\hat{\alpha}}^q}{\nu_{\hat{\alpha}}^q})^\gamma}.$$

$$\left(1-\frac{1}{1+\left(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|(|P_h|-1)}\left(\sum_{i,j\in P_h,i\neq j}\frac{1}{v_{\hat{\alpha}_{ij}}}\right)}}\right)^{1/\gamma}}\right)^{1/q}$$

$$=\left(1-\frac{1}{1+\left(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\frac{1}{r+s}\frac{1}{\frac{1}{(r+s)\left(\frac{1-v_{\hat{\alpha}}^q}{v_{\hat{\alpha}}^q}\right)^{\gamma}}}}\right)^{1/\gamma}}\right)^{1/q}$$

$$=\left(1-\frac{1}{1+\left(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\frac{1}{r+s}(r+s)\left(\frac{1-v_{\hat{\alpha}}^q}{v_{\hat{\alpha}}^q}\right)^{\gamma}}\right)^{1/\gamma}}\right)^{1/q}$$

$$=\left(1-\frac{1}{1+\left(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\left(\frac{1-v_{\hat{\alpha}}^q}{v_{\hat{\alpha}}^q}\right)^{\gamma}}\right)^{1/\gamma}}\right)^{1/q}$$

$$=\left(1-\frac{1}{1+\left(\frac{1}{\left(\frac{1-v_{\hat{\alpha}}^q}{v_{\hat{\alpha}}^q}\right)^{\gamma}}\right)^{1/\gamma}}\right)^{1/q}$$

$$=\left(1-\frac{1}{1+\frac{1}{\frac{1-v_{\hat{\alpha}}^q}{v_{\hat{\alpha}}^q}}}\right)^{1/q}=\left(1-\frac{1}{1+\frac{v_{\hat{\alpha}}^q}{1-v_{\hat{\alpha}}^q}}\right)^{1/q}=v_{\hat{\alpha}}.$$

Hence q-ROFPGBMDA$(\hat{\alpha}_1,\hat{\alpha}_2,\ldots,\hat{\alpha}_n)=\hat{\alpha}$.

Proof of Theorem 26: (Commutativity) Let $\hat{\alpha}_k$ $(k=1,2,\ldots,n)$ be a collection of q-rung orthopair fuzzy numbers. If $(\hat{\alpha}'_1,\hat{\alpha}'_2,\ldots,\hat{\alpha}'_n)$ is any permutation of $(\hat{\alpha}_1,\hat{\alpha}_2,\ldots,\hat{\alpha}_n)$, then

$$\text{q-ROFPGBMDA}(\hat{\alpha}_1,\hat{\alpha}_2,\ldots,\hat{\alpha}_n)=\text{q-ROFPGBMDA}(\hat{\alpha}'_1,\hat{\alpha}'_2,\ldots,\hat{\alpha}'_n).$$

Proof:

$$\begin{aligned}&\text{q-ROFPGBMDA}(\hat{\alpha}_1,\hat{\alpha}_2,\ldots,\hat{\alpha}_n)\\&=\left(\otimes_{i=1}^{m}\left(\frac{1}{r+s}\left(\otimes_{i,j\in P_h,i\neq j}((r\hat{\alpha}_i)\oplus(s\hat{\alpha}_j))^{\frac{1}{|P_h|(|P_h|-1)}}\right)\right)^{\frac{1}{m}}\right)\\&=\left(\otimes_{i=1}^{m}\left(\frac{1}{r+s}\left(\otimes_{i,j\in P_h,i\neq j}((r\hat{\alpha}'_i)\oplus(s\hat{\alpha}'_j))^{\frac{1}{|P_h|(|P_h|-1)}}\right)\right)^{\frac{1}{m}}\right).\end{aligned}$$

Proof of Theorem 27:

Proof: Since $\mu_{\hat{\alpha}_k}\leq\mu_{\hat{\beta}_k}$, $\mu^q_{\hat{\alpha}_k}\leq\mu^q_{\hat{\beta}_k}$, $(\frac{x}{1-x})'=\frac{1}{(1-x)^2}>0$. Hence, $\frac{\mu^q_{\hat{\alpha}_i}}{1-\mu^q_{\hat{\alpha}_i}}\leq\frac{\mu^q_{\hat{\beta}_i}}{1-\mu^q_{\hat{\beta}_i}}$, $\frac{\mu^q_{\hat{\alpha}_j}}{1-\mu^q_{\hat{\alpha}_j}}\leq\frac{\mu^q_{\hat{\beta}_j}}{1-\mu^q_{\hat{\beta}_j}}$ and $\left(r(\frac{\mu^q_{\hat{\alpha}_i}}{1-\mu^q_{\hat{\alpha}_i}})^{\gamma}+s(\frac{\mu^q_{\hat{\alpha}_j}}{1-\mu^q_{\hat{\alpha}_j}})^{\gamma}\right)\leq\left(r(\frac{\mu^q_{\hat{\beta}_i}}{1-\mu^q_{\hat{\beta}_i}})^{\gamma}+s(\frac{\mu^q_{\hat{\beta}_j}}{1-\mu^q_{\hat{\beta}_j}})^{\gamma}\right)$.

$$\frac{1}{|P_h|(|P_h|-1)}\left(\sum_{i,j\in P_h,i\neq j}\frac{1}{r\left(\frac{\mu^q_{\hat{\alpha}_i}}{1-\mu^q_{\hat{\alpha}_i}}\right)^{\gamma}+s\left(\frac{\mu^q_{\hat{\alpha}_j}}{1-\mu^q_{\hat{\alpha}_j}}\right)^{\gamma}}\right)$$
$$\geq\frac{1}{|P_h|(|P_h|-1)}\left(\sum_{i,j\in P_h,i\neq j}\frac{1}{r\left(\frac{\mu^q_{\hat{\beta}_i}}{1-\mu^q_{\hat{\beta}_i}}\right)^{\gamma}+s\left(\frac{\mu^q_{\hat{\beta}_j}}{1-\mu^q_{\hat{\beta}_j}}\right)^{\gamma}}\right),$$

$$\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|(|P_h|-1)}\left(\sum_{i,j\in P_h,i\neq j}\frac{1}{r\left(\frac{\mu^q_{\hat{\alpha}_i}}{1-\mu^q_{\hat{\alpha}_i}}\right)^{\gamma}+s\left(\frac{\mu^q_{\hat{\alpha}_j}}{1-\mu^q_{\hat{\alpha}_j}}\right)^{\gamma}}\right)}$$
$$\leq\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|(|P_h|-1)}\left(\sum_{i,j\in P_h,i\neq j}\frac{1}{r\left(\frac{\mu^q_{\hat{\beta}_i}}{1-\mu^q_{\hat{\beta}_i}}\right)^{\gamma}+s\left(\frac{\mu^q_{\hat{\beta}_j}}{1-\mu^q_{\hat{\beta}_j}}\right)^{\gamma}}\right)},$$

Let $u_{\hat{\alpha}_{ij}}=r(\frac{\mu^q_{\hat{\alpha}_i}}{1-\mu^q_{\hat{\alpha}_i}})^{\gamma}+s(\frac{\mu^q_{\hat{\alpha}_j}}{1-\mu^q_{\hat{\alpha}_j}})^{\gamma}$, $u_{\hat{\beta}_{ij}}=r(\frac{\mu^q_{\hat{\beta}_i}}{1-\mu^q_{\hat{\beta}_i}})^{\gamma}+s(\frac{\mu^q_{\hat{\beta}_j}}{1-\mu^q_{\hat{\beta}_j}})^{\gamma}$.

$$\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|(|P_h|-1)}\left(\sum_{i,j\in P_h,i\neq j}\frac{1}{u_{\hat{\alpha}_{ij}}}\right)}\leq\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|(|P_h|-1)}\left(\sum_{i,j\in P_h,i\neq j}\frac{1}{u_{\hat{\beta}_{ij}}}\right)},$$

$$\left(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|(|P_h|-1)}\left(\sum_{i,j\in P_h,i\neq j}\frac{1}{u_{\hat{\alpha}_{ij}}}\right)}}\right)^{1/\gamma}\geq\left(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|(|P_h|-1)}\left(\sum_{i,j\in P_h,i\neq j}\frac{1}{u_{\hat{\beta}_{ij}}}\right)}}\right)^{1/\gamma},$$

$$\left(\frac{1}{1+\left(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|(|P_h|-1)}\left(\sum_{i,j\in P_h,i\neq j}\frac{1}{u_{\hat{\alpha}_{ij}}}\right)}}\right)^{1/\gamma}}\right)^{1/q}\leq\left(\frac{1}{1+\left(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|(|P_h|-1)}\left(\sum_{i,j\in P_h,i\neq j}\frac{1}{u_{\hat{\beta}_{ij}}}\right)}}\right)^{1/\gamma}}\right)^{1/q}.$$

Since $v_{\hat{\alpha}_k}\geq v_{\hat{\beta}_k}$, $v^q_{\hat{\alpha}_k}\geq v^q_{\hat{\beta}_k}$. $(\frac{1-x}{x})'=-\frac{1}{x^2}$, $\frac{1-v^q_{\hat{\alpha}_i}}{v^q_{\hat{\alpha}_i}}\leq\frac{1-v^q_{\hat{\beta}_i}}{v^q_{\hat{\beta}_i}}$ and $\frac{1-v^q_{\hat{\alpha}_j}}{v^q_{\hat{\alpha}_j}}\leq\frac{1-v^q_{\hat{\beta}_j}}{v^q_{\hat{\beta}_j}}$. $\left(r(\frac{1-v^q_{\hat{\alpha}_i}}{v^q_{\hat{\alpha}_i}})^{\gamma}+s(\frac{1-v^q_{\hat{\alpha}_j}}{v^q_{\hat{\alpha}_j}})^{\gamma}\right)\leq\left(r(\frac{1-v^q_{\hat{\beta}_i}}{v^q_{\hat{\beta}_i}})^{\gamma}+s(\frac{1-v^q_{\hat{\beta}_j}}{v^q_{\hat{\beta}_j}})^{\gamma}\right)$.

Let $v_{\hat{\alpha}_{ij}}=\left(r(\frac{1-v^q_{\hat{\alpha}_i}}{v^q_{\hat{\alpha}_i}})^{\gamma}+s(\frac{1-v^q_{\hat{\alpha}_j}}{v^q_{\hat{\alpha}_j}})^{\gamma}\right)$, $v_{\hat{\beta}_{ij}}=r(\frac{1-v^q_{\hat{\beta}_i}}{v^q_{\hat{\beta}_i}})^{\gamma}+s(\frac{1-v^q_{\hat{\beta}_j}}{v^q_{\hat{\beta}_j}})^{\gamma}$. Then

$$\frac{1}{|P_h|(|P_h|-1)}\left(\sum_{i,j\in P_h,i\neq j}\frac{1}{v_{\hat{\alpha}_{ij}}}\right)\geq\frac{1}{|P_h|(|P_h|-1)}\left(\sum_{i,j\in P_h,i\neq j}\frac{1}{v_{\hat{\beta}_{ij}}}\right),$$

$$\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|(|P_h|-1)}\left(\sum_{i,j\in P_h,i\neq j}\frac{1}{v_{\hat{\alpha}_{ij}}}\right)}$$

$$\leq \frac{1}{r+s}\frac{1}{\frac{1}{|P_h|(|P_h|-1)}\left(\sum_{i,j\in P_h,i\neq j}\frac{1}{v_{\hat{\beta}_{ij}}}\right)},$$

$$\left(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|(|P_h|-1)}\left(\sum_{i,j\in P_h,i\neq j}\frac{1}{v_{\hat{\alpha}_{ij}}}\right)}}\right)^{1/\gamma}$$

$$\geq \left(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|(|P_h|-1)}\left(\sum_{i,j\in P_h,i\neq j}\frac{1}{v_{\hat{\beta}_{ij}}}\right)}\right)^{1/\gamma},$$

$$\frac{1}{1+\left(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|(|P_h|-1)}\left(\sum_{i,j\in P_h,i\neq j}\frac{1}{v_{\hat{\alpha}_{ij}}}\right)}}\right)^{1/\gamma}}$$

$$\leq \frac{1}{1+\left(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|(|P_h|-1)}\left(\sum_{i,j\in P_h,i\neq j}\frac{1}{v_{\hat{\beta}_{ij}}}\right)}\right)^{1/\gamma}},$$

$$\left(1-\frac{1}{1+\left(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|(|P_h|-1)}\left(\sum_{i,j\in P_h,i\neq j}\frac{1}{v_{\hat{\alpha}_{ij}}}\right)}}\right)^{1/\gamma}}\right)^{1/q}$$

$$\geq \left(\frac{1}{1+\left(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{r+s}\frac{1}{\frac{1}{|P_h|(|P_h|-1)}\left(\sum_{i,j\in P_h,i\neq j}\frac{1}{v_{\hat{\beta}_{ij}}}\right)}\right)^{1/\gamma}}\right)^{1/q}.$$

By using the score function, we can get

$$\text{q-ROFPGBMDA}(\hat{\alpha}_1,\hat{\alpha}_2,\ldots,\hat{\alpha}_n) \leq \text{q-ROFPGBMDA}(\hat{\beta}_1,\hat{\beta}_2,\ldots,\hat{\beta}_n).$$

REFERENCES

[1] R. R. Yager, "Pythagorean fuzzy subsets," in *Proc. Joint IFSA World Congr. NAFIPS Annu. Meeting (IFSA/NAFIPS)*, Edmonton, AB, Canada, Jun. 2013, pp. 57–61.

[2] R. R. Yager, "Generalized orthopair fuzzy sets," *IEEE Trans. Fuzzy Syst.*, vol. 25, no. 5, pp. 1222–1230, Oct. 2017.

[3] A. P. Darko and D. Liang, "Some q-rung orthopair fuzzy hamacher aggregation operators and their application to multiple attribute group decision making with modified EDAS method," *Eng. Appl. Artif. Intell.*, vol. 87, Jan. 2020, Art. no. 103259.

[4] P. Liu and P. Wang, "Some q-Rung orthopair fuzzy aggregation operators and their applications to multiple-attribute decision making," *Int. J. Intell. Syst.*, vol. 33, no. 2, pp. 259–280, Feb. 2018.

[5] D. Liang, Y. Zhang, and W. Cao, "q-rung orthopair fuzzy Choquet integral aggregation and its application in heterogeneous multicriteria two-sided matching decision making," *Int. J. Intell. Syst.*, vol. 34, no. 12, pp. 3275–3301, Dec. 2019.

[6] W. S. Du, "Weighted power means of q-rung orthopair fuzzy information and their applications in multiattribute decision making," *Int. J. Intell. Syst.*, vol. 34, no. 11, pp. 2835–2862, Nov. 2019.

[7] Y. Ju, C. Luo, J. Ma, and A. Wang, "A novel multiple-attribute group decision-making method based on q-rung orthopair fuzzy generalized power weighted aggregation operators," *Int. J. Intell. Syst.*, vol. 34, no. 9, pp. 2077–2103, Sep. 2019, doi: 10.1002/int.22132.

[8] G. Wei, C. Wei, J. Wang, H. Gao, and Y. Wei, "Some q-rung orthopair fuzzy maclaurin symmetric mean operators and their applications to potential evaluation of emerging technology commercialization," *Int. J. Intell. Syst.*, vol. 34, no. 1, pp. 50–81, Jan. 2019.

[9] P. Liu and P. Wang, "Multiple-attribute decision-making based on archimedean Bonferroni operators of q-Rung orthopair fuzzy numbers," *IEEE Trans. Fuzzy Syst.*, vol. 27, no. 5, pp. 834–848, May 2019.

[10] W. Yang and Y. Pang, "New q-rung orthopair fuzzy partitioned Bonferroni mean operators and their application in multiple attribute decision making," *Int. J. Intell. Syst.*, vol. 34, no. 3, pp. 439–476, Mar. 2019.

[11] G. Wei, H. Gao, and Y. Wei, "Some q-rung orthopair fuzzy Heronian mean operators in multiple attribute decision making," *Int. J. Intell. Syst.*, vol. 33, no. 7, pp. 1426–1458, Jul. 2018.

[12] J. Wang, R. Zhang, X. Zhu, Z. Zhou, X. Shang, and W. Li, "Some q-rung orthopair fuzzy Muirhead means with their application to multi-attribute group decision making," *J. Intell. Fuzzy Syst.*, vol. 36, no. 2, pp. 1599–1614, Mar. 2019.

[13] J. Wang, G. Wei, J. Lu, F. E. Alsaadi, T. Hayat, C. Wei, and Y. Zhang, "Some q-rung orthopair fuzzy Hamy mean operators in multiple attribute decision-making and their application to enterprise resource planning systems selection," *Int. J. Intell. Syst.*, vol. 34, no. 10, pp. 2429–2458, Oct. 2019.

[14] C. Jana, G. Muhiuddin, and M. Pal, "Some Dombi aggregation of q-rung orthopair fuzzy numbers in multiple-attribute decision making," *Int. J. Intell. Syst.*, vol. 34, no. 12, pp. 3220–3240, Dec. 2019.

[15] D. Liu, D. Peng, and Z. Liu, "The distance measures between q-rung orthopair hesitant fuzzy sets and their application in multiple criteria decision making," *Int. J. Intell. Syst.*, vol. 34, no. 9, pp. 2104–2121, Sep. 2019.

[16] X. D. Peng and L. Liu, "Information measures for q-rung orthopair fuzzy sets," *Int. J. Intell. Syst.*, vol. 34, no. 8, pp. 1795–1834, 2019.

[17] D. Liu, X. Chen, and D. Peng, "Some cosine similarity measures and distance measures between q-rung orthopair fuzzy sets," *Int. J. Intell. Syst.*, vol. 34, no. 7, pp. 1572–1587, Jul. 2019.

[18] X. Peng, R. Krishankumar, and K. S. Ravichandran, "Generalized orthopair fuzzy weighted distance-based approximation (WDBA) algorithm in emergency decision-making," *Int. J. Intell. Syst.*, vol. 34, no. 10, pp. 2364–2402, Oct. 2019.

[19] Z. Liu, H. Xu, Y. Yu, and J. Li, "Some q-rung orthopair uncertain linguistic aggregation operators and their application to multiple attribute group decision making," *Int. J. Intell. Syst.*, vol. 34, no. 10, pp. 2521–2555, Oct. 2019.

[20] Z. Liu, L. Li, and J. Li, "q-rung orthopair uncertain linguistic partitioned Bonferroni mean operators and its application to multiple attribute decision-making method," *Int. J. Intell. Syst.*, vol. 34, no. 10, pp. 2490–2520, Oct. 2019.

[21] P. Liu and W. Liu, "Multiple-attribute group decision-making method of linguistic q-rung orthopair fuzzy power Muirhead mean operators based on entropy weight," *Int. J. Intell. Syst.*, vol. 34, no. 8, pp. 1755–1794, Aug. 2019.

[22] Y. Ju, C. Luo, J. Ma, H. Gao, E. D. R. Santibanez Gonzalez, and A. Wang, "Some interval-valued q-rung orthopair weighted averaging operators and their applications to multiple-attribute decision making," *Int. J. Intell. Syst.*, vol. 34, no. 10, pp. 2584–2606, Oct. 2019.

[23] R. Krishankumar, K. S. Ravichandran, S. Kar, F. Cavallaro, E. K. Zavadskas, and A. Mardani, "Scientific decision framework for evaluation of renewable energy sources under Q-Rung orthopair fuzzy set with partially known weight information," *Sustainability*, vol. 11, no. 15, p. 4202, 2019.

[24] X. Peng and J. Dai, "Research on the assessment of classroom teaching quality with q-rung orthopair fuzzy information based on multiparametric similarity measure and combinative distance-based assessment," *Int. J. Intell. Syst.*, vol. 34, no. 7, pp. 1588–1630, Jul. 2019.

[25] Z. Xu, "Intuitionistic fuzzy aggregation operators," *IEEE Trans. Fuzzy Syst.*, vol. 15, no. 6, pp. 1179–1187, Dec. 2007.

[26] Z. Xu and R. R. Yager, "Intuitionistic fuzzy Bonferroni means," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 41, no. 2, pp. 568–578, Apr. 2011.

[27] Z. Xu and R. R. Yager, "Some geometric aggregation operators based on intuitionistic fuzzy sets," *Int. J. Gen. Syst.*, vol. 35, no. 4, pp. 417–433, 2006.

[28] F. Blanco-Mesa, E. León-Castro, and J. M. Merigó, "A bibliometric analysis of aggregation operators," *Appl. Soft Comput.*, vol. 81, Aug. 2019, Art. no. 105488.

[29] C. Bonferroni, "Sulle medie multiple di potenze," *Bollettino Dell'Unione Matematica Italiana*, vol. 5, nos. 3–4, pp. 267–270, 1950.

[30] R. R. Yager, "On generalized Bonferroni mean operators for multi-criteria aggregation," *Int. J. Approx. Reasoning*, vol. 50, no. 8, pp. 1279–1286, Sep. 2009.

[31] M. Xia, Z. Xu, and B. Zhu, "Geometric Bonferroni means with their application in multi-criteria decision making," *Knowl.-Based Syst.*, vol. 40, pp. 88–100, Mar. 2013.

[32] M. Xia, Z. Xu, and B. Zhu, "Generalized intuitionistic fuzzy Bonferroni means," *Int. J. Intell. Syst.*, vol. 27, no. 1, pp. 23–47, Jan. 2012.

[33] Z.-S. Chen, K.-S. Chin, Y.-L. Li, and Y. Yang, "On generalized extended Bonferroni means for decision making," *IEEE Trans. Fuzzy Syst.*, vol. 24, no. 6, pp. 1525–1543, Dec. 2016.
[34] F. Blanco-Mesa and J. M. Merigó, "Bonferroni distances and their application in group decision making," *Cybern. Syst.*, vol. 51, no. 1, pp. 27–58, Jan. 2020.
[35] A. Mesiarova-Zemankova, S. Kelly, and K. Ahmad, "Bonferroni mean with weighted interaction," *IEEE Trans. Fuzzy Syst.*, vol. 26, no. 5, pp. 3085–3096, Oct. 2018.
[36] P. Liu, J. Liu, and S.-M. Chen, "Some intuitionistic fuzzy Dombi Bonferroni mean operators and their application to multi-attribute group decision making," *J. Oper. Res. Soc.*, vol. 69, no. 1, pp. 1–24, Jan. 2018.
[37] P. Liu, S.-M. Chen, and J. Liu, "Multiple attribute group decision making based on intuitionistic fuzzy interaction partitioned Bonferroni mean operators," *Inf. Sci.*, vol. 411, pp. 98–121, Oct. 2017.
[38] P. Liu, H. Gao, and J. Ma, "Novel green supplier selection method by combining quality function deployment with partitioned Bonferroni mean operator in interval type-2 fuzzy environment," *Inf. Sci.*, vol. 490, pp. 292–316, Jul. 2019.
[39] Y. Yang, K. Chin, H. Ding, H. Lv, and Y. Li, "Pythagorean fuzzy Bonferroni means based on T-norm and its dual T-conorm," *Int. J. Intell. Syst.*, vol. 34, no. 6, pp. 1303–1336, Jun. 2019.
[40] D. Liang, A. P. Darko, and Z. Xu, "Pythagorean fuzzy partitioned geometric Bonferroni mean and its application to multi-criteria group decision making with grey relational analysis," *Int. J. Fuzzy Syst.*, vol. 21, no. 1, pp. 115–128, Feb. 2019.
[41] W. Yang, J. Shi, Y. Liu, Y. Pang, and R. Lin, "Pythagorean fuzzy interaction partitioned Bonferroni mean operators and their application in multiple-attribute decision-making," *Complexity*, vol. 2018, pp. 1–25, Nov. 2018, doi: 10.1155/2018/3606245.
[42] G. W. Wei, "2-tuple intuitionistic fuzzy linguistic aggregation operators in multiple attribute decision making," *Iranian J. Fuzzy Syst.*, vol. 16, no. 4, pp. 159–174, 2019.
[43] L. Wang, Y. Wang, and W. Pedrycz, "Hesitant 2-tuple linguistic Bonferroni operators and their utilization in group decision making," *Appl. Soft Comput.*, vol. 77, pp. 653–664, Apr. 2019.
[44] P. Ji, J.-Q. Wang, and H.-Y. Zhang, "Frank prioritized Bonferroni mean operator with single-valued neutrosophic sets and its application in selecting third-party logistics providers," *Neural Comput. Appl.*, vol. 30, no. 3, pp. 799–823, Aug. 2018.
[45] J. Dombi, "A general class of fuzzy operators, the demorgan class of fuzzy operators and fuzziness measures induced by fuzzy operators," *Fuzzy Sets Syst.*, vol. 8, no. 2, pp. 149–163, Aug. 1982.
[46] J. Dombi, "On a certain class of aggregative operators," *Inf. Sci.*, vol. 245, pp. 313–328, Oct. 2013.
[47] D. Pamucar, M. Deveci, F. Canıtez, and D. Bozanic, "A fuzzy full consistency Method-Dombi-Bonferroni model for prioritizing transportation demand management measures," *Appl. Soft Comput.*, vol. 87, Feb. 2020, Art. no. 105952.
[48] S. Roychowdhury and B. H. Wang, "Composite generalization of Dombi class and a new family of T-operators using additive-product connective generator," *Fuzzy Sets Syst.*, vol. 66, no. 3, pp. 329–346, Sep. 1994.
[49] C. Jana, T. Senapati, M. Pal, and R. R. Yager, "Picture fuzzy Dombi aggregation operators: Application to MADM process," *Appl. Soft Comput.*, vol. 74, pp. 99–109, Jan. 2019.
[50] H. Zhang, R. Zhang, H. Huang, and J. Wang, "Some picture fuzzy Dombi Heronian mean operators with their application to multi-attribute decision-making," *Symmetry*, vol. 10, no. 11, p. 593, 2018.
[51] X. He, "Group decision making based on Dombi operators and its application to personnel evaluation," *Int. J. Intell. Syst.*, vol. 34, no. 7, pp. 1718–1731, Jul. 2019.
[52] L. Wu, G. Wei, H. Gao, and Y. Wei, "Some interval-valued intuitionistic fuzzy Dombi Hamy mean operators and their application for evaluating the elderly tourism service quality in tourism destination," *Mathematics*, vol. 6, no. 12, p. 294, 2018.
[53] Z. Li, H. Gao, and G. Wei, "Methods for multiple attribute group decision making based on intuitionistic fuzzy Dombi Hamy mean operators," *Symmetry*, vol. 10, no. 11, p. 574, 2018.
[54] C. Jana, M. Pal, and J.-Q. Wang, "Bipolar fuzzy Dombi aggregation operators and its application in multiple-attribute decision-making process," *J. Ambient Intell. Hum. Comput.*, vol. 10, no. 9, pp. 3533–3549, Sep. 2019.
[55] J. Chen and J. Ye, "Some single-valued neutrosophic Dombi weighted aggregation operators for multiple attribute decision-making," *Symmetry*, vol. 9, no. 6, p. 82, Jun. 2017, doi: 10.3390/sym9060082.
[56] X. Peng and F. Smarandache, "Novel neutrosophic Dombi Bonferroni mean operators with mobile cloud computing industry evaluation," *Expert Syst.*, vol. 36, no. 4, p. 12411, Aug. 2019.
[57] Q. Khan, P. Liu, T. Mahmood, F. Smarandache, and K. Ullah, "Some interval neutrosophic Dombi power Bonferroni mean operators and their application in multi-attribute decision-making," *Symmetry*, vol. 10, no. 10, p. 459, Oct. 2018, doi: 10.3390/sym10100459.
[58] G. Wei, J. Wu, C. Wei, J. Wang, and J. Lu, "Models for MADM with 2-tuple linguistic neutrosophic Dombi Bonferroni mean operators," *IEEE Access*, vol. 7, pp. 108878–108905, 2019.
[59] P. Liu, Q. Khan, T. Mahmood, F. Smarandache, and Y. Li, "Multiple attribute group decision making based on 2-tuple linguistic neutrosophic Dombi power Heronian mean operators," *IEEE Access*, vol. 7, pp. 100205–100230, 2019.
[60] H. Zhang, R. Zhang, H. Huang, and J. Wang, "Some picture fuzzy Dombi Heronian mean operators with their application to multi-attribute decision-making," *Symmetry*, vol. 10, no. 11, p. 593, Nov. 2018, doi: 10.3390/sym10110593.
[61] J. He, X. Wang, R. Zhang, and L. Li, "Some q-Rung picture fuzzy Dombi Hamy mean operators with their application to project assessment," *Mathematics*, vol. 7, no. 5, p. 468, 2019.
[62] C. Jana, T. Senapati, and M. Pal, "Pythagorean fuzzy Dombi aggregation operators and their application in decision support system," *Int. J. Intell. Syst.*, vol. 34, no. 9, pp. 2019–2038, 2019.

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