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Globally Exponential Stability and Globally Power Stability of Quaternion-Valued Neural Networks With Discrete and Distributed Delays

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ABSTRACT This paper investigates the stability of a class of quaternion-valued neural networks (QVNNs) with discrete and distributed delays. By decomposing the QVNN and forming an equivalent real-valued vector-matrix differential equation (RVVDE), based on the Lyapunov theory and some matrix inequalities, some sufficient conditions are derived to ensure the existence and uniqueness, the globally exponential stability and the globally power stability of the equilibrium of RVVDE. These conditions also apply to the QVNN. Two numerical examples are given to show the advantage and the effectiveness of the main results.

INDEX TERMS Globally exponential stability, globally power stability, quaternion-valued neural network.

I. INTRODUCTION

Neural network is a parallel distributed processor with a large number of connections, and it has an adaptive ability to acquire knowledge through learning [1]. Since the neural network model was proposed by Hopfield in 1982 [2], more and more results on neural networks have been put forward. For the various characteristics of neural networks, many interesting directions of neural networks appear in large numbers [3]–[6]. More cross-studies on neural networks have emerged [7]–[11].

In the last few decades, the stability, the synchronization, the limit cycles, the branches, the oscillation and the chaotic attractors of neural networks have been widely studied [12]–[14]. Many interesting results on the stability of the equilibrium point of the real-valued neural networks (RVNNs), whose state variables, activation functions, link weights and external inputs are all real-valued [3], [5], [6], [15], have been obtained. However, RVNNs can process low-dimensional data well but perform poorly when processing high-dimensional data. Complex-valued neural networks (CVNNs), whose state variables, activation functions, link weights and external inputs are all complex-valued, can deal with high-dimensional data efficiently. CVNNs can be thought as an extension of RVNNs and as a class of important nonlinear complex-valued systems [16], [17]. Moreover, CVNNs have been applied in many fields, such as imaging processing, photoelectron, remote sense and so on [18]. However, we hope there would be more applications for neural networks, so that we extend the CVNNs once again. There come quaternion value neural networks(QVNNs).

A quaternion is a supercomplex number, which is a mathematical concept and discovered in 1843. Because of the diversity of its expression forms, the quaternion is wildly used in various fields, such as machine and structure, robot technology, and satellite attitude adjustment and so on [19].

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By a quaternionic approach, the optimum separation of polarized signals was realized by Buchholz and Bihanthe [20]. Contacting neural network with quaternions, we can obtain a class of more complex neural networks, QVNNs, whose state variables, activation functions, link weights and external inputs are all quaternion-valued. Research on the dynamics characteristics and applications of QVNNs is becoming a hot topic [20]–[26]. By using the Lyapunov theory and homeomorphic mapping theory, the robust stability of delayed QVNNs with parameter uncertainties was analyzed [22]. Based on homeomorphic mapping theory and complex decomposition method, the globally asymptotical stability of continuous-time QVNNs and discrete-time QVNNs was considered in [24]. Based on homeomorphic mapping theorem and linear matrix inequality, several sufficient criteria are proposed to verify that the QVNNs with both discrete and distributed delays is globally asymptotically stable and globally exponentially stable [27].

As is well known, the stability of various neural networks is a key factor in many applications. So it is important to use a suitable method to ascertain the stability of QVNNs. Some results concern the dynamic characteristics of QVNNs by taking QVNNs as an entirety without decomposition [21], [22]. However, some results investigated the stability of QVNNS by equivalently decomposing QVNNs into two CVNNs or four RVNNs [23], [24], which is concise and practical because there are many methods to investigate the stability of RVNNs or CVNNs.

Meanwhile, time delays are probably inevitable in the practical neural networks, because the transmission of information may take some time. Currently, some results on neural network with time delays always fix on the disadvantage to the stability of neural networks [27]–[35]. Perhaps, on the contrary, by time delays, we could control the dynamic behavior of neural networks better. The global convergence of a class of RVNNs with time delays was analyzed [29]. The globally asymptotic robust stability of delayed RVNNs with norm-bounded uncertainties was studied [30]. By a decomposition method and an appropriate Lyapunov–Krasovskii functional, the global μ-stability of a class of QVNNs with unbounded time-varying delays was investigated [35]. When considering globally asymptotical stability and globally exponential stability of RVNNs, Chen and Wang in [33] proposed a new concept – globally power stability. To the best of our knowledge, there are few papers concerning the stability of QVNNs with mixed delays, such as, with both discrete and distributed delays.

Enlightened by the above analysis, this paper investigates the stability problem of a class of QVNNs with discrete and distributed delays, by decomposing the QVNN into four real-valued neural networks and forming a RVVDE. This paper's main contributions are listed as follows. Firstly, to the best of our knowledge, it is the first time to analyze the stability of a class of QVNNs with mixed time delays by a decomposition method. Compared with [23], our model is a more general QVNN with discrete and distributed delays, and compared with [27], to study the stability of the equilibrium of QVNNs, our method is an equivalent decomposition method, not by taking QVNNs as an entirety. Secondly, by constructing some new Lyapunov–Krasovskii functionals and matrix inequalities, three sufficient conditions are proposed to ensure the existence and uniqueness, the globally exponential stability and the globally power stability of the equilibrium of the RVVDE. Note that these criteria, without too many restrictions compared with Theorem 1 in [23], are suitable not only for QVNNs but also for RVNNs. Finally, our criteria formulated by matrix inequalities can be easily checked, and our decomposition methods are of faster convergence speed than the results in [27].

The remaining sections are organized as follows. In Section 2, we present the considered QVNNs model descriptions and some preliminaries. In section 3, the stability analysis of QVNNs with discrete and distributed delays is given. In Section 4, the effectiveness of our results is verified by numerical examples. In Section 5, some conclusions and some future ideas are given.

Notations: Throughout this article, some notations will be used. $\mathbb{R}$ denotes the set of real numbers. $\mathbb{C}$ denotes the set of complex numbers. $\mathbb{R}^{m\times n}$, $\mathbb{Q}^{m\times n}$ denote the set of all $m \times n$ real-valued and quaternion-valued matrices, respectively. $\mathbb{Q}$ denotes the set of quaternions. $\mathbb{Q}^n$ denotes the n-dimensional quaternion space. The conjugate transpose of $q \in \mathbb{Q}$ is $q^* = q^{(r)} - q^{(i)}i - q^{(j)}j - q^{(k)}k$. $\|q\| = \sqrt{qq^*} = \sqrt{\left(q^{(r)}\right)^2 + \left(q^{(i)}\right)^2 + \left(q^{(j)}\right)^2 + \left(q^{(k)}\right)^2}$. $\phi \in C([t_0 - \tau,\ t_0];\ \mathbb{Q}^n)$ denotes a continuous mapping from $[t_0 - \tau,\ t_0]$ to $\mathbb{Q}^n$, $\|\phi\| = sup_{t_0-\tau\le s\le t_0}|\phi(s)|$. $[A]^S$ denotes the symmetric part of matrix A and $[A]^S = \frac{1}{2}(A^T + A)$, where A^T denotes transpose of A. If A is a symmetric matrix, $A > 0$ ($A \geq 0$) means that A is positive definite (positive semidefinite). Similarly, $A < 0$ ($A \leq 0$) means that A is negative definite (negative semidefinite). The matrix norm of A is written as $\|A\| = (\lambda_{max}(A^TA))^{\frac{1}{2}}$, and λ_{max} is the largest eigenvalue of the matrix.

II. PRELIMINARIES

Firstly, some preliminaries are recapitulated. The symbol q denotes a quaternion, if $q = q^{(r)} + q^{(i)}i + q^{(j)}j + q^{(k)}k$, where $q^{(r)},\ q^{(i)},\ q^{(j)},\ q^{(k)} \in \mathbb{R}$. And $i,\ j,\ k$ satisfy the following rules:

$$
\begin{aligned}
i^2 = j^2 = k^2 = -1, \quad ij = -ji = k,\\
jk = -kj = i, \quad ki = -ik = j.
\end{aligned}
$$

Obviously, you have to pay attention to the noncommutativity of quaternion multiplication.

Consider the following QVNN with discrete and distributed delays:

$$
\begin{aligned}
\dot{q}(t) = -Dq(t) + Af(q(t)) + Bf(q(t - \tau(t)))\\
+ C\int_{t-\tau}^{t} f(q(s))ds + U, \qquad (1)
\end{aligned}
$$

where $q(t) = (q_1(t),\ q_2(t),\ \cdots,\ q_n(t)) \in \mathbb{Q}^n$ is the state vector at time t. $D = diag\{d_1, d_2, \cdots, d_n\} \in \mathbb{R}^{n\times n}$

denotes the self-feedback connection weight matrix with $d_l > 0,\ l = 1, 2, \cdots, n$. $A, B, C \in \mathbb{Q}^{n\times n}$ are link weights matrices. $f(q(t)) = (f_1(q_1(t)), f_2(q_2(t)), \cdots, f_n(q_n(t)))^T \in \mathbb{Q}^n$, $f(q(t-\tau(t))) = (f_1(q_1(t-\tau(t))), f_2(q_2(t-\tau(t))), \cdots, f_n(q_n(t-\tau(t))))^T \in \mathbb{Q}^n$ and $\int_{t-\tau}^{t} f(q(s))ds = \left(\int_{t-\tau}^{t} f_1(q_1(s))ds, \int_{t-\tau}^{t} f_2(q_2(s))ds, \cdots, \int_{t-\tau}^{t} f_n(q_n(s))ds\right)^T \in \mathbb{Q}^n$ are vector activation functions without time delays, with discrete delays and with distributed delays respectively. $\tau(t)$ and τ denote the discrete time-varying delay and distributed delay respectively. $\tau(t)$ satisfies $0 < \tau(t) \le \tau$, $\dot{\tau}(t) \le \mu < 1$. $U = (u_1(t), u_2(t), \cdots, u_n(t))^T \in \mathbb{Q}^n$ denotes external input vector. The initial condition is given by $q(s) = \phi(s) \in \mathbb{Q}^n$, $s \in [t_0 - \tau, t_0]$, where $\phi(s) = \phi^{(r)}(s) + \phi^{(i)}(s)i + \phi^{(j)}(s)j + \phi^{(k)}(s)k$. Just as the definition presented in [27], if q^* satisfies $-Dq^* + Cf(q^*) + Af(q^*) + \tau Bf(q^*) + U = 0$, then q^* is an equilibrium point of (1).

Let

$$
\begin{aligned}
q(t) &= q^{(r)}(t) + iq^{(i)}(t) + jq^{(j)}(t) + kq^{(k)}(t),\\
f(q(t)) &= f^{(r)}(q^{(r)}(t)) + if^{(i)}(q^{(i)}(t)) + jf^{(j)}(q^{(j)}(t))\\
&\quad + kf^{(k)}(q^{(k)}(t)),\\
\int_{t-\tau}^{t} f(q(s))ds &= \int_{t-\tau}^{t} f^{(r)}(q^{(r)}(s))ds\\
&\quad + i\int_{t-\tau}^{t} f^{(i)}(q^{(i)}(s))ds\\
&\quad + j\int_{t-\tau}^{t} f^{(j)}(q^{(j)}(s))ds\\
&\quad + k\int_{t-\tau}^{t} f^{(k)}(q^{(k)}(s))ds,\\
A &= A^{(r)} + iA^{(i)} + jA^{(j)} + kA^{(k)},\\
B &= B^{(r)} + iB^{(i)} + jB^{(j)} + kB^{(k)},\\
C &= C^{(r)} + iC^{(i)} + jC^{(j)} + kC^{(k)},
\end{aligned}
$$

where $q^{(l)}(t), f^{(l)}(q^{(l)}(t)), \int_{t-\tau}^{t} f^{(l)}(q^{(l)}(s))ds \in \mathbb{R}^n$, $A^{(l)}, B^{(l)}, C^{(l)} \in \mathbb{R}^{n\times n}$, $l = r, i, j, k$.

QVNN (1) can be separated into four RVNNs as follows:

$$
\begin{aligned}
\dot{q}^{(r)}(t) &= -Dq^{(r)}(t) + A^{(r)}f^{(r)}(q^{(r)}(t)) - A^{(i)}f^{(i)}(q^{(i)}(t))\\
&\quad - A^{(j)}f^{(j)}(q^{(j)}(t)) - A^{(k)}f^{(k)}(q^{(k)}(t))\\
&\quad + B^{(r)}f^{(r)}(q^{(r)}(t-\tau(t)))\\
&\quad - B^{(i)}f^{(i)}(q^{(i)}(t-\tau(t)))\\
&\quad - B^{(j)}f^{(j)}(q^{(j)}(t-\tau(t)))\\
&\quad - B^{(k)}f^{(k)}(q^{(k)}(t-\tau(t)))\\
&\quad + C^{(r)}\int_{t-\tau}^{t} f^{(r)}(q^{(r)}(s))ds\\
&\quad - C^{(i)}\int_{t-\tau}^{t} f^{(i)}(q^{(i)}(s))ds\\
&\quad - C^{(j)}\int_{t-\tau}^{t} f^{(j)}(q^{(r)}(s))ds\\
&\quad - C^{(k)}\int_{t-\tau}^{t} f^{(k)}(q^{(k)}(s))ds + U^{(r)}, \qquad (2)\\
\dot{q}^{(i)}(t) &= -Dq^{(i)}(t) + A^{(r)}f^{(i)}(q^{(i)}(t)) + A^{(i)}f^{(r)}(q^{(r)}(t))\\
&\quad + A^{(j)}f^{(k)}(q^{(k)}(t)) - A^{(k)}f^{(j)}(q^{(j)}(t))\\
&\quad + B^{(r)}f^{(i)}(q^{(i)}(t-\tau(t)))\\
&\quad + B^{(i)}f^{(r)}(q^{(r)}(t-\tau(t)))\\
&\quad + B^{(j)}f^{(k)}(q^{(k)}(t-\tau(t)))\\
&\quad - B^{(k)}f^{(j)}(q^{(j)}(t-\tau(t)))\\
&\quad + C^{(r)}\int_{t-\tau}^{t} f^{(i)}(q^{(i)}(s))ds\\
&\quad + C^{(i)}\int_{t-\tau}^{t} f^{(r)}(q^{(r)}(s))ds\\
&\quad + C^{(j)}\int_{t-\tau}^{t} f^{(k)}(q^{(k)}(s))ds\\
&\quad - C^{(k)}\int_{t-\tau}^{t} f^{(j)}(q^{(j)}(s))ds + U^{(i)}, \qquad (3)\\
\dot{q}^{(j)}(t) &= -Dq^{(j)}(t) + A^{(r)}f^{(j)}(q^{(j)}(t))\\
&\quad + A^{(j)}f^{(r)}(q^{(r)}(t)) - A^{(i)}f^{(k)}(q^{(k)}(t))\\
&\quad + A^{(k)}f^{(i)}(q^{(i)}(t))\\
&\quad + B^{(r)}f^{(j)}(q^{(j)}(t-\tau(t)))\\
&\quad + B^{(j)}f^{(r)}(q^{(r)}(t-\tau(t)))\\
&\quad - B^{(i)}f^{(k)}(q^{(k)}(t-\tau(t)))\\
&\quad + B^{(k)}f^{(i)}(q^{(i)}(t-\tau(t)))\\
&\quad + C^{(r)}\int_{t-\tau}^{t} f^{(j)}(q^{(j)}(s))ds\\
&\quad + C^{(j)}\int_{t-\tau}^{t} f^{(r)}(q^{(r)}(s))ds\\
&\quad - C^{(i)}\int_{t-\tau}^{t} f^{(k)}(q^{(k)}(s))ds\\
&\quad + C^{(k)}\int_{t-\tau}^{t} f^{(i)}(q^{(i)}(s))ds + U^{(j)}, \qquad (4)\\
\dot{q}^{(k)}(t) &= -Dq^{(k)}(t) + A^{(r)}f^{(k)}(q^{(k)}(t))\\
&\quad + A^{(k)}f^{(r)}(q^{(r)}(t)) + A^{(i)}f^{(j)}(q^{(j)}(t))\\
&\quad - A^{(j)}f^{(i)}(q^{(i)}(t))\\
&\quad + B^{(r)}f^{(k)}(q^{(k)}(t-\tau(t)))\\
&\quad + B^{(k)}f^{(r)}(q^{(r)}(t-\tau(t)))\\
&\quad + B^{(i)}f^{(j)}(q^{(j)}(t-\tau(t)))\\
&\quad - B^{(j)}f^{(i)}(q^{(i)}(t-\tau(t)))\\
&\quad + C^{(r)}\int_{t-\tau}^{t} f^{(k)}(q^{(k)}(s))ds\\
&\quad + C^{(k)}\int_{t-\tau}^{t} f^{(r)}(q^{(r)}(s))ds\\
&\quad + C^{(i)}\int_{t-\tau}^{t} f^{(j)}(q^{(j)}(s))ds\\
&\quad - C^{(j)}\int_{t-\tau}^{t} f^{(i)}(q^{(i)}(s))ds + U^{(k)}. \qquad (5)
\end{aligned}
$$

According to (2), (3), (4), (5), one can obtain that

$$
\begin{aligned}
\dot{Q}(t) = -\widehat{D}Q(t) + \widehat{A}\widehat{f}(Q(t)) + \widehat{B}\widehat{f}(Q(t-\tau(t)))&\\
+ \widehat{C}\int_{t-\tau}^{t} \widehat{f}(Q(s))ds + \widehat{U}&, \qquad (6)
\end{aligned}
$$

where

$$
\widehat{D} = diag\{D, D, D, D\} \in \mathbb{R}^{4n\times 4n},
$$

$$
\widehat{A} = \begin{pmatrix} A^{(r)} & -A^{(i)} & -A^{(j)} & -A^{(k)}\\ A^{(i)} & A^{(r)} & -A^{(k)} & A^{(j)}\\ A^{(j)} & A^{(k)} & A^{(r)} & -A^{(i)}\\ A^{(k)} & -A^{(j)} & A^{(i)} & A^{(r)} \end{pmatrix} \in \mathbb{R}^{4n\times 4n},
$$

$$\widehat{B} = \begin{pmatrix} B^{(r)} & -B^{(i)} & -B^{(j)} & -B^{(k)} \\ B^{(i)} & B^{(r)} & -B^{(k)} & B^{(j)} \\ B^{(j)} & B^{(k)} & B^{(r)} & -B^{(i)} \\ B^{(k)} & -B^{(j)} & B^{(i)} & B^{(r)} \end{pmatrix} \in \mathbb{R}^{4n\times 4n},$$

$$\widehat{C} = \begin{pmatrix} C^{(r)} & -C^{(i)} & -C^{(j)} & -C^{(k)} \\ C^{(i)} & C^{(r)} & -C^{(k)} & C^{(j)} \\ C^{(j)} & C^{(k)} & C^{(r)} & -C^{(i)} \\ C^{(k)} & -C^{(j)} & C^{(i)} & C^{(r)} \end{pmatrix} \in \mathbb{R}^{4n\times 4n},$$

$$Q(t) = \left(q^{(r)}(t)^T,\ q^{(i)}(t)^T,\ q^{(j)}(t)^T,\ q^{(k)}(t)^T\right)^T \in \mathbb{R}^{4n},$$

$$\widehat{U} = \left((U^{(r)})^T,\ (U^{(i)})^T,\ (U^{j})^T,\ (U^{k})^T\right)^T \in \mathbb{R}^{4n},$$

$$\widehat{f}(Q(t)) = \Big((f^{(r)}(q^{(r)}(t))^T, (f^{(i)}(q^{(i)}(t))^T, (f^{(j)}(q^{(j)}(t))^T, (f^{(k)}(q^{(k)}(t))^T\Big)^T \in \mathbb{R}^{4n},$$

$$\widehat{f}(Q(t-\tau(t))) = \Big((f^{(r)}(q^{(r)}(t-\tau(t))))^T, (f^{(i)}(q^{(i)}(t-\tau(t))))^T, (f^{(j)}(q^{(j)}(t-\tau(t))))^T, (f^{(k)}(q^{(k)}(t-\tau(t))))^T\Big)^T \in \mathbb{R}^{4n},$$

$$\int_{t-\tau}^{t} \widehat{f}(Q(s))ds = \left(\left(\int_{t-\tau}^{t} f^{(r)}(q^{(r)}(s))ds\right)^T, \left(\int_{t-\tau}^{t} f^{(i)}(q^{(i)}(s))ds\right)^T, \left(\int_{t-\tau}^{t} f^{(j)}(q^{(j)}(s))ds\right)^T, \left(\int_{t-\tau}^{t} f^{(k)}(q^{(k)}(s))ds\right)^T\right)^T \in \mathbb{R}^{4n}.$$

Remark 1: System (6) is a RVNN with mixed delays. Obviously, the equilibrium point and dynamic characteristics of QVNN (1) are the same as those of system (6) by matching $q(t) = q^{(r)}(t) + iq^{(i)}(t) + jq^{(j)}(t) + kq^{(k)}(t)$ with $Q(t) = \left((q^{(r)}(t))^{\mathrm{T}}, (q^{(i)}(t))^{\mathrm{T}}, (q^{(j)}(t))^{\mathrm{T}}, (q^{(k)}(t))^{\mathrm{T}}\right)^{\mathrm{T}}$. Therefore, one can investigate the dynamics characteristics of the equilibrium of the system (6) instead of system (1).

Definition 1 ([29]): $A = (a_{ij})_{n\times n}$ $(a_{ij} \in \mathbb{R})$ is called a Lyapunov Diagonally Stable (LDS) matrix, if there is a matrix P, such that $P^T = P, P > 0$ and $[PA]^S > 0$.

Definition 2 ([29]): A map $H : \mathbb{R}^n \to \mathbb{R}^n$ is said to be a homeomorphism of R^n onto itself, if it satisfies: H is continuous, one-to-one and onto, and its inverse mapping H^{-1} is continuous too.

Definition 3 ([27]): The unique equilibrium point Q^* of real valued system (6) is said to be globally exponentially stable if there exists a constant $\alpha > 0$ $(\alpha \in \mathbb{R})$ such that

$$\|Q(t) - Q^*\| \leq \|\phi(t) - Q^*\|e^{-\alpha t}, t \geq t_0.$$

Definition 4 ([33]): Suppose that there are constants $M > 0$ and $\gamma > 0$ $(M, \gamma \in \mathbb{R})$, such that

$$\|Q(t) - Q^*\| \leq Mt^{-\gamma},$$

then, the system (6) is power-rate globally stable with power convergence rate γ, or globally power stable.

Assumption 1 ([29]): A continuous function $f : \mathbb{R}^n \to \mathbb{R}^n$ of the form $f = (f_1, f_2, ..., f_n)^T$ is said to be of class $G\{G_1, G_2, \cdots, G_n\}$, and $G = diag\{G_1, G_2, \cdots, G_n\}$ with $0 < G_i < +\infty$, $i = 1, 2, ..., n$, if the function $f(x)$ satisfies $0 \leq \frac{f_i(x)-f_i(y)}{x-y} \leq G_i$ for each $x, y \in \mathbb{R}$, $x \neq y$ and for $i = 1, 2, \cdots, n$.

Assumption 2: Under Assumption 1, we have $\widehat{f} = ((f^{(r)})^T, (f^{(i)})^T, (f^{(j)})^T, (f^{(k)})^T)^T = (\widehat{f}_1, \widehat{f}_2, \cdots, \widehat{f}_{4n})^T : \mathbb{R}^{4n} \to \mathbb{R}^{4n}$, $\widehat{f} \in G\{\widehat{G}_1, \widehat{G}_2, \cdots, \widehat{G}_{4n}\}$ and $\widehat{G} = diag\{G, G, G, G\}$.

Lemma 1 ([36]): The LMI $H = \begin{pmatrix} H_{11} & H_{12} \\ H_{12}^T & H_{22} \end{pmatrix} < 0$ with $H_{11} = H_{11}^T, H_{22} = H_{22}^T$ is equivalent to one of the following conditions:

1) $H_{22} < 0, H_{11} - H_{12}^{-1}H_{12}^T < 0$;
2) $H_{11} < 0, H_{22} - H_{12}^T H_{11}^{-1} H_{12} < 0$.

Lemma 2 ([34]): Under Assumption 1 and $DG^{-1} - A \in LDS$, we have:

1) $H(Q) = -DQ + Af(Q) + I$ is a homeomorphism of $\mathbb{R}^n$ onto itself;
2) the system $\frac{dQ(t)}{dt} = -DQ(t) + Af(Q(t)) + U$ has a unique equilibrium point for each $U \in \mathbb{R}^n$.

Lemma 3 ([37]): For any constant matrix $W \in \mathbb{R}^{m\times m}$, $W = W^T > 0$, scalar $\gamma > 0$, vector function $\omega : [0, \gamma] \to \mathbb{R}^m$, such that

$$\left(\int_0^{\gamma} \omega(s)ds\right)^T W \left(\int_0^{\gamma} \omega(s)ds\right) \leq \gamma \int_0^{\gamma} \omega^T(s)W\omega(s)ds. \quad (7)$$

Lemma 4 ([33]): Let P, D, S be real matrices of appropriate dimensions, and P is positive definite. Then for any vectors x and y with appropriate dimensions, it holds that

$$2x^T DSy \leq x^T DSP^{-1}(DS)^T x + y^T Py. \quad (8)$$

III. MAIN RESULTS

In this section we will propose three criteria for the existence and uniqueness, globally exponential stability and globally power stability of the equilibrium of the given system (6) by using matrix inequalities and Lyapunov analysis respectively. At last, two corollaries are given for QVNNs without distributed delays.

Theorem 1: Under Assumption 2, there exist positive diagonal matrices P, Q and R such that:

$$\begin{pmatrix} 2P\widehat{D}\widehat{G}^{-1}-(P\widehat{A}+\widehat{A}^TP)-Q-R & P\widehat{B} & P(\tau\widehat{C}) \\ \widehat{B}^TP & Q & 0 \\ (\tau\widehat{C})^TP & 0 & R \end{pmatrix} > 0, \quad (9)$$

then, for each $U \in R^n$, system (6) has a unique equilibrium point.

Proof: From Assumption 2, Eq.(9) and Lemma 1, one can obtain

$$2P\widehat{D}\widehat{G}^{-1}-(P\widehat{A}+\widehat{A}^TP) > P\widehat{B}Q^{-1}(\widehat{B})^TP+Q + P(\tau\widehat{C})R^{-1}(\tau\widehat{C})^TP+R. \quad (10)$$

By the inequality $[Q^{-\frac{1}{2}}(P\widehat{B})^T-Q^{\frac{1}{2}}]^T[Q^{-\frac{1}{2}}(P\widehat{B})^T-Q^{\frac{1}{2}}] \geq 0$, we have $P\widehat{B}Q^{-1}(\widehat{B})^TP+Q \geq P\widehat{B}+\widehat{B}^TP$. Similarly, we have $P(\tau\widehat{C})R^{-1}(\tau\widehat{C})^TP+R \geq P(\tau\widehat{C})+(\tau\widehat{C})^TP$. So (10) becomes

$$2P\widehat{D}\widehat{G}^{-1} > P(\widehat{A}+\widehat{B}+\tau\widehat{C})+(\widehat{A}+\widehat{B}+\tau\widehat{C})^TP, \quad (11)$$

that is,

$$\{P(\widehat{D}\widehat{G}^{-1}-\widehat{A}-\widehat{B}-\tau\widehat{C})\}^s > 0, \quad (12)$$

which implies $\widehat{D}\widehat{G}^{-1}-(\widehat{A}+\widehat{B}+\tau\widehat{C}) \in LDS$ by Definition 1. From Lemma 2, $H(Q) = -\widehat{D}Q+\widehat{A}\widehat{f}(Q)+\widehat{B}\widehat{f}(Q)+\tau\widehat{C}\widehat{f}(Q)+\widehat{U}$ is a homeomorphism of $\mathbb{R}^n$ onto itself, and hence system (6) has a unique equilibrium point for each $\widehat{U} \in \mathbb{R}^n$. This completes the proof. ∎

Remark 2: Theorem 1 aims to investigate the existence and uniqueness of equilibrium of QVNN, from the proof process of Theorem 1, it is concise compared with the proof process of the Theorem 3.1 in [27]. For requiring fewer variables and instead of considering $\tau(t)$, the criterion can be more easily checked by the LMI toolbox in MATLAB than that in [27], too.

By decomposing the QVVN (1) into RVNNs, one can obtain system (6) which is a high-dimensional RVVDE and whose model is more practical than that in [29], for one more item–distributed delays than [29] and with time-varying delays. However, inspired by the method of [29], next, we will construct a new Lyapunov functional to investigate the globally exponential stability of the system (6).

Theorem 2: Under Assumption 2, there exist positive diagonal matrices P, Q and R such that:

$$\begin{pmatrix} 2P\widehat{D}\widehat{G}^{-1}-(P\widehat{A}+\widehat{A}^TP)-Q-R & P\widehat{B} & P(\tau\widehat{C}) \\ (\widehat{B})^TP & (1-\mu)Q & 0 \\ (\tau\widehat{C})^TP & 0 & R \end{pmatrix} > 0, \quad (13)$$

then, for each $U \in \mathbb{R}^n$, system (6) has a unique equilibrium point which is globally exponentially stable.

Proof: Suppose that $Q^* = \big((q^{*(r)})^T, (q^{*(i)})^T, (q^{*(j)})^T, (q^{*(k)})^T\big)^T = \big(Q_1^*, Q_2^*, \cdots, Q_{4n}^*\big)^T \in \mathbb{R}^{4n}$ is an equilibrium point of real-valued system (6) and $q^* = q^{*(r)}+iq^{*(i)}+jq^{*(j)}+kq^{*(k)}$ is an equilibrium point of system (1). By transformation $\widehat{Q}(t) = Q(t)-Q^*$ and $\widehat{Q}(t) = (\widehat{Q}_1(t), \widehat{Q}_2(t), \cdots, \widehat{Q}_{4n}(t))^T \in \mathbb{R}^{4n}$, one can obtain the following vector form of real-valued system (14):

$$\dot{\widehat{Q}}(t) = -\widehat{D}\widehat{Q}(t)+\widehat{A}F(\widehat{Q}(t))+\widehat{B}F(\widehat{Q}(t-\tau(t))) + \widehat{C}\int_{t-\tau}^{t}F(\widehat{Q}(s))ds, \quad (14)$$

where $F(\widehat{Q}(t)) = \widehat{f}(\widehat{Q}(t)+Q^*)-\widehat{f}(Q^*)$.

Clearly, Q^* is globally exponentially stable for (6) if and only if the trivial solution of (14) is globally exponentially stable. To analyze the global stability of the trivial solution of (14), we consider the following Lyapunov functional:

$$\begin{aligned} V(\widehat{Q}(t),t) &= e^{\epsilon t}\widehat{Q}^T(t)\widehat{Q}(t)+2\alpha\sum_{i=1}^{4n}P_ie^{\epsilon t}\int_0^{\widehat{Q}_i(t)}F_i(\rho)d\rho \\ &\quad +(\alpha+\beta)\int_{t-\tau(t)}^{t}F^T(\widehat{Q}(s))QF(\widehat{Q}(s))e^{\epsilon(s+\tau)}ds \\ &\quad +\frac{\alpha}{\tau}\int_{-\tau}^{0}\int_{t+s}^{t}e^{\epsilon(\theta+\tau)}F^T(\widehat{Q}(\theta))RF(\widehat{Q}(\theta))d\theta ds, \end{aligned} \quad (15)$$

where positive constants α and β are to be decided and $\epsilon > 0$ is a small real number and $P = diag\{P_1, P_2, \cdots, P_{4n}\}$, $Q = diag\{Q_1, Q_2, \cdots, Q_{4n}\}$, $P_i > 0$, $Q_i > 0$, ρ can be regarded as $\widehat{Q}_i(s)$, $i = 1, 2, \cdots, 4n$. Differentiating V along the solution of (14), we have

$$\begin{aligned} \dot{V}(\widehat{Q}(t),t) &= \epsilon e^{\epsilon t}\widehat{Q}^T(t)\widehat{Q}(t)+2e^{\epsilon t}\widehat{Q}^T(t)\Big[-\widehat{D}\widehat{Q}(t) \\ &\quad +\widehat{A}F(\widehat{Q}(t))+\widehat{B}F(\widehat{Q}(t-\tau(t))) \\ &\quad +\widehat{C}\int_{t-\tau}^{t}F(\widehat{Q}(s))ds\Big] \\ &\quad +2\alpha\epsilon e^{\epsilon t}\sum_{i=0}^{4n}P_i\int_0^{Q_i(t)}F_i(\rho)d\rho \\ &\quad +2\alpha e^{\epsilon t}F^T(\widehat{Q}(t))P\Big[-\widehat{D}\widehat{Q}(t)+\widehat{A}F(\widehat{Q}(t)) \\ &\quad +\widehat{B}F(\widehat{Q}(t-\tau(t)))+\widehat{C}\int_{t-\tau}^{t}F(\widehat{Q}(s))ds\Big] \\ &\quad +(\alpha+\beta)e^{\epsilon t}\Big[F^T(\widehat{Q}(t))e^{\epsilon\tau}QF(\widehat{Q}(t)) \\ &\quad -(1-\dot{\tau}(t))e^{\epsilon(t+\tau-\tau(t))}F^T(\widehat{Q}(t-\tau(t))) \\ &\quad \cdot QF(Q(t-\tau(t)))\Big] \\ &\quad +\alpha e^{\epsilon(t+\tau)}F^T(\widehat{Q}(t))RF(\widehat{Q}(t)) \\ &\quad -\frac{\alpha}{\tau}\int_{t-\tau}^{t}e^{\epsilon(s+\tau)}F^T(\widehat{Q}(s))RF(\widehat{Q}(s))ds, \end{aligned} \quad (16)$$

where τ is a constant. For $0 < \tau(t) < \tau$, $\dot{\tau}(t) < \mu < 1$, thus,

$$\begin{aligned} \dot{V}(\widehat{Q}(t),t) &\leq e^{\epsilon t}\Big\{\epsilon\widehat{Q}^T(t)\widehat{Q}(t)+2\widehat{Q}^T(t)\Big[-\widehat{D}\widehat{Q}(t) \\ &\quad +\widehat{A}F(\widehat{Q}(t))+\widehat{B}F(\widehat{Q}(t-\tau(t))) \\ &\quad +\widehat{C}\int_{t-\tau}^{t}F(\widehat{Q}(s))ds\Big] \end{aligned}$$

$$
\begin{aligned}
&+2\alpha\epsilon e^{\epsilon t}\sum_{i=0}^{4n}P_i\int_0^{\widehat{Q}_i(t)}F_i(\rho)d\rho\\
&+2\alpha e^{\epsilon t}F^T(\widehat{Q}(t))P\Big[-\widehat{D}\widehat{Q}(t)+\widehat{A}F(\widehat{Q}(t))\\
&+\widehat{B}F(\widehat{Q}(t-\tau(t)))+\widehat{C}\int_{t-\tau}^{t}F(\widehat{Q}(s))ds\Big]\\
&+(\alpha+\beta)\Big[F^T(\widehat{Q}(t))e^{\epsilon\tau}QF(\widehat{Q}(t))\\
&-(1-\mu)F^T(\widehat{Q}(t-\tau(t)))QF(\widehat{Q}(t-\tau(t)))\Big]\\
&+\alpha e^{\epsilon\tau}F^T(\widehat{Q}(t))RF(\widehat{Q}(t))\\
&-\frac{\alpha}{\tau}\int_{t-\tau}^{t}F^T(\widehat{Q}(s))RF(\widehat{Q}(s))ds\Big\}. \qquad (17)
\end{aligned}
$$

By Assumption 2, $|F_i(\rho)|=|\widehat{f}_i(\rho+Q_i^*)-\widehat{f}_i(Q_i^*)|\le\widehat{G}_i|\rho|$ for each $\rho\in\mathbb{R}$, $i=1,2,\cdots,4n$, and one can obtain

$$
\int_0^{\widehat{Q}_i(t)}F_i(\rho)d\rho\le\frac{1}{2}\widehat{G}_i\widehat{Q}_i^2(t), \qquad (18)
$$

$$
\begin{aligned}
&F^T(\widehat{Q}(t))(-P\widehat{D})\widehat{Q}(t)\\
&\quad\le F^T(\widehat{Q}(t))(-P\widehat{D}\widehat{G}^{-1})F(\widehat{Q}(t)). \qquad (19)
\end{aligned}
$$

By Lemma 3, Eq. (18) and (19),

$$
\begin{aligned}
\dot{V}(\widehat{Q}(t),t)\le e^{\epsilon t}\Big\{&\widehat{Q}^T(t)\Big[\epsilon I-2\widehat{D}+\alpha\epsilon P\widehat{G}\Big]\widehat{Q}(t)\\
&+2\widehat{Q}^T(t)\widehat{A}F(\widehat{Q}(t))\\
&+2\widehat{Q}^T(t)\widehat{B}F(\widehat{Q}(t-\tau(t)))\\
&+2\widehat{Q}^T(t)\widehat{C}\int_{t-\tau}^{t}F(\widehat{Q}(s))ds\\
&+2\alpha F^T(\widehat{Q}(t))(-P\widehat{D}\widehat{G}^{-1})F(\widehat{Q}(t))\\
&+2\alpha F^T(\widehat{Q}(t))P\widehat{A}F(\widehat{Q}(t))\\
&+2\alpha F^T(\widehat{Q}(t))P\widehat{B}F(\widehat{Q}(t-\tau(t)))\\
&+2\alpha F^T(\widehat{Q}(t))P\widehat{C}\int_{t-\tau}^{t}F(\widehat{Q}(s)ds)\\
&+(\alpha+\beta)F^T(\widehat{Q}(t))e^{\epsilon\tau}QF(\widehat{Q}(t))\\
&+\alpha e^{\epsilon\tau}F^T(\widehat{Q}(t))RF(\widehat{Q}(t))\\
&-(\alpha+\beta)(1-\mu)F^T(\widehat{Q}(t-\tau(t)))Q\\
&\cdot F(\widehat{Q}(t-\tau(t)))\\
&-\frac{\alpha}{\tau^2}\Big(\int_{t-\tau}^{t}F(\widehat{Q}(s))ds\Big)^TR\\
&\cdot\Big(\int_{t-\tau}^{t}F(\widehat{Q}(s))ds\Big)\Big\}, \qquad (20)
\end{aligned}
$$

$$
\begin{aligned}
\dot{V}(\widehat{Q}(t),t)\le e^{\epsilon t}\Big\{&-3\widehat{Q}^T(t)[\frac{2}{3}D-\frac{1}{3}\epsilon I-\frac{1}{3}\alpha\epsilon P\widehat{G}]\widehat{Q}(t)\\
&+2\widehat{Q}^T(t)\widehat{A}F(\widehat{Q}(t))\\
&+2\widehat{Q}^T(t)\widehat{B}F(\widehat{Q}(t-\tau(t)))\\
&+2\widehat{Q}^T(t)\widehat{C}\int_{t-\tau}^{t}F(\widehat{Q}(s))ds\\
&+2\alpha F^T(\widehat{Q}(t))(-P\widehat{D}\widehat{G}^{-1})F(\widehat{Q}(t))\\
&+2\alpha F^T(\widehat{Q}(t))P\widehat{A}F(\widehat{Q}(t))\\
&+2\alpha F^T(\widehat{Q}(t))P\widehat{B}F(\widehat{Q}(t-\tau(t)))\\
&+2\alpha F^T(\widehat{Q}(t))P\widehat{C}\int_{t-\tau}^{t}F(\widehat{Q}(s)ds\\
&+(\alpha+\beta)F^T(\widehat{Q}(t))e^{\epsilon\tau}QF(\widehat{Q}(t))\\
&+\alpha e^{\epsilon\tau}F^T(\widehat{Q}(t))RF(\widehat{Q}(t))\\
&-(\alpha+\beta)(1-\mu)F^T(\widehat{Q}(t-\tau(t)))\\
&\cdot QF(\widehat{Q}(t-\tau(t)))\\
&-\frac{\alpha}{\tau^2}\Big(\int_{t-\tau}^{t}F(\widehat{Q}(s))ds\Big)^TR\\
&\cdot\Big(\int_{t-\tau}^{t}F(\widehat{Q}(s))ds\Big)\Big\}. \qquad (21)
\end{aligned}
$$

Let us discuss an inequality:

$$
\begin{aligned}
&-\widehat{Q}^T(t)(\frac{2}{3}D-\frac{1}{3}\epsilon I-\frac{1}{3}\alpha\epsilon P\widehat{G})\widehat{Q}(t)\\
&\quad+2\widehat{Q}^T(t)\widehat{A}F(\widehat{Q}(t))\\
&=-[(\frac{2}{3}D-\frac{1}{3}\epsilon I-\frac{1}{3}\alpha\epsilon P\widehat{G})^{\frac{1}{2}}\widehat{Q}(t)\\
&\quad-(\frac{2}{3}D-\frac{1}{3}\epsilon I-\frac{1}{3}\alpha\epsilon P\widehat{G})^{-\frac{1}{2}}\widehat{A}F(\widehat{Q}(t))]^T\\
&\quad\cdot[(\frac{2}{3}\widehat{D}-\frac{1}{3}\epsilon I-\frac{1}{3}\alpha\epsilon P\widehat{G})^{\frac{1}{2}}\widehat{Q}(t)\\
&\quad-(\frac{2}{3}\widehat{D}-\frac{1}{3}\epsilon I-\frac{1}{3}\alpha\epsilon P\widehat{G})^{-\frac{1}{2}}\widehat{A}F(\widehat{Q}(t))]\\
&\quad+F^T(\widehat{Q}(t))\widehat{A}^T(\frac{2}{3}\widehat{D}-\frac{1}{3}\epsilon I-\frac{1}{3}\alpha\epsilon P\widehat{G})^{-1}\\
&\quad\cdot\widehat{A}F(\widehat{Q}(t))\\
&\le F^T(\widehat{Q}(t))\widehat{A}^T(\frac{2}{3}\widehat{D}-\frac{1}{3}\epsilon I-\frac{1}{3}\alpha\epsilon P\widehat{G})^{-1}\\
&\quad\cdot\widehat{A}F(\widehat{Q}(t)). \qquad (22)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&-\widehat{Q}^T(t)(\frac{2}{3}\widehat{D}-\frac{1}{3}\epsilon I-\frac{1}{3}\alpha\epsilon P\widehat{G})\widehat{Q}(t)\\
&\quad+2\widehat{Q}^T(t)\widehat{B}F(\widehat{Q}(t-\tau(t)))\\
&\le F^T(\widehat{Q}(t-\tau(t)))\widehat{B}^T(\frac{2}{3}D-\frac{1}{3}\epsilon I-\frac{1}{3}\alpha\epsilon P\widehat{G})^{-1}\\
&\quad\cdot\widehat{B}F(\widehat{Q}(t-\tau(t))), \qquad (23)
\end{aligned}
$$

$$
\begin{aligned}
&-\widehat{Q}^T(t)(\frac{2}{3}\widehat{D}-\frac{1}{3}\epsilon I-\frac{1}{3}\alpha\epsilon P\widehat{G})\widehat{Q}(t)\\
&+2\widehat{Q}^T(t)\widehat{C}\int_{t-\tau}^{t}F(\widehat{Q}(s))ds\\
&\le\Big(\int_{t-\tau}^{t}F(\widehat{Q}(s))ds\Big)^T\widehat{C}^T(\frac{2}{3}\widehat{D}-\frac{1}{3}\epsilon I-\frac{1}{3}\alpha\epsilon P\widehat{G})^{-1}\\
&\cdot\widehat{C}\Big(\int_{t-\tau}^{t}F(\widehat{Q}(s))ds\Big), \qquad (24)
\end{aligned}
$$

$$
\begin{aligned}
&-\alpha(1-\mu)F^T(\widehat{Q}(t-\tau(t))QF(\widehat{Q}(t-\tau(t)))\\
&+2\alpha F^T(\widehat{Q}(t))P\widehat{B}F(\widehat{Q}(t-\tau(t)))\\
&\le\alpha F^T(\widehat{Q}(t))P\widehat{B}\big((1-\mu)Q\big)^{-1}\widehat{B}^TPF(\widehat{Q}(t)), \qquad (25)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\alpha\theta}{\tau^2}\Big(\int_{t-\tau}^{t}F(\widehat{Q}(s))ds\Big)^TR\Big(\int_{t-\tau}^{t}F(\widehat{Q}(s))ds\Big)\\
&+2\alpha F^T(\widehat{Q}(t))P\widehat{C}\Big(\int_{t-\tau}^{t}F(\widehat{Q}(s))ds\Big)\\
&\le\frac{\alpha\tau^2}{\theta}F^T(\widehat{Q}(t))P\widehat{C}R^{-1}\widehat{C}^TPF(\widehat{Q}(t)). \qquad (26)
\end{aligned}
$$

where $0<\theta<1$.

By θ, we have

$$
\begin{aligned}
&-\frac{\alpha}{\tau^2}\Big(\int_{t-\tau}^{t}F(\widehat{Q}(s))ds\Big)^TR\Big(\int_{t-\tau}^{t}F(\widehat{Q}(s))ds\Big)\\
&=-\theta\frac{\alpha}{\tau^2}\Big(\int_{t-\tau}^{t}F(\widehat{Q}(s))ds\Big)^TR\Big(\int_{t-\tau}^{t}F(\widehat{Q}(s))ds\Big)
\end{aligned}
$$

$$-(1-\theta)\frac{\alpha}{\tau^2}\Big(\int_{t-\tau}^{t}F(\widehat{Q}(s))ds\Big)^T \cdot R\Big(\int_{t-\tau}^{t}F(\widehat{Q}(s))ds\Big). \tag{27}$$

Thus,

$$\begin{aligned}\dot{V}(\widehat{Q}(t),t)\le e^{\epsilon t}\Big\{&F^T(\widehat{Q}(t))\widehat{A}^T(\frac{2}{3}\widehat{D}-\frac{1}{3}\epsilon I-\frac{1}{3}\alpha\epsilon P\widehat{G})^{-1}\\ &\cdot\widehat{A}F(\widehat{Q}(t))+F^T(\widehat{Q}(t-\tau(t)))\widehat{B}^T\\ &\cdot(\frac{2}{3}\widehat{D}-\frac{1}{3}\epsilon I-\frac{1}{3}\alpha\epsilon P\widehat{G})^{-1}\widehat{B}F(\widehat{Q}(t-\tau(t)))\\ &+\Big(\int_{t-\tau}^{t}F(\widehat{Q}(s))ds\Big)^T\widehat{C}^T\\ &\cdot(\frac{2}{3}\widehat{D}-\frac{1}{3}\epsilon I-\frac{1}{3}\alpha\epsilon P\widehat{G})^{-1}\widehat{C}\\ &\cdot\Big(\int_{t-\tau}^{t}F(\widehat{Q}(s))ds\Big)\\ &+2\alpha F^T(\widehat{Q}(t))(-P\widehat{D}\widehat{G}^{-1})F(\widehat{Q}(t))\\ &+2\alpha F^T(\widehat{Q}(t))P\widehat{A}F(\widehat{Q}(t))\\ &+(\alpha+\beta)F^T(\widehat{Q}(t))e^{\epsilon\tau}QF(\widehat{Q}(t))\\ &-\beta(1-\mu)F^T(\widehat{Q}(t-\tau(t)))QF(\widehat{Q}(t-\tau(t)))\\ &+\alpha F^T(\widehat{Q}(t))P\widehat{B}\,((1-\mu)Q)^{-1}\,\widehat{B}^TPF(\widehat{Q}(t))\\ &+\frac{\alpha\tau^2}{\theta}F^T(\widehat{Q}(t))P\widehat{C}R^{-1}\widehat{C}^TPF(\widehat{Q}(t))\\ &+\alpha e^{\epsilon\tau}F^T(\widehat{Q}(t))RF(\widehat{Q}(t))\\ &-(1-\theta)\frac{\alpha}{\tau^2}\Big(\int_{t-\tau}^{t}F(\widehat{Q}(s))ds\Big)^TR\\ &\cdot\Big(\int_{t-\tau}^{t}F(\widehat{Q}(s))ds\Big)\Big\},\end{aligned} \tag{28}$$

$$\begin{aligned}\dot{V}(\widehat{Q}(t),t)\le e^{\epsilon t}\Big\{&F^T(\widehat{Q}(t))\Big[-\alpha\big[2P\widehat{D}\widehat{G}^{-1}-(P\widehat{A}\\ &+\widehat{A}^TP)-\frac{\alpha+\beta}{\alpha}e^{\epsilon\tau}Q-e^{\epsilon\tau}R\\ &-P\widehat{B}\big((1-\mu)Q\big)^{-1}\widehat{B}^TP\\ &-\frac{1}{\theta}P(\tau\widehat{C})R^{-1}(\tau\widehat{C})^TP\big]+\widehat{A}^T(\frac{2}{3}\widehat{D}-\frac{1}{3}\epsilon I\\ &-\frac{1}{3}\alpha\epsilon P\widehat{G})^{-1}\widehat{A}\Big]F(\widehat{Q}(t))\\ &+F^T(\widehat{Q}(t-\tau(t)))\Big[\widehat{B}^T(\frac{2}{3}D-\frac{1}{3}\epsilon I\\ &-\frac{1}{3}\alpha\epsilon P\widehat{G})^{-1}\widehat{B}-\beta(1-\mu)Q\Big]F(\widehat{Q}(t-\tau(t)))\\ &+\Big(\int_{t-\tau}^{t}F(\widehat{Q}(s))ds\Big)^T\Big[\widehat{C}^T(\frac{2}{3}\widehat{D}-\frac{1}{3}\epsilon I\\ &-\frac{1}{3}\alpha\epsilon P\widehat{G})^{-1}\widehat{C}-(1-\theta)\frac{\alpha}{\tau^2}R\Big]\\ &\cdot\Big(\int_{t-\tau}^{t}F(\widehat{Q}(s))ds\Big)\Big\},\end{aligned} \tag{29}$$

where $F^T(\widehat{Q}(t))2P\widehat{A}F(\widehat{Q}(t)) = F^T(\widehat{Q}(t))P\widehat{A}F(\widehat{Q}(t)) + F^T(\widehat{Q}(t))\widehat{A}^TPF(\widehat{Q}(t))$.

Now, we choose appropriate ϵ, α, β, θ to make $\dot{V}(\widehat{Q}(t),t)\le 0$. Firstly, let $\varepsilon=\epsilon\alpha$, and one can obtain

$$\begin{aligned}\dot{V}(\widehat{Q}(t),t)\le e^{\frac{\varepsilon}{\alpha}t}\Big\{&F^T(\widehat{Q}(t))\Big[-\alpha\big[2P\widehat{D}\widehat{G}^{-1}-(P\widehat{A}\\ &+\widehat{A}^TP)-\frac{(\alpha+\beta)}{\alpha}e^{\frac{\varepsilon}{\alpha}\tau}Q-e^{\frac{\varepsilon}{\alpha}\tau}R\\ &-P\widehat{B}\big((1-\mu)Q\big)^{-1}\widehat{B}^TP\\ &-\frac{1}{\theta}P(\tau\widehat{C})R^{-1}(\tau\widehat{C})^TP\big]+\widehat{A}^T(\frac{2}{3}\widehat{D}-\frac{1}{3}\frac{\varepsilon}{\alpha}I\\ &-\frac{1}{3}\varepsilon P\widehat{G})^{-1}\widehat{A}\Big]F(\widehat{Q}(t))+F^T(\widehat{Q}(t-\tau(t)))\\ &\cdot\Big[\widehat{B}^T(\frac{2}{3}\widehat{D}-\frac{1}{3}\frac{\varepsilon}{\alpha}I-\frac{1}{3}\varepsilon P\widehat{G})^{-1}\widehat{B}\\ &-\beta(1-\mu)Q\Big]F(\widehat{Q}(t-\tau(t)))\\ &+\Big(\int_{t-\tau}^{t}F(\widehat{Q}(s))ds\Big)^T\Big[\widehat{C}^T(\frac{2}{3}\widehat{D}-\frac{1}{3}\frac{\varepsilon}{\alpha}I\\ &-\frac{1}{3}\varepsilon P\widehat{G})^{-1}\widehat{C}-(1-\theta)\frac{\alpha}{\tau^2}R\Big]\\ &\cdot\Big(\int_{t-\tau}^{t}F(\widehat{Q}(s))ds\Big)\Big\},\end{aligned} \tag{30}$$

As described in [29], we choose a fixed positive β such that

$$\beta\ge\frac{\parallel B\parallel^2\parallel(\frac{2}{3}\widehat{D})^{-1}\parallel}{(1-\mu)\min_i Q_i}. \tag{31}$$

Secondly, we choose a sufficiently small $\varepsilon>0$ and a sufficiently large $\alpha>0$ such that

$$\frac{2}{3}\widehat{D}-\frac{\varepsilon}{3\alpha}I-\frac{1}{3}\varepsilon P\widehat{G}>0, \tag{32}$$

$$\frac{\varepsilon}{2\alpha}\left\|\widehat{D}^{-1}\right\|+\frac{\varepsilon}{2}\left\|P\widehat{D}\widehat{G}^{-1}\right\|\le 1-\frac{\parallel\widehat{B}\parallel^2\parallel(\frac{2}{3}\widehat{D})^{-1}\parallel}{\beta(1-\mu)\min_i Q_i}. \tag{33}$$

By the inequality (13) and Lemma 1, we have

$$\begin{aligned}2P\widehat{D}\widehat{G}^{-1}-(P\widehat{A}+\widehat{A}^TP)-P\widehat{B}\big((1-\mu)Q\big)^{-1}(\widehat{B})^TP&\\ -P(\tau\widehat{C})R^{-1}(\tau\widehat{C})^TP-Q-R>0.&\end{aligned} \tag{34}$$

So a sufficiently small $\varepsilon>0$, an appropriate θ and a sufficiently large α can meet the following inequality:

$$\begin{aligned}&-\alpha\Big[2P\widehat{D}\widehat{G}^{-1}-(P\widehat{A}+\widehat{A}^TP)-\frac{(\alpha+\beta)}{\alpha}e^{\frac{\varepsilon}{\alpha}\tau}Q\\ &-e^{\frac{\varepsilon}{\alpha}\tau}R-P\widehat{B}\big((1-\mu)Q\big)^{-1}\widehat{B}^TP\\ &-\frac{1}{\theta}P(\tau\widehat{C})R^{-1}(\tau\widehat{C})^TP\Big]\\ &+\widehat{A}^T(\frac{2}{3}\widehat{D}-\frac{1}{3}\frac{\varepsilon}{\alpha}I-\frac{1}{3}\varepsilon P\widehat{G})^{-1}\widehat{A}\le 0.\end{aligned} \tag{35}$$

From Eq. (33) and $\left\|(I-F)^{-1}\right\|\le\frac{1}{1-\|F\|}$ ([38]), we also have

$$\begin{aligned}\beta&\ge\frac{\|\widehat{B}\|^2\left\|(\frac{2}{3}\widehat{D})^{-1}\right\|}{(1-\mu)\min_i Q_i\big(1-(\frac{\varepsilon}{2\alpha}\left\|\widehat{D}^{-1}\right\|+\frac{\varepsilon}{2}\left\|P\widehat{G}\widehat{D}^{-1}\right\|)\big)}\\ &\ge\frac{\|\widehat{B}\|^2\left\|(\frac{2}{3}\widehat{D})^{-1}\right\|}{(1-\mu)\min_i Q_i}\left\|\Big(I-\frac{\varepsilon}{2\alpha}\widehat{D}^{-1}-\frac{\varepsilon}{2}P\widehat{G}\widehat{D}^{-1}\Big)^{-1}\right\|\\ &\ge\frac{\|\widehat{B}\|^2\left\|(\frac{2}{3}\widehat{D}-\frac{\varepsilon}{3\alpha}I-\frac{\varepsilon}{3}P\widehat{G})^{-1}\right\|}{(1-\mu)\min_i Q_i}.\end{aligned} \tag{36}$$

Thus, by Eq.(32), $\frac{2}{3}\widehat{D}-\frac{\varepsilon}{3\alpha}I-\frac{1}{3}\varepsilon P\widehat{G}$ is a positive diagonal matrix, and we have

$$\begin{aligned}
&F^T(\widehat{Q}(t-\tau(t)))\Big[\widehat{B}^T\left(\frac{2}{3}\widehat{D}-\frac{1}{3}\frac{\varepsilon}{\alpha}I-\frac{1}{3}\varepsilon P\widehat{G}\right)^{-1}\widehat{B}\\
&\quad-\beta(1-\mu)Q\Big]F(\widehat{Q}(t-\tau(t)))\\
&\leq\lambda_{max}\left(\frac{2}{3}\widehat{D}-\frac{\varepsilon}{3\alpha}I-\frac{1}{3}\varepsilon P\widehat{G}\right)^{-1}\\
&\quad\cdot\left\|\widehat{B}F(\widehat{Q}(t-\tau(t))\right\|^2\\
&\quad-\lambda_{min}[\beta(1-\mu)Q]\left\|F(\widehat{Q}(t-\tau(t))\right\|^2\\
&\leq\left\|\left(\frac{2}{3}\widehat{D}-\frac{\varepsilon}{3\alpha}I-\frac{1}{3}\varepsilon P\widehat{G}\right)^{-1}\right\|\|\widehat{B}\|^2\\
&\quad\cdot\left\|F(\widehat{Q}(t-\tau(t)))\right\|^2\\
&\quad-\lambda_{min}[\beta(1-\mu)Q]\left\|F(\widehat{Q}(t-\tau(t))\right\|^2\\
&\leq 0.
\end{aligned}\tag{37}$$

Similarly, given appropriate θ satisfing $0<\theta<1-\frac{\|\widehat{C}\|^2\left\|\frac{2}{3}\widehat{D}-\frac{\varepsilon}{3\alpha}I-\frac{1}{3}\varepsilon P\widehat{G})^{-1}\widehat{C}\right\|}{\frac{\alpha}{\tau^2}\min_i R_i}$, we can obtain

$$\begin{aligned}
&\left(\int_{t-\tau}^{t}F(\widehat{Q}(s))ds\right)^T\left[\widehat{C}^T\left(\frac{2}{3}\widehat{D}-\frac{1}{3}\frac{\varepsilon}{\alpha}I-\frac{1}{3}\varepsilon P\widehat{G}\right)^{-1}\widehat{C}\right.\\
&\quad\left.-(1-\theta)\frac{\alpha}{\tau^2}R\right]\left(\int_{t-\tau}^{t}F(\widehat{Q}(s))ds\right)\leq 0.
\end{aligned}\tag{38}$$

By Eqs. (35), (37) and (38), we can easily get

$$\dot{V}(\widehat{Q}(t),t)\leq 0.\tag{39}$$

Therefore, $V(\widehat{Q}(t),t)\leq V(Q(0),0)$, $e^{\epsilon t}\widehat{Q}^T(t)\widehat{Q}(t)\leq V(\widehat{Q}(t),t)\leq V(\widehat{Q}(0),0)$, and then $\widehat{Q}^T(t)\widehat{Q}(t)\leq V(\widehat{Q}(0),0)e^{-\epsilon t}$. Consequently, the trivial solution of (14) is globally exponentially stable, which implies that the equilibrium point of system (6) is globally exponentially stable. This completes the proof. ■

Remark 3: In Eq. (35), by choosing a sufficiently large α, a fixed β, and a sufficiently small ε, we can make some coefficients of Eq. (35) smaller, that is, $\frac{\alpha+\beta}{\alpha}e^{\frac{\varepsilon}{\alpha}\tau}\to 1, e^{\frac{\varepsilon}{\alpha}\tau}\to 1$. However, for θ, except for satisfying (38), it can't be too small, therefore, an appropriate θ is needed.

Remark 4: In [29], βQ is a diagonal matrix and $B^T(D-\frac{\varepsilon}{2\alpha}I-\frac{\varepsilon}{PG})^{-1}B$ is a symmetrical matrix. So we don't think it's proper to write the inequality, $\beta Q\geq B^T(D-\frac{\varepsilon}{2\alpha}I-\frac{\varepsilon}{PG})^{-1}B$ (Eq.(22) in [29]). In proof of our Theorem 2, we use a transformation method used in (37).

Remark 5: Theorem 1 can be regarded as a corollary of Theorem 2, when $\mu=0$.

In [33], Chen and Wang investigated the globally power stability of RVNNs with unbounded time-varying delays. Next, inspired by their method, we will investigate the globally power stability of RVVDE (6) to obtain the criterion of the globally power stability of QVNN (1). As far as we know, few papers concern the globally power stability of QVNNs.

Theorem 3: Under Assumption 2 and suppose $\tau(t)\leq\eta t$, $0<\eta<1$, system (6) has a unique equilibrium point Q^* which is globally power stable if there exist positive diagonal matrices Y, U and V and a positive definite matrix P_1, such that the following LMI:

$$\begin{pmatrix}
P_1\widehat{D}+\widehat{D}P_1-Y-P_1 & -P_1\widehat{A} & -P_1\widehat{B} & -P_1\widehat{C}\\
-\widehat{A}^TP_1 & Y & 0 & 0\\
-\widehat{B}^TP_1 & 0 & U & 0\\
-\widehat{C}^TP_1 & 0 & 0 & V
\end{pmatrix}>0,$$
$$P_1\geq U, P_1\geq V\tag{40}$$

holds.

Proof: By Eq. (40) we can find three sufficiently small constants $\alpha_1>0$ and $\gamma>0$ and $\varpi>0$, such that

$$\begin{pmatrix}
P_1\widehat{D}+\widehat{D}P_1-Y-\star & -P_1\widehat{A} & -P_1\widehat{B} & P_1\widehat{C}\\
-\widehat{A}^TP_1 & Y & 0 & 0\\
-\widehat{B}^TP_1 & 0 & U & 0\\
-\widehat{C}^TP_1 & 0 & 0 & R
\end{pmatrix}>0,\tag{41}$$

where $*=\alpha_1P_1+(1-\eta)^{-\gamma}P_1+\varpi P_1$. And we can find a sufficiently large T, for all $t\geq T$, such that $\frac{\gamma}{t}<\alpha_1$, $0<\frac{\tau t}{1-\gamma}\left[1-(1-\frac{\tau}{t})^{1-\gamma}\right]<\varpi$.

Define a function $V_1(\widehat{Q}(t))=t^{\gamma}\widehat{Q}(t)^TP_1\widehat{Q}(t)$, $t\geq T$, where $\widehat{Q}(t)=Q(t)-Q^*$, $F(\widehat{Q}(t))=\widehat{f}(\widehat{Q}(t)+Q^*)-\widehat{f}(Q^*)$. And denote $M(t)=\sup_{0\leq s\leq t}V_1(\widehat{Q}(s))$, which is a non-decreasing function and obviously, $V_1(\widehat{Q}(t))\leq M(t)$.

Now, we will prove that $M(t)$ is bounded, that is, for all $t\geq T$, we have $M(t)=M(T)$.

Firstly, for any $t\geq T$, if $V_1(\widehat{Q}(t))<M(t)$, $M(t)$ is a non-increasing function, and then $M(t)=M(T)$.

Secondly, if $V_1(\widehat{Q}(t))=M(t)$, one can obtain

$$\begin{aligned}
&\frac{dV_1(\widehat{Q}(t))}{dt}\\
&=\gamma t^{\gamma-1}\widehat{Q}(t)^TP_1\widehat{Q}(t)+2t^{\gamma}\widehat{Q}(t)^TP_1\Big[-\widehat{D}\widehat{Q}(t)\\
&\quad+\widehat{A}F(\widehat{Q}(t))+\widehat{B}F(\widehat{Q}(t-\tau(t)))\\
&\quad+\widehat{C}\int_{t-\tau}^{t}F(\widehat{Q}(s)ds\Big]\\
&=t^{\gamma}\Big\{\widehat{Q}(t)^T(\frac{\gamma}{t}P_1-2P_1\widehat{D})\widehat{Q}(t)\\
&\quad+2\widehat{Q}(t)^TP_1\widehat{A}F(\widehat{Q}(t))\\
&\quad+2\widehat{Q}(t)^TP_1\widehat{B}F(\widehat{Q}(t-\tau(t)))\\
&\quad+2\widehat{Q}(t)^TP_1\widehat{C}\int_{t-\tau}^{t}F(\widehat{Q}(s)ds\Big\}\\
&\leq t^{\gamma}\Big\{\widehat{Q}(t)^T(\frac{\gamma}{t}P_1-2P_1\widehat{D})\widehat{Q}(t)\\
&\quad+\widehat{Q}(t)^TP_1\widehat{A}Y^{-1}(P_1\widehat{A})^T\widehat{Q}(t)\\
&\quad+F(\widehat{Q}(t))^TYF(\widehat{Q}(t))\\
&\quad+\widehat{Q}(t)^TP_1\widehat{B}U^{-1}(P_1\widehat{B})^T\widehat{Q}(t)\\
&\quad+F(\widehat{Q}(t-\tau(t)))^TUF(\widehat{Q}(t-\tau(t)))\\
&\quad+\widehat{Q}(t)^TP_1\widehat{C}V^{-1}(P_1\widehat{C})^T\widehat{Q}(t)\\
&\quad+\Big(\int_{t-\tau}^{t}F(\widehat{Q}(s))ds\Big)^TV\Big(\int_{t-\tau}^{t}F(\widehat{Q}(s)ds)\Big\}\\
&\leq t^{\gamma}\Big\{\widehat{Q}(t)^T\big[\frac{\gamma}{t}P_1-2P_1\widehat{D}+P_1\widehat{A}Y^{-1}(P_1\widehat{A})^T\\
&\quad+Y+P_1\widehat{B}U^{-1}(P_1\widehat{B})^T\\
&\quad+P_1\widehat{C}V^{-1}(P_1\widehat{C})^T\big]\widehat{Q}(t)
\end{aligned}$$

$$+F(\widehat{Q}(t-\tau(t)))^T UF(\widehat{Q}(t-\tau(t))) + \tau\int_{t-\tau}^{t} F(\widehat{Q}(s))^T VF(\widehat{Q}(s))ds\Bigg\}, \tag{42}$$

where we use Lemma 3, Lemma 4 and Assumption 2 with $\widehat{G}=I$.

Since $M(t)=\sup_{0\le s\le t} V_1(\widehat{Q}(s))$, when $s\in[t-\tau,\ t]\subset[0,\ t]$, $V_1(\widehat{Q}(s))\le M(t)=V_1(\widehat{Q}(t))$. And for $P_1\ge U$, $P_1\ge V$ and Assumption 2, one can obtain

$$\begin{aligned} &t^\gamma F^T(\widehat{Q}(t-\tau(t)))UF(\widehat{Q}(t-\tau(t)))\\ &\quad\le t^\gamma \widehat{Q}(t-\tau(t))^T P_1\widehat{Q}(t-\tau(t))\\ &\quad=\left(\frac{t}{t-\tau(t)}\right)^\gamma V_1\left(\widehat{Q}(t-\tau(t))\right)\\ &\quad\le\left(\frac{t}{t-\tau(t)}\right)^\gamma V_1\left(\widehat{Q}(t)\right)\\ &\quad\le(1-\eta)^{-\gamma}t^\gamma\widehat{Q}(t)^T P_1\widehat{Q}(t), \end{aligned}\tag{43}$$

$$\begin{aligned} &t^\gamma\tau\int_{t-\tau}^{t} F^T(\widehat{Q}(s))VF(\widehat{Q}(s))ds\\ &\quad\le t^\gamma\tau\int_{t-\tau}^{t}\widehat{Q}(s)^T V\widehat{Q}(s)ds\\ &\quad\le t^\gamma\tau\int_{t-\tau}^{t}\widehat{Q}(s)^T P_1\widehat{Q}(s)ds\\ &\quad= t^\gamma\tau\int_{t-\tau}^{t}\frac{1}{s^\gamma}V_1(\widehat{Q}(s))ds\\ &\quad\le t^\gamma\tau\int_{t-\tau}^{t}\frac{1}{s^\gamma}V_1(\widehat{Q}(t))ds\\ &\quad= t^\gamma V_1(\widehat{Q}(t))\tau\int_{t-\tau}^{t}\frac{1}{s^\gamma}ds\\ &\quad=\frac{\tau t}{1-\gamma}[1-(1-\frac{\tau}{t})^{1-\gamma}]V_1(\widehat{Q}(t))\\ &\quad=\frac{\tau t^{\gamma+1}}{1-\gamma}[1-(1-\frac{\tau}{t})^{1-\gamma}]Q(t)^T P_1 Q(t). \end{aligned}\tag{44}$$

Thus, Eq. (42) can be written as follow:

$$\begin{aligned} &\frac{dV_1(\widehat{Q}(t))}{dt}\\ &\quad\le t^\gamma\widehat{Q}(t)^T\Bigg[\frac{\gamma}{t}P_1-P_1\widehat{D}-\widehat{D}P_1+P_1\widehat{A}Y^{-1}(P_1\widehat{A})^T\\ &\qquad+Y+P_1\widehat{B}U^{-1}(P_1\widehat{B})^T+P_1\widehat{C}V^{-1}(P_1\widehat{C})^T\\ &\qquad+(1-\eta)^{-\gamma}P_1+\frac{\tau t}{1-\gamma}[1-(1-\frac{\tau}{t})^{1-\gamma}]P_1\Bigg]\widehat{Q}(t)\\ &\quad\le t^\gamma\widehat{Q}(t)^T\Big[\alpha_1P_1-P_1\widehat{D}-\widehat{D}P_1+P_1\widehat{A}Y^{-1}(P_1\widehat{A})^T\\ &\qquad+Y+P_1\widehat{B}U^{-1}(P_1\widehat{B})^T+P_1\widehat{C}V^{-1}(P_1\widehat{C})^T\\ &\qquad+(1-\eta)^{-\gamma}P_1+\varpi P_1\Big]\widehat{Q}(t). \end{aligned}\tag{45}$$

By Lemma 1, Eq. (41) is equivalent to the following inequality:

$$\begin{aligned} &P_1\widehat{D}+\widehat{D}P_1-\alpha_1P_1-Y-(1-\eta)^{-\gamma}P_1-\varpi P_1\\ &\quad-P_1\widehat{A}Y^{-1}(P_1\widehat{A})^T-P_1\widehat{B}U^{-1}(P_1\widehat{B})^T\\ &\quad-P_1\widehat{C}V^{-1}(P_1\widehat{C})^T>0. \end{aligned}\tag{46}$$

Therefore, $\frac{dV_1(\widehat{Q}(t))}{dt}\le 0$, when $V_1(\widehat{Q}(t))=M(t)$, for all $t\ge T$. Thus, $M(t)$ is also non-increasing at t, and then $M(t)=M(T)$.

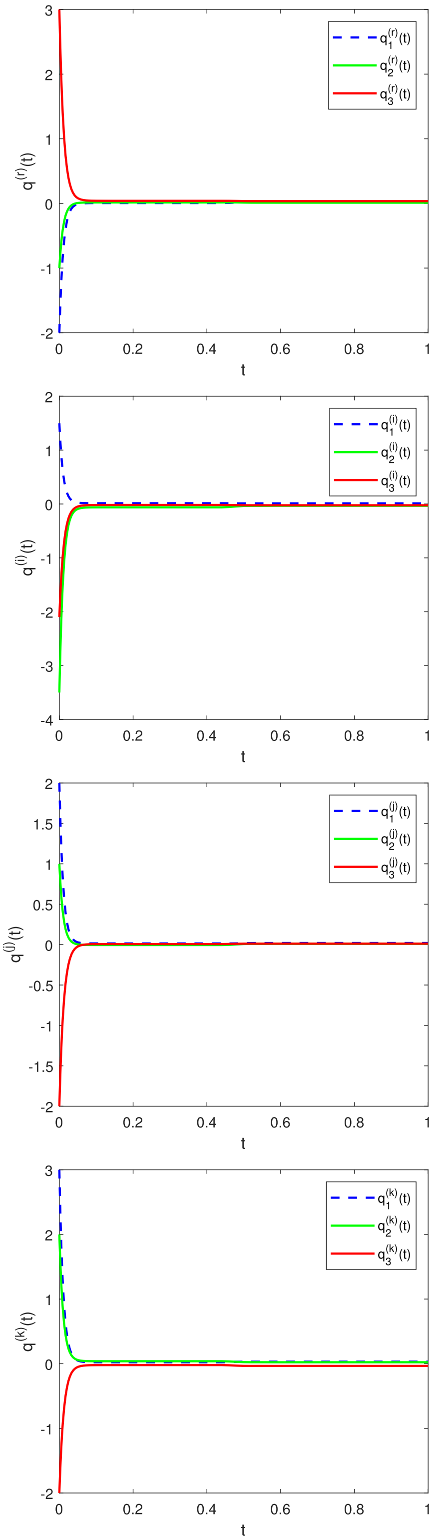

FIGURE 1. The state trajectories of $q^r(t), q^i(t), q^j(t), q^k(t)$ of QVNN (52).

In short, $M(t)=M(T)$ for all $t\ge T$. Therefore, for all $t\ge T$, $V(\widehat{Q}(t))\le M(T)$, and then, $t^\gamma\lambda_{max}(P_1)\left\|\widehat{Q}(t)\right\|^2\le M(T)$, thus, $\left\|\widehat{Q}(t)\right\|\le\lambda_{max}^{-\frac{1}{2}}(P_1)M(T)^{\frac{1}{2}}t^{\frac{-\gamma}{2}}$.

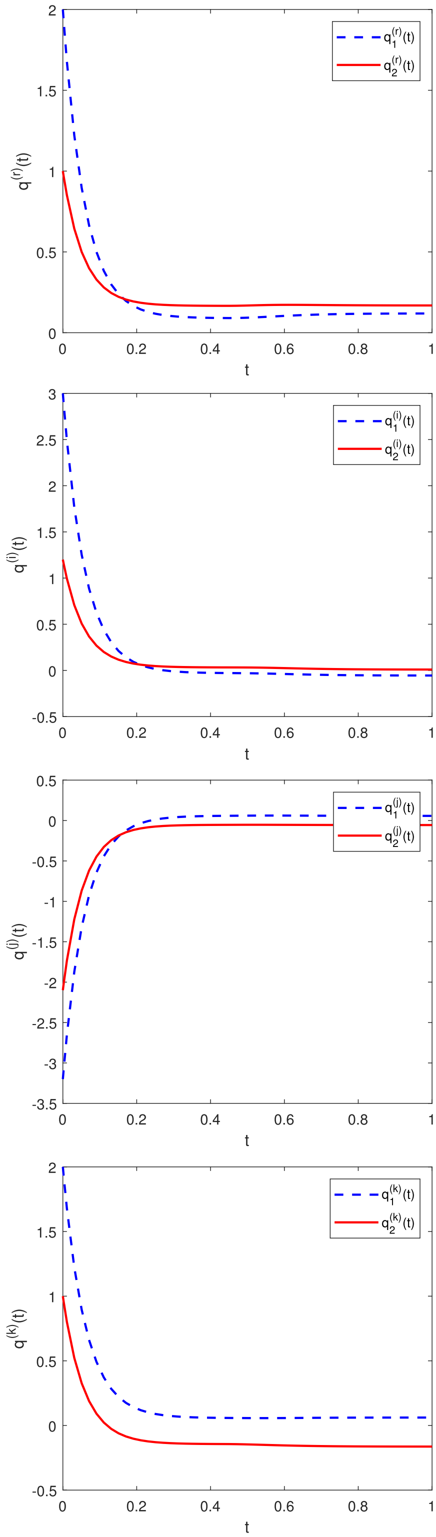

FIGURE 2. **The state trajectories of $q^r(t)$, $q^i(t)$, $q^j(t)$, $q^k(t)$ of QVNN (53).**

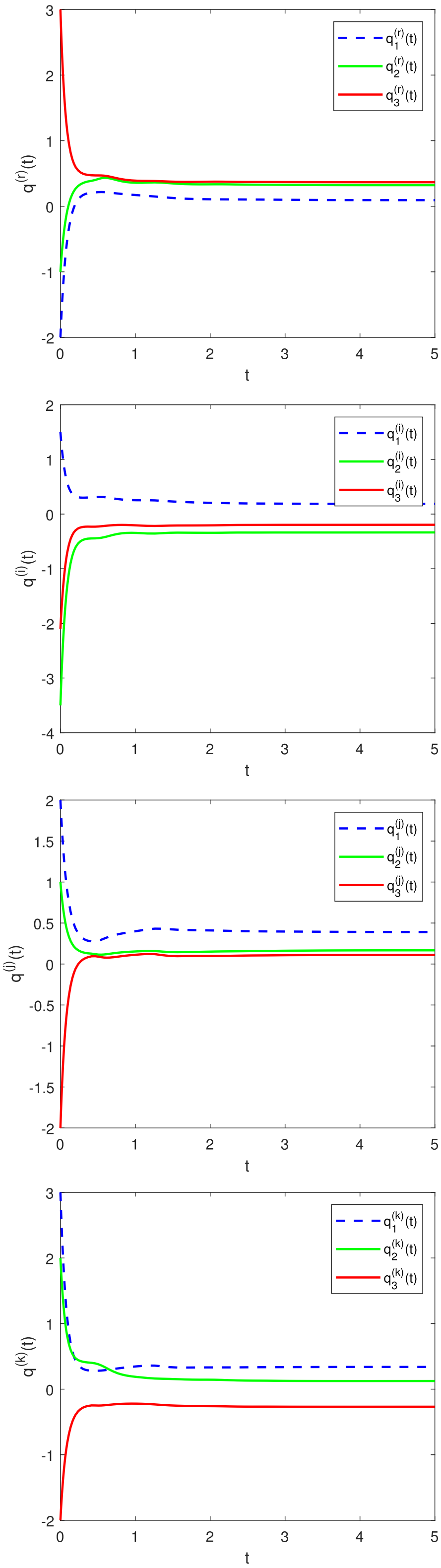

FIGURE 3. **The state trajectories of $q^r(t)$, $q^i(t)$, $q^j(t)$, $q^k(t)$ of Example 4.1 in [27].**

Then, $Q(t)$ converges to Q^* with power convergence rate $-\frac{\gamma}{2}$ in the system (6). This completes the proof. ■

Remark 6: Theorem 3 applies to both QVNNs and RVNNs. And Theorem 3 can also be regarded as a generalization of [33].

Remark 7: For $0 < \frac{\tau t}{1-\gamma}\left[1-(1-\frac{\tau}{t})^{1-\gamma}\right] < \varpi$, let $\varphi(t,\gamma) = \frac{\tau t}{1-\gamma}\left[1-(1-\frac{\tau}{t})^{1-\gamma}\right]$, $0 < \gamma < 1$, $t \geq T$, $t > \tau$, we can easily find that $\varphi(t,\gamma)$ is a monotone increasing function about t with a fixed γ, and $\frac{\tau T}{1-\gamma}\left[1-(1-\frac{\tau}{T})^{1-\gamma}\right] < \varphi(t,\gamma) < \tau^2$. Furthermore, similarly, we can find $\varphi(t,\gamma)$ is a monotone decreasing function about γ with a fixed t, and $\lim\limits_{\gamma\to 1}\frac{\tau T}{1-\gamma}\left[1-(1-\frac{\tau}{T})^{1-\gamma}\right] = -ln(1-\frac{\tau}{T})$. Therefore, we can find a appropriate T and γ, for all $t \geq T$, such that $\frac{\gamma}{t} < \alpha_1$, $0 < \frac{\tau t}{1-\gamma}\left[1-(1-\frac{\tau}{t})^{1-\gamma}\right] < \varpi$.

If the QVNN (1) is studied without distributed delays, i.e. $C = 0$, then the following QVNN model can be obtained

$$\dot{q}(t) = -Dq(t) + Af(q(t)) + Bf(q(t-\tau(t))) + U \tag{47}$$

which is the QVNN model discussed in [23].

By decomposing system (47), one gets a matrix differential equation as follow:

$$\dot{Q}(t) = -\widehat{D}Q(t) + \widehat{A}\widehat{f}(Q(t)) + \widehat{B}\widehat{f}(Q(t-\tau(t))) + \widehat{U}. \tag{48}$$

where $\widehat{D}$, $\widehat{A}$, $\widehat{B}$, $\widehat{U}$, $\widehat{f}(Q(t))$ are same as those in system (6). By transformation $\widehat{Q}(t) = Q(t) - Q^*$, one can get the following vector-matrix form of real-valued system (49):

$$\dot{\widehat{Q}}(t) = -\widehat{D}\widehat{Q}(t) + \widehat{A}F(\widehat{Q}(t)) + \widehat{B}F(\widehat{Q}(t-\tau(t))), \tag{49}$$

where $F(\widehat{Q}(t)) = \widehat{f}(\widehat{Q}(t)+Q^*) - \widehat{f}(Q^*)$.

Moreover, the QVNN (47) shares the same equilibrium point and identical dynamics behavior with system (49) when $q(t) = q^{(r)}(t) + iq^{(i)}(t) + jq^{(j)}(t) + kq^{(k)}(t)$ corresponds to $Q(t) = (q^{(r)}(t)^T,\ q^{(i)}(t)^T,\ q^{(j)}(t)^T,\ q^{(k)}(t)^T)^T$.

Similar to Theorem 2 and Theorem 3, one can get the following corollaries.

Corollary 1: Under Assumption 2, there exist positive diagonal matrices P and Q such that:

$$2P\widehat{D}\widehat{G}^{-1} - (P\widehat{A} + \widehat{A}^T P) - P\widehat{B}Q^{-1}(\widehat{B})^T P - Q > 0, \tag{50}$$

then, for each $U \in \mathbb{R}^n$, system (49) has a unique equilibrium point which is globally exponentially stable, independent of the delays.

Corollary 2: Suppose $\tau(t) \leq \eta t$, $0 < \eta < 1$, $\widehat{G} = I$, system (49) has a unique equilibrium point Q^* and Q^* is globally power stable if there exist positive diagonal matrices Y, U and a positive definite matrix P_1, such that the following LMI:

$$\begin{pmatrix} P_1D + DP_1 - Y - P_1 & -P_1A & -P_1B \\ -A^TP_1 & Y & 0 \\ -B^TP_1 & 0 & U \end{pmatrix} > 0,$$
$$P_1 \geq U \tag{51}$$

holds.

Remark 8: Neural network is of great value to military, image processing, artificial intelligence, and other application fields. In these applications of neural networks, stability is an important factor. For fewer results on the stability of QVNNs with mixed delays, it is necessary to discuss the dynamics behavior of these QVNNs. Based on the rule of quaternion operation, the existence, uniqueness and globally exponential stability of the equilibrium of QVNNs with both discrete and distributed delays are investigated in [27]. However, compared with the method of [27], our decomposing method is simpler and requires fewer conditionalities.

IV. ILLUSTRATIVE EXAMPLES

In this section, two numerical examples are provided to illustrate the effectiveness and superiority of the proposed results.

Example 1: Consider the QVNN model as follow:

$$\dot{q}(t) = -Dq(t) + Af(q(t)) + Bf(q(t-\tau(t))) + C\int_{t-\tau}^{t} f(q(s))ds + U, \tag{52}$$

where

$$A = \begin{pmatrix} 1.3+5.2i-0.5j-1.3k & 1.6+0.5i-3.7j+1.3k & 9.6-2.5i-1.9j+2.4k \\ -1.8+1.1i+1.5j+1.6k & 1.2+3.1i+3.5j+1.2k & -0.6+1.5i+1.9j+2.4k \\ 1.8+1.1i+3.5j+0.6k & -5.1-5.1i+3.6i+1.6k & 1.7-1.6i+1.9j+0.4k \end{pmatrix},$$

$$B = \begin{pmatrix} 1.3+0.3i+1.4j+1.3k & 1.3-2.5i-1.3j+0.6k & 1.4-2.0-3.3j+1.6k \\ -2.3+3.1i-1.5j-1.4k & 1.2+3.3i+1.6j+1.3k & 1.2-1.3i+2.6j+1.3k \\ 1.4+1.3i-1.3j-2.2k & -1.5+2.2i+0.3j-1.6k & -1.3-0.2i-1.1j-1.0k \end{pmatrix},$$

$$C = \begin{pmatrix} 2.4+1.3i+0.1j+1.4k & 1.6+1.1i+1.2j+1.1k & 9.6+1.1i+1.2j+1.1k \\ -1.8+0.1i+1.2j+1.6k & 1.2+3.3i+1.4j+3.2k & -0.6+2.1i+0.2j+0.1k \\ 1.8+1.5i+0.8j+1.7k & -5.1+1.3i+2.4j+1.2k & 1.7+1.1i-1.2j+2.1k \end{pmatrix},$$

$$D = diag\{95, 95, 85\},$$

$$U = (2+1.2\,i+2\,j+3\,k,\ 1-3.2\,i+j+2\,k,\ 3-2.1\,i+j-3\,k)^T,$$

$$f(q(t)) = 0.18\tanh(q(t)), \quad \tau(t) = 0.45\sin t + 0.35,\ \tau = 0.6.$$

By using the LMI toolbox in MATLAB, QVNN (52) has the following feasible solutions:

$$P = \begin{pmatrix} 0.0028 & -0.0000 & 0.0000 & -0.0000 & -0.0000 & 0.0000 \\ -0.0000 & 0.0028 & -0.0000 & 0.0000 & -0.0000 & -0.0000 \\ 0.0000 & -0.0000 & 0.0032 & -0.0000 & 0.0000 & -0.0000 \\ -0.0000 & 0.0000 & -0.0000 & 0.0028 & -0.0000 & 0.0000 \\ -0.0000 & -0.0000 & 0.0000 & -0.0000 & 0.0028 & -0.0000 \\ 0.0000 & -0.0000 & -0.0000 & 0.0000 & -0.0000 & 0.0032 \\ 0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & -0.0000 & 0.0000 & -0.0000 & -0.0000 \\ 0.0000 & 0.0000 & 0.0000 & -0.0000 & -0.0000 & -0.0000 \\ -0.0000 & -0.0000 & 0.0000 & -0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & -0.0000 & -0.0000 & 0.0000 \\ -0.0000 & -0.0000 & 0.0000 & -0.0000 & -0.0000 & 0.0000 \end{pmatrix}$$

$$
\left.\begin{matrix}
0.0000 & 0.0000 & 0.0000 & -0.0000 & 0.0000 & -0.0000\\
-0.0000 & 0.0000 & 0.0000 & -0.0000 & 0.0000 & -0.0000\\
-0.0000 & -0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000\\
0.0000 & 0.0000 & -0.0000 & -0.0000 & -0.0000 & -0.0000\\
0.0000 & -0.0000 & -0.0000 & 0.0000 & -0.0000 & -0.0000\\
0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 & 0.0000\\
0.0028 & -0.0000 & 0.0000 & 0.0000 & -0.0000 & 0.0000\\
-0.0000 & 0.0028 & -0.0000 & -0.0000 & 0.0000 & -0.0000\\
0.0000 & -0.0000 & 0.0032 & -0.0000 & 0.0000 & 0.0000\\
0.0000 & -0.0000 & -0.0000 & 0.0028 & -0.0000 & 0.0000\\
-0.0000 & 0.0000 & 0.0000 & -0.0000 & 0.0028 & -0.0000\\
0.0000 & -0.0000 & 0.0000 & 0.0000 & -0.0000 & 0.0032
\end{matrix}\right),
$$

$$
\begin{aligned}
Q &= diag\{1.0863, 1.0863, 1.0863, 1.0863,\\
&\quad 1.0863, 1.0863, 1.0863, 1.0863\},\\
R &= diag\{0.9390, 0.9390, 0.9390, 0.9390,\\
&\quad 0.9390, 0.9390, 0.9390, 0.9390\}.
\end{aligned}
$$

$$
P_1 = \left(\begin{matrix}
3.8786 & -0.0298 & -0.0318 & 0.0000 & -0.0546 & -0.0466\\
-0.0298 & 3.7339 & -0.1123 & 0.0546 & 0.0000 & 0.0603\\
-0.0318 & -0.1123 & 3.9775 & 0.0466 & -0.0603 & -0.0000\\
0.0000 & 0.0546 & 0.0466 & 3.8786 & -0.0298 & -0.0318\\
-0.0546 & 0.0000 & -0.0603 & -0.0298 & 3.7339 & -0.1123\\
-0.0466 & 0.0603 & -0.0000 & -0.0318 & -0.1123 & 3.9775\\
-0.0000 & 0.1037 & -0.0224 & -0.0000 & -0.0294 & -0.0418\\
-0.1037 & 0.0000 & 0.0986 & 0.0294 & 0.0000 & -0.0318\\
0.0224 & -0.0986 & -0.0000 & 0.0418 & 0.0318 & -0.0000\\
-0.0000 & -0.0294 & -0.0418 & 0.0000 & -0.1037 & 0.0224\\
0.0294 & -0.0000 & -0.0318 & 0.1037 & -0.0000 & -0.0986\\
0.0418 & 0.0318 & 0.0000 & -0.0224 & 0.0986 & 0.0000
\end{matrix}\right.
$$

$$
\left.\begin{matrix}
-0.0000 & -0.1037 & 0.0224 & -0.0000 & 0.0294 & 0.0418\\
0.1037 & 0.0000 & -0.0986 & -0.0294 & -0.0000 & 0.0318\\
-0.0224 & 0.0986 & -0.0000 & -0.0418 & -0.0318 & 0.0000\\
-0.0000 & 0.0294 & 0.0418 & 0.0000 & 0.1037 & -0.0224\\
-0.0294 & 0.0000 & 0.0318 & -0.1037 & -0.0000 & 0.0986\\
-0.0418 & -0.0318 & -0.0000 & 0.0224 & -0.0986 & 0.0000\\
3.8786 & -0.0298 & -0.0318 & -0.0000 & -0.0546 & -0.0466\\
-0.0298 & 3.7339 & -0.1123 & 0.0546 & 0.0000 & 0.0603\\
-0.0318 & -0.1123 & 3.9775 & 0.0466 & -0.0603 & -0.0000\\
-0.0000 & 0.0546 & 0.0466 & 3.8786 & -0.0298 & -0.0318\\
-0.0546 & 0.0000 & -0.0603 & -0.0298 & 3.7339 & -0.1123\\
-0.0466 & 0.0603 & -0.0000 & -0.0318 & -0.1123 & 3.9775
\end{matrix}\right),
$$

$$
\begin{aligned}
U &= diag\{2.8586, 2.8586, 2.8586, \cdots, 2.8586\},\\
V &= diag\{2.8101, 2.8101, 2.8101, \cdots, 2.8101\}.
\end{aligned}
$$

Therefore, consider the above P, Q, R, P_1, U, V, the conditions in Theorem 2 and Theorem 3 are satisfied. Note that Theorem 1 can be regarded as a special case of Theorem 2, when $\mu = 0$. So no need to verify Theorem 1, that is, QVNN (52) has a unique equilibrium point and it is globally exponentially stable and globally power stable. And solution trajectories of the QVNN (52) with distributed delay are shown in Figure 1.

Example 2: Consider the QVNN model as follow:

$$
\dot{q}(t) = -Dq(t) + Af(q(t)) + Bf(q(t-\tau(t))) + u, \quad (53)
$$

where

$$
A = \begin{pmatrix} 1.2-0.5i-0.2j+0.5k & 1.2-0.2i-1.4j+0.3k\\ 0.3-0.3i+0.6j+0.7k & 0.9-0.5i-1.0j+0.6k \end{pmatrix},
$$

$$
B = \begin{pmatrix} 0.2+0.5i+0.3j+1.0k & -0.4+1.0i-0.4j+0.5k\\ 1.0-0.9i+1.0j+0.5k & 0.1+1.0i-1.1j+1.0k \end{pmatrix},
$$

$$
D = diag\{17, 18\}, \quad u = (2-i+j+k, 3+0.1i-j-3k)^T,
$$

$$
f(q(t)) = 0.18\tanh(q(t)), \quad \tau(t) = 0.45\sin t + 0.35, \ \tau = 0.6.
$$

By using the LMI toolbox in MATLAB, QVNN (53) has the following feasible solutions:

$$
P = \left(\begin{matrix}
0.3880 & -0.0003 & -0.0000 & 0.0024 & 0.0000 & 0.0056\\
-0.0003 & 0.3605 & -0.0024 & -0.0000 & -0.0056 & -0.0000\\
-0.0000 & -0.0024 & 0.3880 & -0.0003 & -0.0000 & 0.0024\\
0.0024 & -0.0000 & -0.0003 & 0.3605 & -0.0024 & 0.0000\\
0.0000 & -0.0056 & -0.0000 & -0.0024 & 0.3880 & -0.0003\\
0.0056 & -0.0000 & 0.0024 & 0.0000 & -0.0003 & 0.3605\\
-0.0000 & -0.0024 & 0.0000 & 0.0056 & 0.0000 & -0.0024\\
0.0024 & 0.0000 & -0.0056 & 0.0000 & 0.0024 & -0.0000
\end{matrix}\right.
$$

$$
\left.\begin{matrix}
-0.0000 & 0.0024\\
-0.0024 & 0.0000\\
0.0000 & -0.0056\\
0.0056 & 0.0000\\
0.0000 & 0.0024\\
-0.0024 & -0.0000\\
0.3880 & -0.0003\\
-0.0003 & 0.3605
\end{matrix}\right),
$$

$$
\begin{aligned}
Q = {} & diag\{35.3877, 35.3877, 35.3877, 35.3877,\\
& 35.3877, 35.3877, 35.3877, 35.3877\},
\end{aligned}
$$

$$
P_1 = \left(\begin{matrix}
7.1603 & -0.2056 & -0.0013 & 0.1046 & -0.0289 & -0.0059\\
-0.2056 & 6.9692 & -0.1090 & -0.0038 & -0.0498 & -0.0418\\
-0.0013 & -0.1090 & 7.1576 & -0.2109 & -0.0728 & -0.0778\\
0.1046 & -0.0038 & -0.2109 & 6.9547 & -0.0751 & -0.1555\\
-0.0289 & -0.0498 & -0.0728 & -0.0751 & 7.1676 & -0.2049\\
-0.0059 & -0.0418 & -0.0778 & -0.1555 & -0.2049 & 6.9467\\
-0.0024 & -0.0317 & -0.0062 & 0.0067 & -0.0198 & -0.0392\\
0.0317 & 0.0019 & -0.0079 & 0.0059 & 0.0701 & 0.0603
\end{matrix}\right.
$$

$$
\left.\begin{matrix}
-0.0024 & 0.0317\\
-0.0317 & 0.0019\\
-0.0062 & -0.0079\\
0.0067 & 0.0059\\
-0.0198 & 0.0701\\
-0.0392 & 0.0603\\
7.1690 & -0.2009\\
-0.2009 & 6.9743
\end{matrix}\right),
$$

$$
\begin{aligned}
Y = {} & diag\{4.8490, 4.8490, 4.8490, 4.8490,\\
& 4.8490, 4.8490, 4.8490, 4.8490\},\\
U = {} & diag\{4.7941, 4.7941, 4.7941, 4.7941,\\
& 4.7941, 4.7941, 4.7941, 4.7941\}.
\end{aligned}
$$

Therefore, consider the above P, Q, P_1, Y, U, the conditions in Corollary 1 and Corollary 2 are satisfied, that is, QVNN (53) has a unique equilibrium point and it is globally exponentially stable and globally power stable. And solution trajectories of the QVNN (53) with distributed delay are shown in Figure 2. On the other hand, the parameters in Example 2 do not meet the theorem 1 in [23]. So the method of [23] is invalid for QVNN (53). Accordingly, our results are more adaptable to all kinds of QVNN than that of [23].

Remark 9: Because we don't have to consider the multiplication of quaternions, our decomposing method is of higher converging speed than that in [27]. By our decomposing method, the state trajectories of Example 4.1 of [27] are shown in Figure 3.

V. CONCLUSION

In this paper, the globally exponential stability and the globally power stable of QVNNs with discrete and distributed delays are investigated respectively. Firstly, we decompose the considered QVNN into four real-valued systems, and then form a real-valued matrix differential equation, by which we equivalently discuss the dynamics behavior of QVNN, instead of considering the noncommutativity for the multiplication of quaternion. Some criteria are given by some matrix inequalities to affirm the existence and uniqueness of the equilibrium of QVNNs, which is globally exponentially stable and power stable too. By using LMI toolbox in MATLAB, the proposal criteria can be verified easily. Theorems 1 and 2 can be regarded as a simplification of [27], and Theorem 3 can be regarded as a generalization of [33]. In the future, we consider that synchronization is also an important dynamic characteristic of various network problems, and we will focus on the synchronization of QVNNs, especially finite-time synchronization or fixed-time synchronization of QVNNs.

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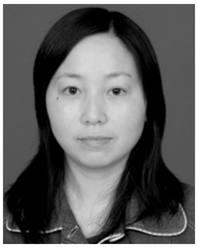

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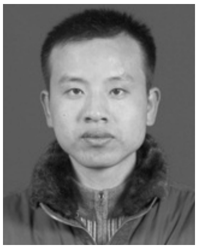

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