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A Novel Measure of Uncertainty in the Dempster-Shafer Theory

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ABSTRACT In the Dempster-Shafer theory, how to quantitatively evaluate the quality of information is an essential issue and also an open issue. Many of measures of uncertainty have been proposed in previous work, whereas some measures among them had been proved to have a few shortcomings. The validity and rationality of the measures proposed in recent years have been explored and analyzed preliminarily, and then an empirical measure of uncertainty with exponential function form which is directly based on the framework of the evidence theory is proposed to overcome the shortcomings. Several numerical examples have been presented to illustrate the validity and rationality of the empirical measure.

INDEX TERMS The Dempster-Shafer theory, measure of uncertainty, exponential function, validity and rationality.

I. INTRODUCTION

How to deal with uncertain information has been one of the research domains in which people are interested all the while. As a kind of methods to deal with uncertain information, the Dempster-Shafer theory (DST, for short) proposed by Dempster [1] firstly and then further developed by Shafer [2], also referred to as the evidence theory [3], [4] or the theory of evidence [5], [6] or belief function theory [7], [8], has attracted considerable critical attention by reason of its validity in expressing and processing uncertain information. DST has been broadly applied in various domains, such as decision-making [9], [10], pattern recognition [11]–[13], cluster analysis [8], [14]–[16], controller modeling [17], [18] and so on. However, DST is not a perfect theory, due to there are still some issues to be solved. One of the major issues is that the combination of highly conflicting evidence between each other by using Dempster's combination rule may bring forth a counter-intuitive result [19]–[21]. Another major issue of great concern is how to quantitatively evaluate the quality of a basic probability assignment (BPA, for short), or can be said how to gauge the uncertainty of a BPA [22], [23]. The latter issue is that we are going to discuss in the rest of this paper.

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In the research history of the measure of uncertainty in the DST, Höhle [24] put forward the first measure of uncertainty of a BPA, namely Confusion. Yager [25] held primarily that the uncertainty of a BPA can be indicated through two types of uncertainty: dissonance and dispersion, denoted as the two-tuple (S(m), E(m)). The specific calculation formula of the entropy measure E(m) regarded as an indicatrix of the dissonance of a BPA and the specificity measure S(m) regarded as an indicatrix of the dispersion of a BPA also were given. The new improved two-tuple (Sp(m), G(m))has already been presented by Yager in his following research [26]. The greatest difference between the original two-tuple and the new improved two-tuple is that the value of two measures in the latter ranges from 0 to 1. Nevertheless, it may be impossible to make a clearly differentiate between different BPAs compared by using either the original two-tuple or the new improved two-tuple. Soon afterwards, Klir and Yuan [27] summarized that there may be three different types of uncertainty existing in various uncertainty theories, illustrated by a tree diagram shown in Figure 1. As a matter of fact, the two types of uncertainty coexisting in a BPA, namely discord and nonspecificity, presented by Klir are equivalent to the two types of uncertainty presented by Yager [25]. In order to use a unified measure of uncertainty to quantitatively evaluate the uncertainty of a BPA, a great many measures of uncertainty

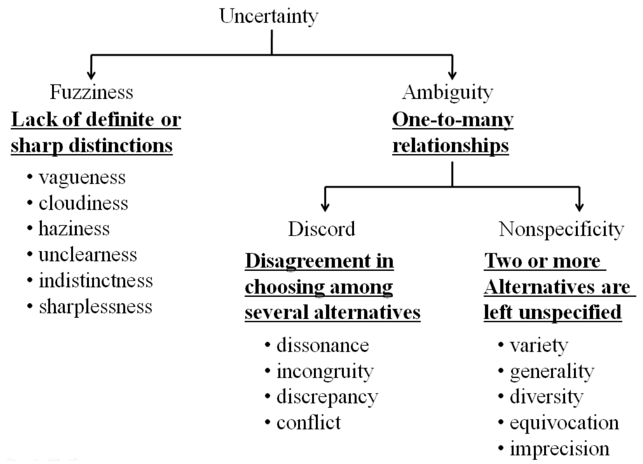

FIGURE 1. Three different types of uncertainty proposed by Klir.

which integrates the two types of uncertainty coexisting in a BPA were developed. Up to present, all measures of uncertainty proposed by researchers are mainly categorized into two categories. In the opinion of some researchers, the DST is a kind of generalized information theory [28], or can be said a generalization of probability theory [29], thus one of two categories whose main characteristic is that the measure has the form of logarithmic function is the natural generalization of Shannon entropy, also known as Entropy-like measure, such as Klir's entropy NS(m) [30], [31], Pal's entropy H_p(m) [32], Deng's entropy E_d(m) [33], etc. The other kind of measure are functions with other forms, such as Yager's two-tuple (Sp(m), G(m)) [26], Yang's TU^I(m) [34], Deng and Xiao's iTU^I(m) [35], etc. However, the existing measures of uncertainty cannot be able to effectively measure the uncertainty in all cases. The details will be discussed in Section 3 and Section 5 of this paper.

The structural framework of this paper is described as follows. In Section 2, we give a brief introduction to the terminology and notation involved in this paper. The main existing measures of total uncertainty including two categories and preliminary analysis on validity are given in Section 3. A novel measure of uncertainty is proposed in Section 4. In Section 5, the validity and rationality of the novel measure compared with other measures can be further reflected by numerical examples. The conclusion is presented in Section 6.

II. PRELIMINARIES

A. BASIC CONCEPTS IN THE DST

Definition 1 (Basic Probability Assignment): Let $\Theta = \{\theta_1, \theta_2, \ldots\ldots, \theta_n\}$ be a nonempty set consisted of finite and mutually exclusive elements, which is designated as the frame of discernment (FOD). 2^Θ represents the set of all subsets of Θ, also called the power set of Θ. A function m is called a basic probability assignment (BPA) [2] defining over FOD Θ as a mapping from the domain 2^Θ to the range [0,1], denoted as $m : 2^\Theta \to [0, 1]$, whenever satisfying the following condition:

$$m(\varnothing) = 0 \quad and \quad \sum_{A\subseteq\Theta} m(A) = 1 \tag{1}$$

The function m defined as above is also called such asa body of evidence (BOE) [2], a partial belief function (PBF) [36], a belief structure (BS) [37], a basic belief assignment (BBA) [38], [39], and so on. The value $m(A)$ is called a basic probability number of the set A, and it can be understood as the degree of belief that is assigned exactly to the set A.

Definition 2 (Belief Function and Plausibility Function): Corresponding to a given BPA, there are two functions, namely belief function (*Bel*) and plausibility function (*Pl*), which can be one-to-one denoted as:

$$Bel(A) = \sum_{B\subseteq A} m(B) \tag{2}$$

$$Pl(A) = 1 - Bel(\bar{A}) = \sum_{B\cap A=\varnothing} m(B) \tag{3}$$

It can also be expressed as a mapping, as follows:

$$Bel : 2^\Theta \to [0, 1] \tag{4}$$

$$Pl : 2^\Theta \to [0, 1] \tag{5}$$

Bel(A) represents the degree of total belief, consisting of the sum of the basic probability number assigned to the set A and its subsets. *Bel*(A) is also called the lower probability of the set A. *Pl*(A) represents the possible degree of belief assigned to the set A, which is called the upper probability of the set A. Hence a belief interval, denoted as $BI(A) = [Bel(A), Pl(A)]$, is used to represent the degree of belief over the set A, that is also called interval probability. The length of the belief interval could be seen as the degree of imprecision over the set A.

Since the three functions can be converted to each other, given any function, the other functions can be obtained. For more information about specific transformation relationships, see Reference [2].

Definition 3 (Focal Element and Core): Let m be a BPA defining over FOD Θ. A, as a nonempty subset of Θ, is called a focal element of *Bel* if $m(A) > 0$. $|A|$ represents the number of elements in the set A, also known as the cardinality of the set A.

Core is defined as the union of all the focal elements of a BPA, denoted as C [2]. It can be written as follows:

$$C = \bigcup_{i=1}^{q} \{A_i \,|m(A_i) > 0\} \tag{6}$$

where q represents the number of focal elements corresponding to the m, $q \leqslant 2^{|\Theta|}$-1.

It should be emphasized that there is a concept denoted as F [40] which is easily confused with C. F is defined as the set of all the focal elements of BPA, denoted as:

$$F = \{A\,|m(A) > 0\} = \left\{A_1, A_2, \cdots\cdots, A_q\right\} \tag{7}$$

The difference between the two definitions is that C is a subset of Θ, while F is a subset of 2^{Θ}.

Based on the above definition, two special belief functions are listed as follows: (1) a Bayesian belief function satisfies the condition that all the focal elements of Bel are singletons (i.e., if A is a focal element of Bel, then $|A| = 1$); (2) a vacuous belief function meets the following conditions: $m(\Theta) = 1$.

B. THE PIGNISTIC TRANSFORMATION

Let m be a BPA defining over FOD Θ. the pignistic transformation is defined as a mapping from the framework of the DST to the framework of probability theory, the pignistic probability function $BetP$ on Θ based on a given m is denoted as [38]:

$$BetP_m\left(\{\theta_i\}\right) = \sum_{\theta_i \in A \subseteq \Theta} \frac{1}{|A|}\frac{m\left(A\right)}{1 - m\left(\varnothing\right)}, \quad \forall \theta_i \in \Theta \tag{8}$$

All of the following discussions in this paper are limited to the closed world assuming (i.e., FOD is complete). That means $m(\emptyset) = 0$, thus Eq. (8) can be rewritten as:

$$BetP_m\left(\{\theta_i\}\right) = \sum_{\theta_i \in A \subseteq \Theta} \frac{m\left(A\right)}{|A|}, \quad \forall \theta_i \in \Theta \tag{9}$$

C. THE PLAUSIBILITY TRANSFORMATION

The plausibility transformation is another method that transforms the framework of the DST to the framework of probability theory. Given a m defining over FOD Θ, the plausibility probability function Pl_P_m [41] can be derivated from the plausibility function with respect to the m by using the plausibility transformation, Pl_P_m can be denoted as:

$$Pl\_P_m\left(\{\theta_i\}\right) = \frac{Pl\left(\{\theta_i\}\right)}{\sum_{\theta_i \in \Theta} Pl\left(\{\theta_i\}\right)}, \quad \forall \theta_i \in \Theta \tag{10}$$

where $Pl(\{\theta_i\})$ is the plausibility function of $\{\theta_i\}$ corresponding to the m.

III. THE EXISTING MEASURES OF TOTAL UNCERTAINTY IN THE DST

A. ENTROPY-LIKE MEASURES OF UNCERTAINTY

Shannon entropy [42] plays a crucial role as a measure of uncertainty in the probabilistic framework and the expected value of information content containing in a message in information theory. Suppose X is a random variable over the discrete sample space Ω, Shannon entropy, denoted as H, is defined as:

$$H\left(X\right) = -\sum_{i=1}^{n} p_i \log_2 p_i \tag{11}$$

where n is the cardinality of Ω, and p_i is the probability of sample i.

Entropy-like measures of uncertainty can be further subdivided into two different types based on the framework of the DST and the probabilistic framework, as shown below.

1) MEASURES OF UNCERTAINTY BASED ON THE FRAMEWORK OF THE DST

Klir's entropy $NS(m)$ [30], [31]

$$NS(m) = \sum_{A \in F} m(A) \log_2 \sum_{B \in F} m(B) \frac{|A|^2}{|A \cap B|} \tag{12}$$

Pal's entropy $H_p(m)$ [32]

$$H_p\left(m\right) = \sum_{A \in F} m(A) \log_2 \frac{|A|}{m(A)} \tag{13}$$

Deng's entropy $E_d(m)$ [33]

$$E_d\left(m\right) = -\sum_{A \subseteq \Theta} m(A) \log_2 \frac{m(A)}{2^{|A|} - 1} \tag{14}$$

Zhou's entropy $E_{Id}(m)$ [43]

$$E_{Id}\left(m\right) = -\sum_{A \subseteq \Theta} m(A) \log_2 \left(\frac{m(A)}{2^{|A|} - 1} e^{\frac{|A|-1}{|\Theta|}}\right) \tag{15}$$

Tang's entropy $E_{Wd}(m)$ [44]

$$E_{Wd}\left(m\right) = -\sum_{A \subseteq \Theta} \frac{|A|}{|\Theta|} m(A) \log_2 \frac{m(A)}{2^{|A|} - 1} \tag{16}$$

Wang's entropy $SU(m)$ [29]

$$\begin{aligned} SU\left(m\right) = \sum_{i=1}^{|\Theta|} \Bigg[& \frac{Pl\left(\{\theta_i\}\right) - Bel\left(\{\theta_i\}\right)}{2} \\ & - \frac{Bel\left(\{\theta_i\}\right) + Pl\left(\{\theta_i\}\right)}{2} \log_2 \frac{Bel\left(\{\theta_i\}\right) + Pl\left(\{\theta_i\}\right)}{2} \Bigg] \end{aligned} \tag{17}$$

Pan and Deng's entropy $H_{bel}(m)$ [45]

$$H_{bel}\left(m\right) = -\sum_{A \subseteq \Theta} \frac{Bel\left(A\right) + Pl\left(A\right)}{2} \log_2 \frac{Bel\left(A\right) + Pl\left(A\right)}{2\left(2^{|A|} - 1\right)} \tag{18}$$

Chen's entropy $E_i(m)$ [46]

$$E_i\left(m\right) = -\sum_{A \subseteq \Theta} m(A) \log_2 \left(\frac{m(A)}{2^{|A|} - 1} \frac{|A|}{|C|}\right) \tag{19}$$

2) MEASURES OF UNCERTAINTY BASED ON THE PROBABILISTIC FRAMEWORK

Harmanec's $AU(m)$ [47]

$$\begin{aligned} AU(m) = & \max \left[-\sum_{\theta_i \in \Theta} p_{\theta_i} \log_2 p_{\theta_i} \right] \\ & s.t. \begin{cases} p_{\theta_i} \in [0, 1], & \forall \theta_i \in \Theta \\ \sum_{\theta_i \in \Theta} p_{\theta_i} = 1 \\ \mathrm{Bel}\left(A\right) \le \sum_{\theta_i \in \Theta} p_{\theta_i} \le 1 - Bel\left(\bar{A}\right), & \forall A \subseteq \Theta \end{cases} \end{aligned} \tag{20}$$

Smith's $TU_1(m, \delta)$ [48]

$$TU_1\,(m,\delta)=\delta AM\,(m)+(1-\delta)\sum_{A\subseteq\Theta} m(A)\log_2|A| \tag{21}$$

Jousselme's $AM(m)$ [40]

$$AM\,(m)=-\sum_{\theta_i\in\Theta} BetP_m\,(\{\theta_i\})\log_2 BetP_m\,(\{\theta_i\}) \tag{22}$$

Jiroušek's $H_j(m)$ [49]

$$H_j\,(m)=\sum_{\theta_i\in\Theta} Pl\_P_m\,(\{\theta_i\})\log_2\frac{1}{Pl\_P_m\,(\{\theta_i\})}+\sum_{A\in 2^{|\Theta|}} m(A)\log_2|A| \tag{23}$$

Pan and Zhou's entropy $H_{PQ}(m)$ [50]

$$H_{PQ}\,(m)=\sum_{A\in\,2^{|\Theta|}} m(A)\log_2\frac{1}{Pm(A)}+\sum_{A\in 2^{|\Theta|}} m(A)\log_2|A| \tag{24}$$

where $Pm\,(A)=\sum_{\theta_i\in A} Pl\_P_m\,(\{\theta_i\})$

B. MEASURES OF UNCERTAINTY IN OTHER FUNCTIONAL FORMS

Yager's two-tuple $(Sp(m), G(m))$ [26]

$$(Sp\,(m)\,,G\,(m))=\left(1-\frac{\sum_{j=1}^{|F|} m\,(A_j)\times(|A_j|-1)}{|\Theta|-1},\quad 1+\frac{\sum_{j=1}^{|F|} m\,(A_j)\log_2 Pl(A_j)}{\log_2|\Theta|}\right) \tag{25}$$

Yang's $TU^I(m)$ [34] equates to the value in (26), where (26) as shown at the bottom of this page.

Deng and Xiao's $iTU^I(m)$ [35]

$$iTU^I\,(m)=\sum_{i=1}^{|\Theta|}\left[1-\sqrt{(Bel\,(\{\theta_i\}))^2+(Pl\,(\{\theta_i\})-1)^2}\right] \tag{27}$$

Gao's $T_{D-S}(m)$ [51]

$$T_{D-S}\,(m)=-\sum_{|A|=1} m(A)\log_2 m(A)+\sum_{|A|\neq 1}\frac{\left(2^{|A|}-1\right)\times m(A)\times\left(1-\left(\frac{m(A)}{2^{|A|}-1}\right)^{|A|-1}\right)}{|A|-1} \tag{28}$$

C. ANALYSIS ON THE EXISTING MEASURE OF UNCERTAINTY

In the previous studies, the shortcomings of some existing measures of total uncertainty have been verified by researchers. Klir [52] had proved that three main shortcomings including high computational complexity, less sensitivity to the variations of BPA as well as distinction between discord and nonspecificity, lied in the measure AU presented by Harmanec and Klir [47]. Jousselme *et al.* [40] claimed that the measure $TU_1(m, \delta)$ proposed by Smith not only failed to solve the shortcoming of high computational complexity, but also brought subjectivity due to the selection of parameter δ. The ambiguity measure $AM(m)$ proposed by Jousselme cannot be able to differentiate the uncertainty of different BPA with the same pignistic probability distribution proved by Wang and Song [29] and satisfy the monotonicity requirement defined by Abellán and Masegosa [53]. On the basis of the definition in the DST, a vacuous belief function represents the total ignorance/unknown about the system/object, and that means if and only if a BPA is a vacuous belief function, its uncertainty must be the largest. However, some measures, such as $NS(m)$ proposed by Klir, $H_p(m)$ proposed by Pal, $E_d(m)$ proposed by Deng, $E_i(m)$ proposed by Chen, $H_{bel}(m)$ proposed by Pan, $H_{PQ}(m)$ proposed by Pan and Zhou, and T_{D-S}(m) proposed by Gao, had been proved not to meet this requirement [31], [33], [45], [46], [50], [54], [55]. The measure $TU^I(m)$ proposed by Yang had been pointed out that it is insensitive to the changes of BPA with illustrative examples [35]. This inadequacy also exists in the measure $H_{bel}(m)$ proposed by Pan and the measure $E_{Id}(m)$ proposed by Cui *et al.* [56].

In addition to the above mentioned measures of uncertainty that had been proved to have some shortcomings, some new measures of uncertainty had been proposed gradually by some scholars in recent years, and the question whether they have shortcomings has not been found yet. We try to make a preliminary analysis on these measures through cases.

Example 1: Here is given nine BPAs established on the same FOD $\Theta=\{\theta_1,\theta_2,\ldots\ldots,\theta_n\}$, as shown below.

m1: $m(\{\theta_i\})=1$;
m2: $m(\{\theta_1\})=m(\{\theta_2\})=0.5$;
m3: $m(\{\theta_1\})=m(\{\theta_1,\theta_2\})=0.5$;
m4: $m(\{\theta_1,\theta_2\})=1$;
m5: $m(\{\theta_1,\theta_2\})=m(\{\theta_2,\theta_3\})=0.5$;
m6: $m(\{\theta_1,\theta_2\})=m(\{\theta_3,\theta_4\})=0.5$;
m7: $m(\{\theta_1,\theta_2\})=m(\{\theta_2,\theta_3\})=m(\{\theta_3,\theta_4\})=m(\{\theta_1,\theta_4\})=0.25$;

$$TU^I\,(m)=1-\frac{\sqrt{3}}{|\Theta|}\sum_{i=1}^{|\Theta|} d^I\,([Bel\,(\{\theta_i\})\,,Pl\,(\{\theta_i\})]\,,[0,1])$$

$$d^I\,([Bel\,(\{\theta_i\})\,,Pl\,(\{\theta_i\})]\,,[0,1])=\sqrt{\left(\frac{Bel\,(\{\theta_i\})+Pl\,(\{\theta_i\})}{2}-\frac{1}{2}\right)^2+\frac{1}{3}\left(\frac{Pl\,(\{\theta_i\})-Bel\,(\{\theta_i\})}{2}-\frac{1}{2}\right)^2} \tag{26}$$

TABLE 1. Values of different measures associated with nine BPAs.

	m1	*m2*	*m3*	*m4*	*m5*	*m6*	*m7*	*m8*	*m9*
$E_{Wd}(m)$	0.0000	0.2000	0.6170	0.6340	1.0340	1.0340	1.4340	1.6680	4.9542
$SU(m)$	0.0000	1.0000	1.3113	2.0000	2.5000	3.0000	3.0000	3.0000	5.0000
$H_j(m)$	0.0000	1.0000	1.4183	2.0000	2.5000	3.0000	3.0000	3.0000	4.6439
$iTU^I(m)$	0.0000	0.5858	1.0000	2.0000	2.0000	1.7500	1.7500	1.7500	5.0000

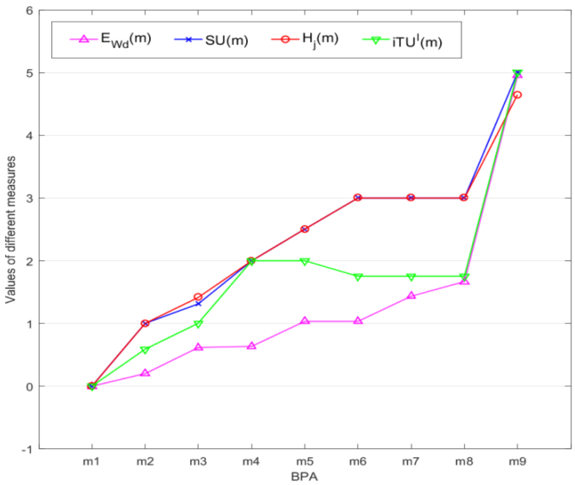

FIGURE 2. Values of different measures associated with nine BPAs.

m8: $m(\{\theta_1, \theta_2\}) = m(\{\theta_1, \theta_3\}) = m(\{\theta_1, \theta_4\}) = m(\{\theta_2, \theta_3\}) = m(\{\theta_2, \theta_4\}) = m(\{\theta_3, \theta_4\}) = 1/6$;

m9: $m(\Theta) = 1$;

Based on the definition of BPA in the DST, we can intuitively conclude that the total uncertainty(TU) ingrained in these nine BPAs should be satisfied with the following inequalities: $TU(m1) < TU(m2) < TU(m3) < TU(m4) < TU(m5) < TU(m6) < TU(m7) < TU(m8) < TU(m9)$. The results respectively calculated by the functions of the measure, including $E_{Wd}(m)$, $SU(m)$, $H_j(m)$ and $iTU^I(m)$, are listed in Table 1 and shown in Figure 2.

As we can see from Table 1 and Figure 2, none of these four measures satisfies the inequalities mentioned above. Among them, the measure $E_{Wd}(m)$ cannot distinguish the uncertainty of *m5* from that of *m6*. The measure $SU(m)$, as well as $H_j(m)$ and $iTU^I(m)$, have the same performance on the three BPAs(*m6*, *m7* and *m8*). But beyond that, it is clearly shown that the uncertainty of *m6*(same as *m7* and *m8*) calculated by the measure $iTU^I(m)$ is obviously lower than that of *m5*. All the counter-intuitive signs indicate that all the four measures proposed in recent years lose its validity and rationality in measuring the uncertainty of evidence in some cases.

IV. A NOVEL MEASURE OF UNCERTAINTY

As listed and discussed in Section 3, plenty of researchers deemed that the DST is a special kind of the probability theory (or the information theory), therefore the measure

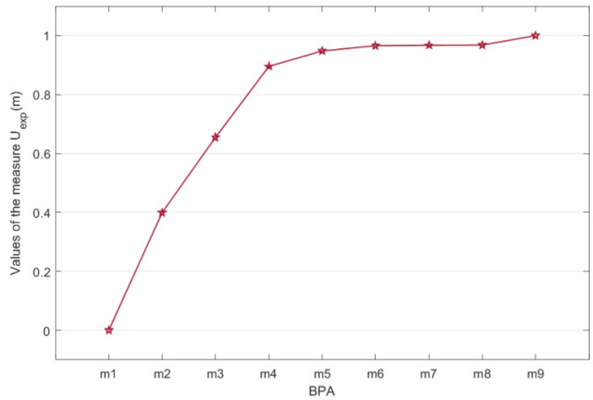

FIGURE 3. Values of the measure $U_{exp}(m)$ associated with nine BPAs.

of uncertainty of a BPA in the DST, i.e. Entropy-like measures either directly based on the framework of the DST or indirectly based on the probabilistic framework with the transformation such as the pignistic transformation and the plausibility transformation, should have the similar functional form as that of random variable denoted as Shannon entropy in the information theory. But at least for now, it turns out that the generalization is not very successful as stated previously. Some other researchers attempted to put forward other functional forms of the measure. Although this attempt cannot completely solve the problem, but provide a very good research idea. And it is for this idea that we come up with a novel measure of uncertainty in the form of exponential function which is completely different from the existing measures, as shown below.

$$U_{\exp}(m) = \frac{e - \sum\limits_{\substack{A \subseteq \Theta \\ |A|=1}} m(A) e^{m(A)} - \sum\limits_{\substack{A \subseteq \Theta \\ |A| \neq 1}} \frac{m(A)}{|A| * |C|} e^{\frac{m(A)}{|A| * |C|}}}{e - \frac{1}{|\Theta|^2} e^{\frac{1}{|\Theta|^2}}} \tag{29}$$

An significant characteristic of this measure is that the value varies from 0 to 1. The measure $U_{exp}(m)$ is defined directly in the framework of the DST, avoiding the additional increase or loss of information due to the transformation. Additionally, Multi-Scale information, including the cardinality of the focal element, the cardinality of Core and the cardinality of FOD, is brought in the measure $U_{exp}(m)$ so as to the uncertainty of a BPA can be estimated in a more comprehensive measure.

TABLE 2. Values of the measure $U_{exp}(m)$ associated with nine BPAs.

	m1	*m2*	*m3*	*m4*	*m5*	*m6*	*m7*	*m8*	*m9*
$U_{exp}(m)$	0.0000	0.3996	0.6547	0.8956	0.9479	0.9658	0.9674	0.9679	1.0000

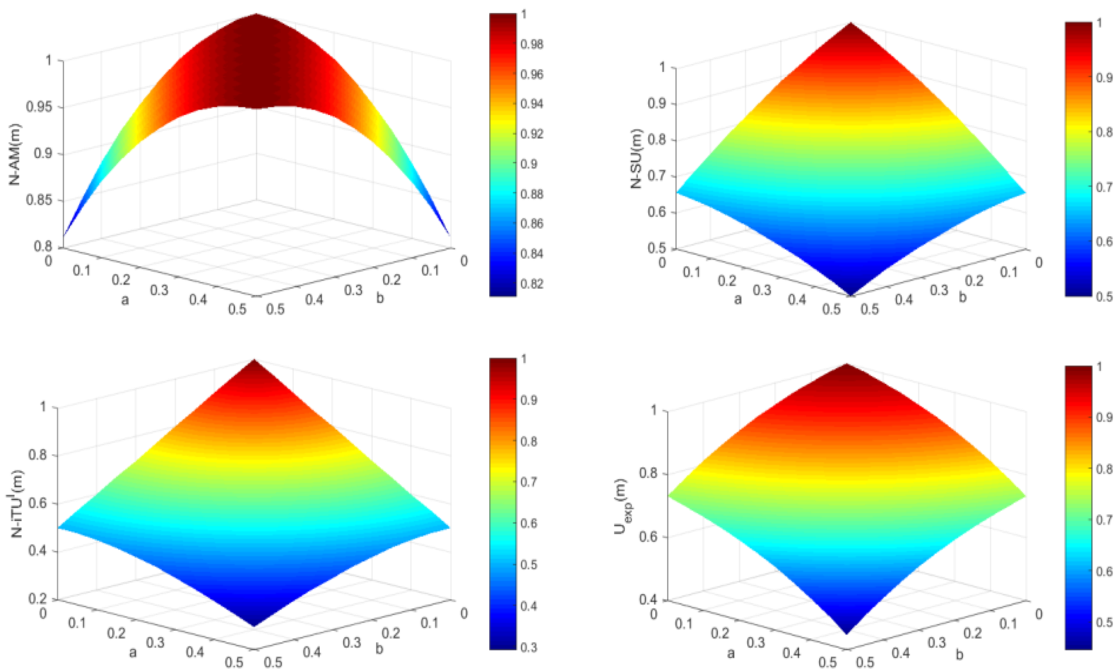

FIGURE 4. Values of the measures with respect to different a and b.

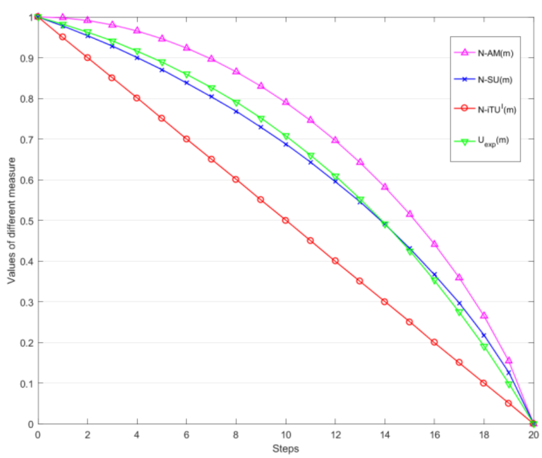

FIGURE 5. Change in value of different measures in Example 4.

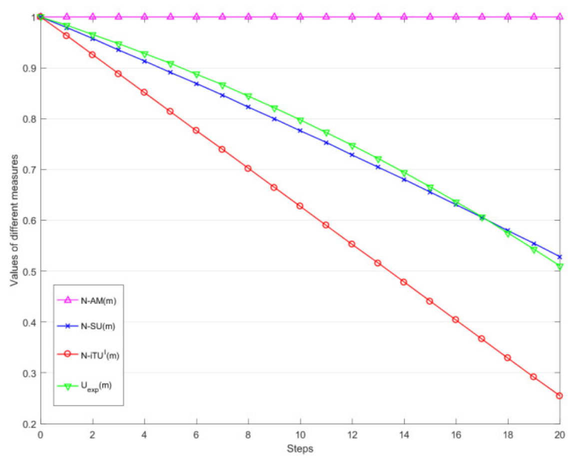

FIGURE 6. Change in value of different measures in Example 5.

The validity and rationality of the measure $U_{exp}(m)$ will be demonstrated by illustrative examples in the following section.

V. NUMERICAL EXAMPLES

Example 2: Let's go back to Example 1, the result calculated by Eq. (29) is listed in Table 2 and shown in Figure 3. It is easily seen that the calculated result overcomes the shortcomings of the existing measures and conforms to our intuitive feeling.

Example 3: Example 3 originated from [57] can be used for verification here. Assume that $\Theta = \{\theta_1, \theta_2\}$ be the FOD. Given a BPA, described as follows: $m(\{\theta_1\}) = a$, $m(\{\theta_2\}) = b, m(\Theta) = 1\text{-a-b}$, a, b $\in [0, 0.5]$. For a more

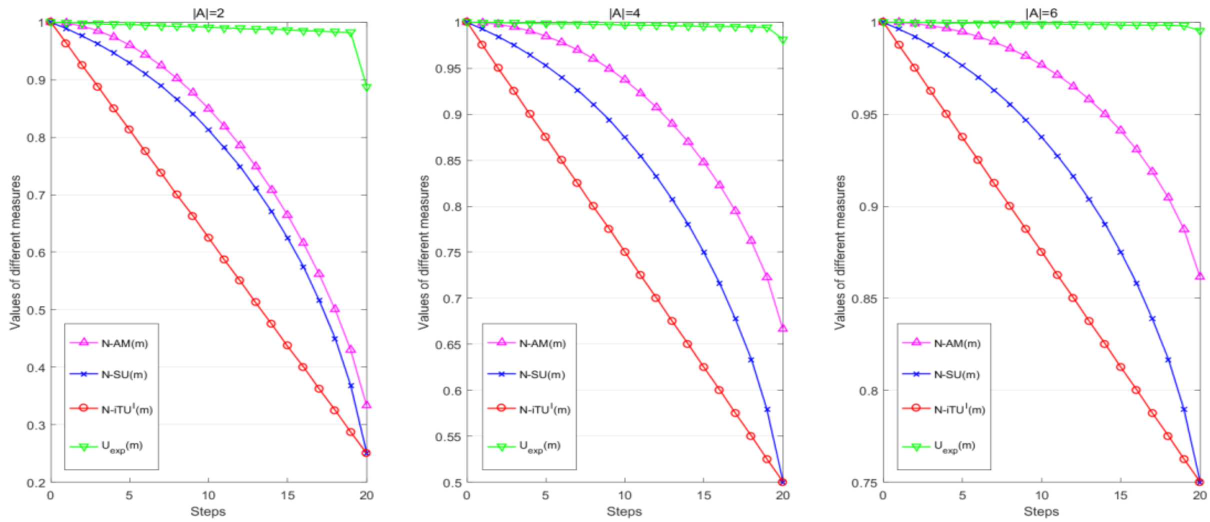

FIGURE 7. Change in value of different measures with respect to different $|A|$.

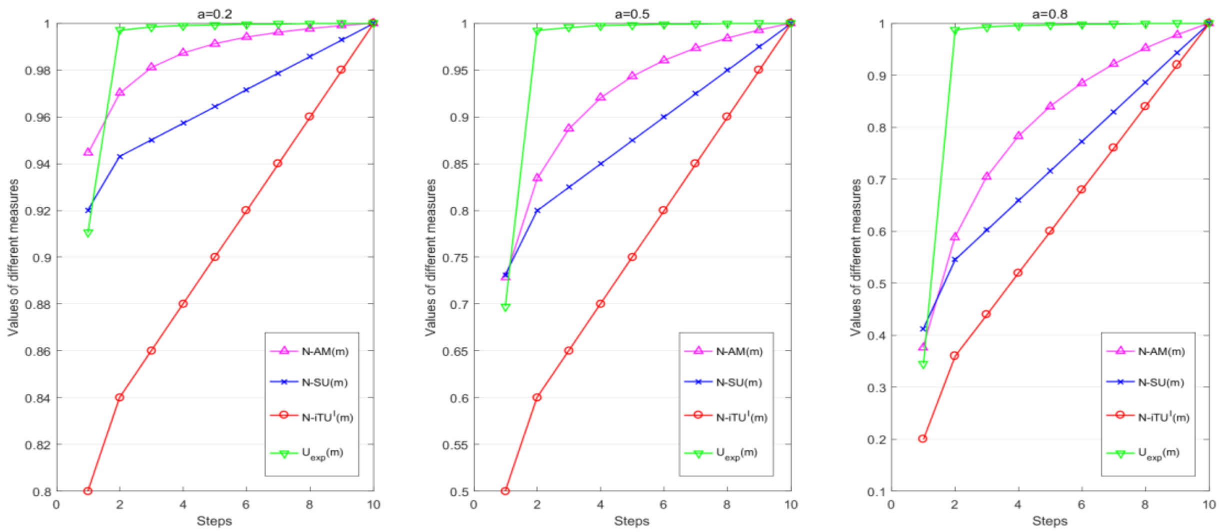

FIGURE 8. Change in value of different measures with respect to different a.

comprehensive comparison, we choose the measure $SU(m)$ as a typical representative of the measures based on the framework of the DST, the measure $AM(m)$ as a typical representative of the measures based on the probabilistic framework and the measure $iTU^I(m)$ as a typical representative of the measures in other functional forms, comparing with the measure $U_{exp}(m)$ which we put forward. In order to make a better comparison, the calculated results of the measures (including $SU(m)$, $AM(m)$, $iTU^I(m)$) will be normalized to the unit interval in the following contents. The normalization method is to divide the current calculated result of the measure by the maximum value of the measure claimed by relevant proposer. The details can be denoted as follows:

$N\text{-}SU(m) = SU(m)/|\Theta|$
$N\text{-}AM(m) = AM(m)/log_2|\Theta|$
$N\text{-}iTU^I(m) = iTU^I(m)/|\Theta|$

With respect to different values of a and b, the performance of these four measures are shown in Figure 4.

In Figure 4, we can see that $N\text{-}AM(m)$ attains its maximum value as long as the condition of a = b is met, no matter how much the value of a and b is. For if a = b, $BetP_{\mathrm{m}}(\{\theta_1\})$ and $BetP_{\mathrm{m}}(\{\theta_2\})$ will always be equal to 1/2 following by the pignistic transformation, it also means that the maximum value of $N\text{-}AM(m)$ can be obtained in some other cases except that BPA is a vacuous belief function. Thus the counter-intuitive signs imply that $AM(m)$ is also not sensitive to different BPA to some extent and violates the requirement when the measure attains its maximum value. The other three measures can better reflect the change of uncertainty caused by the change of values of a or b in this example.

Example 4: Example 4-5 originated from [58] are used for reference. Assume that $\Theta = \{\theta_1, \theta_2, \theta_3\}$ be the FOD. Given a BPA in the very beginning: $m(\Theta) = 1$, changes associated with the BPA take place one step at a time. The value of $m(\Theta)$ decreases by 0.05, while that of $m(\{\theta_2\})$ increases by 0.05 at each step, until $m(\Theta) = 0$ and $m(\{\theta_2\}) = 1$ at the last step. The aforementioned four measures are calculated at every step and their calculated results are shown in Figure 5.

As we can see from Figure 5, all four measures can better reflect the variation tendency of uncertainty with the step-by-step change in this example.

Example 5: Assume that $\Theta = \{\theta_1, \theta_2, \theta_3\}$ be the FOD. Let $m(\Theta) = 1$ be a BPA at the start. Changes to the BPA made by us arise step by step, the value of $m(\Theta)$ decreases by $\Delta = 0.05$, while $\Delta/3$ is added to each singleton $m(\{\theta_i\})$, $i \in \{1, 2, 3\}$, until $m(\Theta)$ attains 0 and $m(\{\theta_i\})$ attains 1/3, $i \in \{1, 2, 3\}$ at the final step. The calculated results of the four measures at every step are shown in Figure 6.

As shown in Figure 6, we can come to a conclusion again that $AM(m)$ is insensitive to different BPA in some cases, and the other three measures are relatively valid and rational in this example.

Example 6: Example 6-7 originated from [40] also can be used for further verification here. Assume that $\Theta = \{\theta_1, \theta_2, \ldots\ldots, \theta_8\}$ be the FOD. Given a BPA in the very beginning: $m(\Theta) = 1$, changes associated with the BPA take place one step at a time. The value of $m(\Theta)$ decreases by 0.05, while that of $m(A)$ increases by 0.05 at each step, until $m(\Theta) = 0$ and $m(A) = 1$ at the last step, where $A \neq \Theta$. When the cardinality $|A|$ take different values, such as 2, 4, 6 in here, the performance of the four measures are shown in Figure 7.

As we can see from Figure 7, though all four measures can reflect the variation tendency of uncertainty step-by-step with respect to different $|A|$, our proposed measure $U_{exp}(m)$ is obviously different from the other three measures. Intuitively, it means that the cardinality of the set that might contain the unique target is equal to 8(i.e. $|C| = 8$) as long as the value of $m(\Theta)$ is not equal to 0, thus the uncertainty of the BPA should be relatively large in general. And at the last step, the uncertainty should also have a corresponding mutation when $|C|$ suddenly changes from 8 to 2. However, the measures except for our proposed measure $U_{exp}(m)$ can not reflect these two characteristics in this example.

Example 7: Assume that $\Theta = \{\theta_1, \theta_2, \ldots\ldots, \theta_{10}\}$ be the FOD. Given a BPA at the start: $m(\Theta) =$ 1-a, $m(A) = a$, $|A| \leqslant 9$. Corresponding to different values of a such as 0.2, 0.5, 0.8 in here, changes to the cardinality $|A|$ made by us take place step by step, $|A|$ increases by 1 one step at a time until $|A|$ reaches 9. A vacuous belief function is taken into consideration as a special step for changes in final. At every step, the calculated results of the four measures based on the different a are shown in Figure 8.

Based on the definition of BPA, the degree of belief is exactly assigned to a single element of the FOD when $|A|$ is equal to 1, the smaller the cardinality of focal element the smaller the uncertainty of the BPA. Therefore, the amplitude of variation as $|A|$ changes from 1 to 2 should be far greater than the amplitude of variation as $|A|$ changes from 2 to other values($|A| \neq 1, 2$) in this example. From Figure 8, we can see that only our proposed measure $U_{exp}(m)$ can better reflect this characteristic.

VI. CONCLUSION

In this paper, we propose a novel measure of uncertainty containing Multi-Scale information is proposed to overcome the shortcoming of the existing measures. Comparing with Entropy-like measures of uncertainty based on the generalization of information theory and measures of uncertainty in other functional forms such as the distance between interval values, the main characteristic of our proposed measure is that it has the form of exponential function. The calculated results of some numerical examples show the validity and rationality of our proposed measure. In the future, we will devote to expanding practical application of our proposed measure to multiple fields as an effective quantitative evaluation tool of the BPA.

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