IEEE Access
Multidisciplinary : Rapid Review : Open Access Journal

Received February 14, 2020, accepted February 29, 2020, date of publication March 6, 2020, date of current version March 24, 2020.

Digital Object Identifier 10.1109/ACCESS.2020.2978976

Interval-Valued Pythagorean Normal Fuzzy Information Aggregation Operators for Multi-Attribute Decision Making

ZAOLI YANG[1] AND JINPING CHANG[2]
[1]College of Economics and Management, Beijing University of Technology, Beijing 100124, China
[2]College of Management, Beijing Union University, Beijing 100101, China

Corresponding author: Jinping Chang (gltjinping@buu.edu.cn)

This work was supported in part by the Natural Science Foundation of China under Grant 71704007, in part by the Beijing Social Science Foundation of China under Grant 18GLC082, and in part by the University Nursing Program for Young Scholars With Creative Talents in Heilongjiang Province under Grant 2017103.

ABSTRACT The interval-valued Pythagorean fuzzy (IVPF) sets, describing the membership and non-membership degrees from interval values, can address uncertain information, while the normal fuzzy number (NFN) can depict normal distribution information in anthropogenic activity and natural environment. By combining the advantages of both operations, in this study, we proposed the interval-valued Pythagorean normal fuzzy (IVPNF) sets by introducing the NFN into IVPF environment. Firstly, we defined the conception, the operational laws, score function, accuracy function of IVPNF sets. Secondly, we presented four information aggregation operators to aggregate IVPNF information, including the IVPNF weighted averaging (IVPNFWA) operator, IVPNF weighted geometric (IVPNFWG) operator, the generalized IVPNFWA operator, and the generalized IVPNFWG operator. In addition, we analyzed some desirable properties of monotonicity, commutativity, and idempotency for the proposed four operators. Finally, a numerical example on multi-attribute decision-making problem is given to verify the practicality of the proposed operators, and the comparative and sensitive analysis are used to show the effectiveness and flexibility of our proposed approach.

INDEX TERMS Normal fuzzy number, interval-valued Pythagorean normal fuzzy, information aggregation operators, multi-attribute decision-making.

I. INTRODUCTION

In our daily life, most of human beings are often faced with multiple attribute decision-making (MADM) problems, which may involve in multiple alternatives and multiple evaluation elements. Due to the complexity of human social activities and the uncertainty of natural environment, the way to deal with such uncertain information has become the key to solve the MADM problems. Zadeh [1] proposed a membership-based fuzzy set (FS), which effectively characterized the fuzzy information and uncertain environment, and thus benefit to recommend a better decision. Furthermore, Atanassov [2] extended the Zadeh's FS to intuitionistic fuzzy sets (IFSs) containing three elements, i.e., membership degree, non-membership degree and hesitancy degree. The above intuitionistic fuzzy set has been extensively studied and extended into different types of intuitionistic fuzzy sets, including interval IFSs [3], hesitant IFSs [4], triangular IFSs [5], trapezoidal IFSs [6] and normal IFSs [7], etc. IFSs indicate support, opposition and neutrality of the decision-makers to the same attribute depending on membership degree, non-membership degree and hesitancy degree, and characterize fuzzy information by integrating the above three aspects. The characterization of fuzzy information by Atanassov's IFSs is more comprehensive and detailed than that by Zadeh's FS for the single aspect of membership degree. However, IFSs has certain deficiencies shown as follows. As stipulated by IFS, the sum of membership degree and non-membership degree should be less than or equal to the value number 1. When people independently assign membership degree and non-membership degree to the same attribute in the actual decision-making process, the sum will

The associate editor coordinating the review of this manuscript and approving it for publication was Muhammad Imran Tariq.

be greater than the value number 1, while the square sum will not exceed the value number 1, but the IFSs is inapplicable in this case. For the purpose of characterization of such fuzzy information, Yager and Abbasov [8] and Yager [9] extended intuitionistic fuzzy sets into Pythagorean fuzzy sets (PFSs), so that the decision-makers could characterize fuzzy information more effective and make more accurate decisions without modifying original fuzzy information into information in the form of intuitionistic fuzzy.

Since Yager put forward PFSs, lots of studies investigated to combine the PFSs and other types of fuzzy sets to extend the PFSs. In the terms of basic theory, Hussain *et al.* [10] presented some Rough Pythagorean fuzzy ideals to extend the PFS. Verma and Merigo [11] proposed two new generalized similarity measures between PFSs using cosine and cotangent functions. Hussian and Yang [12] developed a method to calculate the distance between PFSs based on the Hasudorff metric. Xian *et al.* [13] defined a new trapezoidal Pythagorean fuzzy linguistic PRs (TrPFL-PRs), and proved some properties of TrPFLPRs. Zhou and Yang [14] defined the concept of single granulation hesitant Pythagorean fuzzy rough sets (SGHPFRSs). In the terms of information aggregation operators, Liang *et al.* [15] presented Pythagorean fuzzy geometric weighted Bonferroni mean PYGWBM operator, and developed a MADM based on PYGWBM operator and projection. Tang *et al.* [16] proposed dual hesitant Pythagorean fuzzy sets (DHPFSs), and developed the dual hesitant Pythagorean fuzzy (DHPF) generalized weighted Heronian mean (DHPFGWHM) operator and DHPF generalized geometric weighted Heronian mean (DHPFGGWHM) operator. Garg [17] proposed the family of generalized Pythagorean fuzzy (HPF) Einstein operator. Khan *et al.* [18] presented the Pythagorean trapezoidal uncertain linguistic fuzzy (PTULF) Einstein weighted averaging (PTULFEWA) operator, the PTULF Einstein ordered weighted averaging (PTULFEOWA) operator, and the PTULF Einstein hybrid weighted averaging (PTULFEHWA) operator. Wei *et al.* [19] developed dual hesitant Pythagorean fuzzy (DHPF) Hamy mean (DHPFHM) operators, such as the DHPF weighted Hamy mean (DHPFWHM) operator and the DHPF weighted dual Hamy mean (DHPFWDHM) operator. Abdullah and Mohd [20] proposed the Pythagorean fuzzy Hamacher Choquet integral (PFHCI) average (PFH-CIA) operators and PFHCI geometric PFHCIG) operators. Abbas *et al.* [21] defined the concept of Cubic Pythagorean fuzzy numbers (CPFNs), and presented Cubic Pythagorean fuzzy (CPF) weighted averaging (CPFWA) operator, and CPF weighted geometric (CPFWG) operator. Shakeel *et al.* [22] proposed Pythagorean trapezoidal fuzzy (PTF) ordered weighted averaging (PTFOWA) operator and PTF hybrid averaging (PTFHA) operator. Deng *et al.* [23] by introducing the Hamy mean (HM) operator into the 2-tuple linguistic Pythagorean fuzzy numbers (2TLPFNs), developed the family of the 2-tuple linguistic Pythagorean fuzzy information aggregation operator. Xian *et al.* [24] developed a new trapezoidal Pythagorean fuzzy linguistic entropic, and analyzed its operational rules and information aggregation operators. Teng *et al.* [25] by introducing the power average (PA) operator and Maclaurin symmetric mean (MSM) operator into Pythagorean fuzzy linguistic (PFL), presented some power MSM aggregation operators for PFL information. Garg [26] presented some neutrality operation-based Pythagorean fuzzy geometric aggregation operators. Sarkar and Biswas [27] presented archimedean t-conorm and t-norm-based Pythagorean hesitant fuzzy weighted averaging operator and weighted geometric operator. Jana *et al.* [28] introduced the Dombi operations into PFS, developed the family of Pythagorean fuzzy Dombi aggregation operators.

However, owing to the limitation of human cognition and the complexity of the objective world, it is difficult for a human to exactly express the membership and non-membership degrees by crisp numbers, but can be shown by the interval numbers [29]. Interval value PFS (IVPFS), as the expansion of PFS, was presented by Peng and Yang [29], and caused widespread attention of many scholars. According to the basic theory of Interval value PFS, many scholarships extended and fulfil the theory of IVPFS. Garg [30] introduced the exponential operational laws into the interval-valued Pythagorean fuzzy set (IVPFS), and proposed some new exponential operational rules and information aggregation operators of IVPFS. Du *et al.* [31] defined the interval-valued Pythagorean fuzzy linguistic variable set (IVPFLVS), and presented the interval-valued Pythagorean fuzzy linguistic (IVPFL) weighted averaging (IVPFLWA), IVPFL ordered weighted averaging (IVPFLOWA), IVPFL hybrid averaging, and generalized IVPFL ordered weighted average operators. Tang *et al.* [32] combined the Muirhead Mean (MM) operator and dual MM (DMM) with the interval-valued Pythagorean fuzzy numbers (IVPFNs), and proposed the family of interval-valued Pythagorean fuzzy Muirhead mean operators based on MM and DMM operators. Yang and Pang [33] developed the concepts of the hesitant interval-valued Pythagorean fuzzy set (HIVPFS) are defined, and presented a series of aggregation operators based on HIVPFS. Wang *et al.* [34] developed a series of the interval-valued 2-tuple linguistic Pythagorean fuzzy Maclaurin symmetric mean operator. Liang *et al.* [35] developed a series of interval-valued Pythagorean fuzzy Frank power (IVPFFP) aggregation operators. Wei *et al.* [36] by using Maclaurin symmetric mean (MSM) operator, developed IVPF Maclaurin symmetric mean and IVPF weighted Maclaurin symmetric mean operators. Rahman and Abdullah [37] developed some operators under interval-valued Pythagorean fuzzy (IVPF) environment, including induced generalized IVPF Einstein ordered weighted geometric (I-GIVPFEOWG) operator and induced generalized IVPF Einstein hybrid weighted geometric (I-GIVPFEHWG) aggregation operator. Liu *et al.* [38] introduced a new decision-making method based on interval-valued Pythagorean hesitant fuzzy sets to select third-party reverse logistics providers (3PRLs). Haktanir and Kahraman [39] presented an MDAM method combining Quality function deployment (QFD) with IVPFS to evaluate solar photovoltaic technology development. Chen [40] developed an inferior ratio(IR)-based assignment method under IVPF information environment to evaluate risk level of enterprise technological innovation. Yu *et al.* [41] presented a group MADM method for sustainable supplier selection using extended TOPSIS based on IVPF sets.

Furthermore, in the process of actual decision-making, people will also be exposed to much information obeying normal distribution which is derived from a large number of human activities and natural phenomena obeying normal distribution, e.g., "the service life of different products", "measurement errors", and "the law of weather changing with the seasons", etc. However, it is impossible to characterize such kind of fuzzy information with the existing hesitant fuzzy numbers, triangular fuzzy numbers and trapezoidal fuzzy numbers, etc. For this reason, Yang and Ko [42] put forward a concept of normal fuzzy number (NFN) to describe the above-mentioned fuzzy phenomena. As shown by the comparison results, normal fuzzy numbers boasted high-order derivative continuity and were closer to human thinking in decision-making [43]. Based on this, some scholars developed some new notions by combining the NFN and intuitionistic fuzzy sets. Wang *et al.* [44] defined the concept of intuitionistic normal fuzzy (INF) sets and presented some information aggregation under INF environment. Wang *et al.* [45] developed a series of induced ordered weighted aggregation operators for INF. Liu and Teng [46] defined some concepts of normal interval-valued intuitionistic fuzzy numbers (NIVIFNs). Liu and Liu [47] proposed some INF operators based on Bonferroni mean. Yang *et al.* [48] proposed two dynamic intuitionistic normal fuzzy weighted operators. Moreover, Li *et al.* [49] presented dynamic interval-valued INF aggregation operators. Liu [50] proposed some NIF operators with power interaction. Zhang *et al.* [51] proposed some INF Heronian mean operators.

Inspired by the above survey of related studies, we understand that PFS, as an extension of IFS, describes fuzzy information in a wider way than IFS. Compared with hesitant fuzzy numbers, triangular fuzzy numbers and trapezoidal fuzzy numbers, NFN is closer to human thinking in decision-making. INF sets have been presented in previous studies, but PFS or IVPFS based on NFN has not yet been reported.

In this study, we proposed a new fuzzy set, called interval value Pythagorean normal fuzzy set (IVPNFS). We assume that this new set for operation can support the multi-attribute decision making and improve the decision performance. The main contributions of this paper are summarized as follows:

(1) The concept of IVPNFN and its based operational rules are defined, the related properties of the operational rules are proved, and the score function and accuracy function under IVPNF environment is proposed.

(2) The method for measurement of the distance between IVPNFNs is defined, including the method for measurement of the distance between IVPNFNs based on Euclidean distance and Hamming distance, and the compliance of the distance measurement method to the three elements of distance is proved.

(3) Some information aggregation operators in IVPNF environment are proposed, including IVPNF weighted averaging (IVPNFWA) operator, IVPNF weighted geometric (IVPNFWG) operator, the generalized IVPNFWA (GIVPNFWA) operator, and the generalized IVPNFWG (GIVPNFWG) operator, and commutativity, idempotency, and boundedness properties of the above aggregation operators are demonstrated.

(4) A MADM method based on IVPNFW aggregation operator and TOPSIS in IVPNF environment is proposed.

The rest parts are arranged as follows: In Part 2, the basic concepts of NFN and IVPFS are reviewed. In Part 3, PNFS and some of its operational rules are proposed. In Part 4, Euclidean distance and Hamming distance between IVPNFSs are put forward. In Part 5, some information aggregation operators under IVPNFS environment are presented. In Part 6, a MADM method based on IVPNFS weighted information aggregation operators and TOPSIS is proposed. In Part 7, an example is given to demonstrate the effectiveness of the proposed method. In Part 8, some conclusions are made.

II. PRELIMLINARIES

A. THE NORMAL FUZZY NUMBER

Definition 1 [42]: Let R be a real number set, the membership function of fuzzy number

$$\tilde{Q}(x) = e^{-\left(\frac{x-\alpha}{\sigma}\right)^2} (\sigma > 0) \tag{1}$$

is called as a normal fuzzy number (NFN) $\tilde{Q} = (\alpha, \sigma)$, the normal fuzzy number set (NFNS) is denoted by $\tilde{N}$.

Definition 2 [52]: let $\tilde{Q}_1, \tilde{Q}_2 \in \tilde{N}$, denoted by $\tilde{Q}_1 = (\alpha, \sigma)$, $\tilde{Q}_2 = (\beta, \tau)$, then

(1) $\lambda \tilde{Q}_1 = \lambda(\alpha, \sigma) = (\lambda\alpha, \lambda\sigma)$, $\lambda > 0$

(2) $\tilde{Q}_1 + \tilde{Q}_2 = (\alpha, \sigma) + (\beta, \tau) = (\alpha+\beta, \sigma+\tau)$

Definition 3 [52]: let $\tilde{Q}_1, \tilde{Q}_2 \in \tilde{N}$, denoted by $\tilde{Q}_1 = (\alpha, \sigma)$, $\tilde{Q}_2 = (\alpha, \sigma)$, then the distance between $\tilde{A}$ and $\tilde{B}$ can be defined as

$$d^2\left(\tilde{A}, \tilde{B}\right) = (\alpha-\beta)^2 + \frac{1}{2}(\sigma-\tau)^2 \tag{2}$$

B. THE INTERVAL-VALUED PYTHAGOREAN FUZZY NUMBER

Definition 4 [29]: Let X be a non-empty set of the universe, an interval-valued Pythagorean fuzzy A in X defined by

$$A = \left\langle x, \left[\mu_A^L(x), \mu_A^U(x)\right], \left[\nu_A^L(x), \nu_A^U(x)\right]\right\rangle$$

where $\left[\mu_A^L(x), \mu_A^U(x)\right]$ and $\left[\nu_A^L(x), \nu_A^U(x)\right]$ respectively, represent the membership and non-membership degree of A, $\left[\mu_A^L(x), \mu_A^U(x)\right] \in [0, 1]$, $\left[\nu_A^L(x), \nu_A^U(x)\right] \in [0, 1]$, and $0 \le u_A^u(x)^2 + \nu_A^u(x)^2 \le 1$, the degree of indeterminacy is determined as

$$\pi_A(x) = \left[\pi_A^L(x), \pi_A^U(x)\right] = \begin{bmatrix} \sqrt{1-\left(\mu_A^U(x)\right)^2-\left(\nu_A^U(x)\right)^2}, \\ \sqrt{1-\left(\mu_A^L(x)\right)^2-\left(\nu_A^L(x)\right)^2} \end{bmatrix}$$

Peng and Yang called $A = \langle [\mu_A^L, \mu_A^U], [\nu_A^L, \nu_A^U] \rangle$ as an interval-valued Pythagorean fuzzy number (IVPFN).

Definition 5 [29]: Let $A = \langle [\mu_A^L, \mu_A^U], [\nu_A^L, \nu_A^U] \rangle$, $A_1 = \langle [\mu_1^L, \mu_1^U], [\nu_1^L, \nu_1^U] \rangle$, and $A_2 = \langle [\mu_2^L, \mu_2^U], [\nu_2^L, \nu_2^U] \rangle$ be any three IVPFNs, λ be a non-negative real number, then

$$(1)\ A_1 \oplus A_2 = \left(\begin{array}{l} \left[\begin{array}{l} \sqrt{(\mu_1^L)^2 + (\mu_2^L)^2 - (\mu_1^L)^2 \cdot (\mu_2^L)^2}, \\ \sqrt{(\mu_1^U)^2 + (\mu_2^U)^2 - (\mu_1^U)^2 \cdot (\mu_2^U)^2} \end{array} \right], \\ [\nu_1^L \nu_2^L, \nu_1^U \nu_2^U] \end{array} \right),$$

$$(2)\ A_1 \otimes A_2 = \left(\begin{array}{l} [\mu_1^L \mu_2^L, \mu_1^U \mu_2^U], \\ \sqrt{(\nu_1^L)^2 + (\nu_2^L)^2 - (\nu_1^L)^2 \cdot (\nu_2^L)^2}, \\ \sqrt{(\nu_1^U)^2 + (\nu_2^U)^2 - (\nu_1^U)^2 \cdot (\nu_2^U)^2} \end{array} \right),$$

$$(3)\ \lambda A = \left(\begin{array}{l} \left[\sqrt{1 - \left(1 - (\mu_A^L)^2\right)^\lambda}, \sqrt{1 - \left(1 - (\mu_A^U)^2\right)^\lambda} \right], \\ \left[(\nu_A^L)^\lambda, (\nu_A^U)^\lambda \right] \end{array} \right),$$

$$(4)\ A^\lambda = \left(\begin{array}{l} \left[(\mu_A^L)^\lambda, (\mu_A^U)^\lambda \right], \\ \left[\sqrt{1 - \left(1 - (\nu_A^L)^2\right)^\lambda}, \sqrt{1 - \left(1 - (\nu_A^U)^2\right)^\lambda} \right] \end{array} \right),$$

Definition 6 [29]: Let $A = \langle [\mu_A^L, \mu_A^U], [\nu_A^L, \nu_A^U] \rangle$ be an IVPFN, its score function is defined as

$$S(A) = \frac{1}{2}\left(\left(\mu_A^L\right)^2 + \left(\mu_A^U\right)^2 - \left(\nu_A^L\right)^2 - \left(\nu_A^U\right)^2 \right)$$

and its accuracy function is defined as

$$H(A) = \frac{1}{2}\left(\left(\mu_A^L\right)^2 + \left(\mu_A^U\right)^2 + \left(\nu_A^L\right)^2 + \left(\nu_A^U\right)^2 \right)$$

for any two IVPFNs, $A_1 = \langle [\mu_1^L, \mu_1^U], [\nu_1^L, \nu_1^U] \rangle$, and $A_2 = \langle [\mu_2^L, \mu_2^U], [\nu_2^L, \nu_2^U] \rangle$, then

(1) If $S(A_1) > S(A_2)$, then $A_1 > A_2$;
(2) If $S(A_1) = S(A_2)$, then
If $H(A_1) > H(A_2)$, then $A_1 > A_2$
If $H(A_1) = H(A_2)$, then $A_1 = A_2$

III. THE INTERVAL-VALUED PYTHAGOREAN NORMAL FUZZY NUMBER AND ITS OPERATIONS

Based on the conceptions and operations of IVPFN and NFN, we defined the interval-valued Pythagorean normal fuzzy number (IVPNFN) and its operations.

Definition 7: Let X be an ordinary fixed non-empty set and $(\alpha, \sigma) \in \tilde{N}$, $\tilde{A} = \langle (\alpha, \sigma); [\mu_A^L, \mu_A^U], [\nu_A^L, \nu_A^U] \rangle$ is a IVPNFN when its membership function is defined as

$$\left[\mu_A^L, \mu_A^U\right] = \left[\mu_A^L e^{-(\frac{x-\alpha}{\sigma})^2}, \mu_A^U e^{-(\frac{x-\alpha}{\sigma})^2}\right], \quad x \in X \tag{3}$$

and non-membership function is defined as

$$\left[\nu_A^L, \nu_A^U\right] = \left[\begin{array}{l} \left(1 - (1 - \nu_A^L) e^{-(\frac{x-\alpha}{\sigma})^2}\right), \\ \left(1 - (1 - \nu_A^U) e^{-(\frac{x-\alpha}{\sigma})^2}\right) \end{array} \right], \quad x \in X \tag{4}$$

where $\left[\mu_A^L(x), \mu_A^U(x)\right] \in [0, 1]$, $\left[\nu_A^L(x), \nu_A^U(x)\right] \in [0, 1]$, and $0 \leq u_A^u(x)^2 + v_A^u(x)^2 \leq 1$.

Definition 8: Let $\tilde{A} = \langle (\alpha, \sigma); [\mu_A^L, \mu_A^U], [\nu_A^L, \nu_A^U] \rangle$, $\tilde{A}_1 = \langle (\alpha_1, \sigma_1); [\mu_1^L, \mu_1^U], [\nu_1^L, \nu_1^U] \rangle$, and $\tilde{A}_2 = \langle (\alpha_2, \sigma_2); [\mu_2^L, \mu_2^U], [\nu_2^L, \nu_2^U] \rangle$ be any three IVPFNs, λ be a non-negative real number, then

$$(1) \tilde{A}_1 \oplus \tilde{A}_2 = \left(\begin{array}{l} (\alpha_1 + \alpha_2, \sigma_1 + \sigma_2); \\ \left[\begin{array}{l} \sqrt{(\mu_1^L)^2 + (\mu_2^L)^2 - (\mu_1^L)^2 \cdot (\mu_2^L)^2}, \\ \sqrt{(\mu_1^U)^2 + (\mu_2^U)^2 - (\mu_1^U)^2 \cdot (\mu_2^U)^2} \end{array} \right], \\ [\nu_1^L \nu_2^L, \nu_1^U \nu_2^U] \end{array} \right),$$

$$(2) \tilde{A}_1 \otimes \tilde{A}_2 = \left(\begin{array}{l} (\alpha_1 \cdot \alpha_2, \sigma_1 \cdot \sigma_2); [\mu_1^L \mu_2^L, \mu_1^U \mu_2^U], \\ \left[\begin{array}{l} \sqrt{(\nu_1^L)^2 + (\nu_2^L)^2 - (\nu_1^L)^2 \cdot (\nu_2^L)^2}, \\ \sqrt{(\nu_1^U)^2 + (\nu_2^U)^2 - (\nu_1^U)^2 \cdot (\nu_2^U)^2} \end{array} \right] \end{array} \right),$$

$$(3) \lambda \tilde{A} = \left(\begin{array}{l} (\lambda\alpha, \lambda\sigma); \\ \left[\sqrt{1 - \left(1 - (\mu_A^L)^2\right)^\lambda}, \sqrt{1 - \left(1 - (\mu_A^U)^2\right)^\lambda} \right], \\ \left[(\nu_A^L)^\lambda, (\nu_A^U)^\lambda \right] \end{array} \right),$$

$$(4) \tilde{A}^\lambda = \left(\begin{array}{l} (\alpha^\lambda, \sigma^\lambda); \left[(\mu_A^L)^\lambda, (\mu_A^U)^\lambda \right], \\ \left[\sqrt{1 - \left(1 - (\nu_A^L)^2\right)^\lambda}, \sqrt{1 - \left(1 - (\nu_A^U)^2\right)^\lambda} \right] \end{array} \right);$$

Proposition 1: Let $\tilde{A} = \langle (\alpha, \sigma); [\mu_A^L, \mu_A^U], [\nu_A^L, \nu_A^U] \rangle$, $\tilde{A}_1 = \langle (\alpha_1, \sigma_1); [\mu_1^L, \mu_1^U], [\nu_1^L, \nu_1^U] \rangle$, and $\tilde{A}_2 = \langle (\alpha_2, \sigma_2); [\mu_2^L, \mu_2^U], [\nu_2^L, \nu_2^U] \rangle$ be any three IVPFNs, and $\lambda, \lambda_1, \lambda_2$ be non-negative real numbers, we can obtain that:

$$(1)\ \tilde{A}_1 \oplus \tilde{A}_2 = \tilde{A}_2 \oplus \tilde{A}_1,$$
$$(2)\ \left(\tilde{A}_1 \oplus \tilde{A}_2\right) \oplus \tilde{A}_3 = \tilde{A}_1 \oplus \left(\tilde{A}_2 \oplus \tilde{A}_3\right),$$
$$(3)\ \tilde{A}_1 \otimes \tilde{A}_2 = \tilde{A}_2 \otimes \tilde{A}_1,$$
$$(4)\ \left(\tilde{A}_1 \otimes \tilde{A}_2\right) \otimes \tilde{A}_3 = \tilde{A}_1 \otimes \left(\tilde{A}_2 \otimes \tilde{A}_3\right),$$
$$(5)\ \lambda\left(\tilde{A}_1 \otimes \tilde{A}_2\right) = \lambda\tilde{A}_2 \otimes \lambda\tilde{A}_1,$$
$$(6)\ (\lambda_1 + \lambda_2)\tilde{A} = \lambda_1\tilde{A} + \lambda_2\tilde{A},$$
$$(7)\ \left(\tilde{A}_1 \otimes \tilde{A}_2\right)^\lambda = \tilde{A}_1^\lambda \otimes \tilde{A}_2^\lambda.$$

Proof: Based on definition 8, we can easily infer that (1), (3), (5), (6) and (7) are right, (2) and (4) need be proved as follows:

$$\text{For (2) } \left(\tilde{A}_1 \oplus \tilde{A}_2\right) \oplus \tilde{A}_3 = \tilde{A}_1 \oplus \left(\tilde{A}_2 \oplus \tilde{A}_3\right)$$

Let the NFN of IVPFN $\tilde{Q}$ be $\tilde{N}_{\tilde{Q}}$, the degree of membership of $\left(\tilde{A}_1 \oplus \tilde{A}_2\right) \oplus \tilde{A}_3$ and $\tilde{A}_1 \oplus \left(\tilde{A}_2 \oplus \tilde{A}_3\right)$ be $\left[\mu^L_{(A_1 \oplus A_2) \oplus A_3}, \mu^U_{(A_1 \oplus A_2) \oplus A_3}\right]$ and $\left[\mu^L_{A_1 \oplus (A_2 \oplus A_3)}, \mu^U_{A_1 \oplus (A_2 \oplus A_3)}\right]$, and their degree of non-membership be $\left[\nu^L_{(A_1 \oplus A_2) \oplus A_3}, \nu^U_{(A_1 \oplus A_2) \oplus A_3}\right]$ and $\left[\nu^L_{A_1 \oplus (A_2 \oplus A_3)}, \nu^U_{A_1 \oplus (A_2 \oplus A_3)}\right]$, respectively, and we can get

$$\tilde{N}_{(A_1\oplus A_2)\oplus A_3} = \tilde{N}_{A_1\oplus(A_2\oplus A_3)} = (\alpha_1+\alpha_2+\alpha_3,\ \sigma_1+\sigma_2+\sigma_3)$$

$$\left[\mu^L_{(A_1\oplus A_2)\oplus A_3},\ \mu^U_{(A_1\oplus A_2)\oplus A_3}\right]$$
$$= \left[\sqrt{\begin{array}{l}(\mu_1^L)^2+(\mu_2^L)^2-(\mu_1^L)^2\cdot(\mu_2^L)^2+(\mu_3^L)^2\\ -\left((\mu_1^L)^2+(\mu_2^L)^2-(\mu_1^L)^2\cdot(\mu_2^L)^2\right)\cdot(\mu_3^L)^2\end{array}},\ \sqrt{\begin{array}{l}(\mu_1^U)^2+(\mu_2^U)^2-(\mu_1^U)^2\cdot(\mu_2^U)^2+(\mu_3^U)^2\\ -\left((\mu_1^U)^2+(\mu_2^U)^2-(\mu_1^U)^2\cdot(\mu_2^U)^2\right)\cdot(\mu_3^U)^2\end{array}}\right]$$
$$= \left[\sqrt{\begin{array}{l}(\mu_1^L)^2+(\mu_2^L)^2+(\mu_3^L)^2-(\mu_1^L)^2\cdot(\mu_2^L)^2\\ -(\mu_1^L)^2\cdot(\mu_3^L)^2-(\mu_2^L)^2\cdot(\mu_3^L)^2+(\mu_1^L)^2\cdot(\mu_3^L)^2(\mu_2^L)^2\end{array}},\ \sqrt{\begin{array}{l}(\mu_1^U)^2+(\mu_2^U)^2+(\mu_3^U)^2-(\mu_1^U)^2\cdot(\mu_2^U)^2\\ -(\mu_1^U)^2\cdot(\mu_3^U)^2-(\mu_2^U)^2\cdot(\mu_3^U)^2+(\mu_1^U)^2\cdot(\mu_2^U)^2\cdot(\mu_3^U)^2\end{array}}\right]$$

$$\left[\mu^L_{A_1\oplus(A_2\oplus A_3)},\ \mu^U_{A_1\oplus(A_2\oplus A_3)}\right]$$
$$= \left[\sqrt{\begin{array}{l}(\mu_2^L)^2+(\mu_3^L)^2-(\mu_2^L)^2\cdot(\mu_3^L)^2+(\mu_1^L)^2\\ -\left((\mu_2^L)^2+(\mu_3^L)^2-(\mu_2^L)^2\cdot(\mu_3^L)^2\right)\cdot(\mu_1^L)^2\end{array}},\ \sqrt{\begin{array}{l}(\mu_2^U)^2+(\mu_3^U)^2-(\mu_2^U)^2\cdot(\mu_3^U)^2+(\mu_1^U)^2\\ -\left((\mu_2^U)^2+(\mu_3^U)^2-(\mu_2^U)^2\cdot(\mu_3^U)^2\right)\cdot(\mu_1^U)^2\end{array}}\right]$$
$$= \left[\sqrt{\begin{array}{l}(\mu_1^L)^2+(\mu_2^L)^2+(\mu_3^L)^2-(\mu_1^L)^2\cdot(\mu_2^L)^2\\ -(\mu_1^L)^2\cdot(\mu_3^L)^2-(\mu_2^L)^2\cdot(\mu_3^L)^2+(\mu_1^L)^2\cdot(\mu_3^L)^2\cdot(\mu_2^L)^2\end{array}},\ \sqrt{\begin{array}{l}(\mu_1^U)^2+(\mu_2^U)^2+(\mu_3^U)^2-(\mu_1^U)^2(\mu_2^U)^2\\ -(\mu_1^U)^2\cdot(\mu_3^U)^2-(\mu_2^U)^2\cdot(\mu_3^U)^2+(\mu_1^U)^2\cdot(\mu_2^U)^2\cdot(\mu_3^U)^2\end{array}}\right]$$

Then

$$\left[\mu^L_{(A_1\oplus A_2)\oplus A_3},\ \mu^U_{(A_1\oplus A_2)\oplus A_3}\right] = \left[\mu^L_{A_1\oplus(A_2\oplus A_3)},\ \mu^U_{A_1\oplus(A_2\oplus A_3)}\right]$$

Similarly, we can get

$$\left[\nu^L_{(A_1\oplus A_2)\oplus A_3},\ \nu^U_{(A_1\oplus A_2)\oplus A_3}\right] = \left[\nu^L_{A_1\oplus(A_2\oplus A_3)},\ \nu^U_{A_1\oplus(A_2\oplus A_3)}\right]$$

Therefore,

$$\left(\tilde{A}_1\oplus\tilde{A}_2\right)\oplus\tilde{A}_3 = \tilde{A}_1\oplus\left(\tilde{A}_2\oplus\tilde{A}_3\right)$$

For (4)

$$\left(\tilde{A}_1\otimes\tilde{A}_2\right)\otimes\tilde{A}_3 = \tilde{A}_1\otimes\left(\tilde{A}_2\otimes\tilde{A}_3\right)$$

Let the NFN of IVPFNs $\tilde{Q}$ be $\tilde{N}_{\tilde{Q}}$, the degree of membership of $\left(\tilde{A}_1\otimes\tilde{A}_2\right)\otimes\tilde{A}_3$ and $\tilde{A}_1\otimes\left(\tilde{A}_2\otimes\tilde{A}_3\right)$ be $\left[\mu^L_{(A_1\otimes A_2)\otimes A_3}, \mu^U_{(A_1\otimes A_2)\otimes A_3}\right]$ and $\left[\mu^L_{A_1\otimes(A_2\otimes A_3)}, \mu^U_{A_1\otimes(A_2\otimes A_3)}\right]$, and their degree of non-membership be $\left[\nu^L_{(A_1\otimes A_2)\otimes A_3}, \nu^U_{(A_1\otimes A_2)\otimes A_3}\right]$ and $\left[\nu^L_{A_1\otimes(A_2\otimes A_3)}, \nu^U_{A_1\otimes(A_2\otimes A_3)}\right]$, respectively, and we can get

$$\tilde{N}_{(A_1\otimes A_2)\otimes A_3} = \tilde{N}_{A_1\otimes(A_2\otimes A_3)} = (\alpha_1+\alpha_2+\alpha_3,\ \sigma_1+\sigma_2+\sigma_3)$$

$$\left[\nu^L_{(A_1\otimes A_2)\otimes A_3},\ \nu^U_{(A_1\otimes A_2)\otimes A_3}\right]$$
$$= \left[\sqrt{\begin{array}{l}(\nu_1^L)^2+(\nu_2^L)^2-(\nu_1^L)^2\cdot(\nu_2^L)^2+(\nu_3^L)^2\\ -\left((\nu_1^L)^2+(\nu_2^L)^2-(\nu_1^L)^2\cdot(\nu_2^L)^2\right)\cdot(\nu_3^L)^2\end{array}},\ \sqrt{\begin{array}{l}(\nu_1^U)^2+(\nu_2^U)^2-(\nu_1^U)^2\cdot(\nu_2^U)^2+(\nu_3^U)^2\\ -\left((\nu_1^U)^2+(\nu_2^U)^2-(\nu_1^U)^2\cdot(\nu_2^U)^2\right)\cdot(\nu_3^U)^2\end{array}}\right]$$
$$= \left[\sqrt{\begin{array}{l}(\nu_1^L)^2+(\nu_2^L)^2+(\nu_3^L)^2-(\nu_1^L)^2\cdot(\nu_2^L)^2\\ -(\nu_1^L)^2\cdot(\nu_3^L)^2-(\nu_2^L)^2\cdot(\nu_3^L)^2+(\nu_1^L)^2\cdot(\nu_3^L)^2\cdot(\nu_2^L)^2\end{array}},\ \sqrt{\begin{array}{l}(\nu_1^U)^2+(\nu_2^U)^2+(\nu_3^U)^2-(\nu_1^U)^2\cdot(\nu_2^U)^2\\ -(\nu_1^U)^2\cdot(\nu_3^U)^2-(\nu_2^U)^2\cdot(\nu_3^U)^2+(\nu_1^U)^2\cdot(\nu_2^U)^2\cdot(\nu_3^U)^2\end{array}}\right]$$

$$\left[\nu^L_{A_1\otimes(A_2\otimes A_3)},\ \nu^U_{A_1\otimes(A_2\otimes A_3)}\right]$$
$$= \left[\sqrt{\begin{array}{l}(\nu_2^L)^2+(\nu_3^L)^2-(\nu_2^L)^2\cdot(\nu_3^L)^2+(\nu_1^L)^2\\ -\left((\nu_2^L)^2+(\nu_3^L)^2-(\nu_2^L)^2\cdot(\nu_3^L)^2\right)\cdot(\nu_1^L)^2\end{array}},\ \sqrt{\begin{array}{l}(\nu_2^U)^2+(\nu_3^U)^2-(\nu_2^U)^2\cdot(\nu_3^U)^2+(\nu_1^U)^2\\ -\left((\nu_2^U)^2+(\nu_3^U)^2-(\nu_2^U)^2\cdot(\nu_3^U)^2\right)\cdot(\nu_1^U)^2\end{array}}\right]$$
$$= \left[\sqrt{\begin{array}{l}(\nu_1^L)^2+(\nu_2^L)^2+(\nu_3^L)^2-(\nu_1^L)^2\cdot(\nu_2^L)^2\\ -(\nu_1^L)^2\cdot(\nu_3^L)^2-(\nu_2^L)^2\cdot(\nu_3^L)^2+(\nu_1^L)^2\cdot(\nu_3^L)^2\cdot(\nu_2^L)^2\end{array}},\ \sqrt{\begin{array}{l}(\nu_1^U)^2+(\nu_2^U)^2+(\nu_3^U)^2-(\nu_1^U)^2\cdot(\nu_2^U)^2\\ -(\nu_1^U)^2\cdot(\nu_3^U)^2-(\nu_2^U)^2\cdot(\nu_3^U)^2+(\nu_1^U)^2\cdot(\nu_2^U)^2\cdot(\nu_3^U)^2\end{array}}\right]$$

Then

$$\left[\nu^L_{(A_1\otimes A_2)\otimes A_3},\ \nu^U_{(A_1\otimes A_2)\otimes A_3}\right] = \left[\nu^L_{A_1\otimes(A_2\otimes A_3)},\ \nu^U_{A_1\otimes(A_2\otimes A_3)}\right]$$

Similarly, we can get that

$$\left[\mu^L_{(A_1\otimes A_2)\otimes A_3},\ \mu^U_{(A_1\otimes A_2)\otimes A_3}\right] = \left[\mu^L_{A_1\otimes(A_2\otimes A_3)},\ \mu^U_{A_1\otimes(A_2\otimes A_3)}\right].$$

Therefore, $\left(\tilde{A}_1\otimes\tilde{A}_2\right)\otimes\tilde{A}_3 = \tilde{A}_1\otimes\left(\tilde{A}_2\otimes\tilde{A}_3\right)$

Definition 9: Let $\tilde{A} = \left\langle(\alpha,\sigma); \left[\mu^L,\mu^U\right], \left[\nu^L,\nu^U\right]\right\rangle$ be an IVPFN, its score function is determined as

$$S_1(\tilde{A}) = \frac{\alpha}{2}\left(\frac{(\mu^L)^2+(\mu^U)^2}{2}+1-\frac{(\nu^L)^2+(\nu^U)^2}{2}\right),$$
$$S_2(\tilde{A}) = \frac{\sigma}{2}\left(\frac{(\mu^L)^2+(\mu^U)^2}{2}+1-\frac{(\nu^L)^2+(\nu^U)^2}{2}\right).$$

its accuracy function is determined as

$$H_1(\tilde{A}) = \frac{\alpha}{2}\left(\frac{(\mu^L)^2+(\mu^U)^2}{2}+\frac{(\nu^L)^2+(\nu^U)^2}{2}\right),$$
$$H_2(\tilde{A}) = \frac{\sigma}{2}\left(\frac{(\mu^L)^2+(\mu^U)^2}{2}+\frac{(\nu^L)^2+(\nu^U)^2}{2}\right).$$

Definition 10: Let $\tilde{A}_1 = \langle(\alpha_1,\sigma_1);[\mu_1^L,\mu_1^U],[\nu_1^L,\nu_1^U]\rangle$, $\tilde{A}_2 = \langle(\alpha_2,\sigma_2);[\mu_2^L,\mu_2^U],[\nu_2^L,\nu_2^U]\rangle$ be any two IVPFNs, their score functions are $S_1(a)$, $S_2(a)$, their accuracy functions are $H_1(a)$, $H_2(a)$, respectively, then we can get

(1) If $S_1(\tilde{A}_1) > S_1(\tilde{A}_2)$, then $\tilde{A}_1 > \tilde{A}_2$;
(2) If $S_1(\tilde{A}_1) = S_1(\tilde{A}_2)$ and $H_1(\tilde{A}_1) > H_1(\tilde{A}_2)$, $\tilde{A}_1 > \tilde{A}_2$;
(3) If $S_1(\tilde{A}_1) = S_1(\tilde{A}_2)$ and $H_1(\tilde{A}_1) = H_1(\tilde{A}_2)$
(a) If $S_2(\tilde{A}_1) > S_2(\tilde{A}_2)$, then $\tilde{A}_1 < \tilde{A}_2$
(b) If $S_2(\tilde{A}_1) = S_2(\tilde{A}_2)$ and $H_2(\tilde{A}_1) > H_2(\tilde{A}_2)$, then $\tilde{A}_1 < \tilde{A}_2$
(c) If $S_2(\tilde{A}_1) = S_2(\tilde{A}_2)$ and $H_2(\tilde{A}_1) = H_2(\tilde{A}_2)$, then $\tilde{A}_1 = \tilde{A}_2$

IV. THE DISTANCE MEASURE BETWEEN IVPNFNs

Definition 11: Assume $\tilde{A}_1 = \langle(\alpha_1,\sigma_1);[\mu_1^L,\mu_1^U],[\nu_1^L,\nu_1^U]\rangle$, $\tilde{A}_2 = \langle(\alpha_2,\sigma_2);[\mu_2^L,\mu_2^U],[\nu_2^L,\nu_2^U]\rangle$, be any two IVPFNs in X, and the Euclidean distance and Hamming distance between IVPFNs are defined as follows:

$$D_E\left(\tilde{A}_1,\tilde{A}_2\right) = \frac{1}{2}\sqrt{\begin{array}{l}\left(\frac{1+(\mu_1^L)^2-(\nu_1^L)^2+1+(\mu_1^U)^2-(\nu_1^U)^2}{2}\alpha_1 - \frac{1+(\mu_2^L)^2-(\nu_2^L)^2+1+(\mu_2^U)^2-(\nu_2^U)^2}{2}\alpha_2\right)^2 \\ +\frac{1}{2}\left(\frac{1+(\mu_1^L)^2-(\nu_1^L)^2+1+(\mu_1^U)^2-(\nu_1^U)^2}{2}\sigma_1 - \frac{1+(\mu_2^L)^2-(\nu_2^L)^2+1+(\mu_2^U)^2-(\nu_2^U)^2}{2}\sigma_2\right)^2 \\ +\left(\frac{1+(\pi_1^L)^2-(\pi_2^L)^2+1+(\pi_1^U)^2-(\pi_2^U)^2}{2}\right)^2\end{array}} \quad (5)$$

$$D_H\left(\tilde{A}_1,\tilde{A}_2\right) = \frac{1}{2}\left(\begin{array}{l}\left|\frac{1+(\mu_1^L)^2-(\nu_1^L)^2+1+(\mu_1^U)^2-(\nu_1^U)^2}{2}\alpha_1 - \frac{1+(\mu_2^L)^2-(\nu_2^L)^2+1+(\mu_2^U)^2-(\nu_2^U)^2}{2}\alpha_2\right| \\ +\frac{1}{2}\left|\frac{1+(\mu_1^L)^2-(\nu_1^L)^2+1+(\mu_1^U)^2-(\nu_1^U)^2}{2}\sigma_1 - \frac{1+(\mu_2^L)^2-(\nu_2^L)^2+1+(\mu_2^U)^2-(\nu_2^U)^2}{2}\sigma_2\right| \\ +\left|\frac{1+(\pi_1^L)^2-(\pi_2^L)^2+1+(\pi_1^U)^2-(\pi_2^U)^2}{2}\right|\end{array}\right) \quad (6)$$

Theorem 1: Let $\tilde{A}_1 = \langle(\alpha_1,\sigma_1);[\mu_1^L,\mu_1^U],[\nu_1^L,\nu_1^U]\rangle$, $\tilde{A}_2 = \langle(\alpha_2,\sigma_2);[\mu_2^L,\mu_2^U],[\nu_2^L,\nu_2^U]\rangle$, and $\tilde{A}_3 = \langle(\alpha_3,\sigma_3);[\mu_3^L,\mu_3^U],[\nu_3^L,\nu_3^U]\rangle$ be any three IVPFNs in X, then $D_E(A_1,A_2)$ satisfies the following properties:

(1) $D_E\left(\tilde{A}_1,\tilde{A}_2\right) \geq 0$, only if $\tilde{A}_1 = \tilde{A}_2$, then $D_E\left(\tilde{A}_1,\tilde{A}_2\right) = 0$;

(2) $D_E\left(\tilde{A}_1,\tilde{A}_2\right) = D_E\left(\tilde{A}_2,\tilde{A}_1\right)$;

(3) $\tilde{A}_1 = \langle(\alpha_1,\sigma_1);[\mu_1^L,\mu_1^U],[\nu_1^L,\nu_1^U]\rangle$ be an IVPFN, $D_E\left(\tilde{A}_1,\tilde{A}_3\right) \leq D_E\left(\tilde{A}_1,\tilde{A}_2\right)+D_E\left(\tilde{A}_2,\tilde{A}_3\right)$.

Proof: Based on the operational rules of IVPFNs in definition 8, we know that (1), (2) are right, (3) need be proved as follows:

Since we can get

$$(D(A_1,A_2)+D(A_2,A_3))^2$$
$$= \left(\begin{array}{l}\frac{1}{2}\sqrt{\begin{array}{l}\left(\frac{1+(\mu_1^L)^2-(\nu_1^L)^2+1+(\mu_1^U)^2-(\nu_1^U)^2}{2}\alpha_1 - \frac{1+(\mu_2^L)^2-(\nu_2^L)^2+1+(\mu_2^U)^2-(\nu_2^U)^2}{2}\alpha_2\right)^2 \\ +\frac{1}{2}\left(\frac{1+(\mu_1^L)^2-(\nu_1^L)^2+1+(\mu_1^U)^2-(\nu_1^U)^2}{2}\sigma_1 - \frac{1+(\mu_2^L)^2-(\nu_2^L)^2+1+(\mu_2^U)^2-(\nu_2^U)^2}{2}\sigma_2\right)^2 \\ +\left(\frac{1+(\pi_1^L)^2-(\pi_2^L)^2+1+(\pi_1^U)^2-(\pi_2^U)^2}{2}\right)^2\end{array}} \\ +\frac{1}{2}\sqrt{\begin{array}{l}\left(\frac{1+(\mu_2^L)^2-(\nu_2^L)^2+1+(\mu_2^U)^2-(\nu_2^U)^2}{2}\alpha_2 - \frac{1+(\mu_3^L)^2-(\nu_3^L)^2+1+(\mu_3^U)^2-(\nu_3^U)^2}{2}\alpha_2\right)^2 \\ +\frac{1}{2}\left(\frac{1+(\mu_2^L)^2-(\nu_2^L)^2+1+(\mu_2^U)^2-(\nu_2^U)^2}{2}\sigma_2 - \frac{1+(\mu_3^L)^2-(\nu_3^L)^2+1+(\mu_3^U)^2-(\nu_3^U)^2}{2}\sigma_3\right)^2 \\ +\left(\frac{1+(\pi_2^L)^2-(\pi_3^L)^2+1+(\pi_2^U)^2-(\pi_3^U)^2}{2}\right)^2\end{array}}\end{array}\right)^2$$

According to the above formula, we can get

$$\begin{aligned}&(D(A_1,A_2)+D(A_2,A_3))^2 \\ &= \frac{1}{4}\left((\Gamma_1\alpha_1-\Gamma_2\alpha_2)^2+\frac{1}{2}(\Gamma_1\sigma_1-\Gamma_2\sigma_2)^2+(\Phi_{12})^2\right) \\ &\quad+\frac{1}{4}\left((\Gamma_2\alpha_2-\Gamma_3\alpha_3)^2+\frac{1}{2}(\Gamma_2\sigma_2-\Gamma_3\sigma_3)^2+(\Phi_{23})^2\right) \\ &\quad+\frac{1}{2}\left(\begin{array}{l}\sqrt{(\Gamma_1\alpha_1-\Gamma_2\alpha_2)^2+\frac{1}{2}(\Gamma_1\sigma_1-\Gamma_2\sigma_2)^2+(\Phi_{12})^2} \\ \times\sqrt{(\Gamma_2\alpha_2-\Gamma_3\alpha_3)^2+\frac{1}{2}(\Gamma_2\sigma_2-\Gamma_3\sigma_3)^2+(\Phi_{23})^2}\end{array}\right)\end{aligned}$$

where

$$\begin{aligned}\Gamma_1 &= \frac{1+\left(\mu_1^L\right)^2-\left(\nu_1^L\right)^2+1+\left(\mu_1^U\right)^2-\left(\nu_1^U\right)^2}{2}, \\ \Gamma_2 &= \frac{1+\left(\mu_2^L\right)^2-\left(\nu_2^L\right)^2+1+\left(\mu_2^U\right)^2-\left(\nu_2^U\right)^2}{2}, \\ \Gamma_3 &= \frac{1+\left(\mu_3^L\right)^2-\left(\nu_3^L\right)^2+1+\left(\mu_3^U\right)^2-\left(\nu_3^U\right)^2}{2}, \\ \Phi_{12} &= \frac{1+\left(\pi_1^L\right)^2-\left(\pi_2^L\right)^2+1+\left(\pi_1^U\right)^2-\left(\pi_2^U\right)^2}{2}, \\ \Phi_{23} &= \frac{1+\left(\pi_2^L\right)^2-\left(\pi_3^L\right)^2+1+\left(\pi_2^U\right)^2-\left(\pi_3^U\right)^2}{2}, \\ \Phi_{13} &= \frac{1+\left(\pi_1^L\right)^2-\left(\pi_3^L\right)^2+1+\left(\pi_1^U\right)^2-\left(\pi_3^U\right)^2}{2}.\end{aligned}$$

Based on the above formula's expansion, we can obtain

$$\begin{aligned}&(D(A_1,A_2)+D(A_2,A_3))^2 \\ &\geq \frac{1}{4}\left((\Gamma_1\alpha_1-\Gamma_2\alpha_2)^2+\frac{1}{2}(\Gamma_1\sigma_1-\Gamma_2\sigma_2)^2+(\Phi_{12})^2\right) \\ &\quad+\frac{1}{4}\left((\Gamma_2\alpha_2-\Gamma_3\alpha_3)^2+\frac{1}{2}(\Gamma_2\sigma_2-\Gamma_3\sigma_3)^2+(\Phi_{23})^2\right) \\ &\quad+\frac{1}{2}\left(\begin{array}{l}(\Gamma_1\alpha_1-\Gamma_2\alpha_2)\times(\Gamma_2\alpha_2-\Gamma_3\alpha_3)+\frac{1}{2}(\Gamma_1\sigma_1-\Gamma_2\sigma_2) \\ \times(\Gamma_2\sigma_2-\Gamma_3\sigma_3)+(\Phi_{12})\times(\Phi_{23})\end{array}\right)\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{4}\begin{pmatrix} (\Gamma_1\alpha_1-\Gamma_2\alpha_2)^2+(\Gamma_2\alpha_2-\Gamma_3\alpha_3)^2 \\ +2(\Gamma_1\alpha_1-\Gamma_2\alpha_2)\times(\Gamma_2\alpha_2-\Gamma_3\alpha_3) \end{pmatrix} \\
&\quad +\frac{1}{4}\begin{pmatrix} \frac{1}{2}(\Gamma_1\sigma_1-\Gamma_2\sigma_2)^2+\frac{1}{2}(\Gamma_2\sigma_2-\Gamma_3\sigma_3)^2 \\ +(\Gamma_1\sigma_1-\Gamma_2\sigma_2)(\Gamma_2\sigma_2-\Gamma_3\sigma_3) \end{pmatrix} \\
&\quad +\frac{1}{4}\left((\Phi_{12})^2+(\Phi_{23})^2+2(\Phi_{12})\times(\Phi_{23})\right) \\
&\geq \frac{1}{4}(\Gamma_1\alpha_1-\Gamma_2\alpha_2+\Gamma_2\alpha_2-\Gamma_3\alpha_3)^2 \\
&\quad +\frac{1}{8}(\Gamma_1\sigma_1-\Gamma_2\sigma_2+\Gamma_2\sigma_2-\Gamma_3\sigma_3)^2+\frac{1}{4}(\Phi_{12}+\Phi_{23})^2 \\
&\geq \frac{1}{4}(\Gamma_1\alpha_1-\Gamma_3\alpha_3)^2+\frac{1}{8}(\Gamma_1\sigma_1-\Gamma_3\sigma_3)^2+\frac{1}{4}(\Phi_{13})^2 \\
&= D(A_1, A_3)^2
\end{aligned}
$$

The proof is completed.

So $D\left(\tilde{A}_1, \tilde{A}_3\right) \leq D\left(\tilde{A}_1, \tilde{A}_2\right)+D\left(\tilde{A}_2, \tilde{A}_3\right)$ is maintained, then we can infer that the $D_E\left(\tilde{A}_1, \tilde{A}_2\right)$ is kept.

Similarly, we can proof that the $D_H\left(\tilde{A}_1, \tilde{A}_2\right)$ is also kept.

In formula (5) and (6), when $\left\langle\left[\mu_1^L, \mu_1^U\right], \left[\nu_1^L, \nu_1^U\right]\right\rangle = \langle[1, 1], [0, 0]\rangle$, $\left\langle\left[\mu_2^L, \mu_2^U\right], \left[\nu_2^L, \nu_2^U\right]\right\rangle = \langle[1, 1], [0, 0]\rangle$, the IVPNFNs $\tilde{A}_1$ and $\tilde{A}_2$ are reduced to two NFs, the distance between IVPNFNs is transferred to the distance between NFs.

V. SOME AGGREGATION OPERATORS FOR IVPNFNs

There are some aggregation operators for IVPNFNs are presented, including interval-valued Pythagorean normal fuzzy weighed averaging operators and their generalized form.

A. INTERVAL-VALUED PYTHAGOREAN NORMAL FUZZY WEIGHTED AVERAGING OPERATORS

Based on the operational rules of IVPNFs, the weighed averaging operators for IVPNFs are presented as follows:

Definition 12: Let $\tilde{A}_i = \left\langle(\alpha_i, \sigma_i); \left[\mu_i^L, \mu_i^U\right], \left[\nu_i^L, \nu_i^U\right]\right\rangle$ be a collection of IVPNFN, $W = (w_1, w_2, \ldots, w_n)$ be a weight vector of $\tilde{A}_i$, and $w_i \geq 0$, $\sum_{i=1}^n w_i = 1$. Then the interval-valued Pythagorean normal fuzzy weighed averaging (IVPNFWA) operator can be defined as

$$IVPNFWA\left(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n\right) = \sum_{i=1}^{n} w_i\tilde{A}_i \; (i = 1, 2, \ldots, n) \tag{7}$$

Theorem 2: Let $\tilde{A}_i = \left\langle(\alpha_i, \sigma_i); \left[\mu_i^L, \mu_i^U\right], \left[\nu_i^L, \nu_i^U\right]\right\rangle$ be a collection of IVPNFN, then the aggregated value using IVPNFWA operator is still an IVPNFN, that is

$$IVPNFWA\left(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n\right) = \left(\begin{array}{l} \left(\sum_{i=1}^n w_i\alpha_i, \sum_{i=1}^n w_i\sigma_i\right); \\ \left[\begin{array}{l} \sqrt{1-\prod_{i=1}^n \left(1-\left(\mu_i^L\right)^2\right)^{w_i}}, \\ \sqrt{1-\prod_{i=1}^n \left(1-\left(\mu_i^U\right)^2\right)^{w_i}} \end{array}\right], \\ \left[\prod_{i=1}^n \left(\nu_i^L\right)^{w_i}, \prod_{i=1}^n \left(\nu_i^U\right)^{w_i}\right] \end{array}\right) \tag{8}$$

Proof: We use the method of mathematical induction to prove the theorem 2 as follows:

If $n = 2$, then

$$IVPNFWA\left(\tilde{A}_1, \tilde{A}_2\right)$$

$$= w_1\tilde{A}_1 \oplus w_1\tilde{A}_1$$

$$w_1\tilde{A}_1 = \left(\begin{array}{l} (w_1\alpha_1, w_1\sigma_1); \\ \left[\sqrt{1-\left(1-\left(\mu_1^L\right)^2\right)^{w_1}}, \sqrt{1-\left(1-\left(\mu_1^U\right)^2\right)^{w_1}}\right], \\ \left[\left(\nu_1^L\right)^{w_1}, \left(\nu_1^U\right)^{w_1}\right] \end{array}\right)$$

$$w_2\tilde{A}_2 = \left(\begin{array}{l} (w_2\alpha_2, w_2\sigma_2); \\ \left[\sqrt{1-\left(1-\left(\mu_2^L\right)^2\right)^{w_2}}, \sqrt{1-\left(1-\left(\mu_2^U\right)^2\right)^{w_2}}\right], \\ \left[\left(\nu_2^L\right)^{w_2}, \left(\nu_2^U\right)^{w_2}\right] \end{array}\right)$$

$$w_1\tilde{A}_1 \oplus w_2\tilde{A}_2 = \left(\begin{array}{l} (w_1\alpha_1+w_2\alpha, w_1\sigma_2+w_2\sigma_2); \\ \left[\begin{array}{l} \sqrt{\begin{array}{l}\left(1-\left(1-\left(\mu_1^L\right)^2\right)^{w_1}\right)+\left(1-\left(1-\left(\mu_2^L\right)^2\right)^{w_2}\right) \\ -\left(1-\left(1-\left(\mu_1^L\right)^2\right)^{w_1}\right)\cdot\left(1-\left(1-\left(\mu_2^L\right)^2\right)^{w_2}\right)\end{array}}, \\ \sqrt{\begin{array}{l}\left(1-\left(1-\left(\mu_1^U\right)^2\right)^{w_1}\right)+\left(1-\left(1-\left(\mu_2^U\right)^2\right)^{w_2}\right) \\ -\left(1-\left(1-\left(\mu_1^U\right)^2\right)^{w_1}\right)\cdot\left(1-\left(1-\left(\mu_2^U\right)^2\right)^{w_2}\right)\end{array}} \end{array}\right], \\ \left[\left(\nu_1^L\right)^{w_1}\left(\nu_2^L\right)^{w_2}, \left(\nu_1^U\right)^{w_1}\left(\nu_2^U\right)^{w_2}\right] \end{array}\right)$$

$$= \left(\begin{array}{l} (w_1\alpha_1+w_2\alpha, w_1\sigma_2+w_2\sigma_2); \\ \left[\begin{array}{l} \sqrt{1-\left(1-\left(\mu_1^L\right)^2\right)^{w_1}\left(1-\left(\mu_2^L\right)^2\right)^{w_2}}, \\ \sqrt{1-\left(1-\left(\mu_1^u\right)^2\right)^{w_1}\left(1-\left(\mu_2^u\right)^2\right)^{w_2}} \end{array}\right], \\ \left[\left(\nu_1^L\right)^{w_1}\left(\nu_2^L\right)^{w_2}, \left(\nu_1^U\right)^{w_1}\left(\nu_2^U\right)^{w_2}\right] \end{array}\right)$$

We suppose that formula holds for $n = k$ $(k \geq 3)$, that is

$$IVPNFWA\left(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_k\right) = \left(\begin{array}{l} \left(\sum_{i=1}^k w_i\alpha_i, \sum_{i=1}^n w_i\sigma_i\right); \\ \left[\sqrt{1-\prod_{i=1}^k \left(1-\left(\mu_i^L\right)^2\right)^{w_i}}, \sqrt{1-\prod_{i=1}^k \left(1-\left(\mu_i^U\right)^2\right)^{w_i}}\right], \\ \left[\prod_{i=1}^k \left(\nu_i^L\right)^{w_i}, \prod_{i=1}^k \left(\nu_i^U\right)^{w_i}\right] \end{array}\right)$$

Then if $n = k+1$, we can use the operational rules of IVPNFNs to obtain the

$$IVPNFWA\left(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_k, \tilde{A}_{k+1}\right)$$

$$= \left(\begin{array}{l} \left(\sum_{i=1}^{k} w_i\alpha_i + w_{k+1}\alpha_{k+1}, \sum_{i=1}^{k} w_i\sigma_i + w_{k+1}\sigma_{k+1}\right); \\ \left[\begin{array}{l} \sqrt{ \begin{array}{l} 1-\prod_{i=1}^{k}\left(1-(\mu_i^L)^2\right)^{w_i} + 1-\left(1-(\mu_{k+1}^L)^2\right)^{w_{k+1}} - \\ \left(1-\prod_{i=1}^{k}\left(1-(\mu_i^L)^2\right)^{w_i}\right) \\ \left(1-\left(1-(\mu_{k+1}^L)^2\right)^{w_{k+1}}\right) \end{array} }, \\ \sqrt{ \begin{array}{l} 1-\prod_{i=1}^{k}\left(1-(\mu_i^U)^2\right)^{w_i} + 1-\left(1-(\mu_{k+1}^U)^2\right)^{w_{k+1}} - \\ \left(1-\prod_{i=1}^{k}\left(1-(\mu_i^U)^2\right)^{w_i}\right) \\ \left(1-\left(1-(\mu_{k+1}^U)^2\right)^{w_{k+1}}\right) \end{array} } \end{array} \right], \\ \left[\prod_{i=1}^{k}(\nu_i^L)^{w_i}(\nu_{k+1}^L)^{w_{k+1}}, \prod_{i=1}^{k}(\nu_i^U)^{w_i}(\nu_{k+1}^U)^{w_{k+1}}\right] \end{array} \right)$$

$$= \left(\begin{array}{l} \left(\sum_{i=1}^{k+1} w_i\alpha_i, \sum_{i=1}^{k+1} w_i\sigma_i\right); \\ \left[\begin{array}{l} \sqrt{ \begin{array}{l} 1+\left(1-\prod_{i=1}^{k}\left(1-(\mu_i^L)^2\right)^{w_i}-1\right) \\ \left(1-1+\left(1-(\mu_{k+1}^L)^2\right)^{w_{k+1}}\right) \end{array} }, \\ \sqrt{ \begin{array}{l} 1+\left(1-\prod_{i=1}^{k}\left(1-(\mu_i^U)^2\right)^{w_i}-1\right) \\ \left(1-1+\left(1-(\mu_{k+1}^U)^2\right)^{w_{k+1}}\right) \end{array} } \end{array} \right], \left[\begin{array}{l} \prod_{i=1}^{k+1}(\nu_i^L)^{w_i}, \\ \prod_{i=1}^{k+1}(\nu_{k+1}^U)^{w_i} \end{array} \right] \end{array} \right)$$

$$= \left(\begin{array}{l} \left(\sum_{i=1}^{k+1} w_i\alpha_i, \sum_{i=1}^{k+1} w_i\sigma_i\right); \\ \left[\begin{array}{l} \sqrt{1-\prod_{i=1}^{k+1}\left(1-(\mu_i^L)^2\right)^{w_i}}, \\ \sqrt{1-\prod_{i=1}^{k+1}\left(1-(\mu_i^U)^2\right)^{w_i}} \end{array} \right], \left[\prod_{i=1}^{k+1}(\nu_i^L)^{w_i}, \prod_{i=1}^{k+1}(\nu_{k+1}^U)^{w_i}\right] \end{array} \right)$$

The IVPNFWA operator satisfies the following three properties.

Theorem 3 (Idempotency): If all $\tilde{A}_i = \langle(\alpha_i, \sigma_i); [\mu_i^L, \mu_i^U], [\nu_i^L, \nu_i^U]\rangle (i = 1, 2, \ldots, n)$ are equal with $\tilde{A}_i = \tilde{A}$, then $IVPNFWA\left(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n\right) = \tilde{A}$.

Proof: Since $\alpha_i = \alpha$, $\sigma_i = \sigma$, $\mu_i^L = \mu^L$, $\mu_i^U = \mu^U$, $\nu_i^L = \nu^L$, $\nu_i^U = \nu^U$,

$$IVPNFWA\left(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n\right)$$

$$= \left(\begin{array}{l} \left(\sum_{i=1}^{n} w_i\alpha_i, \sum_{i=1}^{n} w_i\sigma_i\right); \left[\begin{array}{l} \sqrt{1-\prod_{i=1}^{n}\left(1-(\mu_i^L)^2\right)^{w_i}}, \\ \sqrt{1-\prod_{i=1}^{n}\left(1-(\mu_i^U)^2\right)^{w_i}} \end{array} \right], \\ \left[\prod_{i=1}^{n}(\nu_i^L)^{w_i}, \prod_{i=1}^{n}(\nu_i^U)^{w_i}\right] \end{array} \right)$$

$$= \left(\begin{array}{l} \left(\alpha\sum_{i=1}^{n} w_i, \sigma\sum_{i=1}^{n} w_i\right); \left[\begin{array}{l} \sqrt{1-\left(1-(\mu_i^L)^2\right)^{\sum_{i=1}^{n} w_i}}, \\ \sqrt{1-\left(1-(\mu_i^U)^2\right)^{\sum_{i=1}^{n} w_i}} \end{array} \right], \\ \left[(\nu_i^L)^{\sum_{i=1}^{n} w_i}, (\nu_i^U)^{\sum_{i=1}^{n} w_i}\right] \end{array} \right)$$

Since $\sum_{i=1}^{n} w_i = 1$, we can get

$$IVPNFWA\left(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n\right) = \left(\begin{array}{l} (\alpha, \sigma); \\ \left[\sqrt{1-\left(1-(\mu_i^L)^2\right)}, \sqrt{1-\left(1-(\mu_i^U)^2\right)}\right], \\ \left[\nu_i^L, \nu_i^U\right] \end{array} \right) = \tilde{A}$$

Theorem 4 (Boundedness); Let $\tilde{A}_i = \left\langle(\alpha_{ij}, \sigma_{ij}); \left[\mu_{ij}^L, \mu_{ij}^U\right], \left[\nu_{ij}^L, \nu_{ij}^U\right]\right\rangle (i = 1, 2, \ldots, n; j = 1, 2, \ldots i_j)$ be a collection of IVPNFN,

If

$$\tilde{A}_i = \left\langle(\alpha_{ij}, \sigma_{ij}); \left[\mu_{ij}^L, \mu_{ij}^U\right], \left[\nu_{ij}^L, \nu_{ij}^U\right]\right\rangle \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots i_j)$$

$$\alpha^- = \min_{1\le i\le n, j=1,2,\ldots,i_j} \alpha_{ij}, \alpha^+ = \max_{1\le i\le n, j=1,2,\ldots,i_j} \alpha_{ij},$$
$$\sigma^- = \max_{1\le i\le n, j=1,2,\ldots,i_j} \sigma_{ij}, \sigma^+ = \min_{1\le i\le n, j=1,2,\ldots,i_j} \sigma_{ij}.$$
$$\mu^{L-} = \min_{1\le i\le n, j=1,2,\ldots,i_j} \mu_{ij}^L, \mu^{L+} = \max_{1\le i\le n, j=1,2,\ldots,i_j} \mu_{ij}^L,$$
$$\mu^{U-} = \min_{1\le i\le n, j=1,2,\ldots,i_j} \mu_{ij}^U, \mu^{U+} = \max_{1\le i\le n, j=1,2,\ldots,i_j} \mu_{ij}^U,$$
$$\nu^{L-} = \min_{1\le i\le n, j=1,2,\ldots,i_j} \nu_{ij}^L, \nu^{L+} = \max_{1\le i\le n, j=1,2,\ldots,i_j} \nu_{ij}^L,$$
$$\nu^{U-} = \min_{1\le i\le n, j=1,2,\ldots,i_j} \nu_{ij}^U, \nu^{U+} = \max_{1\le i\le n, j=1,2,\ldots,i_j} \nu_{ij}^U.$$

Then

$$\left\langle(\alpha^-, \sigma^-); \left[\mu^{L-}, \mu^{U-}\right], \left[\nu^{L+}, \nu^{U+}\right]\right\rangle$$
$$\le IVPNFWA\left(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n\right)$$
$$\le \left\langle(\alpha^+, \sigma^+); \left[\mu^{L+}, \mu^{U+}\right], \left[\nu^{L-}, \nu^{U-}\right]\right\rangle$$

Proof: Since $\nu^{L-} = \min_{1\le i\le n, j=1,2,\ldots,i_j} \nu_{ij}^L$, $\nu^{U-} = \min_{1\le i\le n, j=1,2,\ldots,i_j} \nu_{ij}^U$, $\nu^{L+} = \max_{1\le i\le n, j=1,2,\ldots,i_j} \nu_{ij}^L$, $\nu^{U+} = \max_{1\le i\le n, j=1,2,\ldots,i_j} \nu_{ij}^U$, we have $\nu^{L-} \le \nu_{ij}^L \le \nu^{L+}$, $\nu^{U-} \le \nu_{ij}^U \le \nu^{U+}$, and

$$\nu^{L-}+\nu^{U-}$$
$$= \prod_{i=1}^{n}\left(\nu^{L-}\right)^{w_i} + \prod_{i=1}^{n}\left(\nu^{U-}\right)^{w_i} \le \prod_{i=1}^{n}\left(\nu_{ij}^L\right)^{w_i} + \prod_{i=1}^{n}\left(\nu_{ij}^U\right)^{w_i}$$
$$\le \prod_{i=1}^{n}\left(\nu^{L+}\right)^{w_i} + \prod_{i=1}^{n}\left(\nu^{U+}\right)^{w_i} = \nu^{L+}+\nu^{U+}$$

Since $\mu^{L-} = \min_{1\le i\le n, j=1,2,\ldots,i_j} \mu_{ij}^L$, $\mu^{L+} = \max_{1\le i\le n, j=1,2,\ldots,i_j} \mu_{ij}^L$, $\mu^{U-} = \min_{1\le i\le n, j=1,2,\ldots,i_j} \mu_{ij}^U$,

$\mu^{U+} = \max_{1\le i\le n, j=1,2,\ldots,i_j} \mu_{ij}^{U}$, we have $\mu^{L-} \le \mu_{ij}^{L} \le \mu^{L+}$, $\mu^{U-} \le \mu_{ij}^{U} \le \mu^{U+}$ and

$$\begin{aligned}\mu^{L-}&+\mu^{U-}\\ &=\sqrt{1-\prod_{i=1}^{n}\left(1-(\mu^{L-})^2\right)^{w_i}}+\sqrt{1-\prod_{i=1}^{n}\left(1-(\mu^{U-})^2\right)^{w_i}}\\ &\le\sqrt{1-\prod_{i=1}^{n}\left(1-\left(\mu_{ij}^{L}\right)^2\right)^{w_i}}+\sqrt{1-\prod_{i=1}^{n}\left(1-\left(\mu_{ij}^{U}\right)^2\right)^{w_i}}\\ &\le\sqrt{1-\prod_{i=1}^{n}\left(1-(\mu^{L+})^2\right)^{w_i}}+\sqrt{1-\prod_{i=1}^{n}\left(1-(\mu^{U+})^2\right)^{w_i}}\\ &=\mu^{L+}+\mu^{U+}\end{aligned}$$

Since $\alpha^{-} = \min_{1\le i\le n, j=1,2,\ldots,i_j} \alpha_{ij}$, $\alpha^{+} = \max_{1\le i\le n, j=1,2,\ldots,i_j} \alpha_{ij}$, $\sigma^{-} = \max_{1\le i\le n, j=1,2,\ldots,i_j} \sigma_{ij}$, $\sigma^{+} = \min_{1\le i\le n, j=1,2,\ldots,i_j} \sigma_{ij}$, we have $\alpha^{-} \le \alpha_{ij} \le \alpha^{+}$, $\alpha^{+} \le \alpha_{ij} \le \alpha^{-}$ and

$$\sum_{i=1}^{n} w_i\alpha^{-} \le \sum_{i=1}^{n} w_i\alpha_{ij} \le \sum_{i=1}^{n} w_i\alpha^{+},\quad \sum_{i=1}^{n} w_i\sigma^{+} \le \sum_{i=1}^{n} w_i\sigma_{ij} \le \sum_{i=1}^{n} w_i\sigma^{-}$$

Therefore, based on the score function and accurate function, we can infer that

$$\begin{aligned}&\frac{\sum_{i=1}^{n} w_i\alpha^{-}}{2}\left(\frac{\left(\sqrt{1-\prod_{i=1}^{n}\left(1-(\mu^{L-})^2\right)^{w_i}}\right)^2+\left(\sqrt{1-\prod_{i=1}^{n}\left(1-(\mu^{U-})^2\right)^{w_i}}\right)^2}{2} +1-\frac{\left(\prod_{i=1}^{n}(\nu^{L+})^{w_i}\right)^2+\left(\prod_{i=1}^{n}(\nu^{U+})^{w_i}\right)^2}{2}\right)\\ &\le\frac{\sum_{i=1}^{n} w_i\alpha_{ij}}{2}\times\left(\frac{\left(\sqrt{1-\prod_{i=1}^{n}\left(1-\left(\mu_{ij}^{L}\right)^2\right)^{w_i}}\right)^2+\left(\sqrt{1-\prod_{i=1}^{n}\left(1-\left(\mu_{ij}^{U}\right)^2\right)^{w_i}}\right)^2}{2} +1-\frac{\left(\prod_{i=1}^{n}\left(\nu_{ij}^{L}\right)^{w_i}\right)^2+\left(\prod_{i=1}^{n}\left(\nu_{ij}^{U}\right)^{w_i}\right)^2}{2}\right)\\ &\le\frac{\sum_{i=1}^{n} w_i\alpha^{+}}{2}\times\left(\frac{\left(\sqrt{1-\prod_{i=1}^{n}\left(1-(\mu^{L+})^2\right)^{w_i}}\right)^2+\left(\sqrt{1-\prod_{i=1}^{n}\left(1-(\mu^{U+})^2\right)^{w_i}}\right)^2}{2} +1-\frac{\left(\prod_{i=1}^{n}(\nu^{L-})^{w_i}\right)^2+\left(\prod_{i=1}^{n}(\nu^{U-})^{w_i}\right)^2}{2}\right)\end{aligned}$$

That is

$$\begin{aligned}&\left\langle(\alpha^{-},\sigma^{-});\left[\mu^{L-},\mu^{U-}\right],\left[\nu^{L+},\nu^{U+}\right]\right\rangle\\ &\quad\le IVPNFWA\left(\tilde{A}_1,\tilde{A}_2,\ldots,\tilde{A}_n\right)\\ &\quad\le\left\langle(\alpha^{+},\sigma^{+});\left[\mu^{L+},\mu^{U+}\right],\left[\nu^{L-},\nu^{U-}\right]\right\rangle\end{aligned}$$

Theorem 5 (Monotonicity): Suppose $\left(\tilde{A}_1,\tilde{A}_2,\ldots,\tilde{A}_n\right)$ and $\left(\tilde{B}_1,\tilde{B}_2,\ldots,\tilde{B}_n\right)$ are two sets of IVPNFN, and $\tilde{A}_i = \left\langle(\alpha_{a_{ij}},\sigma_{a_{ij}});\left[\mu_{a_{ij}}^{L},\mu_{a_{ij}}^{U}\right],\left[\nu_{a_{ij}}^{L},\nu_{a_{ij}}^{U}\right]\right\rangle$ and $\tilde{B}_i = \left\langle(\alpha_{b_{ij}},\sigma_{b_{ij}});\left[\mu_{b_{ij}}^{L},\ \mu_{b_{ij}}^{U}\right],\ \left[\nu_{b_{ij}}^{L},\nu_{b_{ij}}^{U}\right]\right\rangle$, $(i = 1,2,\ldots,n;$ $j = 1,2,\ldots i_j)$ For any i, if there is $\alpha_{aij} \le \alpha_{bij}$, $\left(\mu_{aij}^{L}\right)^2+\left(\mu_{aij}^{U}\right)^2 \le \left(\mu_{bij}^{L}\right)^2+\left(\mu_{bij}^{U}\right)^2$ and $\left(\nu_{aij}^{L}\right)^2+\left(\nu_{aij}^{U}\right)^2 \ge \left(\nu_{bij}^{L}\right)^2+\left(\nu_{bij}^{U}\right)^2$ or $A_i \le B_i$ then

$$IVPNFWA\left(\tilde{A}_1,\tilde{A}_2,\ldots,\tilde{A}_n\right)\le IVPNFWA\left(\tilde{B}_1,\tilde{B}_2,\ldots,\tilde{B}_n\right).$$

Proof: For any i, there is $\alpha_{aij} \le \alpha_{bij}$,
Therefore

$$\sum_{i}^{n}\alpha_{aij} \le \sum_{i}^{n}\alpha_{bij}$$

For any i, there is $\left(\mu_{aij}^{L}\right)^2+\left(\mu_{aij}^{U}\right)^2 \le \left(\mu_{bij}^{L}\right)^2+\left(\mu_{bij}^{U}\right)^2$,
Therefore

$$\begin{aligned}1-\left(\mu_{ai}^{L}\right)^2+1-\left(\mu_{ai}^{U}\right)^2 &\ge 1-\left(\mu_{bi}^{L}\right)^2+1-\left(\mu_{bi}^{U}\right)^2\\ &\times\prod_{i=1}^{n}\left(1-\left(\mu_{ai}^{L}\right)^2\right)^{w_i}+\prod_{i=1}^{n}\left(1-\left(\mu_{ai}^{U}\right)^2\right)^{w_i}\\ &\ge\prod_{i=1}^{n}\left(1-\left(\mu_{bi}^{L}\right)^2\right)^{w_i}+\prod_{i=1}^{n}\left(1-\left(\mu_{bi}^{U}\right)^2\right)^{w_i}\end{aligned}$$

and

$$\begin{aligned}&\sqrt{1-\prod_{i=1}^{n}\left(1-(\mu_{ai}^{L})^2\right)^{w_i}+\prod_{i=1}^{n}\left(1-\left(\mu_{ai}^{U}\right)^2\right)^{w_i}}\\ &\quad\le\sqrt{1-\prod_{i=1}^{n}\left(1-(\mu_{bi}^{L})^2\right)^{w_i}}+\sqrt{1-\prod_{i=1}^{n}\left(1-(\mu_{bi}^{U})^2\right)^{w_i}}\end{aligned}$$

Since there is $\left(\nu_{aij}^{L}\right)^2+\left(\nu_{aij}^{U}\right)^2 \ge \left(\nu_{bij}^{L}\right)^2+\left(\nu_{bij}^{U}\right)^2$,
Therefore

$$1-\frac{\left(\prod_{i=1}^{n}\nu_{aij}^{L}\right)^2+\left(\prod_{i=1}^{n}\nu_{aij}^{U}\right)^2}{2}\le 1-\frac{\left(\prod_{i=1}^{n}\nu_{bij}^{L}\right)^2+\left(\prod_{i=1}^{n}\nu_{bij}^{U}\right)^2}{2},$$

And

$$\frac{\sum_{i=1}^{n}\alpha_{aij}}{2}\times\left(\frac{\left(\sqrt{1-\prod_{i=1}^{n}\left(1-(\mu_{ai}^{L})^{2}\right)^{w_i}}\right)^{2}+\left(\sqrt{1-\prod_{i=1}^{n}\left(1-(\mu_{ai}^{U})^{2}\right)^{w_i}}\right)^{2}}{2}+1-\frac{\left(\prod_{i=1}^{n}\nu_{aij}^{L}\right)^{2}+\left(\prod_{i=1}^{n}\nu_{aij}^{U}\right)^{2}}{2}\right)$$

$$\leq\frac{\sum_{i=1}^{n}\alpha_{bij}}{2}\left(\frac{\left(\sqrt{1-\prod_{i=1}^{n}\left(1-(\mu_{bi}^{L})^{2}\right)^{w_i}}\right)^{2}+\left(\sqrt{1-\prod_{i=1}^{n}\left(1-(\mu_{bi}^{U})^{2}\right)^{w_i}}\right)^{2}}{2}+1-\frac{\left(\prod_{i=1}^{n}\nu_{bij}^{L}\right)^{2}+\left(\prod_{i=1}^{n}\nu_{bij}^{U}\right)^{2}}{2}\right).$$

Therefore

$$IVPNFWA\left(\tilde{A}_1,\tilde{A}_2,\ldots,\tilde{A}_n\right)\leq IVPNFWA\left(\tilde{B}_1,\tilde{B}_2,\ldots,\tilde{B}_n\right),$$

B. INTERVAL-VALUED PYTHAGOREAN NORMAL FUZZY WEIGHTED GEOMETRIC OPERATOR

Definition 13: Let $\tilde{A}_i = \langle(\alpha_i,\sigma_i);[\mu_i^L,\mu_i^U],[\nu_i^L,\nu_i^U]\rangle$ $(i = 1,2,\ldots,n)$ be a collection of IVPNFN, $W = (w_1,w_2,\ldots,w_n)$ be a weight vector of $\tilde{A}_i$, and $w_i \geq 0$, $\sum_{i=1}^{n} w_i = 1$. Then the interval-valued Pythagorean normal fuzzy weighed geometric (IVPNFWG) operator can be defined as

$$IVPNFWG\left(\tilde{A}_1,\tilde{A}_2,\ldots,\tilde{A}_n\right)=\prod_{i=1}^{n}\tilde{A}_i^{w_i}\ (i=1,2,\ldots,n) \tag{9}$$

Theorem 6: Let $\tilde{A}_i = \langle(\alpha_i,\sigma_i);[\mu_i^L,\mu_i^U],[\nu_i^L,\nu_i^U]\rangle$ $(i = 1,2,\ldots,n)$ be a collection of IVPNFN, then the aggregated value of the IVPNFWG operator is still an IVPNFN, that is

$$IVPNFWG\left(\tilde{A}_1,\tilde{A}_2,\ldots,\tilde{A}_n\right)=\left(\begin{array}{l}\left(\prod_{i=1}^{n}\alpha_i^{w_i},\prod_{i=1}^{n}\sigma_i^{w_i}\right);\left[\prod_{i=1}^{n}(\mu_i^L)^{w_i},\prod_{i=1}^{n}(\mu_i^U)^{w_i}\right],\\ \left[\begin{array}{l}\sqrt{1-\prod_{i=1}^{n}\left(1-(\nu_i^L)^2\right)^{w_i}},\\ \sqrt{1-\prod_{i=1}^{n}\left(1-(\nu_i^U)^2\right)^{w_i}}\end{array}\right]\end{array}\right) \tag{10}$$

Proof: The method of mathematical induction is used to prove the theorem 6 as follows:

(1) If $n = 2$, then

$$IVPNFWG\left(\tilde{A}_1,\tilde{A}_2\right)=\tilde{A}_1^{w_1}\otimes\tilde{A}_2^{w_2}$$

$$\tilde{A}_1^{w_1}=\left(\begin{array}{l}(\alpha_1^{w_1},\sigma_1^{w_1});\left[(\mu_1^L)^{w_1},(\mu_1^U)^{w_1}\right],\\ \left[\sqrt{1-\left(1-(\nu_1^L)^2\right)^{w_1}},\sqrt{1-\left(1-(\nu_1^U)^2\right)^{w_1}}\right]\end{array}\right)$$

$$\tilde{A}_2^{w_2}=\left(\begin{array}{l}(\alpha_2^{w_2},\sigma_2^{w_2});\left[(\mu_2^L)^{w_2},(\mu_2^U)^{w_2}\right],\\ \left[\sqrt{1-\left(1-(\nu_2^L)^2\right)^{w_2}},\sqrt{1-\left(1-(\nu_2^U)^2\right)^{w_2}}\right]\end{array}\right)$$

$$\tilde{A}_1^{w_1}\otimes\tilde{A}_2^{w_2}=\left(\begin{array}{l}(\alpha_1^{w_1}\times\alpha_2^{w_3},\sigma_1^{w_1}\times\sigma_2^{w_2});\left[\begin{array}{l}(\mu_1^L)^{w_1}(\mu_2^L)^{w_2},\\ (\mu_1^U)^{w_1}(\mu_2^U)^{w_2}\end{array}\right],\\ \left[\begin{array}{l}\sqrt{\begin{array}{l}\left(1-\left(1-(\nu_1^L)^2\right)^{w_1}\right)+\left(1-\left(1-(\nu_2^L)^2\right)^{w_2}\right)\\ -\left(1-\left(1-(\nu_1^L)^2\right)^{w_1}\right)\cdot\left(1-\left(1-(\nu_2^L)^2\right)^{w_2}\right)\end{array}},\\ \sqrt{\begin{array}{l}\left(1-\left(1-(\nu_1^U)^2\right)^{w_1}\right)+\left(1-\left(1-(\nu_2^U)^2\right)^{w_2}\right)\\ -\left(1-\left(1-(\nu_1^U)^2\right)^{w_1}\right)\cdot\left(1-\left(1-(\nu_2^U)^2\right)^{w_2}\right)\end{array}}\end{array}\right]\end{array}\right)$$

$$=\left(\begin{array}{l}(\alpha_1^{w_1}\times\alpha_2^{w_3},\sigma_1^{w_1}\times\sigma_2^{w_2});\left[\begin{array}{l}(\mu_1^L)^{w_1}(\mu_2^L)^{w_2},\\ (\mu_1^U)^{w_1}(\mu_2^U)^{w_2}\end{array}\right],\\ \left[\begin{array}{l}\sqrt{1-\left(1-(\nu_1^L)^2\right)^{w_1}\left(1-(\nu_2^L)^2\right)^{w_2}},\\ \sqrt{1-\left(1-(\nu_1^u)^2\right)^{w_1}\left(1-(\nu_2^u)^2\right)^{w_2}}\end{array}\right]\end{array}\right)$$

(2) We suppose that formula holds for $n = k$ $(k \geq 3)$, that is

$$IVPNFWG\left(\tilde{A}_1,\tilde{A}_2,\ldots,\tilde{A}_k\right)=\left(\begin{array}{l}\left(\prod_{i=1}^{k}\alpha_i^{w_i},\prod_{i=1}^{k}\sigma_i^{w_i}\right);\\ \left[\prod_{i=1}^{k}(\mu_i^L)^{w_i},\prod_{i=1}^{k}(\mu_i^U)^{w_i}\right],\left[\begin{array}{l}\sqrt{1-\prod_{i=1}^{k}\left(1-(\nu_i^L)^2\right)^{w_i}},\\ \sqrt{1-\prod_{i=1}^{k}\left(1-(\nu_i^U)^2\right)^{w_i}}\end{array}\right]\end{array}\right)$$

(3) Then if $n = k+1$, we can use the operational rules of IVPNFNs to obtain the $IVPNFWG\left(\tilde{A}_1,\tilde{A}_2,\ldots,\tilde{A}_k,\tilde{A}_{k+1}\right)$, as shown at the bottom of the next page.

(4) Based on steps (1), (2) and (3), we can get Theorem 6 holds for any k.

According to Theorems 3, 4, 5, we can similarly prove the properties of idempotency, monotonicity and boundedness for IVPNFWG operator

C. GENERALIZED INTERVAL-VALUED PYTHAGOREAN NORMAL FUZZY WEIGHTED AVERAGING OPERATORS

As generalizations of the IVPNFNWA and IVPNFNWG operators, some generalized Interval-valued Pythagorean normal fuzzy weighed averaging operators are developed in the following.

Definition 14: Let $\tilde{A}_i = \langle(\alpha_i,\sigma_i);[\mu_i^L,\mu_i^U],[\nu_i^L,\nu_i^U]\rangle$ $(i = 1,2,\ldots,n)$ be a collection of IVPNFN,

$W=(w_1,w_2,\ldots,w_n)$ be a weight vector of $\tilde{A}_i$, and $w_i\geq 0$, $\sum_{i=1}^{n} w_i=1$, λ be a parameter and $\lambda\in(-\infty,0)\cup(0,+\infty)$ then

$$GIVPNFWA\left(\tilde{A}_1,\tilde{A}_2,\ldots,\tilde{A}_n\right)=\left(\sum_{i=1}^{n} w_i\tilde{A}_i^{\lambda}\right)^{1/\lambda}\quad(i=1,2,\ldots,n)\quad(11)$$

is called a generalized IVPNF weighted averaging (GIVPNFWA) operator.

Theorem 7: Let $\tilde{A}_i=\left\langle(\alpha_i,\sigma_i);\left[\mu_i^L,\mu_i^U\right],\left[\nu_i^L,\nu_i^U\right]\right\rangle$ $(i=1,2,\ldots,n)$ be a collection of the IVPNFN, based on the operational rules of IVPNF, the GIVPNFWA operator is still an IVPNFN, that is Eq. (12), as shown at the bottom of this page.

$$IVPNFWG\left(\tilde{A}_1,\tilde{A}_2,\ldots,\tilde{A}_k,\tilde{A}_{k+1}\right)=\left(\begin{array}{l}\left(\begin{array}{l}\prod_{i=1}^{k}\alpha_i^{w_i}\times\alpha_{k+1}^{w_{k+1}},\\ \prod_{i=1}^{k}\sigma_i^{w_i}\times\sigma_{k+1}^{w_{k+1}}\end{array}\right);\left[\begin{array}{l}\prod_{i=1}^{k}\left(\mu_i^L\right)^{w_i}\left(\mu_{k+1}^L\right)^{w_{k+1}},\\ \prod_{i=1}^{k}\left(\mu_i^U\right)^{w_i}\left(\mu_{k+1}^U\right)^{w_{k+1}}\end{array}\right],\\ \left[\begin{array}{l}\sqrt{\begin{array}{l}1-\prod_{i=1}^{k}\left(1-\left(\nu_i^L\right)^2\right)^{w_i}+1-\left(1-\left(\nu_{k+1}^L\right)^2\right)^{w_{k+1}}\\ -\left(1-\prod_{i=1}^{k}\left(1-\left(\nu_i^L\right)^2\right)^{w_i}\right)\left(1-\left(1-\left(\nu_{k+1}^L\right)^2\right)^{w_{k+1}}\right)\end{array}},\\ \sqrt{1-\prod_{i=1}^{k}\begin{array}{l}\left(1-\left(\nu_i^U\right)^2\right)^{w_i}+1-\left(1-\left(\nu_{k+1}^U\right)^2\right)^{w_{k+1}}\\ -\left(1-\prod_{i=1}^{k}\left(1-\left(\nu_i^U\right)^2\right)^{w_i}\right)\left(1-\left(1-\left(\nu_{k+1}^U\right)^2\right)^{w_{k+1}}\right)\end{array}}\end{array}\right]\end{array}\right)$$

$$=\left(\begin{array}{l}\left(\prod_{i=1}^{k+1}\alpha_i^{w_i},\prod_{i=1}^{k+1}\sigma_i^{w_i}\right);\left[\prod_{i=1}^{k+1}\left(\mu_i^L\right)^{w_i},\prod_{i=1}^{k+1}\left(\mu_{k+1}^U\right)^{w_i}\right],\\ \left[\begin{array}{l}\sqrt{1+\left(1-\prod_{i=1}^{k}\left(1-\left(\nu_i^L\right)^2\right)^{w_i}-1\right)\left(1-1+\left(1-\left(\nu_{k+1}^L\right)^2\right)^{w_{k+1}}\right)},\\ \sqrt{1+\left(1-\prod_{i=1}^{k}\left(1-\left(\nu_i^U\right)^2\right)^{w_i}-1\right)\left(1-1+\left(1-\left(\nu_{k+1}^U\right)^2\right)^{w_{k+1}}\right)}\end{array}\right]\end{array}\right)$$

$$=\left(\begin{array}{l}\left(\prod_{i=1}^{k+1}\alpha_i^{w_i},\prod_{i=1}^{k+1}\sigma_i^{w_i}\right);\left[\prod_{i=1}^{k+1}\left(\mu_i^L\right)^{w_i},\prod_{i=1}^{k+1}\left(\mu_{k+1}^U\right)^{w_i}\right],\\ \left[\begin{array}{l}\sqrt{1-\prod_{i=1}^{k+1}\left(1-\left(\nu_i^L\right)^2\right)^{w_i}},\\ \sqrt{1-\prod_{i=1}^{k+1}\left(1-\left(\nu_i^U\right)^2\right)^{w_i}}\end{array}\right]\end{array}\right)$$

$$GIVPNFWA\left(\tilde{A}_1,\tilde{A}_2,\ldots,\tilde{A}_n\right)=\left(\begin{array}{l}\left(\begin{array}{l}\left(\sum_{i=1}^{n} w_i\alpha_i^{\lambda}\right)^{1/\lambda},\\ \left(\sum_{i=1}^{n} w_i\sigma_i^{\lambda}\right)^{1/\lambda}\end{array}\right),\left[\begin{array}{l}\left(\sqrt{1-\prod_{i=1}^{n}\left(1-\left(\left(\mu_i^L\right)^{\lambda}\right)^2\right)^{w_i}}\right)^{1/\lambda},\\ \left(\sqrt{1-\prod_{i=1}^{n}\left(1-\left(\left(\mu_i^U\right)^{\lambda}\right)^2\right)^{w_i}}\right)^{1/\lambda}\end{array}\right],\\ \left[\begin{array}{l}\sqrt{1-\left(1-\left(\prod_{i=1}^{n}\left(\sqrt{1-\left(1-\left(\nu_i^L\right)^2\right)^{\lambda}}\right)^{w_i}\right)^2\right)^{1/\lambda}},\\ \sqrt{1-\left(1-\left(\prod_{i=1}^{n}\left(\sqrt{1-\left(1-\left(\nu_i^U\right)^2\right)^{\lambda}}\right)^{w_i}\right)^2\right)^{1/\lambda}}\end{array}\right]\end{array}\right)\quad(12)$$

Proof: We use the mathematical induction method to prove the follow formula firstly:

$$\sum_{i=1}^{n} w_i\tilde{A}_i^\lambda = \left(\begin{array}{l}\left(\begin{array}{l}\sum_{i=1}^{n} w_i\alpha_i^\lambda,\\ \sum_{i=1}^{n} w_i\sigma_i^\lambda\end{array}\right);\left[\begin{array}{l}\sqrt{1-\prod_{i=1}^{n}\left(1-\left(\left(\mu_i^L\right)^\lambda\right)^2\right)^{w_i}},\\ \sqrt{1-\prod_{i=1}^{n}\left(1-\left(\left(\mu_i^U\right)^\lambda\right)^2\right)^{w_i}}\end{array}\right],\\ \left[\begin{array}{l}\prod_{i=1}^{n}\left(\sqrt{1-\left(1-\left(\nu_i^L\right)^2\right)^\lambda}\right)^{w_i},\\ \prod_{i=1}^{n}\left(\sqrt{1-\left(1-\left(\nu_i^U\right)^2\right)^\lambda}\right)^{w_i}\end{array}\right]\end{array}\right)$$

(1) When $n = 2$,

Since

$$w_1\tilde{A}_1^\lambda = \left(\begin{array}{l}\left(\begin{array}{l}w_1\alpha_1^\lambda,\\ w_1\sigma_1^\lambda\end{array}\right);\left[\begin{array}{l}\sqrt{1-\left(1-\left(\left(\mu_1^L\right)^\lambda\right)^2\right)^{w_1}},\\ \sqrt{1-\left(1-\left(\left(\mu_1^U\right)^\lambda\right)^2\right)^{w_1}}\end{array}\right],\\ \left[\begin{array}{l}\left(\sqrt{1-\left(1-\left(\nu_1^L\right)^2\right)^\lambda}\right)^{w_1},\\ \left(\sqrt{1-\left(1-\left(\nu_1^U\right)^2\right)^\lambda}\right)^{w_1}\end{array}\right]\end{array}\right),$$

$$w_2\tilde{A}_2^\lambda = \left(\begin{array}{l}\left(\begin{array}{l}w_2\alpha_2^\lambda,\\ w_2\sigma_2^\lambda\end{array}\right);\left[\begin{array}{l}\sqrt{1-\left(1-\left(\left(\mu_2^L\right)^\lambda\right)^2\right)^{w_1}},\\ \sqrt{1-\left(1-\left(\left(\mu_2^U\right)^\lambda\right)^2\right)^{w_1}}\end{array}\right],\\ \left[\begin{array}{l}\left(\sqrt{1-\left(1-\left(\nu_2^L\right)^2\right)^\lambda}\right)^{w_1},\\ \left(\sqrt{1-\left(1-\left(\nu_2^U\right)^2\right)^\lambda}\right)^{w_1}\end{array}\right]\end{array}\right),$$

Then $w_1A_1 \oplus w_2A_2$, as shown at the bottom of this page.

(2) Supposing $n = k,\ k > 2$, that is

$$\sum_{i=1}^{k} w_i\tilde{A}_i^\lambda = \left(\begin{array}{l}\left(\sum_{i=1}^{k} w_i\alpha_i^\lambda, \sum_{i=1}^{k} w_i\sigma_i^\lambda\right);\left[\begin{array}{l}\sqrt{1-\prod_{i=1}^{k}\left(1-\left(\left(\mu_i^L\right)^\lambda\right)^2\right)^{w_i}},\\ \sqrt{1-\prod_{i=1}^{k}\left(1-\left(\left(\mu_i^U\right)^\lambda\right)^2\right)^{w_i}}\end{array}\right],\\ \left[\begin{array}{l}\prod_{i=1}^{k}\left(\sqrt{1-\left(1-\left(\nu_i^L\right)^2\right)^\lambda}\right)^{w_i},\\ \prod_{i=1}^{k}\left(\sqrt{1-\left(1-\left(\nu_i^U\right)^2\right)^\lambda}\right)^{w_i}\end{array}\right]\end{array}\right)$$

$$w_1A_1 \oplus w_2A_2 = \left(\begin{array}{l}\left(\begin{array}{l}w_1\alpha_1^\lambda+w_2\alpha_2^\lambda,\\ w_1\sigma_1^\lambda+w_2\sigma_2^\lambda\end{array}\right);\left[\begin{array}{l}\sqrt{\begin{array}{l}\left(\sqrt{1-\left(1-\left(\left(\mu_1^L\right)^\lambda\right)^2\right)^{w_1}}\right)^2+\left(\sqrt{1-\left(1-\left(\left(\mu_2^L\right)^\lambda\right)^2\right)^{w_1}}\right)^2\\ -\left(\sqrt{1-\left(1-\left(\left(\mu_1^L\right)^\lambda\right)^2\right)^{w_1}}\right)^2\cdot\left(\sqrt{1-\left(1-\left(\left(\mu_2^L\right)^\lambda\right)^2\right)^{w_1}}\right)^2\end{array}},\\ \sqrt{\begin{array}{l}\left(\sqrt{1-\left(1-\left(\left(\mu_1^U\right)^\lambda\right)^2\right)^{w_1}}\right)^2+\left(\sqrt{1-\left(1-\left(\left(\mu_2^U\right)^\lambda\right)^2\right)^{w_1}}\right)^2\\ -\left(\sqrt{1-\left(1-\left(\left(\mu_1^U\right)^\lambda\right)^2\right)^{w_1}}\right)^2\cdot\left(\sqrt{1-\left(1-\left(\left(\mu_2^U\right)^\lambda\right)^2\right)^{w_1}}\right)^2\end{array}}\end{array}\right],\\ \left[\begin{array}{l}\left(\sqrt{1-\left(1-\left(\nu_1^L\right)^2\right)^\lambda}\right)^{w_1}\left(\sqrt{1-\left(1-\left(\nu_2^L\right)^2\right)^\lambda}\right)^{w_1},\\ \left(\sqrt{1-\left(1-\left(\nu_1^U\right)^2\right)^\lambda}\right)^{w_1}\left(\sqrt{1-\left(1-\left(\nu_2^U\right)^2\right)^\lambda}\right)^{w_1}\end{array}\right]\end{array}\right)$$

$$= \left(\begin{array}{l}\left(\sum_{i=1}^{2} w_i\alpha_i^\lambda, \sum_{i=1}^{2} w_i\sigma_i^\lambda\right);\left[\begin{array}{l}\sqrt{1-\prod_{i=1}^{2}\left(1-\left(\left(\mu_i^L\right)^\lambda\right)^2\right)^{w_i}},\\ \sqrt{1-\prod_{i=1}^{2}\left(1-\left(\left(\mu_i^U\right)^\lambda\right)^2\right)^{w_i}}\end{array}\right],\\ \left[\prod_{i=1}^{2}\left(\sqrt{1-\left(1-\left(\nu_i^L\right)^2\right)^\lambda}\right)^{w_i}, \prod_{i=1}^{2}\left(\sqrt{1-\left(1-\left(\nu_i^U\right)^2\right)^\lambda}\right)^{w_i}\right]\end{array}\right)$$

If $n = k+1$, according to the operational laws of IVPNFN, we can get $\sum_{i=1}^{k} w_i\tilde{A}_i^\lambda + w_{k+1}\tilde{A}_{ik+1}^\lambda$, as shown at the bottom of this page, then $\left(\sum_{i=1}^{k+1} w_i\tilde{A}_i^\lambda\right)^{1/\lambda}$, as shown at the bottom of this page.

(3) According to above steps, we can get Theorem 7 holds for any k.

Where, if $\lambda = 1$, the GIVPNFWA operator is reduced to the IVPNFWA operator. Furthermore, based on the Theorems 3, 4 and 5, we know that the GIVPNFWA operator has the properties of boundedness, and idempotency and monotonicity.

$$
\begin{aligned}
\sum_{i=1}^{k} w_i\tilde{A}_i^\lambda + w_{k+1}\tilde{A}_{ik+1}^\lambda &= w_1\tilde{A}_1^\lambda \oplus w_2\tilde{A}_2^\lambda \oplus \ldots \oplus w_k\tilde{A}_k^\lambda \oplus w_{k+1}\tilde{A}_{k+1}^\lambda \\
&= \left(\begin{array}{l}
\left(\begin{array}{l}
\sum_{i=1}^{k} w_i\alpha_i^\lambda + w_{k+1}\alpha_{k+1}^\lambda, \\
\sum_{i=1}^{k} w_i\sigma_i^\lambda + w_{k+1}\sigma_{k+1}^\lambda
\end{array}\right), \\
\left[\begin{array}{l}
\sqrt{\begin{array}{l}
\left(\sqrt{1-\prod_{i=1}^{k}\left(1-\left(\left(\mu_i^L\right)^\lambda\right)^2\right)^{w_i}}\right)^2 + \left(\sqrt{1-\left(1-\left(\left(\mu_{k+1}^L\right)^\lambda\right)^2\right)^{w_1}}\right)^2 \\
-\left(\sqrt{1-\prod_{i=1}^{k}\left(1-\left(\left(\mu_i^L\right)^\lambda\right)^2\right)^{w_i}}\right)^2 \cdot \left(\sqrt{1-\left(1-\left(\left(\mu_{k+1}^L\right)^\lambda\right)^2\right)^{w_1}}\right)^2
\end{array}}, \\
\sqrt{\begin{array}{l}
\left(\sqrt{1-\prod_{i=1}^{k}\left(1-\left(\left(\mu_i^U\right)^\lambda\right)^2\right)^{w_i}}\right)^2 + \left(\sqrt{1-\left(1-\left(\left(\mu_{k+1}^U\right)^\lambda\right)^2\right)^{w_1}}\right)^2 \\
-\left(\sqrt{1-\prod_{i=1}^{k}\left(1-\left(\left(\mu_i^U\right)^\lambda\right)^2\right)^{w_i}}\right)^2 \cdot \left(\sqrt{1-\left(1-\left(\left(\mu_{k+1}^U\right)^\lambda\right)^2\right)^{w_1}}\right)^2
\end{array}}
\end{array}\right], \\
\left[\begin{array}{l}
\prod_{i=1}^{k}\left(\sqrt{1-\left(1-\left(\nu_i^L\right)^2\right)^\lambda}\right)^{w_i}\left(\sqrt{1-\left(1-\left(\nu_{k+1}^L\right)^2\right)^\lambda}\right)^{w_1}, \\
\prod_{i=1}^{k}\left(\sqrt{1-\left(1-\left(\nu_i^U\right)^2\right)^\lambda}\right)^{w_i}\left(\sqrt{1-\left(1-\left(\nu_{k+1}^U\right)^2\right)^\lambda}\right)^{w_1}
\end{array}\right]
\end{array}\right) \\
&= \left(\begin{array}{l}
\left(\sum_{i=1}^{k+1} w_i\alpha_i^\lambda, \sum_{i=1}^{k+1} w_i\sigma_i^\lambda\right); \left[\begin{array}{l}
\sqrt{1-\prod_{i=1}^{k+1}\left(1-\left(\left(\mu_i^L\right)^\lambda\right)^2\right)^{w_i}}, \\
\sqrt{1-\prod_{i=1}^{k+1}\left(1-\left(\left(\mu_i^U\right)^\lambda\right)^2\right)^{w_i}}
\end{array}\right], \\
\left[\prod_{i=1}^{k+1}\left(\sqrt{1-\left(1-\left(\nu_i^L\right)^2\right)^\lambda}\right)^{w_i}, \prod_{i=1}^{k+1}\left(\sqrt{1-\left(1-\left(\nu_i^U\right)^2\right)^\lambda}\right)^{w_i}\right]
\end{array}\right)
\end{aligned}
$$

$$
\left(\sum_{i=1}^{k+1} w_i\tilde{A}_i^\lambda\right)^{1/\lambda} = \left(\begin{array}{l}
\left(\left(\sum_{i=1}^{k+1} w_i\alpha_i^\lambda\right)^{1/\lambda}, \left(\sum_{i=1}^{k+1} w_i\sigma_i^\lambda\right)^{1/\lambda}\right); \left[\begin{array}{l}
\left(\sqrt{1-\prod_{i=1}^{k+1}\left(1-\left(\left(\mu_i^L\right)^\lambda\right)^2\right)^{w_i}}\right)^{1/\lambda}, \\
\left(\sqrt{1-\prod_{i=1}^{k+1}\left(1-\left(\left(\mu_i^U\right)^\lambda\right)^2\right)^{w_i}}\right)^{1/\lambda}
\end{array}\right], \\
\left[\begin{array}{l}
\sqrt{1-\left(1-\left(\prod_{i=1}^{k+1}\left(\sqrt{1-\left(1-\left(\nu_i^L\right)^2\right)^\lambda}\right)^{w_i}\right)^2\right)^{1/\lambda}}, \\
\sqrt{1-\left(1-\left(\prod_{i=1}^{k+1}\left(\sqrt{1-\left(1-\left(\nu_i^U\right)^2\right)^\lambda}\right)^{w_i}\right)^2\right)^{1/\lambda}}
\end{array}\right]
\end{array}\right)
$$

D. GENERALIZED INTERVAL-VALUED PYTHAGOREAN NORMAL FUZZY WEIGHTED GEOMETRIC OPERATOR

Definition 15: Let $\tilde{A}_i = \langle(\alpha_i,\sigma_i);[\mu_i^L,\mu_i^U],[\nu_i^L,\nu_i^U]\rangle$ $(i=1,2,\ldots,n)$ be a collection of IVPNFN, $W=(w_1,w_2,\ldots,w_n)$ be a weight vector of $\tilde{A}_i$, and $w_i\geq 0$, $\sum_{i=1}^n w_i=1$, λ be a parameter and $\lambda\in(-\infty,0)\cup(0,+\infty)$ then

$$GIVPNFWG\left(\tilde{A}_1,\tilde{A}_2,\ldots,\tilde{A}_n\right)=\frac{1}{\lambda}\left(\prod_{i=1}^n\left(\lambda\tilde{A}_i\right)^{w_i}\right) \tag{13}$$

is called a generalized IVPNF weighted geometric (G IVPNFWG) operator.

Theorem 8: Let $\tilde{A}_i = \langle(\alpha_i,\sigma_i);[\mu_i^L,\mu_i^U],[\nu_i^L,\nu_i^U]\rangle$ $(i=1,2,\ldots,n)$ be a collection of the IVPNFN, the GIVPNFWG operator is still an IVPNFN, that is (14), as shown at the bottom of this page.

Proof: Firstly, we use the mathematical induction method to prove the follow formula:

$$\prod_{i=1}^n\left(\lambda\tilde{A}_i\right)^{w_i}=\left(\begin{array}{l}\left(\begin{array}{l}\prod_{i=1}^n(\lambda\alpha_i)^{w_i},\\ \prod_{i=1}^n(\lambda\sigma_i)^{w_i}\end{array}\right);\left[\begin{array}{l}\sqrt{1-\prod_{i=1}^n\left(1-\left(\left(\mu_i^L\right)^\lambda\right)^2\right)^{w_i}},\\ \sqrt{1-\prod_{i=1}^n\left(1-\left(\left(\mu_i^U\right)^\lambda\right)^2\right)^{w_i}}\end{array}\right],\\ \left[\prod_{i=1}^n\left(\sqrt{1-\left(1-\left(\nu_i^L\right)^2\right)^\lambda}\right)^{w_i},\prod_{i=1}^n\left(\sqrt{1-\left(1-\left(\nu_i^U\right)^2\right)^\lambda}\right)^{w_i}\right]\end{array}\right)$$

(1) When $n=2$,
Since

$$\left(\lambda\tilde{A}_1\right)^{w_1}=\left(\begin{array}{l}\left(\begin{array}{l}(\lambda\alpha_1)^{w_1},\\ (\lambda\sigma_1)^{w_1}\end{array}\right);\left[\begin{array}{l}\left(\sqrt{1-\left(1-\left(\mu_1^L\right)^2\right)^\lambda}\right)^{w_i},\\ \left(\sqrt{1-\left(1-\left(\mu_1^U\right)^2\right)^\lambda}\right)^{w_i}\end{array}\right],\\ \left[\sqrt{1-\left(1-\left(\left(\nu_1^L\right)^\lambda\right)^2\right)^{w_i}},\sqrt{1-\left(1-\left(\left(\nu_1^U\right)^\lambda\right)^2\right)^{w_i}}\right]\end{array}\right),$$

$$\left(\lambda\tilde{A}_2\right)^{w_2}=\left(\begin{array}{l}\left(\begin{array}{l}(\lambda\alpha_2)^{w_2},\\ (\lambda\sigma_2)^{w_2}\end{array}\right);\left[\begin{array}{l}\left(\sqrt{1-\left(1-\left(\mu_2^L\right)^2\right)^\lambda}\right)^{w_i},\\ \left(\sqrt{1-\left(1-\left(\mu_2^U\right)^2\right)^\lambda}\right)^{w_i}\end{array}\right],\\ \left[\sqrt{1-\left(1-\left(\left(\nu_2^L\right)^\lambda\right)^2\right)^{w_i}},\sqrt{1-\left(1-\left(\left(\nu_2^U\right)^\lambda\right)^2\right)^{w_i}}\right]\end{array}\right),$$

Then $\left(\lambda\tilde{A}_1\right)^{w_1}\otimes\left(\lambda\tilde{A}_2\right)^{w_2}$, as shown at the bottom of the next page.

(2) Supposing $n=k,\ k>2$, that is $\prod_{i=1}^k\left(\lambda\tilde{A}_i\right)^{w_i}$, as shown at the bottom of the next page.

If $n=k+1$, according to the operational laws of IVPNFN, we can get $\prod_{i=1}^n\left(\lambda\tilde{A}_i\right)^{w_i}\times\left(\lambda\tilde{A}_{k+1}\right)^{w_{k+1}}$, as shown at the bottom of page 51310, then $\frac{1}{\lambda}\left(\prod_{i=1}^n\left(\lambda\tilde{A}_i\right)^{w_i}\right)$, as shown at the bottom of page 51310.

(3) According to above steps, we can get Theorem 8 holds for any k.

Where, if $\lambda=1$, the GIVPNFWG operator is reduced to the IVPNFWG operator. The GIVPNFWG operator has the properties of boundedness, and idempotency and monotonicity.

VI. A MULTI- ATTRIBUTE DECISION MAKING METHOD BASED ON IVPNF INFORMATION

In the IVPNF information environment, let $A=\{A_1,A_2,\ldots,A_n\}$ represent n alternative sets, $C=\{C_1,C_2,\ldots,C_m\}m$ attribute sets, $w=\{w_1,w_2,\ldots,w_m\}$ weights of attributes and $\tilde{A}_{ij}=\left\langle(\alpha_{ij},\sigma_{ij});\left[\mu_{ij}^L,\mu_{ij}^U\right],\left[\nu_{ij}^L,\nu_{ij}^U\right]\right\rangle(i=1,2,\ldots,n;$ $j=1,2,\ldots,m)$ is a IVPNFN of alternative A_i in attribute C_j. Wherein, μ_{ij}^L and μ_{ij}^U are the lower limit and upper limit of the membership degree of alternative A_i to normal fuzzy numbers $(\alpha_{ij},\sigma_{ij})$ in attribute C_j, respectively;ν_{ij}^L and ν_{ij}^U are the lower limit and upper limit of the non-membership degree of alternative A_i to normal fuzzy numbers $(\alpha_{ij},\sigma_{ij})$ in attribute C_j, respectively;$\left[\mu_{ij}^L(x),\mu_{ij}^U(x)\right]\in[0,1]$, $\left[\nu_{ij}^L(x),\nu_{ij}^U(x)\right]\in[0,1]$, and $0\leq u_{ij}^U(x)^2+\nu_{ij}^U(x)^2\leq 1$. n alternative sets

$$GIVPNFWG\left(\tilde{A}_1,\tilde{A}_2,\ldots,\tilde{A}_n\right)=\left(\begin{array}{l}\left(\begin{array}{l}\frac{1}{\lambda}\prod_{i=1}^n(\lambda\alpha_i)^{w_i},\\ \frac{1}{\lambda}\prod_{i=1}^n(\lambda\sigma_i)^{w_i}\end{array}\right),\left[\begin{array}{l}\sqrt{1-\left(1-\left(\prod_{i=1}^n\left(\sqrt{1-\left(1-\left(\mu_i^L\right)^2\right)^\lambda}\right)^{w_i}\right)^2\right)^{1/\lambda}},\\ \sqrt{1-\left(1-\left(\prod_{i=1}^n\left(\sqrt{1-\left(1-\left(\mu_i^U\right)^2\right)^\lambda}\right)^{w_i}\right)^2\right)^{1/\lambda}}\end{array}\right],\\ \left[\left(\sqrt{1-\prod_{i=1}^n\left(1-\left(\left(\nu_i^L\right)^\lambda\right)^2\right)^{w_i}}\right)^{1/\lambda},\left(\sqrt{1-\prod_{i=1}^n\left(1-\left(\left(\nu_i^U\right)^\lambda\right)^2\right)^{w_i}}\right)^{1/\lambda}\right]\end{array}\right) \tag{14}$$

and m attribute sets constitute an $n\times m$ decision matrix $D=\left(\tilde{A}_{ij}\right)_{n\times m}$ to determine the decision results.

The steps of MADM in IVPNF information environment are given below:

Step 1: normalize the decision matrix.

To eliminate the influence of different dimensions on the decision results, the decision matrix $D=\left(\tilde{A}_{ij}\right)_{n\times m}$ is normalized into $\bar{D}=\left(\bar{\tilde{A}}_{ij}\right)_{n\times m}$; wherein, $\bar{\tilde{A}}_{ij}=\left\langle(\bar{\alpha}_{ij},\bar{\sigma}_{ij});\left[\bar{\mu}^L_{ij},\bar{\mu}^U_{ij}\right],\left[\bar{\nu}^L_{ij},\bar{\nu}^U_{ij}\right]\right\rangle$.

For benefit attributes [54]:

$$\bar{\alpha}_{ij}=\frac{\alpha_{ij}}{\max\limits_i(\alpha_{ij})},\quad \bar{\sigma}_{ij}=\frac{\sigma_{ij}}{\max\limits_i(\sigma_{ij})}\cdot\frac{\sigma_{ij}}{\alpha_{ij}},\quad \bar{\mu}^L_{ij}=\mu^L_{ij},\bar{\mu}^U_{ij}=\mu^U_{ij} \tag{15}$$

For cost attributes [54]:

$$\bar{\alpha}_{ij}=\frac{\min\limits_i(\alpha_{ij})}{\alpha_{ij}},\quad \bar{\sigma}_{ij}=\frac{\sigma_{ij}}{\max\limits_i(\sigma_{ij})}\cdot\frac{\sigma_{ij}}{\alpha_{ij}},\quad \bar{\nu}^L_{ij}=\nu^L_{ij},\bar{\nu}^U_{ij}=\nu^U_{ij} \tag{16}$$

Step 2: aggregate the values of alternative attributes.

On the basis of IVPNF information aggregation operators, the information set with attribute C_j in $\tilde{A}_i,\bar{\tilde{A}}_{ij}=\left\langle(\bar{\alpha}_{ij},\bar{\sigma}_{ij});\left[\bar{\mu}^L_{ij},\bar{\mu}^U_{ij}\right],\left[\bar{\nu}^L_{ij},\bar{\nu}^U_{ij}\right]\right\rangle$, is aggregated into $\bar{\tilde{A}}_i=\left\langle(\bar{\alpha}_i,\bar{\sigma}_i);\left[\bar{\mu}^L_i,\bar{\mu}^U_i\right],\left[\bar{\nu}^L_i,\bar{\nu}^U_i\right]\right\rangle$.

Step 3: determine the positive and negative ideal points of the alternatives, and then calculate the distances between each alternative and the two ideal points.

Let the positive ideal point be

$$\bar{\tilde{A}}^+=\left\langle\left(\max_{1\le i\le n}(\bar{\alpha}_{ij}),\ \min_{1\le i\le n}(\bar{\sigma}_{ij})\right);[1,1],[0,0]\right\rangle.$$

Let the negative ideal point be

$$\bar{\tilde{A}}^-=\left\langle\left(\min_{1\le i\le n}(\bar{\alpha}_{ij}),\ \max_{1\le i\le n}(\bar{\sigma}_{ij})\right);[0,0],[1,1]\right\rangle.$$

Then, the distances between each alternative and the two ideal points:

$$D_i^+=D_E\left(\bar{\tilde{A}}_i,\bar{\tilde{A}}^+\right);\ D_i^-=D_E\left(\bar{\tilde{A}}_i,\bar{\tilde{A}}^-\right)$$

Step 4: determine the ranking of alternatives.

Calculate the relative nearness $D_i^*=D_i^-/D_i^++D_i^-$

$$\left(\lambda\tilde{A}_1\right)^{w_1}\otimes\left(\lambda\tilde{A}_2\right)^{w_2}=\left(\begin{array}{l}\left(\begin{array}{l}(\lambda\alpha_1)^{w_1}(\lambda\alpha_2)^{w_2},\\(\lambda\sigma_1)^{w_1}(\lambda\sigma_2)^{w_2}\end{array}\right);\left[\begin{array}{l}\left(\sqrt{1-\left(1-(\mu_1^L)^2\right)^\lambda}\right)^{w_1}\left(\sqrt{1-\left(1-(\mu_2^L)^2\right)^\lambda}\right)^{w_1},\\\left(\sqrt{1-\left(1-(\mu_1^U)^2\right)^\lambda}\right)^{w_1}\left(\sqrt{1-\left(1-(\mu_2^U)^2\right)^\lambda}\right)^{w_1}\end{array}\right]\\\left[\begin{array}{l}\sqrt{\begin{array}{l}\left(\sqrt{1-\left(1-\left((\nu_1^L)^\lambda\right)^2\right)^{w_1}}\right)^2+\left(\sqrt{1-\left(1-\left((\nu_2^L)^\lambda\right)^2\right)^{w_1}}\right)^2\\-\left(\sqrt{1-\left(1-\left((\nu_1^L)^\lambda\right)^2\right)^{w_1}}\right)^2\cdot\left(\sqrt{1-\left(1-\left((\nu_2^L)^\lambda\right)^2\right)^{w_1}}\right)^2\end{array}},\\\sqrt{\begin{array}{l}\left(\sqrt{1-\left(1-\left((\nu_1^U)^\lambda\right)^2\right)^{w_1}}\right)^2+\left(\sqrt{1-\left(1-\left((\nu_2^U)^\lambda\right)^2\right)^{w_1}}\right)^2\\-\left(\sqrt{1-\left(1-\left((\nu_1^U)^\lambda\right)^2\right)^{w_1}}\right)^2\cdot\left(\sqrt{1-\left(1-\left((\nu_2^U)^\lambda\right)^2\right)^{w_1}}\right)^2\end{array}}\end{array}\right]\end{array}\right)$$

$$=\left(\begin{array}{l}\left(\begin{array}{l}\prod\limits_{i=1}^2(\lambda\alpha_i)^{w_i},\\\prod\limits_{i=1}^2(\lambda\sigma_i)^{w_i}\end{array}\right);\left[\begin{array}{l}\prod\limits_{i=1}^2\left(\sqrt{1-\left(1-(\mu_i^L)^2\right)^\lambda}\right)^{w_i},\\\prod\limits_{i=1}^2\left(\sqrt{1-\left(1-(\mu_i^U)^2\right)^\lambda}\right)^{w_i}\end{array}\right],\\\left[\sqrt{1-\prod\limits_{i=1}^2\left(1-\left((\nu_i^L)^\lambda\right)^2\right)^{w_i}},\sqrt{1-\prod\limits_{i=1}^2\left(1-\left((\nu_i^U)^\lambda\right)^2\right)^{w_i}}\right]\end{array}\right)$$

$$\prod_{i=1}^k\left(\lambda\tilde{A}_i\right)^{w_i}=\left(\begin{array}{l}\left(\prod\limits_{i=1}^k(\lambda\alpha_i)^{w_i},\prod\limits_{i=1}^k(\lambda\sigma_i)^{w_i}\right);\left[\begin{array}{l}\sqrt{1-\prod\limits_{i=1}^k\left(1-\left((\mu_i^L)^\lambda\right)^2\right)^{w_i}},\\\sqrt{1-\prod\limits_{i=1}^k\left(1-\left((\mu_i^U)^\lambda\right)^2\right)^{w_i}}\end{array}\right],\\\left[\prod\limits_{i=1}^k\left(\sqrt{1-\left(1-(\nu_i^L)^2\right)^\lambda}\right)^{w_i},\prod\limits_{i=1}^k\left(\sqrt{1-\left(1-(\nu_i^U)^2\right)^\lambda}\right)^{w_i}\right]\end{array}\right)$$

Rank the alternatives according to the value of D_i^*; the larger D_i^* is, the better the alternative will be.

VII. NUMERICAL EXAMPLE

A. DECISION FINDINGS

With the development of e-commerce platforms, online shopping has become a common consumption habit of consumers. A consumer intends to buy a mobile phone on an e-commerce platform. Five mobile phones are selected as alternatives, and the alternative set is $A = \{A_1, A_2, A_3, A_4, A_5\}$. Four attributes are considered, namely, the performance of mobile phone system (C_1), credibility of merchant (C_2), online satisfaction rate (C_3) and price preference (C_4), constituting the attribute set $C = \{C_1, C_2, C_3, C_4\}$. All are benefit attributes and their corresponding weights are $w = \{0.3, 0.25, 0.25, 0.2\}^T$. According to the decision information, the decision information matrix shown in Table 1 is constructed.

$$\prod_{i=1}^{n}\left(\lambda\tilde{A}_i\right)^{w_i}\times\left(\lambda\tilde{A}_{k+1}\right)^{w_{k+1}} = \left(\begin{array}{l} \left(\begin{array}{l}\prod_{i=1}^{k}(\lambda\alpha_i)^{w_i}\times(\lambda\alpha_{k+1})^{w_{k+1}},\\ \prod_{i=1}^{k}(\lambda\sigma_i)^{w_i}\times(\lambda\sigma_{k+1})^{w_{k+1}}\end{array}\right), \left[\begin{array}{l} \prod_{i=1}^{k}\left(\sqrt{1-\left(1-(\mu_i^L)^2\right)^{\lambda}}\right)^{w_i} \\ \left(\sqrt{1-\left(1-(\mu_{k+1}^L)^2\right)^{\lambda}}\right)^{w_1}, \\ \prod_{i=1}^{k}\left(\sqrt{1-\left(1-(\nu_i^U)^2\right)^{\lambda}}\right)^{w_i} \\ \left(\sqrt{1-\left(1-(\nu_{k+1}^U)^2\right)^{\lambda}}\right)^{w_1} \end{array}\right], \\ \left[\begin{array}{l} \sqrt{\begin{array}{l} \left(\sqrt{1-\prod_{i=1}^{k}\left(1-\left((\nu_i^L)^{\lambda}\right)^2\right)^{w_i}}\right)^2 + \left(\sqrt{1-\left(1-\left((\nu_{k+1}^L)^{\lambda}\right)^2\right)^{w_1}}\right)^2 \\ -\left(\sqrt{1-\prod_{i=1}^{k}\left(1-\left((\nu_i^L)^{\lambda}\right)^2\right)^{w_i}}\right)^2\cdot\left(\sqrt{1-\left(1-\left((\nu_{k+1}^L)^{\lambda}\right)^2\right)^{w_1}}\right)^2 \end{array}}, \\ \sqrt{\begin{array}{l} \left(\sqrt{1-\prod_{i=1}^{k}\left(1-\left((\nu_i^U)^{\lambda}\right)^2\right)^{w_i}}\right)^2 + \left(\sqrt{1-\left(1-\left((\nu_{k+1}^U)^{\lambda}\right)^2\right)^{w_1}}\right)^2 \\ -\left(\sqrt{1-\prod_{i=1}^{k}\left(1-\left((\nu_i^U)^{\lambda}\right)^2\right)^{w_i}}\right)^2\cdot\left(\sqrt{1-\left(1-\left((\nu_{k+1}^U)^{\lambda}\right)^2\right)^{w_1}}\right)^2 \end{array}} \end{array} \right] \end{array} \right)$$

$$= \left(\begin{array}{l} \prod_{i=1}^{k+1}(\lambda\alpha_i)^{w_i}, \prod_{i=1}^{k+1}(\lambda\sigma_i)^{w_i}; \\ \left[\prod_{i=1}^{k+1}\left(\sqrt{1-\left(1-(\mu_i^L)^2\right)^{\lambda}}\right)^{w_i}, \prod_{i=1}^{k+1}\left(\sqrt{1-\left(1-(\mu_i^U)^2\right)^{\lambda}}\right)^{w_i}\right], \\ \left[\sqrt{1-\prod_{i=1}^{k+1}\left(1-\left((\nu_i^L)^{\lambda}\right)^2\right)^{w_i}}, \sqrt{1-\prod_{i=1}^{k+1}\left(1-\left((\nu_i^U)^{\lambda}\right)^2\right)^{w_i}}\right] \end{array} \right)$$

$$\frac{1}{\lambda}\left(\prod_{i=1}^{n}\left(\lambda\tilde{A}_i\right)^{w_i}\right) = \left(\begin{array}{l} \left(\begin{array}{l} \frac{1}{\lambda}\prod_{i=1}^{k+1}(\lambda\alpha_i)^{w_i}, \\ \frac{1}{\lambda}\prod_{i=1}^{k+1}(\lambda\sigma_i)^{w_i} \end{array}\right); \left[\begin{array}{l} \sqrt{1-\left(1-\left(\prod_{i=1}^{k+1}\left(\sqrt{1-\left(1-(\mu_i^L)^2\right)^{\lambda}}\right)^{w_i}\right)^2\right)^{1/\lambda}}, \\ \sqrt{1-\left(1-\left(\prod_{i=1}^{k+1}\left(\sqrt{1-\left(1-(\mu_i^U)^2\right)^{\lambda}}\right)^{w_i}\right)^2\right)^{1/\lambda}} \end{array} \right], \\ \left[\begin{array}{l} \left(\sqrt{1-\prod_{i=1}^{k+1}\left(1-\left((\nu_i^L)^{\lambda}\right)^2\right)^{w_i}}\right)^{1/\lambda}, \\ \left(\sqrt{1-\prod_{i=1}^{k+1}\left(1-\left((\nu_i^U)^{\lambda}\right)^2\right)^{w_i}}\right)^{1/\lambda} \end{array} \right] \end{array} \right)$$

TABLE 1. Original decision matrix.

	C_1	C_2	C_3	C_4
A_1	<(9,0.8), ([0.6,0.7], [0.3,0.4]>	<(4,0.5), ([0.2,0.4], [0.45,0.65])>	<(80,7), ([0.15,0.25], [0.65,0.8])>	<(50,4), ([0.6,0.75], [0.3,0.45])>
A_2	<(8,0.7), ([0.7,0.9], [0.2,0.35])>	<(7,0.6), ([0.4,0.45], [0.5,0.6])>	<(90,8), ([0.4,0.55], [0.6,0.7])>	<(45,5), ([0.6,0.7], [0. 5,0.55])>
A_3	<(6,0.6), ([0.5,0.55], [0.5,0.6])>	<(9,0.8), ([0.7,0.75], [0.3,0.35])>	<(75,6.5), ([0.6,0.65], [0.3,0.4])>	<(60,6), ([0.3,0.5], [0.6,0.65])>
A_4	<(7,0.6), ([0.7,0.8], [0.35,0.4])>	<(8,0.7), ([0.5,0.6] [0.6,0.7])>	<(85,7.5), ([0.5,0.6], [0.7,0.75])>	<(70,7), ([0.7,0.8], [0.4,0.45])>
A_5	<(6,0.5), ([0.3,0.4], [0.55,0.7])>	<(7,0.6), ([0.5,0.55] [0.3,0.35])>	<(95,6), ([0.6,0.7], [0.3,0.45])>	<(65,6.5), ([0.6,0.65], [0.3,0.35])>

TABLE 2. Normalized decision matrix.

	C_1	C_2	C_3	C_4
A_1	<(1,0.089), ([0.6,0.7], [0.3,0.4]>	<(0.444,0.078), ([0.2,0.4],[0.45,0.65])>	<(0.842,0.077), ([0.15,0.25],[0.65,0.8])>	<(0.714,0.046), ([0.6,0.75],[0.3,0.45])>
A_2	<(0.889,0.077), ([0.7,0.9], [0.2,0.35])>	<(0.778,0.0.064), ([0.4,0.45], [0.5,0.6])>	<(0.947,0.089), ([0.4,0.55], [0.6,0.7])>	<(0.643,0.079), ([0.6,0.7], [0. 5,0.55])>
A_3	<(0.667,0.075), ([0.5,0.55], [0.5,0.6])>	<(1,0.089), ([0.7,0.75],[0.3,0.35])>	<(0.789,0.07), ([0.6,0.65], [0.3,0.4])>	<(0.857,0.086), ([0.3,0.5], [0.6,0.65])>
A_4	<(0.778,0.064), ([0.7,0.8], [0.35,0.4])>	<(0.889,0.077), ([0.5,0.6], [0.6,0.7])>	<(0.895,0.083), ([0.5,0.6], [0.7,0.75])>	<(1,0.1), ([0.7,0.8], [0.4,0.45])>
A_5	<(0.667,0.052), ([0.3,0.4], [0.55,0.7])>	<(0.778,0.064), ([0.5,0.55],[0.3,0.35])>	<(1,0.047), ([0.6,0.7], [0.3,0.45])>	<(0.929,0.093), ([0.6,0.65],[0.3,0.35])>

Step 1: normalize the data listed in Table 1 according to formula (15) and (16) for the normalized matrix $\bar{D} = \left(\bar{\tilde{A}}_{ij}\right)_{5\times 4}$, with the results shown in Table 2.

Step 2: aggregate the information in Table 2 with IVPNFWA operator to get the integrated IVPNF of each alternative.

$$\bar{\tilde{A}}_1 = <(0.764, 0.074), ([0.461, 0.59], [0.403, 0.55])>;$$
$$\bar{\tilde{A}}_2 = <(0.827, 0.077), ([0.561, 0.736], [0.398, 0.521])>;$$
$$\bar{\tilde{A}}_3 = <(0.819, 0.079), ([0.565, 0.632], [0.402, 0.481])>;$$
$$\bar{\tilde{A}}_4 = <(0.879, 0.079), ([0.618, 0.721], [0.489, 0.551])>;$$
$$\bar{\tilde{A}}_5 = <(0.83, 0.062), ([0.51, 0.586], [0.36, 0.459])>;$$

Step 3: determine the positive and negative ideal points of the alternative, and then calculate the distances between each alternative and the two ideal points:

$$\bar{\tilde{A}}^{+} = <(0.879, 0.062), ([1, 1], [0, 0])>;$$
$$\bar{\tilde{A}}^{-} = <(0.764, 0.079), ([0, 0], [1, 1])>;$$

The distance between each alternative and the positive ideal point:

$D_1^+ = 0.4839, D_2^+ = 0.3874, D_3^+ = 0.4051, D_4^+ = 0.3645, D_5^+ = 0.412$

The distance between each alternative and the negative ideal point:

$D_1^- = 0.4073, D_2^- = 0.5096, D_3^- = 0.4787, D_4^- = 0.5219, D_5^- = 0.4722$

Step 4: conduct calculation according to the nearness formula.

$D_1^* = 0.457, D_2^* = 0.5681, D_3^* = 0.5416, D_4^* = 0.5888, D_5^* = 0.534$

Ranking of 5 alternatives: $A_4 > A_2 > A_3 > A_5 > A_1$; therefore, alternative A_4 is the optimal choice.

B. COMPARATIVE ANALYSIS

Firstly, Euclidean distance and Hamming distance of IVPNF proposed in this paper are compared. As shown in Table 3, the ranking of alternatives with IVPNFW operator based on IVPNF Euclidean distance and Hamming distance is $A_4 > A_2 > A_3 > A_5 > A_1$. When such methods proposed other scholars as grey relational analysis, cosine similarity and project are applied the ranking, the result is the same, $A_4 > A_2 > A_3 > A_5 > A_1$. The ranking result calculated with score function in Definition 6 is also the same, $A_4 > A_2 > A_3 > A_5 > A_1$. What's more, the ranking result based on IVPNFWG operator, GIVPNFWA operator and GIVPNFWG operator presented by this paper under different distance measures are consistent. It can be concluded that Euclidean and Hamming distances of IVPNF proposed in this paper are effective.

Then, the four types of operators proposed in this paper are compared with existed IVPF information aggregation operators. When weighted interval-valued Pythagorean fuzzy extended Bonferroni mean (WIVPFEBM) in [36] is applied to IVPNF environment, and let the parameters p and q be 2, the result show that A_4 is the optimal choice while A_1 is the worst choice. When normal intuitionistic fuzzy Bonferroni mean operators proposed by Liu and Liu [47] is applied to IVPNF environment, the parameters p = q = 2, A_4 is

TABLE 3. Comparison of ranking results based on different distance measures.

	IVPNFWA	IVPNFWG	GIVPNFWA(λ=2)	GIVPNFWG(λ=2)
TOPSIS based on Euclidean distance	$A_4 > A_2 > A_3 > A_5 > A_1$	$A_4 > A_3 > A_2 > A_5 > A_1$	$A_4 > A_2 > A_3 > A_5 > A_1$	$A_4 > A_3 > A_2 > A_5 > A_1$
TOPSIS based on Hamming distance	$A_4 > A_2 > A_3 > A_5 > A_1$	$A_4 > A_3 > A_2 > A_5 > A_1$	$A_4 > A_2 > A_3 > A_5 > A_1$	$A_4 > A_3 > A_2 > A_5 > A_1$
Score function	$A_4 > A_2 > A_3 > A_5 > A_1$	$A_4 > A_3 > A_2 > A_5 > A_1$	$A_4 > A_2 > A_3 > A_5 > A_1$	$A_4 > A_3 > A_2 > A_5 > A_1$
Grey relational analysis[53]	$A_4 > A_2 > A_3 > A_5 > A_1$	$A_4 > A_3 > A_2 > A_5 > A_1$	$A_4 > A_2 > A_3 > A_5 > A_1$	$A_4 > A_3 > A_2 > A_5 > A_1$
Cosine similarity[54]	$A_4 > A_2 > A_3 > A_5 > A_1$	$A_4 > A_3 > A_2 > A_5 > A_1$	$A_4 > A_2 > A_3 > A_5 > A_1$	$A_4 > A_3 > A_2 > A_5 > A_1$
Projection method [15]	$A_4 > A_2 > A_3 > A_5 > A_1$	$A_4 > A_3 > A_2 > A_5 > A_1$	$A_4 > A_2 > A_3 > A_5 > A_1$	$A_4 > A_3 > A_2 > A_5 > A_1$

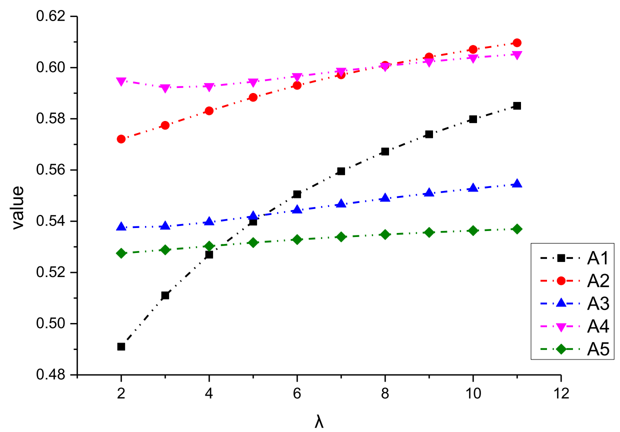

FIGURE 1. Changes in ranking based on GIVPNFWA operator and score function.

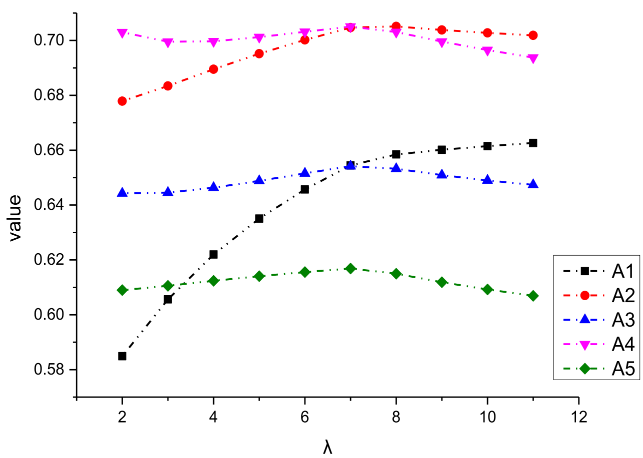

FIGURE 2. Changes in ranking based on GIVPNFWA operator and TOPSIS.

still the optimal choice. When normal Intuitionistic Fuzzy Hamacher weighted Heronian mean (NIFHWHM) proposed by Zhang *et al.* [51] is applied to IVPNF environment, and the parameters $\gamma = 1$, p = 2, q = 2, the optimal alternative is also A_4. It can be seen that the four aggregation operators proposed in this paper are effective and rational.

C. SENSITIVITY ANALYSIS

The influence of generalized parameter λ on the alternative ranking is analyzed. In Figure 1, the alternative ranking is calculated based on GIVPNFWA operator and score function. If $\lambda < 5$, the ranking of alternatives is $A_4 > A_2 > A_3 > A_5 > A_1$; if $\lambda = 5$, the ranking of alternatives changes into $A_4 > A_2 > A_3 > A_1 > A_5$; if $5 < \lambda < 8$, the ranking of alternatives changes again, $A_4 > A_2 > A_1 > A_3 > A_5$; if $8 < \lambda$, the alternatives are ranked in a new order, $A_2 > A_4 > A_1 > A_3 > A_5$, the optimal alternative changes from A_4 to A_2. Similarly, in Figure 2, the ranking of alternatives calculated based on GIVPNFWA operator and TOPSIS varies with the change of parameter λ. The above analysis shows that the generalized parameter λ has a great influence on the ranking of alternatives. The decision-makers may set the parameter λ according to the actual situation for the most reasonable ranking result, and then make appropriate decisions.

As suggested by the above analysis, the method proposed in this paper has the following advantages:

(1) It combines the concept of NFN and IVPFN, puts forward the concept of IVPNFN. IVPNFN interprets human activities and natural phenomena that obey normal distribution in real life, and describes the fuzzy information with

the sum of membership degree and non-membership degree greater than one, while the square sum less than 1; thus, it characterizes the fuzzy information in a wider way and is closer to human thinking in decision-making.

(2) Different ranking results of alternatives can be obtained flexibly with GIVPNFWA operator and GIVPNFWG operator based on parameter λ, and the decision-maker may determine decision result based on parameter λ according to own preferred. Therefore, the method proposed in this paper is featured by strong flexibility.

VIII. CONCLUSION

IVPFN characterizes interval-valued fuzzy information better than IVIFS, but IVPFN cannot describe normal distribution of social and natural phenomena. To solve such problems, this paper combines NFN and IVPFN for advantage complementary, puts forward the concept of IVPNFN, defines some basic theories of IVPNFN, proposes several aggregation operators in IVPFN information environment, and applies them.

There is still space for further development in this paper. For example, in terms of basic theory, the addition and subtraction between IVPFNs can be further proposed, and some measure methods for the similarity between IVPFNs can be extended. In terms of information aggregation, it can be extended to information aggregation model based on Bonferroni mean or Einstein; in terms of application, it can be extended to system control and logistics system.

CONFLICT OF INTEREST STATEMENT

There are no other relationships or activities that could appear to have influenced the submitted work.

REFERENCES

[1] L. A. Zadeh, "Fuzzy sets," *Inf. Control*, vol. 8, no. 3, pp. 338–353, 1965.

[2] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets Syst.*, vol. 20, no. 1, pp. 87–96, Aug. 1986.

[3] K. Atanassov and G. Gargov, "Interval valued Intuitionistic fuzzy-sets," *Fuzzy Sets Syst.*, vol. 31, no. 6, pp. 343–349, 1989.

[4] Z. M. Zhang, "Interval-valued intuitionistic hesitant fuzzy aggregation operators and their application in group decision-making," *J. Appl. Math.*, vol. 2013, Aug. 2013, Art. no. 670285.

[5] D. F. Li, J. X. Nan, and M. J. Zhang, "A ranking method of triangular intuitionistic fuzzy numbers and application to decision making," *Int. J. Comput. Int. Sys.*, vol. 3, no. 5, pp. 522–530, 2010.

[6] J. Q. Wang and Z. Zhang, "Aggregation operators on intuitionistic trapezoidal fuzzy number and its application to multi-criteria decision making problems," *J. Syst. Eng. Electron.*, vol. 20, no. 2, pp. 321–326, 2009.

[7] J.-Q. Wang, K.-J. Li, H.-Y. Zhang, and X.-H. Chen, "A score function based on relative entropy and its application in intuitionistic normal fuzzy multiple criteria decision making," *J. Intell. Fuzzy Syst.*, vol. 25, no. 3, pp. 567–576, 2013.

[8] R. R. Yager and A. M. Abbasov, "Pythagorean membership grades, complex numbers, and decision making," *Int. J. Intell. Syst.*, vol. 28, no. 5, pp. 436–452, May 2013.

[9] R. R. Yager, "Pythagorean membership grades in multicriteria decision making," *IEEE Trans. Fuzzy Syst.*, vol. 22, no. 4, pp. 958–965, Aug. 2014.

[10] A. Hussain, T. Mahmood, and M. I. Ali, "Rough pythagorean fuzzy ideals in semigroups," *Comput. Appl. Math.*, vol. 38, no. 2, Jun. 2019, Art. no. 67.

[11] R. Verma and J. M. Merigó, "On generalized similarity measures for Pythagorean fuzzy sets and their applications to multiple attribute decision-making," *Int. J. Intell. Syst.*, vol. 34, no. 10, pp. 2556–2583, Oct. 2019.

[12] Z. Hussian and M. S. Yang, "Distance and similarity measures of Pythagorean fuzzy sets based on the Hausdorff metric with application to fuzzy TOPSIS," *Int. J. Intell. Syst.*, vol. 34, no. 10, pp. 2633–2654, 2019.

[13] S. Xian, Y. Xiao, Z. Yang, Y. Li, and Z. Han, "A new trapezoidal pythagorean fuzzy linguistic entropic combined ordered weighted averaging operator and its application for enterprise location," *Int. J. Intell. Syst.*, vol. 33, no. 9, pp. 1880–1899, Sep. 2018.

[14] J.-J. Zhou and H.-L. Yang, "Multigranulation hesitant pythagorean fuzzy rough sets and its application in multi-attribute decision making," *J. Intell. Fuzzy Syst.*, vol. 36, no. 6, pp. 5631–5644, Jun. 2019.

[15] D. Liang, Z. Xu, and A. P. Darko, "Projection model for fusing the information of pythagorean fuzzy multicriteria group decision making based on geometric Bonferroni mean," *Int. J. Intell. Syst.*, vol. 32, no. 9, pp. 966–987, Sep. 2017.

[16] Tang, Wang, Lu, Wei, Wei, and Wei, "Dual hesitant Pythagorean fuzzy Heronian mean operators in multiple attribute decision making," *Mathematics*, vol. 7, no. 4, Art. no. 344, 2019.

[17] H. Garg, "Generalized Pythagorean fuzzy geometric aggregation operators using Einstein t-norm and t-conorm for multicriteria decision-making process," *Int. J. Intell. Syst.*, vol. 32, no. 6, pp. 597–630, Jun. 2017.

[18] A. Khan, S. Abdullah, M. Shakeel, F. Khan, N. Amin, and J. Luo, "A new ranking methodology for Pythagorean trapezoidal uncertain linguistic fuzzy sets based on Einstein operations," *Symmetry*, vol. 11, no. 3, p. 440, 2019.

[19] G. Wei, J. Wang, C. Wei, Y. Wei, and Y. Zhang, "Dual hesitant Pythagorean fuzzy Hamy mean operators in multiple attribute decision making," *IEEE Access*, vol. 7, pp. 86697–86716, 2019.

[20] L. Abdullah and W. R. W. Mohd, "Pythagorean Fuzzy Hamacher Choquet integral operators and their application to multi-criteria decision making," *J. Intell. Fuzzy Syst.*, vol. 37, no. 1, pp. 1259–1274, Jul. 2019.

[21] S. Z. Abbas, M. S. Ali Khan, S. Abdullah, H. Sun, and F. Hussain, "Cubic Pythagorean fuzzy sets and their application to multi-attribute decision making with unknown weight information," *J. Intell. Fuzzy Syst.*, vol. 37, no. 1, pp. 1529–1544, Jul. 2019.

[22] M. Shakeel, S. Abduulah, M. Shahzad, T. Mahmood, and N. Siddiqui, "Averaging aggregation operators with Pythagorean trapezoidal fuzzy numbers and their application to group decision making," *J. Intell. Fuzzy Syst.*, vol. 36, no. 2, pp. 1899–1915, Mar. 2019.

[23] X. Deng, J. Wang, G. Wei, and M. Lu, "Models for multiple attribute decision making with some 2-Tuple linguistic Pythagorean fuzzy Hamy mean operators," *Mathematics*, vol. 6, no. 11, p. 236, 2018.

[24] S. Xian, Y. Xiao, L. Li, and D. Yu, "Trapezoidal Pythagorean fuzzy linguistic entropic combined ordered weighted minkowski distance operator based on preference relations," *Int. J. Intell. Syst.*, vol. 34, no. 9, pp. 2196–2224, Sep. 2019.

[25] F. Teng, Z. Liu, and P. Liu, "Some power maclaurin symmetric mean aggregation operators based on Pythagorean fuzzy linguistic numbers and their application to group decision making," *Int. J. Intell. Syst.*, vol. 33, no. 9, pp. 1949–1985, Sep. 2018.

[26] H. Garg, "Novel neutrality operation–based Pythagorean fuzzy geometric aggregation operators for multiple attribute group decision analysis," *Int. J. Intell. Syst.*, vol. 34, no. 10, pp. 2459–2489, Oct. 2019.

[27] A. Sarkar and A. Biswas, "Multicriteria decision-making using Archimedean aggregation operators in Pythagorean hesitant fuzzy environment," *Int. J. Intell. Syst.*, vol. 34, no. 7, pp. 1361–1386, Jul. 2019.

[28] C. Jana, T. Senapati, and M. Pal, "Pythagorean fuzzy Dombi aggregation operators and its applications in multiple attribute decision-making," *Int. J. Intell. Syst.*, vol. 34, no. 9, pp. 2019–2038, Sep. 2018.

[29] X. Peng and Y. Yang, "Fundamental properties of interval-valued Pythagorean fuzzy aggregation operators," *Int. J. Intell. Syst.*, vol. 31, no. 5, pp. 444–487, May 2016.

[30] H. Garg, "New exponential operational laws and their aggregation operators for interval-valued pythagorean fuzzy multicriteria decision-making," *Int. J. Intell. Syst.*, vol. 33, no. 3, pp. 653–683, Mar. 2018.

[31] Y. Du, F. Hou, W. Zafar, Q. Yu, and Y. Zhai, "A novel method for multiattribute decision making with interval-valued Pythagorean fuzzy linguistic information," *Int. J. Intell. Syst.*, vol. 32, no. 10, pp. 1085–1112, Oct. 2017.

[32] X. Y. Tang, G. W. Wei, and H. Gao, "Models for multiple attribute decision making with interval-valued pythagorean fuzzy Muirhead mean operators and their application to green suppliers selection," *Informatica*, vol. 30, no. 1, pp. 153–186, 2019.

[33] W. Yang and Y. Pang, "Hesitant interval-valued Pythagorean fuzzy VIKOR method," *Int. J. Intell. Syst.*, vol. 34, no. 5, pp. 754–789, May 2019.

[34] J. Wang, G. Wei, and H. Gao, "Approaches to multiple attribute decision making with interval-valued 2-Tuple linguistic Pythagorean fuzzy information," *Mathematics*, vol. 6, no. 10, p. 201, 2018.
[35] D. Liang, A. P. Darko, and Z. Xu, "Interval-valued Pythagorean fuzzy extended Bonferroni mean for dealing with heterogenous relationship among attributes," *Int. J. Intell. Syst.*, vol. 33, no. 7, pp. 1381–1411, Jul. 2018.
[36] G. Wei, H. Garg, H. Gao, and C. Wei, "Interval-valued Pythagorean fuzzy Maclaurin symmetric mean operators in multiple attribute decision making," *IEEE Access*, vol. 6, pp. 67866–67884, 2018.
[37] K. Rahman and S. Abdullah, "Some induced generalized interval-valued Pythagorean fuzzy Einstein geometric aggregation operators and their application to group decision-making," *Comput. Appl. Math.*, vol. 38, no. 3, Sep. 2019, Art. no. 139.
[38] A. Liu, X. Ji, H. Lu, and H. Liu, "The selection of 3PRLs on self-service mobile recycling machine: Interval-valued Pythagorean hesitant fuzzy best-worst multi-criteria group decision-making," *J. Cleaner Prod.*, vol. 230, pp. 734–750, Sep. 2019.
[39] E. Haktanir and C. Kahraman, "A novel interval-valued Pythagorean fuzzy QFD method and its application to solar photovoltaic technology development," *Comput. Ind. Eng.*, vol. 132, pp. 361–372, Jun. 2019.
[40] T.-Y. Chen, "A novel risk evaluation method of technological innovation using an inferior ratio-based assignment model in the face of complex uncertainty," *Expert Syst. Appl.*, vol. 95, pp. 333–350, Apr. 2018.
[41] C. Yu, Y. Shao, K. Wang, and L. Zhang, "A group decision making sustainable supplier selection approach using extended TOPSIS under interval-valued Pythagorean fuzzy environment," *Expert Syst. Appl.*, vol. 121, pp. 1–17, May 2019.
[42] M.-S. Yang and C.-H. Ko, "On a class of fuzzy c-numbers clustering procedures for fuzzy data," *Fuzzy Sets Syst.*, vol. 84, no. 1, pp. 49–60, Nov. 1996.
[43] D. Y. Li and C. Y. Liu, "Study on the universality of the normal cloud model," *Eng. Sci.*, vol. 6, no. 8, pp. 28–34, 2004.
[44] J.-Q. Wang, K.-J. Li, and H.-Y. Zhang, "Multi-criteria decision-making method based on induced intuitionistic normal fuzzy related aggregation operators," *Int. J. Uncertainty, Fuzziness Knowl.-Based Syst.*, vol. 20, no. 4, pp. 559–578, Aug. 2012.
[45] J.-Q. Wang, P. Zhou, K.-J. Li, H.-Y. Zhang, and X.-H. Chen, "Multi-criteria decision-making method based on normal intuitionistic fuzzy-induced generalized aggregation operator," *TOP*, vol. 22, no. 3, pp. 1103–1122, Oct. 2014.
[46] P. Liu and F. Teng, "Multiple criteria decision making method based on normal interval-valued intuitionistic fuzzy generalized aggregation operator," *Complexity*, vol. 21, no. 5, pp. 277–290, May 2016.
[47] Z. Liu and P. Liu, "Normal intuitionistic fuzzy Bonferroni mean operators and their applications to multiple attribute group decision making," *J. Intell. Fuzzy Syst.*, vol. 29, no. 5, pp. 2205–2216, Oct. 2015.
[48] Z. Yang, J. Li, L. Huang, and Y. Shi, "Developing dynamic intuitionistic normal fuzzy aggregation operators for multi-attribute decision-making with time sequence preference," *Expert Syst. Appl.*, vol. 82, pp. 344–356, Oct. 2017.
[49] J. Li, W. Chen, Z. Yang, C. Li, and J. S. Sellers, "Dynamic interval-valued intuitionistic normal fuzzy aggregation operators and their applications to multi-attribute decision-making," *J. Intell. Fuzzy Syst.*, vol. 35, no. 4, pp. 3937–3954, Oct. 2018.
[50] P. Liu, "Multiple attribute decision-making methods based on normal intuitionistic fuzzy interaction aggregation operators," *Symmetry*, vol. 9, no. 11, p. 261, 2017.
[51] G. Zhang, Z. Zhang, and H. Kong, "Some normal intuitionistic fuzzy heronian mean operators using hamacher operation and their application," *Symmetry*, vol. 10, no. 6, p. 199, 2018.
[52] R. N. Xu and C. L. Li, "Regression prediction for fuzzy time series," *Appl. Math. J. Chinese Univ. A*, vol. 16, pp. 451–461, Dec. 2001.
[53] G. Sun, X. Guan, X. Yi, and Z. Zhou, "Grey relational analysis between hesitant fuzzy sets with applications to pattern recognition," *Expert Syst. Appl.*, vol. 92, pp. 521–532, Feb. 2018.
[54] H. Liao and Z. Xu, "Approaches to manage hesitant fuzzy linguistic information based on the cosine distance and similarity measures for HFLTSs and their application in qualitative decision making," *Expert Syst. Appl.*, vol. 42, no. 12, pp. 5328–5336, Jul. 2015.

ZAOLI YANG was born in Jiangxi, China, in 1986. He received the B.Sc. degree from the Harbin University of Commerce, Harbin, China, in 2010, and the M.Sc. and Ph.D. degrees from Harbin Engineering University, Harbin, in 2013 and 2016, respectively. He has been an Associate Professor in management science and engineering with the College of Economics and Management, Beijing University of Technology, since 2017. His main research interests include multidecision making, fuzzy set theory, and data mining. His research has been published in some academic journals, including the *International Journal of Intelligent Systems*, the IEEE TRANSACTIONS ON ENGINEERING MANAGEMENT, *Expert Systems With Applications*, *Computers & Industrial Engineering*, *Cluster Computing*, the IEEE ACCESS, *Neural Computing and Applications*, and so on. He is the Lead Guest Editor of *Mathematical Problems in Engineering*, in 2020.

JINPING CHANG received the Ph.D. degree from the College of Economics and Management, Beijing University of Technology, in 2012. He is currently an Associate Professor with the College of Management, Beijing Union University. His research interests include E-commerce and decision science. He has published over 20 journals (including Chinese academic journals) and conference papers in his research field.