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Power Law and Dimension of the Maximum Value for Belief Distribution With the Maximum Deng Entropy

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ABSTRACT Deng entropy is a novel and efficient uncertainty measure to deal with imprecise phenomenon, which is an extension of Shannon entropy. In this paper, power law and dimension of the maximum value for belief distribution with the max Deng entropy are presented, which partially uncover the inherent physical meanings of Deng entropy from the perspective of statistics. This indicated some work related to power law or scale-free can be analyzed using Deng entropy. The results of some numerical simulations are used to support the new views.

INDEX TERMS Deng entropy, power law, maximum Deng entropy, dimension.

I. INTRODUCTION

Uncertainty is a pervasive phenomenon in the real world, and with it, most of the information on which decisions are based is uncertain. Therefore, the processing of uncertain information has attracted much attention. Until now, various mathematical models are proposed to express uncertainties, such as probability theory [1], Dempster-shafer evidence theory [2]–[4], fuzzy mathematics [5], [6], Z-number [7]–[9], and so on.

Uncertainty measure can be represented as the quality of the information, which has been applied in complex networks [10], [11], pattern recognition [12], target recognition [13], decision making [14], machine learning [15] and information fusion [16]. How to measure the uncertainty of the basic probability assignment (BPA) accurately and efficiently is significant and also an open issue in Dempster-Shafer theory (DST). Plenty of functions have been developed for uncertainty modeling, such as Hohle's confusion measure [17], Kullback-Leibler's divergence measure [18], Yeger's dissonance measure [19], Klir & Ramer's discord measure [20], Klir & Parviz's strife measure [21], George & Pal's conflict measure [22], Wang & Song's interval measure [23], Higashi & Klir's entropy [24], etc., and lots of further and improved works have been made on them, e.g. Liu's new uncertainty measure for belief networks [25], Song's uncertainty measure for interval-valued belief structures [26], and some inequalities for different divergences with applications in information theory [27]–[29].

Entropy is a method of uncertainty measures, which can be used to measure uncertainty degree as well as information quality. Since firstly proposed by Clausius in 1865 for thermodynamics, various kinds of entropies have been proposed, such as information entropy [30], Tsallis entropy [31], [32], Gini Entropy, and Shannon entropy [33], which have been applied to real engineering [34]–[37].

A new entropy, named Deng entropy, has been presented by Prof. Deng to manage the uncertain information in the frame of Dempster-Shafer evidence theory (DST) [38], which has achieved plenty of attention in recent years [12], [39]–[48]. Some analyzed the properties of Deng entropy [39], some made improved work based on Deng entropy [12], [40]–[46], and some applied Deng entropy into different aspects, e.g. pattern recognition [12], fault diagnosis [47], sensor fusion [48], etc. Presently, there are many criteria for judging entropy. From different perspectives, the results of judging entropy are also different. For example, Shannon entropy can measure uncertain degree with probability

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distribution efficiently and has been used widely, but it can't measure uncertain degree with basic probability assignment. From the perspective of classical entropy theory, Deng entropy doest not verify the requirements of set consistency, range, subadditivity, additivity and monotonicity, which are defined by Klir & Wierman [49] and extended by Abellán & Masegosa [50]. However, Deng entropy, considered as an extension of Shannon entropy, can not only deal with uncertain phenomenon in the probability field, but also be applied to absorb the complex imprecise (or unknown) phenomenon in the belief filed (frame of DST) efficiently. When the BPA is degenerated as probability distribution, Deng entropy is degenerated as Shannon entropy. In this paper, we focus on the Deng entropy to discover some interesting results.

The maximum value of entropy is a problem worth studying. In [51], the condition of the maximum of Deng entropy is discussed and obtained the analytic solution of the maximum Deng entropy, which lays a foundation for further research. In this paper, the work focuses two investigations based on the maximum values of the belief distribution via the max Deng entropy with different scales of frame of discernment (FOD). One is the relation between the maximum value of belief distribution subjecting to the max Deng entropy and the scale of Deng information correspondingly. The other is dimension of the maximum value for belief distribution with the max Deng entropy. Some numerical simulations have been made to achieve the two discoveries, i.e., approximate power law and approximate constant dimension.

The rest of the paper is organized as follows. The preliminaries briefly introduce some concepts about Dempster-Shafer evidence theory, Deng entropy, max Deng entropy, power law and its distribution, self-similar and fractal dimension in Section II. In Section III, the new views about max Deng entropy are presented. One is the relation between the maximum value of belief distribution subjecting to the max Deng entropy and the scale of Deng information correspondingly. The other is dimension of the maximum value for belief distribution with the max Deng entropy. Finally, this paper is concluded in Section IV.

II. PRELIMINARIES

In this section, some preliminaries are briefly introduced.

A. FRAME OF DEMPSTER-SHAFER EVIDENCE THEORY

Let X be a set of mutually exclusive and collectively exhaustive events, indicated by

$$X = \{\theta_1, \theta_2, \cdots, \theta_i, \cdots, \theta_{|X|}\} \tag{1}$$

where set X is called a frame of discernment (FOD). The power set of X is indicated by 2^X, namely

$$2^X = \{\emptyset, \{\theta_1\}, \cdots, \{\theta_{|X|}\}, \{\theta_1, \theta_2\}, \cdots, \{\theta_1, \theta_2, \cdots, \theta_i\}, \cdots, X\} \tag{2}$$

For a frame of discernment $X = \{\theta_1, \theta_2, \cdots, \theta_{|X|}\}$, a mass function is a mapping m from 2^X to [0, 1], formally defined by:

$$m: \quad 2^X \rightarrow [0, 1] \tag{3}$$

which satisfies the following condition:

$$m(\emptyset) = 0 \quad and \quad \sum_{A \in 2^X} m(A) = 1 \tag{4}$$

where A is a focal element if $m(A)$ is not 0.

B. DENG ENTROPY

With the range of uncertainty mentioned above, Deng entropy [38] can be presented as follows

$$E_d = -\sum_i m(F_i) \log \frac{m(F_i)}{2^{|F_i|} - 1} \tag{5}$$

where, F_i is a proposition in mass function m, and $|F_i|$ is the cardinality of F_i. As shown in the above definition, Deng entropy, formally, is similar with the classical Shannon entropy, but the belief for each proposition F_i is divided by a term $(2^{|F_i|} - 1)$ which represents the potential number of states in F_i (of course, the empty set is not included).

Specially, Deng entropy can definitely degenerate to the Shannon entropy if the belief is only assigned to single elements. Namely,

$$E_d = -\sum_i m(\theta_i) \log \frac{m(\theta_i)}{2^{|\theta_i|} - 1} = -\sum_i m(\theta_i) \log m(\theta_i)$$

Next, the condition of the maximum Deng entropy is discussed [51].

C. THE MAXIMUM DENG ENTROPY

Assume F_i is the focal element and $m(F_i)$ is the basic probability assignment for F_i, then the maximum Deng entropy for a belief function happens when the basic probability assignment satisfy the condition $m\left(F_i\right) = \frac{2^{|F_i|}-1}{\sum_i 2^{|F_i|}-1}$, where $i = 1, 2, \ldots, 2^X - 1$, and X is the scale of the frame of discernment.

Theorem 1 (The Maximum Deng Entropy): The maximum Deng entropy: $E_d = -\sum_i m(F_i) \log \frac{m(F_i)}{2^{|F_i|}-1}$ *if and only if* $m\left(F_i\right) = \frac{2^{|F_i|}-1}{\sum_i 2^{|F_i|}-1}$.

More information refers to the part of APPENDIX. As shown in Fig. 1, belief distributions with the maximum Deng entropy are changing with the scale of FOD, $|X| = 1, \ldots 8$. The point in this paper lies in the maximum value of each belief distribution.

D. POWER LAW AND POWER LAW DISTRIBUTION

Zipf law is one of the fundamental laws in information science, and it is very often used in linguistics. Apart from its use in information science and linguistics, Zipf law is also used in city populations, solar flare intensity, website traffic, earthquake magnitude, and the size of moon craters, etc. This distribution in economics is known as the Pareto

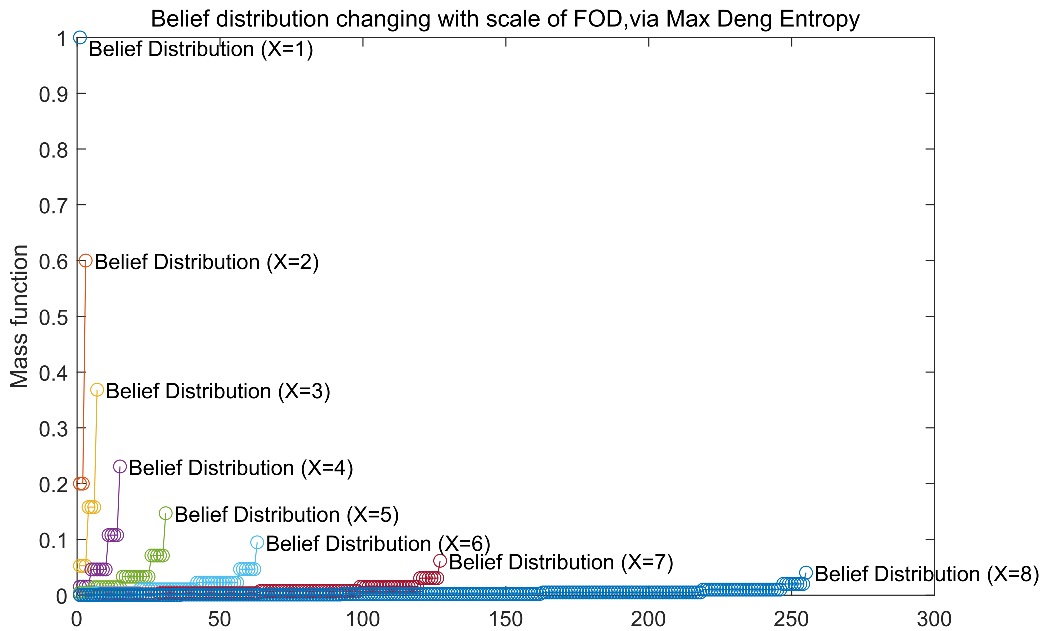

FIGURE 1. **Belief distribution with the maximum Deng entropy changing with the scale of FOD, $|X| = 1, \ldots 8$.**

law (also called the 80-20 rule) [52], [53], which analyzes the distribution of the wealthiest members of the community. It states that generally 80% of all effects result from 20% of all causes. These two laws are the same in the mathematical sense, but they are applied in different contexts [54]–[56]. And the famous Zipf law and Pareto law are both examples of power law distribution.

The power law (also called the scaling law) states that a relative change in one quantity results in a proportional relative change in another, independent of the initial size of those quantities: one quantity varies as a power of another.

A power law distribution has the form $F(x) = kx^{\alpha}$, where: F is a function (the result) and x is the variable (the thing you can change), α is the law's exponent, k is a constant.

Power law distributions exist widely in many fields such as physics, earth and planetary sciences, computer science, biology, ecology, demographics and social sciences, economics and finance, and they have various forms of expression. In nature and daily life, the distribution of earthquake magnitudes, the distribution of computer file sizes, the distribution of the number of cited papers, the distribution of clicks on web pages, etc. are all typical power law distributions. Fig. 2 is a simple power-law distribution graph, which shows the approximate power-law distribution graphically, and the meaning of its axes is various in different specific studies. For example, Pareto distributions are typical scale-probability distributions and Zipf distributions are typical ranking-frequency distributions. Then if you plot two quantities against each other with logarithmic axes and they show a linear relationship, this indicates that the two quantities have a power law distribution.

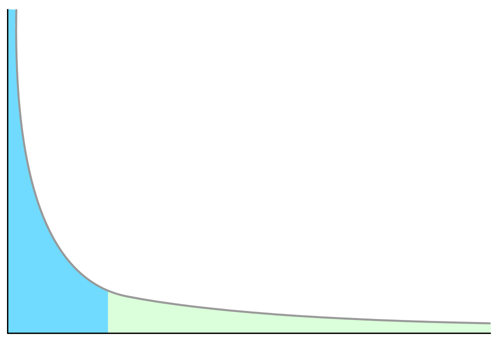

FIGURE 2. **A simple power law distribution.**

E. SELF-SIMILAR AND FRACTAL DIMENSION

In mathematics, a self-similar object is exactly or approximately similar to a part of itself (i.e. the whole has the same shape as one or more of the parts) [57]. And self-similarity is also an important characteristic of power-law distributions.

In fractal geometry, a fractal dimension is a ratio providing a statistical index of complexity comparing how detail in a pattern (strictly speaking, a fractal pattern) changes with the scale at which it is measured. A fractal dimension does not have to be an integer. And a fractal dimension can be presented as follows

$$d = \frac{\log N(a)}{\log(a)} \tag{6}$$

where, d is the fractal dimension, a is the magnification factor, and $N(a)$ is the number of self-similar pieces [58].

Next, Two focuses are presented. One is the relation between the maximum value of belief distribution subjecting

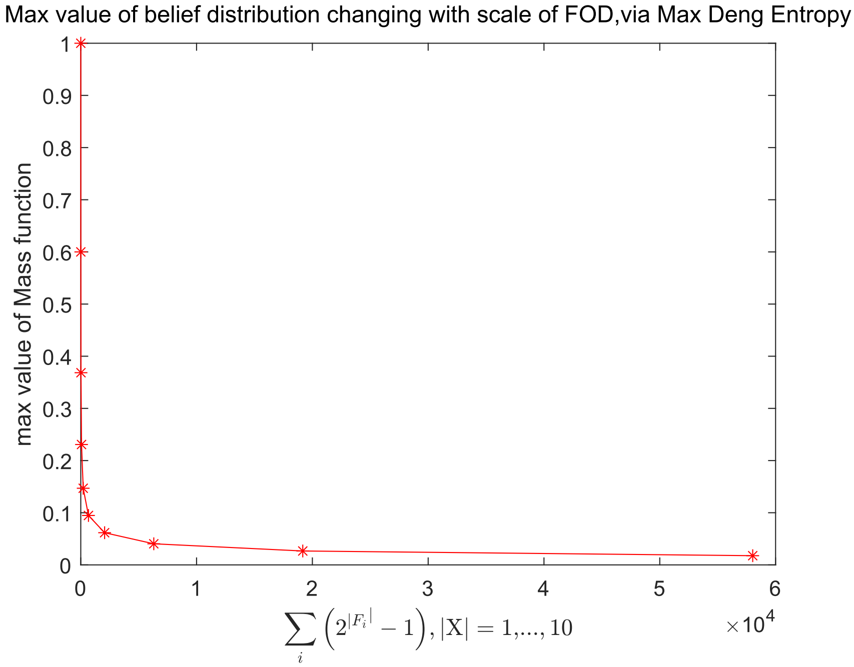

FIGURE 3. The maximum value of belief distribution with the maximum Deng entropy changing with the maximum scale of Deng information, the scale of FOD, $|X| = 1, \ldots 10$. This is an approximate power law distribution.

to the max Deng entropy and the scale of Deng information correspondingly. The other is dimension of the maximum value for belief distribution with the max Deng entropy.

III. POWER LAW AND DIMENSION OF THE MAXIMUM VALUE FOR BELIEF DISTRIBUTION WITH THE MAX DENG ENTROPY

In statistic, a power law is a relationship in which a relative change in one quantity gives rise to a proportional relative change in the other quantity, independent of the initial size of those quantities. Power law is a pervasive phenomenon in many fields, such as complex network (scale-free network) [57], Fractals [59].

Firstly, the power law function is established between the maximum value of belief distribution via max Deng entropy and the maximum Deng information scale correspondingly.

A. POWER LAW OF THE MAXIMUM VALUE FOR BELIEF DISTRIBUTION WITH THE MAX DENG ENTROPY

Suppose the maximum value for belief distribution via max Deng entropy $\max[m(F_i)]$ relates to a function $P(r)$. In addition, assume the corresponding maximum Deng information scale $\sum_i \left(2^{|F_i|}-1\right)$ relates to the variable r. A power law function $P\left(\sum_i \left(2^{|F_i|}-1\right)\right)$ with a scale invariance ($d \approx 0.37$) is established using Eq. (7).

$$P(r) = r^{-d} \tag{7}$$

where $r = \sum_i \left(2^{|F_i|}-1\right)$, $P(r) = \max[m(F_i)]$, $m(F_i) = \frac{2^{|F_i|}-1}{\sum_i \left(2^{|F_i|}-1\right)}$, $d \approx 0.37$.

As shown in Fig. 3, when the scales of FOD ($|X|$) change from 1 to 10, all the few high belief (the maximum values of the belief distribution via max Deng entropy, $\max[m(F_i)]$) are contained in the front of the plane, most of the other low belief (the maximum values of the belief distribution via max Deng entropy, $\max[m(F_i)]$) are distributed in the following wide plane. This is an approximate power law distribution. A power law function $P\left(\sum_i \left(2^{|F_i|}-1\right)\right)$ with a scale invariance (d) is easily observed by Fig. 3. What is more, the scale invariance $d \approx 0.37$, which will be discussed in the next section.

Next, dimension of the maximum value for belief distribution with the max Deng entropy is presented.

B. DIMENSION OF THE MAXIMUM VALUE FOR BELIEF DISTRIBUTION WITH THE MAX DENG ENTROPY

The scale invariance d in Eq.(7) is equal to the dimension of the maximum value for belief distribution with the max Deng entropy. By the *polyfit* function in the Matlab, the dimension $d \approx 0.37$ after investigating the data of belief distributions with max Deng entropy (scale of FOD, $|X| = 1, 2, \ldots, 30$). The result is shown in Fig. 4, which indicates the log values trending between the maximum value of belief distribution with the maximum Deng entropy and the maximum amount of Deng information, the scale of FOD, $|X| = 1, \ldots 30$.

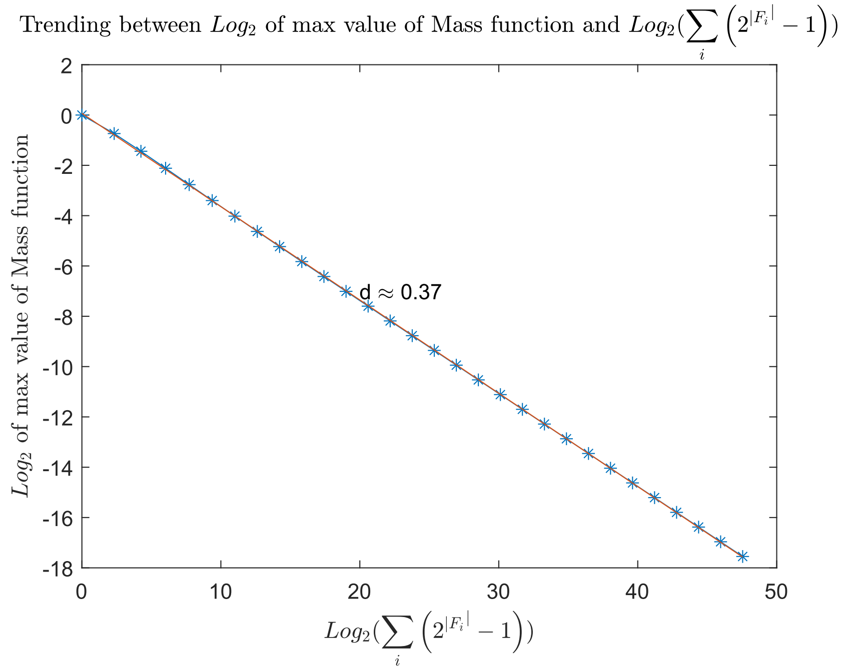

FIGURE 4. Dimension of the maximum value of belief distribution with the maximum Deng entropy and the maximum amount of Deng information, the scale of FOD, $|X| = 1, \ldots 30$.

As shown in Fig. 4, an approximate linear relation is obtained, which indicate the scale-free and power law.

$$d = -\lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(\varepsilon)} = \lim_{i \to \infty} \frac{\log_2 \max[m(F_i)]}{\log_2 \sum_i \left(2^{|F_i|}-1\right)} \approx 0.37 \tag{8}$$

$$s.t.\ m(F_i) = \frac{2^{|F_i|}-1}{\sum_i \left(2^{|F_i|}-1\right)} \tag{9}$$

IV. CONCLUSION

Deng entropy can not only deal with uncertain phenomenon in the probability field, but also measure uncertain degree with basic probability assignment in the belief filed (frame of DST) efficiently. Since it was proposed, lots of researches have been done based on it and it has been applied in pattern recognition, fault diagnosis, sensor fusion, etc. In this paper, power law and dimension of the maximum value for belief distribution with the max Deng entropy are presented, which partially uncover the inherent physical meanings of Deng entropy from the perspective of statistics. The results of some numerical simulations are used to support the new views. The discovery of the power law of the maximum value for belief distribution with the max Deng entropy means that Deng entropy can be used in the applications of fractals. Plenty of researches about entropy associated with fractal have been done, e.g. information entropy and fractal dimension [60] and some other work of entropy associated with fractal [61], [62]. This indicated some work related to power law or scale-free or fractal can be analyzed using Deng entropy and the results of this paper may stimulate some further research.

APPENDIX
THE MAXIMUM DENG ENTROPY

Assume F_i is the focal element and $m(F_i)$ is the basic probability assignment for F_i, then the maximum Deng entropy for a belief function happens when the basic probability assignment satisfy the condition $m(F_i) = \frac{2^{|F_i|}-1}{\sum_i 2^{|F_i|}-1}$, where $i = 1, 2, \ldots, 2^X - 1$, and X is the scale of the frame of discernment.

Theorem 2 (The Maximum Deng Entropy): The maximum Deng entropy: $E_d = -\sum_i m(F_i) \log \frac{m(F_i)}{2^{|F_i|}-1}$ *if and only if* $m(F_i) = \frac{2^{|F_i|}-1}{\sum_i 2^{|F_i|}-1}$

Proof: Let

$$D = -\sum_i m(F_i) \log \frac{m(F_i)}{2^{|F_i|}-1} \tag{10}$$

$$\sum_i m(F_i) = 1 \tag{11}$$

Then the Lagrange function can be defined as

$$D_0 = -\sum_i m(F_i) \log \frac{m(F_i)}{2^{|F_i|}-1} + \lambda \left(\sum_i m(F_i) - 1 \right) \tag{12}$$

Now we can calculate the gradient,

$$\frac{\partial D_0}{\partial m(F_i)} = -\log \frac{m(F_i)}{2^{|F_i|}-1} - m(F_i) \frac{1}{\frac{m(F_i)}{2^{|F_i|}-1} \ln a} \cdot \frac{1}{2^{|F_i|}-1} + \lambda = 0 \tag{13}$$

Then Eq. (13) can be simplified as

$$-\log \frac{m(F_i)}{2^{|F_i|}-1} - \frac{1}{\ln a} + \lambda = 0 \tag{14}$$

From Eq. (14), we can get

$$\frac{m(F_1)}{2^{|F_1|}-1} = \frac{m(F_2)}{2^{|F_2|}-1} = \cdots = \frac{m(F_n)}{2^{|F_n|}-1} \tag{15}$$

Let

$$\frac{m(F_1)}{2^{|F_1|}-1} = \frac{m(F_2)}{2^{|F_2|}-1} = \cdots = \frac{m(F_n)}{2^{|F_n|}-1} = k \tag{16}$$

Then

$$m(F_i) = k\left(2^{|F_i|}-1\right) \tag{17}$$

According to Eq. (11), we can get

$$k = \frac{1}{\sum_i 2^{|F_i|}-1} \tag{18}$$

According to Eq. (16), we can get

$$m(F_i) = \frac{2^{|F_i|}-1}{\sum_i 2^{|F_i|}-1} \tag{19}$$

Hence, the maximum Deng entropy $E_d = -\sum_i m(F_i) \log \frac{m(F_i)}{2^{|F_i|}-1}$ if and only if $m(F_i) = \frac{2^{|F_i|}-1}{\sum_i 2^{|F_i|}-1}$ □

CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

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