

Received February 8, 2020, accepted March 2, 2020, date of publication March 5, 2020, date of current version March 16, 2020.

Digital Object Identifier 10.1109/ACCESS.2020.2978594

Monotone Finite-Difference Schemes With Second Order Approximation Based on Regularization Approach for the Dirichlet Boundary Problem of the Gamma Equation

LE MINH HIEU¹, TRUONG THI HIEU HANH², AND DANG NGOC HOANG THANH³

¹Department of Economics, University of Economics–The University of Danang, Da Nang 550000, Vietnam

²Department of E-commerce, University of Economics–The University of Danang, Da Nang 550000, Vietnam

³Department of Information Technology, School of Business Information Technology, University of Economics Ho Chi Minh City, Ho Chi Minh City 700000, Vietnam

Corresponding author: Dang Ngoc Hoang Thanh (thanhdnh@ueh.edu.vn)

ABSTRACT We investigate the initial boundary value problem for the Gamma equation transformed from the nonlinear Black-Scholes equation for pricing option to a quasilinear parabolic equation of second derivative. Furthermore, two-side estimates for the exact solution are also provided. By using regularization principle, the unconditionally monotone second order approximation finite-difference scheme on uniform and nonuniform grids is generalized, in that the maximum principle is satisfied without depending on relations of the coefficients and grid parameters. By using the difference maximum principle, we acquired two-side estimates for difference solution for the arbitrary non-sign-constant input data. Finally, we also provide a proof for a priori estimate. It can be confirmed that the two-side estimates for difference solution are completely consistent with the differential problem. Otherwise, the maximal and minimal values of the difference solution is independent from the diffusion and convection coefficients.

INDEX TERMS Gamma equation, Black-Scholes equation, financial engineering, stock price prediction, numerical solution, numerical algorithm, maximum principle, two-side estimates, finite-difference scheme, regularization principle.

I. INTRODUCTION

Over the last decades, not only financial engineers but also mathematicians have paid special attention to the valuation of derivative financial instruments. Indeed, since being introduced by Fischer Black and Myron Scholes in 1973, the Black-Scholes model based on partial differential equation has been widely employed in modern mathematical finance and become a common-sense approach for pricing options as well as many other financial securities [25]. This mathematical model was derived from the principle that yielding profits from making portfolios of both short and long positions in options as well as their underlying stocks should not be possible, if option prices are rightly priced in the market [15]. These scholars indicated that a European option's value on a stock, whose price or the log return of underlying price is supposed to follow a geometric Brownian motion with constant volatility and drift, is determined by a second-order parabolic

The associate editor coordinating the review of this manuscript and approving it for publication was Jenny Mahoney.

equation concerning time and stock price. Nevertheless, the assumptions of Black-Scholes equation based on perfectly liquid market are so idealistic in comparison with the high illiquidity recently. There are many numerical methods used for studying properties of typical non-linear Black-Scholes equations [1]–[12], [14], [16], [17].

Not only in mathematical physics, but also in economics, there is a need to solve partial differential equations containing lower derivatives. For example, in financial mathematics, it is of interest to study the Gamma equation obtained by a transformation of the nonlinear Black-Scholes equation to a quasilinear parabolic equation [26], [27], [30], [42]. The approximate solution of the Gamma equation is the main goal of this study.

We consider the following quasilinear parabolic equation, which is called the Gamma equation [26], [27]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 \beta(u)}{\partial x^2} + \frac{\partial \beta(u)}{\partial x} + c \frac{\partial u}{\partial x}, \quad u = u(x, t),$$

$$0 < t \leq T, \quad x \in \mathbb{R}, \quad c - \text{const}, \quad (1)$$

$$u(-\infty, t) = u(+\infty, t) = 0, \quad u(x, 0) = u_0(x), \quad (2)$$

Problem (1)–(2) is obtained by transforming the nonlinear Black-Scholes equation for $V(S, \tau)$

$$V_\tau + 0.5\sigma^2(t, S, V_{SS})S^2V_{SS} + (c - q)SV_S - cV = 0, \quad 0 \leq S < \infty, \quad 0 \leq \tau \leq T. \quad (3)$$

The present paper will focus on some models related to nonlinear Black-Scholes equations for European option whose volatility relies upon various factors like the stock price, the option price, the time as well as its derivatives due to the presence of transaction cost. The option’s behaviour would be disclosed by higher derivative of its price which is mentioned as the Greeks in the financial literature. Not only giving a good approximation for the pricing option, reliable numerical methods are also essential for its derivatives because of the relevance of the Greeks to quantitative analysis.

For the case of European Call Options [27], i.e. value $V(S, \tau)$ is a solution of equation (3) with $q = 0$ and $0 \leq S < \infty, 0 \leq \tau \leq T$, the initial and the boundary conditions of the problem (3) will be

$$\begin{aligned} V(S, T) &= \max\{0, S - E\}, \quad 0 \leq S < \infty, \quad E > 0, \\ V(0, \tau) &= 0, \quad 0 \leq \tau \leq T, \\ V(S, \tau) &= S - Ee^{-c(T-\tau)}, \quad S \rightarrow \infty. \end{aligned}$$

Note that σ – non-const depending on the model, for example, $\sigma^2 = \sigma_{JS}^2$ (Jandacka-Sevcovic model [26]) or $\sigma^2 = \sigma_F^2$ (Frey model [21])

$$\sigma_{JS}^2 = \sigma_0^2 \left(1 + \mu(SV_{SS})^{\frac{1}{3}}\right), \quad \sigma_F^2 = \frac{\sigma_0^2}{1 - \rho SV_{SS}},$$

where $\mu = 3(C^2M / (2\pi))^{1/3}$, σ_0, M, C – const. By using an independent variables $x = \ln(S/E), x \in \mathbb{R}, t = T - \tau, t \in (0, T)$ and putting $u(x, t) = SV_{SS}$ in (3) for the two above models, we acquire the problem (1)–(2). Then the function $\beta(u)$ and the initial condition $u_0(x)$ for the corresponding models will also become

$$\begin{aligned} \beta_{JS} &= \frac{\sigma_0^2}{2} \left(1 + \mu(u)^{1/3}\right) u, \quad u_0(x) = \delta(x), \\ \beta_F &= \frac{\sigma_0^2}{2} \frac{u}{(1 - \rho u)^2}, \quad u_0(x) = \delta(x), \end{aligned}$$

where $\delta(x)$ is the Delta function.

In order to find the difference solution of problem (1)–(2), we must restrict it to a finite spatial interval $x \in (-L, L)$, with $L > 0$ being a sufficiently large number. Since $S = Ee^x$, we limit $S \in (0, +\infty)$ the interval by interval $S \in (Ee^{-L}, Ee^L)$. In practical calculations, we can choose $L \approx 1.5$ to contain important values of S . Thus, instead of (2), we consider the Gamma equation (1) with Dirichlet boundary conditions at $x = \pm L$ [26], i.e.,

$$u(-L, t) = u(L, t) = 0, \quad u(x, 0) = u_0(x). \quad (4)$$

Because the Gamma equation has no exact solutions (analytical solutions), to assess efficiency of the proposed difference scheme and to maintain the equality of the Gamma

equation, we must add a residual term $f(x, t)$ to the right-hand side of (1). Thus, we consider equation (1) in the form (13).

With no loss of generality, we construct a differential scheme for quasi-linear parabolic equation with the initial condition of $u_0(x)$. Depending on the specific model, the initial condition will have different forms. In this work, we have introduced two above models (Jandacka-Sevcovic and Frey) and the initial condition has the form of a Delta function. In order to get a suitable initial condition for computation, we consider the regularization of $u_0(x) = \delta(x)$, given by the function $u_0 = N'(d) / (\sigma_0\sqrt{\pi t^*})$, where $0 < t^* \ll 1, N'(d) = e^{-d^2/2} / \sqrt{2\pi}, d = (x + (c - q - \sigma_0^2/2)t^*) / (\sigma_0\sqrt{t^*})$ [27].

In order to solve the Black-Scholes equation, as well as other nonlinear partial differential equations (PDEs), there are some effective methods such as the finite difference methods, the Galerkin method [54], the Monte-Carlo method [55], reduced basis methods [53], etc. Nowadays, with many achievements of computing techniques, novel methods utilizing the machine learning technique such as the deep Galerkin method [52] are effective for solving nonlinear PDEs. The reduced basis methods and machine-learning-based methods are especially effective in the case of grids containing a huge number of nodes. For this case, methods such as the finite difference methods and the Galerkin method usually require a lot of memory resources. However, the finite difference methods have an advantage: easy to process parallelly. This is effective to be implemented on high-performance computing systems. In this article, we only focus on finite difference methods to develop new finite difference schemes and analyze their two-sided estimates.

The maximum principle has attracted a lot of attention in the theory of difference schemes [39]. In particular, this principle is used to study the stability and convergence of a difference solution on a uniform norm. Computational methods satisfying the maximum principle are called to be monotone. Monotone schemes play a vital role in experiments, because the considered discrete problems are usually well-posed [24]. Furthermore they provide a numerical solution not being oscillating even if solutions are non-smooth [37].

It must be noticed that we can acquire lower estimates of the solutions of differential–difference problems or of two-sided estimates problems. This is very important to investigate theoretical properties of the approximate methods with unbounded nonlinearities, in that it is necessary to prove that numerical solution lays in a neighbourhood of the exact solution. In the case of linear problems, the estimates is to find the range of values of the desired solution in terms of the problem input data (the coefficients and right-hand side of the equation as well as the initial and boundary conditions). In the case of nonlinearity, the estimates provide an approach to confirm the nonnegativity of the exact solution. This feature is very important in physical problems, as well as to find conditions of the input data to let the problem being parabolic or elliptic, for example, investigating the Gamma equation in financial mathematics to model pricing of options.

Finding nontrivial solutions estimates of the initial-boundary value problems based on a special method is provided by Ladyzhenskaya [28] (see also the monograph [29, p. 22]), whereby we can introduce a parameter-dependent change of variables and then minimize or maximize some functions depending on this parameter; the resulting extremal values will provide the corresponding estimates for the numerical solution. Naturally, it is necessary to acquire such estimates for the approximate solution. The theory of finite-difference schemes [39, p. 229] includes the technique, well developed for linear problems on the grid satisfying the maximum principle, which provides two-sided estimates for the approximate solution. The accuracy of the acquired estimates of the solutions of finite-difference problems is not higher [19] than one of the corresponding estimates of the solutions of differential problems [29, p. 22]. There is an interesting work [18], [20], in that two-sided estimates for the finite element method is used for approximating the Dirichlet boundary problem for the linear parabolic equation in the discrete and continuous cases.

On uniform grids [35], for the case of the one-dimensional convection–diffusion equation, there is the simplest approximations of first derivatives leading to upwind difference schemes. Such schemes are unconditionally monotone. However, they have only the first order of approximation. Moreover, the convective term can be approximated using central difference schemes. In the case of the second order approximation, the monotonicity property of the schemes is satisfied based on constraints of the spatial grid step. Unconditionally monotone second-order accurate difference schemes for convection–diffusion problems on uniform grids were constructed by applying the regularization principle (see [39], [44]).

In problems of mathematical physics, it is very important to increase the order of accuracy of approximate methods without enlarging the standard stencil. In applied multidimensional problems with singularities in domains of complex geometry, mathematical simulation frequently relies on nonuniform grids. Nevertheless, in the transition from uniform to nonuniform grids, the order of the local approximation error is usually reduced. For example, approximating the second derivative on the usual three-point stencil (see [39]) ensures only the first order of accuracy in the uniform norm and in the grid norm of L_2 :

$$u''(x_i) - u_{\bar{x}\bar{x},i} = O\left(h_{i+1} - h_i + \bar{h}_i^2\right),$$

where $u_{\bar{x}\bar{x},i} = (u_{x,i} - u_{\bar{x},i})/\bar{h}_i$, $u_{x,i} = (u_{i+1} - u_i)/h_{i+1}$, $u_{\bar{x},i} = (u_i - u_{i-1})/h_i$, $\bar{h}_i = 0.5(h_{i+1} + h_i)$, h_i is the step size of the non-uniform grid. The second order approximation accuracy of the corresponding difference schemes on nonuniform grids can be proved only by applying a negative norm.

To increase the order approximation accuracy of a difference method, we must approximate the original differential equation not at grid nodes, but rather at some intermediate points of the computational domain. Specifically, it turns

out that the usual approximation of the second difference derivative preserves the second order with respect to the point $\bar{x}_i = (x_{i-1} + x_i + x_{i+1})/3 = x_i + (h_{i+1} - h_i)/3$:

$$u''(\bar{x}_i) - u_{\bar{x}\bar{x},i} = O\left(\bar{h}_i^2\right).$$

This simple idea was further developed by A.A. Samarskii, P.N. Vabishchevich, and P.P. Matus. For example, various high-order approximation schemes for a second-order ordinary differential equation and one-dimensional parabolic and hyperbolic equations were constructed in [41]. On an arbitrary rectangular nonuniform grid, monotone conservative schemes of second-order local approximation for the multidimensional Poisson equation were designed in [38], [40]. High-order accurate schemes for various equations of mathematical physics were constructed and investigated in [11]. The construction and study of high-order accurate monotone difference schemes on arbitrary nonuniform grids for stationary and nonstationary convection–diffusion equations were considered in [31], [43], [47], [51].

In the present paper, the Gamma equation is considered, on the basis of the technique from [29], two-sided estimates are obtained for its exact solution. The acquired results are generalized to the construction of monotone finite-difference schemes of second-order of local approximation on uniform and non-uniform grids for a given equation. The construction of such schemes is based on the appropriate choice of the perturbed coefficient, similarly to [39]. Using the difference maximum principle, two-sided and a priori estimates are obtained in the C -norm for the difference solution. It is interesting to note that the provided two-sided estimates of the difference solution are completely consistent with the estimates of exact solution of differential problem. Monotone schemes are very effective for well-posed problem. They also provide a numerical solution not being oscillating even if solutions are non-smooth. Moreover, two-side estimates not only provide a manner to prove the nonnegativity of the exact solution, but it is also helpful to find out sufficient conditions based on the input data if the nonlinear problem is parabolic. Consequently, a priori estimate of the approximate solution in the grid norm C depending on the initial and boundary conditions only is proved.

II. MAXIMUM PRINCIPLE FOR DIFFERENCE SCHEMES WITH VARIABLE SIGN INPUT DATA

Suppose that in the n -dimensional Euclidean space a finite number of points of the grid is given and denoted by Ω_h . For each point $x \in \Omega_h$ we connect one and only one stencil $\mathcal{M}(x)$ - a subset of Ω_h , holding this point. We call a set $\mathcal{M}'(x) = \mathcal{M}(x) \setminus x$ to be *neighborhood* of the point x . Suppose that real-valued functions $A(x)$, $B(x, \xi)$, $F(x)$ are given at $x \in \Omega_h$, $\xi \in \Omega_h$. For each point $x \in \Omega_h$, we consider one and only one respective equation of the form [39]

$$A(x)y(x) = \sum_{\xi \in \mathcal{M}'(x)} B(x, \xi)y(\xi) + F(x), \quad x \in \Omega_h, \quad (5)$$

which is called to be the *canonical form* of the finite-difference scheme [39, p. 226]. Must notice that $\mathcal{M}'(x)$ could be an empty set, for example, for Dirichlet boundary condition. A system of linear algebraic equations obtained by discretizing the boundary problem is often called *difference scheme*. For the grid Ω_h , we consider its arbitrary subset ω_h and we denote

$$\bar{\omega}_h = \bigcup_{x \in \omega_h} \mathcal{M}(x).$$

For example, let ω_h be a set of the internal nodes to approximate the Poisson equation. In this case $\bar{\omega}_h = \Omega_h$. According to [36], [39], the point x is called to be a *grid boundary node*. In the case of Dirichlet boundary condition is posed, we have

$$y(x) = \mu(x), \quad x \in \gamma,$$

where γ is a collection of boundary nodes. We must notice that the grid may not hold boundary nodes in the case of approximating boundary conditions of second/third kind, i.e. all grid nodes will become internal nodes. In this case, we suppose that the usual positivity conditions for the coefficients is fulfilled

$$A(x) > 0, \quad B(x, \xi) > 0 \quad \text{for all } \xi \in \mathcal{M}'(x), \quad (6)$$

$$D(x) = A(x) - \sum_{\xi \in \mathcal{M}'(x)} B(x, \xi) > 0. \quad (7)$$

The conditions will guarantee the unique solvability, monotonicity and stability for the difference scheme (5) in the uniform norm for small perturbation of input data. Now, we formulate the basic results before establishing two-sided estimates for the numerical solution at non-sign-definite input data of the problem $F(x)$.

Lemma 1 ([34], [36]): Assume that conditions (6)–(7) that the coefficients are positive are satisfied. Then the maximum and minimum values of the solution of the finite-difference scheme (5) belong to the range of the input data

$$\min_{x \in \Omega_h} \frac{F(x)}{D(x)} \leq y(x) \leq \max_{x \in \Omega_h} \frac{F(x)}{D(x)}, \quad x \in \Omega_h. \quad (8)$$

Corollary 1 ([39, p. 231]): Assume that conditions of the lemma are satisfied. Then in the grid analog of the C-norm, the solution of finite-difference problem (5) satisfies the estimate

$$\|y\|_C = \max_{x \in \Omega_h} |y(x)| \leq \left\| \frac{F}{D} \right\|_C. \quad (9)$$

Similar to the scalar case we introduce the canonical form of the vector-difference schemes

$$A(x) \vec{Y}(x) = \sum_{\xi \in \mathcal{M}'(x)} B(x, \xi) \vec{Y}(\xi) + \vec{F}(x), \quad x \in \Omega_h. \quad (10)$$

Here the matrices $A(x) = \{a_{ij}(x)\}_{m \times m}$, $B(x, \xi) = \{b_{ij}(x, \xi)\}_{m \times m}$ and right-hand side vector $\vec{F}(x) = (f_1(x), f_2(x), \dots, f_m(x))^T$ are given, unknown vector grid function $\vec{Y}(x) = (y_1(x), y_2(x), \dots, y_m(x))^T$ takes

real values. The point x is called to be a grid boundary node, on condition that the Dirichlet condition is posed

$$\vec{Y}(x) = \vec{\mu}(x), \quad x \in \gamma, \quad (11)$$

where γ is a collection of the boundary nodes, $\vec{\mu}(x) = (\mu_1(x), \mu_2(x), \dots, \mu_m(x))^T$.

Definition 1: Vector-difference scheme (10)–(11) is called *monotone* if its solution satisfies the conditions:

- If $\vec{F}(x) \geq 0$, $x \in \Omega_h$ and $\vec{\mu}(x) \geq 0$, $x \in \gamma$, then $\vec{Y}(x) \geq 0$, $x \in \bar{\Omega}_h$, $\bar{\Omega}_h = \Omega_h \cup \gamma$;
- If $\vec{F}(x) \leq 0$, $x \in \Omega_h$ and $\vec{\mu}(x) \leq 0$, $x \in \gamma$, then $\vec{Y}(x) \leq 0$, $x \in \bar{\Omega}_h$.

We introduce matrices $D^{(1)}(x) = \{d_{ij}^{(1)}(x)\}_{m \times m}$ and $D(x) = \text{diag}\{d_{11}(x), d_{22}(x), \dots, d_{mm}(x)\}$, which defined as follows

$$D^{(1)}(x) = A(x) - \sum_{\xi \in \mathcal{M}'(x)} B(x, \xi),$$

$$d_{ii}(x) = \sum_{j=1}^m d_{ij}^{(1)}(x), \quad i = \overline{1, m}.$$

Matrix $A(x)$ could be rewrite in the form $A(x) = A^{(1)}(x) - A^{(2)}(x)$, where

$$A^{(1)}(x) = \text{diag}\{a_{11}^{(1)}(x), a_{22}^{(1)}(x), \dots, a_{mm}^{(1)}(x)\},$$

$$a_{ii}^{(1)}(x) = a_{ii}(x), \quad i = \overline{1, m},$$

$$A^{(2)}(x) = \{a_{ij}^{(2)}(x)\}_{m \times m}, \quad a_{ii}^{(2)}(x) = 0,$$

$$a_{ij}^{(2)}(x) = -a_{ij}(x), \quad i \neq j, \quad i, j = \overline{1, m}.$$

Then we write vector equation (10) in the form

$$A^{(1)}(x) \vec{Y}(x) = \sum_{\xi \in \mathcal{M}'(x)} B(x, \xi) \vec{Y}(\xi) + A^{(2)}(x) \vec{Y}(x) + \vec{F}(x), \quad x \in \Omega_h.$$

We will assume that the positivity conditions for matrix coefficients is satisfied (i.e. all elements of matrix are positive [23])

$$A^{(1)}(x) > 0, \quad A^{(2)}(x) \geq 0,$$

$$B(x, \xi) > 0, \quad D(x) > 0 \quad \text{for all } \xi \in \mathcal{M}'(x), \quad (12)$$

and define

$$\|\vec{V}\|_{\bar{C}} = \max_{1 \leq j \leq m} \left(\max_{x \in \Omega_h} |v_j(x)| \right),$$

$$\max_{x \in \Omega_h} \vec{V}(x) = \max_{1 \leq j \leq m} \left(\max_{x \in \Omega_h} v_j(x) \right),$$

$$\min_{x \in \Omega_h} \vec{V}(x) = \min_{1 \leq j \leq m} \left(\min_{x \in \Omega_h} v_j(x) \right).$$

Lemma 2: ([33], [36]) Suppose that the positivity conditions for matrix coefficients (12) are fulfilled. Then the maximal and minimal values of the solution of the

vector-difference scheme (10)–(11) belong to value interval of the input data

$$m_1 \leq y_j(x) \leq m_2, \quad x \in \bar{\Omega}_h, \quad j = \overline{1, m},$$

where

$$m_1 = \min \left\{ \min_{x \in \gamma} \bar{\mu}(x), \min_{x \in \Omega_h} \left(D^{-1}(x) \bar{F}(x) \right) \right\},$$

$$m_2 = \max \left\{ \max_{x \in \gamma} \bar{\mu}(x), \max_{x \in \Omega_h} \left(D^{-1}(x) \bar{F}(x) \right) \right\}.$$

Corollary 2 ([33]): Let conditions of Lemma 2 be fulfilled. Then vector-difference schemes (10)–(11) is monotone and for her the following estimate holds

$$\|\bar{Y}\|_{\bar{C}} \leq \max \left\{ \|\bar{\mu}\|_{C_\gamma}, \|D^{-1}\bar{F}\|_C \right\}.$$

III. STATEMENT OF THE PROBLEM AND TWO-SIDED ESTIMATE OF THE EXACT SOLUTION

Without loss of generality in a rectangle $\bar{Q}_T = \{(x, t) : l_1 \leq x \leq l_2, 0 \leq t \leq T\}$ we consider the following initial boundary value problem for a quasilinear parabolic equation (the so called Gamma equation), that is a generalization of Jandacka and Sevcovic model in [26]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 \beta(u)}{\partial x^2} + \frac{\partial \beta(u)}{\partial x} + c \frac{\partial u}{\partial x} - q(x)u + f(x, t),$$

$$u = u(x, t), \quad c = \text{const}, \quad q(x) \geq 0, \quad (13)$$

with boundary conditions

$$u(l_1, t) = \mu_1(t), \quad u(l_2, t) = \mu_2(t), \quad t > 0, \quad (14)$$

and initial conditions

$$u(x, 0) = u_0(x), \quad l_1 \leq x \leq l_2. \quad (15)$$

Equation (13) can be written as

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(k(u) \frac{\partial u}{\partial x} \right) + r(u) \frac{\partial u}{\partial x} - q(x)u + f(x, t), \quad (16)$$

with coefficients

$$k(u) = \beta'(u), \quad r(u) = k(u) + c. \quad (17)$$

We assume that parabolicity condition of equation (16) on the solution [22] is satisfied

$$0 < k_1 \leq k(u) \leq k_2, \quad \forall u \in \bar{D}_u, \quad k_1, k_2 = \text{const}, \quad (18)$$

where

$$\bar{D}_u = \{u(x, t) : m_1 \leq u(x, t) \leq m_2, (x, t) \in \bar{Q}_T\}.$$

We assume in what follows that there exists a unique solution of problem (13)–(15) and all coefficients in Eq. (16) and the desired function have continuous bounded derivatives of order that is required as the presentation proceeds. Let $t_1 \leq T$ and $Q_{t_1} = \{(x, t) \in Q_T : t \leq t_1\}$.

Using the technique from [29], we prove two-sided estimates for the exact solution of problem (13)–(15).

Theorem 1: Let condition (18) be satisfied. Then for solution $u(x, t)$ of problem (13)–(15) at any point $(x, t_1) \in \bar{Q}_T$ the following two-sided estimates are valid:

$$m_1 \leq u(x, t_1) \leq m_2, \quad (19)$$

where

$$m_1 = \sup_{\lambda > \lambda_0} \min \left\{ 0, \min_{Q_{t_1}} \{\mu_1(t), \mu_2(t), u_0(x)\} e^{\lambda(t_1-t)}, \min_{Q_{t_1}} \frac{f(x, t) e^{\lambda(t_1-t)}}{\lambda + q(x)} \right\},$$

$$m_2 = \inf_{\lambda > \lambda_0} \max \left\{ 0, \max_{Q_{t_1}} \{\mu_1(t), \mu_2(t), u_0(x)\} e^{\lambda(t_1-t)}, \max_{Q_{t_1}} \frac{f(x, t) e^{\lambda(t_1-t)}}{\lambda + q(x)} \right\},$$

$$\lambda_0 = \max_{l_1 \leq x \leq l_2} [-q(x)] = - \min_{l_1 \leq x \leq l_2} q(x).$$

IV. UNCONDITIONALLY MONOTONE FINITE-DIFFERENCE SCHEME OF SECOND-ORDER APPROXIMATION ON UNIFORM GRIDS

In this section we construct a new finite difference scheme on uniform grid in space and in time for the problem (13)–(15). Some important properties for the solution of this schemes are studied, namely, order of approximation and monotonicity of scheme, two-sided estimate and a priori estimate on C-norm for difference solution.

A. DIFFERENCE SCHEME

Using the principle of regularization [39] on a regular uniform grid in space and time $\bar{\omega} = \bar{\omega}_h \times \bar{\omega}_\tau$, where

$$\bar{\omega}_h = \{x_i = l_1 + ih, \quad i = \overline{0, N}, \quad hN = l_2 - l_1\},$$

$$\bar{\omega}_h = \omega_h \cup \{x_0 = l_1, x_N = l_2\},$$

$$\bar{\omega}_\tau = \{t_n = n\tau, \quad n = \overline{0, N_0}, \quad \tau N_0 = T\},$$

$$\bar{\omega}_\tau = \omega_\tau \cup \{t_{N_0} = T\},$$

we approximate equation (13) with a difference scheme of the form

$$\frac{y_i^{n+1} - y_i^n}{\tau}$$

$$= \frac{\kappa_i^n(y)}{h} \left(a_{i+1}^n(y) \frac{y_{i+1}^{n+1} - y_i^{n+1}}{h} - a_i^n(y) \frac{y_i^{n+1} - y_{i-1}^{n+1}}{h} \right)$$

$$+ b_i^+(y) a_{i+1}^n(y) \frac{y_{i+1}^{n+1} - y_i^{n+1}}{h}$$

$$+ b_i^-(y) a_i^n(y) \frac{y_i^{n+1} - y_{i-1}^{n+1}}{h} - q_i y_i^{n+1} + f_i^{n+1},$$

$$i = 1, 2, \dots, N - 1,$$

$$y_i^0 = u_0(x_i), \quad i = 1, 2, \dots, N - 1,$$

$$y_0^{n+1} = \mu_1(t_{n+1}), \quad y_N^{n+1} = \mu_2(t_{n+1}), \quad (20)$$

where

$$\kappa_i^n(y) = (1 + R_i^n(y))^{-1},$$

$$\begin{aligned}
 R_i^n(y) &= 0.5h |r(y_i^n)|/k(y_i^n) \geq 0, \\
 b_i^+(y) &= r^+(y_i^n)/k(y_i^n) \geq 0, \\
 b_i^-(y) &= r^-(y_i^n)/k(y_i^n) \leq 0, \\
 r^+(y_i^n) &= 0.5(r(y_i^n) + |r(y_i^n)|) \geq 0, \\
 r^-(y_i^n) &= 0.5(r(y_i^n) - |r(y_i^n)|) \leq 0, \\
 a_{i+1}^n(y) &= 0.5(k(y_{i+1}^n) + k(y_i^n)), \\
 a_i^n(y) &= 0.5(k(y_{i-1}^n) + k(y_i^n)).
 \end{aligned}$$

B. APPROXIMATION ERROR

The difference scheme (20) has the following approximation error

$$\begin{aligned}
 \psi &= -u_t + \kappa(u)(a(u)\hat{u}_{\bar{x}})_x \\
 &+ b^+(u)a^{(+1)}(u)\hat{u}_x + b^-(u)a(u)\hat{u}_{\bar{x}} - qu + f, \quad (21)
 \end{aligned}$$

where

$$\begin{aligned}
 v &= v^n = v(t_n), \quad \hat{v} = v^{n+1} = v(t_{n+1}), \\
 v_x &= (v_{i+1} - v_i)/h, \quad v_{\bar{x}} = (v_i - v_{i-1})/h, \\
 a^{(+1)}(u) &= a_{i+1}(u), \quad a(u) = a_i(u).
 \end{aligned}$$

Taking into account

$$\begin{aligned}
 b^+(u) &= r^+(u)/k(u), \quad b^-(u) = r^-(u)/k(u), \\
 r^+(u) + r^-(u) &= r(u), \quad r^+(u) - r^-(u) = |r(u)|, \\
 u_t &= \frac{\partial u}{\partial t} + O(\tau), \\
 (a(u)\hat{u}_{\bar{x}})_x &= \frac{\partial}{\partial x} \left(k(u) \frac{\partial u}{\partial x} \right) + O(h^2 + \tau), \\
 a^{(+1)}(u)\hat{u}_x &= k(u) \frac{\partial u}{\partial x} + 0.5h \frac{\partial}{\partial x} \left(k(u) \frac{\partial u}{\partial x} \right) + O(h^2 + \tau), \\
 a(u)\hat{u}_{\bar{x}} &= k(u) \frac{\partial u}{\partial x} - 0.5h \frac{\partial}{\partial x} \left(k(u) \frac{\partial u}{\partial x} \right) + O(h^2 + \tau),
 \end{aligned}$$

we get

$$\begin{aligned}
 b^+(u)a^{(+1)}(u)\hat{u}_x + b^-(u)a(u)\hat{u}_{\bar{x}} \\
 = r(u) \frac{\partial u}{\partial x} + R(u) \frac{\partial}{\partial x} \left(k(u) \frac{\partial u}{\partial x} \right) + O(h^2 + \tau).
 \end{aligned}$$

It follows from (21) that

$$\psi = \frac{(R(u))^2}{1 + R(u)} \frac{\partial}{\partial x} \left(k(u) \frac{\partial u}{\partial x} \right) + O(h^2 + \tau) = O(h^2 + \tau).$$

Hence, order of spatial approximation of the difference scheme (20) is two and for temporal approximation - one.

C. MONOTONICITY, TWO-SIDED AND A PRIORI ESTIMATES

We write the difference scheme (20) in the canonical form (5)

$$A_i^n y_{i-1}^{n+1} - C_i^n y_i^{n+1} + B_i^n y_{i+1}^{n+1} = -F_i^n, \quad i = \overline{1, N-1}, \quad (22)$$

$$y_0^{n+1} = \mu_1^{n+1}, \quad y_N^{n+1} = \mu_2^{n+1}, \quad (23)$$

with coefficients defined as follows

$$\begin{aligned}
 A_i^n &= \frac{\tau}{h^2} a_i^n(y) (\kappa_i^n(y) - hb_i^-(y)), \\
 B_i^n &= \frac{\tau}{h^2} a_{i+1}^n(y) (\kappa_i^n(y) + hb_i^+(y)), \\
 C_i^n &= 1 + \tau q_i + A_i^n + B_i^n, \quad F_i^n = y_i^n + \tau f_i^{n+1}, \\
 D_i^n &= C_i^n - A_i^n - B_i^n = 1 + \tau q_i, \quad i = \overline{1, N-1}.
 \end{aligned}$$

The scheme (22)–(23) is monotone if the positivity conditions of the coefficients (6)–(7) are satisfied [39], i.e. if

$$A_i^n > 0, \quad B_i^n > 0, \quad D_i^n = C_i^n - A_i^n - B_i^n > 0.$$

Let us prove that $y_i^n \in \bar{D}_u$ for all i, n . We take an auxiliary grid function $z(x, t) = z_i^n = y_i^n e^{-\lambda t_n}$, $\lambda \neq 0$. The function $z(x, t)$ satisfies the finite-difference equation

$$\begin{aligned}
 C_{(*)i}^n z_i^{n+1} - A_{(*)i}^n z_{i-1}^{n+1} + B_{(*)i}^n z_{i+1}^{n+1} + K_i^n z_i^n + F_{(*)i}^n, \\
 i = 1, 2, \dots, N-1,
 \end{aligned}$$

where

$$\begin{aligned}
 A_{(*)i}^n &= e^{\lambda \tau} A_i^n, \quad B_{(*)i}^n = e^{\lambda \tau} B_i^n, \quad K_i^n = 1, \\
 C_{(*)i}^n &= e^{\lambda \tau} (1 + \tau q_i) + A_{(*)i}^n + B_{(*)i}^n, \quad F_{(*)i}^n = \tau f_i^{n+1} e^{-\lambda t_n},
 \end{aligned}$$

We introduce the coefficients $D_{(*)i}^n$ as follows:

$$D_{(*)i}^n = C_{(*)i}^n - A_{(*)i}^n - B_{(*)i}^n - K_i^n = e^{\lambda \tau} (1 + \tau q_i) - 1 > 0$$

for all $\lambda \tau > 0$. Take an arbitrary $t_n \in \omega_\tau$. The following three cases are possible for the function $z(x, t)$.

1. $\max_{\omega_{t_n}} z(x, t)$ is nonpositive (i.e., $z(x, t) \leq 0, (x, t) \in \omega_{t_n}$);
2. $\max_{\omega_{t_n}} z(x, t)$ is located on the base $t = 0$ or on the boundary (i.e., the inequality $z(x, t) \leq \max_{\omega_{t_n}} e^{-\lambda t} \{\mu_1(t), \mu_2(t), u_0(x)\}, (x, t) \in \omega_{t_n}$, holds);
3. A positive maximum is attained at some interior point $(x^0, t^0): z(x, t) \leq z(x^0, t^0) = \max_{\omega_{t_n}} z(x, t)$.

Obviously, for $n = 0$ we have $y_i^0 = u_{0,i} \in \bar{D}_u$ for all $i = \overline{1, N-1}$. Assume that, for any arbitrary n , the inclusion $y_i^n \in \bar{D}_u$ is also true. We need prove that $y_i^{n+1} \in \bar{D}_u$ is true. From this assumption we have $A_{(*)i}^n > 0, B_{(*)i}^n > 0, C_{(*)i}^n > 0$. According to Lemma 1 on the base of the estimate (8) for arbitrary $t = t_n \in \omega_\tau$ and all $i = 0, 1, \dots, N$, we have

$$\begin{aligned}
 z(x, t) &\leq z(x^0, t^0) \leq \frac{\tau}{e^{\lambda \tau} (1 + \tau q(x^0)) - 1} f(x^0, t^0) e^{-\lambda t^0} \\
 &\leq \max_{\omega_{t_n}} \frac{\tau f(x, t) e^{-\lambda t}}{e^{\lambda \tau} (1 + \tau q(x)) - 1}, \quad \lambda > 0.
 \end{aligned}$$

Then, in all cases (1.)–(3.), the function $z(x, t)$ satisfies the estimate

$$\begin{aligned}
 z(x, t) &\leq \max \left\{ 0, \max_{\omega_{t_n}} e^{-\lambda t} \{\mu_1(t), \mu_2(t), u_0(x)\}, \right. \\
 &\quad \left. \max_{\omega_{t_n}} \frac{\tau f(x, t) e^{-\lambda t}}{e^{\lambda \tau} (1 + \tau q) - 1} \right\}, \quad t \leq t_n.
 \end{aligned}$$

Next we consider the case where a positive maximum is attained at some interior point (x, t_{n+1}) , then once again with the help of Lemma 1 we obtain

$$z(x, t_{n+1}) \leq \max \left\{ 0, \max_{\omega_{n+1}} e^{-\lambda t} \{ \mu_1(t), \mu_2(t), u_0(x) \}, \max_{\omega_{n+1}} \frac{\tau f(x, t) e^{-\lambda t}}{e^{\lambda \tau} (1 + \tau q) - 1} \right\},$$

which implies that

$$y(x, t_{n+1}) \leq m_2^{n+1}, \tag{24}$$

where

$$m_2^{n+1} = \inf_{\lambda > 0} \max \left\{ 0, \max_{\omega_{n+1}} e^{\lambda(t_{n+1}-t)} \{ \mu_1(t), \mu_2(t), u_0(x) \}, \max_{\omega_{n+1}} \frac{\tau f(x, t) e^{\lambda(t_{n+1}-t)}}{e^{\lambda \tau} (1 + \tau q) - 1} \right\}.$$

In a similar way, we obtain the lower bound

$$y(x, t_{n+1}) \geq m_1^{n+1}, \tag{25}$$

where

$$m_1^{n+1} = \sup_{\lambda > 0} \min \left\{ 0, \min_{\omega_{n+1}} e^{\lambda(t_{n+1}-t)} \{ \mu_1(t), \mu_2(t), u_0(x) \}, \min_{\omega_{n+1}} \frac{\tau f(x, t) e^{\lambda(t_{n+1}-t)}}{e^{\lambda \tau} (1 + \tau q) - 1} \right\}.$$

Since

$$\frac{\tau}{e^{\lambda \tau} (1 + \tau q) - 1} \leq \frac{1}{\lambda + q} \quad \text{for all } \lambda, \tau > 0,$$

we see that the estimates (19) and (24)–(25) imply the inequalities

$$m_1 \leq m_1^{n+1}, \quad m_2^{n+1} \leq m_2,$$

i.e., $y_i^{n+1} \in [m_1, m_2]$, $i = \overline{1, N-1}$. In this sense, the finite-difference estimates inherit the properties of the differential problem. Moreover, because positivity conditions for the coefficients (6)–(7) are fulfilled, the difference scheme (20) is monotone for all values of h and τ (i.e. the difference scheme is unconditionally monotone).

Thus, we have proved the following theorem.

Theorem 2: Suppose that the conditions (18) are fulfilled. Then the finite-difference scheme (20) is unconditionally monotone (without constraints on the steps τ and h) and for its solution $y \in \bar{D}_u$ the above two-sided estimates (24)–(25) hold at any point $(x, t) \in \omega$.

On the base of the maximum principle in a standard way [36] we acquire the a priori estimate in the C -norm:

Theorem 3: Let the condition (18) be fulfilled. Then for the solution of the difference problem (20) the following a priori estimate holds

$$\|y(t_{n+1})\|_{\bar{C}} \leq \max \left\{ \|u_0\|_{\bar{C}}, \max_{1 \leq k \leq n+1} \{ |\mu_1(t_k)|, |\mu_2(t_k)| \}, + t_{n+1} \max_{1 \leq k \leq n+1} \|f(t_k)\|_{\bar{C}} \right\}$$

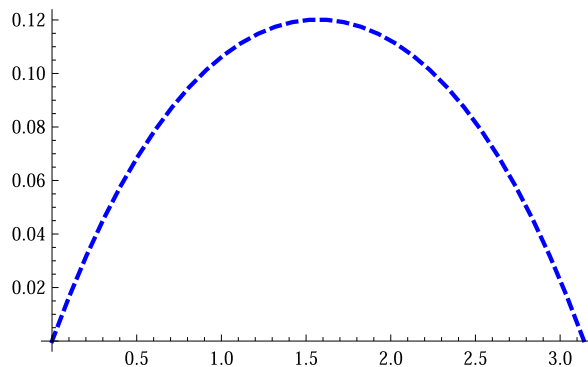


FIGURE 1. Numerical solution at $t = 1$ with step $h = \pi/31 \approx 0.1$ and $\tau = 0.1$.

Remark 1: It is interesting to note that the maximal and minimal values of the difference solution do not depend on the diffusion coefficient $k(u)$ and the convection coefficient $r(u)$.

Remark 2: For the case of $c = q(x) = f(x, t) \equiv 0$, equation (16) can be written as

$$\frac{\partial u}{\partial t} = e^{-x} \frac{\partial}{\partial x} \left(\bar{k}(x, u) \frac{\partial u}{\partial x} \right),$$

$$\bar{k}(x, u) = e^x k(u), \quad k(u) = \beta'(u).$$

Then, as construction of monotone difference schemes for it, we do not need to use the regularization principle.

Remark 3: The estimates obtained in (24)–(25) are fully consistent with the estimates of exact solution of differential problem (19).

Remark 4: For the case of the Frey model [21] $\beta(u) = u/(1 - \rho u)^2$, $\rho > 0$ from (17) we obtain the coefficient $k(u)$ of the form $k(u) = (1 + \rho u)/(1 - \rho u)^3$. Then, by virtue of (18), equation (13) will be parabolic if $k(u) > 0, \forall u \in \bar{D}_u$, i.e., if

$$-\frac{1}{\rho} < u(x, t) < \frac{1}{\rho}. \tag{26}$$

With the help of the estimates (19) we can choose such input data that conditions (26) are satisfied. Then for solution of the difference scheme (20), which approximates problem (13)–(15), conditions (26) are also fulfilled, because by Theorem 2 we always have $m_1 \leq m_1^{n+1}, m_2^{n+1} \leq m_2$.

Remark 5: We consider the particular case of Gamma equation with homogeneous boundary conditions

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(\frac{1+u}{(1-u)^3} \frac{\partial u}{\partial x} \right),$$

$$\rho = 1, \quad 0 < x < \pi, \quad 0 < t \leq 1,$$

$$u(x, 0) = \sin x, \quad u(0, t) = u(\pi, t) = 0. \tag{27}$$

As the coefficient $k(u) = (1+u)/(1-u)^3$ is not defined at $u = 1$, then it is not defined for the initial function $u_0(x) = \sin x$ at $x = x^* = \pi/2$. So we build uniform grid with step $h = \pi/(2N+1)$ in order to $x_i \neq x^*$. The approximate solution of the problem (27) at $t = 1$, obtained by the difference scheme (20), is shown on Fig. 1.

The best numerical results are obtained if the extremal point is not a node of grid. The numerical solution is not defined, if $x = x^*$ is a node of the grid. The solution presented on Fig. 1 is not mathematically correct because the solution of (27) is not defined for such choice of the initial data. Therefore it is important to construct grid domain so that an extremal point is a grid node.

V. MONOTONE FINITE-DIFFERENCE SCHEME OF SECOND-ORDER APPROXIMATION ON NON-UNIFORM SPATIAL GRIDS

In this section, a new monotone difference schemes for the convection–diffusion problem for Gamma equation (13)–(15) will be constructed and analysed on non-uniform grids in space. The construction of such schemes is based on the appropriate choice of the perturbed coefficient (regularization principle), similar to [31], [39]. Using the difference maximum principle, two-sided estimate and a priori estimates in the C -normal are obtained for the solution of the difference scheme that approximates the above equation.

A. DIFFERENCE SCHEME

We introduce an arbitrary non-uniform spatial grid

$$\begin{aligned} \hat{\omega} &= \hat{\omega}_h \cup \gamma_h, \\ \hat{\omega} &= \{x_i = x_{i-1} + h_i, i = 1, 2, \dots, N - 1\}, \\ \gamma_h &= \{x_0 = l_1, x_N = l_2\}, \end{aligned}$$

and uniform grid by time variable

$$\bar{\omega}_\tau = \{t_n = n\tau, n = 0, 1, \dots, N_0, \tau N_0 = T\} = \omega_\tau \cup \{T\}.$$

Taking into account the identity

$$(ku)' = 0, 5((ku)'' + ku'' - k''u), \tag{28}$$

and using standard notation [39]

$$\begin{aligned} h_+ &= h_{i+1}, \quad h = h_i, \quad \tilde{h} = (h_+ + h) / 2, \\ \tilde{h} &= (h_+ - h) / 3, \quad v = v_i = v(x_i), \\ v_\pm &= v_{i\pm 1} = v(x_{i\pm 1}), \quad v_x = (v_+ - v) / h_+, \\ v_{\bar{x}} &= (v - v_-) / h, \quad v_{\tilde{x}\tilde{x}} = (v_x - v_{\bar{x}}) / \tilde{h}, \\ t &= t_n, \quad \hat{t} = t_{n+1}, \quad v = v^n = v(t_n), \\ \hat{v} &= v^{n+1} = v(t_{n+1}), \end{aligned}$$

we construct a new difference scheme for a quasilinear parabolic equation (16) on a non-uniform grid $\omega = \hat{\omega}_h \times \omega_\tau$

$$\begin{aligned} y_{i(\beta_1\beta_2)} &= \kappa(y) A_1 \hat{y} + b^+(y) a_+(y) \hat{y}_x + \\ &\quad + b^-(y) a_-(y) \hat{y}_{\bar{x}} - \tilde{d} \hat{y}_{(\beta_5\beta_6)} + \varphi, \\ y_0^{n+1} &= \mu_1(t_{n+1}), \quad y_N^{n+1} = \mu_2(t_{n+1}), \\ y_i^0 &= u_0(x_i), \end{aligned} \tag{29}$$

where

$$\begin{aligned} v_{(\beta_k\beta_{k+1})} &= \beta_k v_+ + (1 - \beta_k - \beta_{k+1}) v + \beta_{k+1} v_-, \\ A_1 \hat{y} &= 0.5 \left[(k(y) \hat{y})_{\tilde{x}\tilde{x}} + k_{(\beta_1\beta_2)}(y) \hat{y}_{\tilde{x}\tilde{x}} - k_{\tilde{x}\tilde{x}}(y) \hat{y}_{(\beta_3\beta_4)} \right], \end{aligned}$$

$$\begin{aligned} \beta_1 &= 0.5 \left(|\tilde{h}| + \tilde{h} \right) / h_+, \quad \beta_2 = 0.5 \left(|\tilde{h}| - \tilde{h} \right) / h, \\ \beta_3 &= 0.5 \left(\tilde{h} k_{\tilde{x}\tilde{x}} - |\tilde{h} k_{\tilde{x}\tilde{x}}| \right) / (h_+ k_{\tilde{x}\tilde{x}}), \\ \beta_4 &= -0.5 \left(\tilde{h} k_{\tilde{x}\tilde{x}} + |\tilde{h} k_{\tilde{x}\tilde{x}}| \right) / (h k_{\tilde{x}\tilde{x}}), \\ \beta_5 &= 0.5 \left(\tilde{h} - |\tilde{h}| \right) / h_+, \quad \beta_6 = -0.5 \left(\tilde{h} + |\tilde{h}| \right) / h, \\ b^\pm(y) &= \frac{1}{3} \left(\frac{r^\pm(y_-)}{k(y_-)} + \frac{r^\pm(y)}{k(y)} + \frac{r^\pm(y_+)}{k(y_+)} \right), \\ r^\pm(y) &= 0.5 (r(y) \pm |r(y)|), \quad \kappa(y) = \frac{1}{1 + R(y)}, \\ R(y) &= \frac{h_+ + 2h}{6} b^+(y) - \frac{2h_+ + h}{6} b^-(y) \geq 0, \\ a(y) &= 0.5 (k(y) + k(y_-)), \\ a_+(y) &= 0.5 (k(y_+) + k(y)), \\ \tilde{d} &= q(\bar{x}), \quad \varphi = f(\bar{x}, \hat{t}), \quad \bar{x} = x + \tilde{h}. \end{aligned}$$

B. APPROXIMATION ERROR

Let us prove that the scheme (29) approximates the problem (13)–(15) with the second order with respect to the non-calculated point $\tilde{x}_i = x_i + \tilde{h}_i$ (in the case of a uniform grid $\tilde{x}_i = x_i$). To do this, we focus on the relationship [40]

$$v_{\tilde{x}\tilde{x}} - v''(\bar{x}) = O(\tilde{h}^2), \tag{30}$$

$$v_{(\beta_k\beta_{k+1})} - v(\bar{x}) = O(\tilde{h}^2), \quad k = 1, 3, 5, \tag{31}$$

when the condition of variable in space weight factors is fulfilled β_k, β_{k+1}

$$\beta_k h_+ - \beta_{k+1} h = \frac{h_+ - h}{3} = \tilde{h}, \quad k = 1, 3.$$

By virtue of (30)

$$\begin{aligned} (k(u) \hat{u})_{\tilde{x}\tilde{x}} - \frac{\partial^2 (k(u) u)(\bar{x}, \hat{t})}{\partial x^2} &= O(\tilde{h}^2 + \tau), \\ k_{\tilde{x}\tilde{x}}(u) - \frac{\partial^2 k(\bar{x})}{\partial x^2} &= O(\tilde{h}^2). \end{aligned} \tag{32}$$

In view of (31) we obtain

$$k_{(\beta_1\beta_2)}(u) - k(\bar{x}) = O(\tilde{h}^2), \tag{33}$$

$$u_{t(\beta_1\beta_2)} - \frac{\partial u(\bar{x}, \hat{t})}{\partial t} = O(\tilde{h}^2 + \tau). \tag{34}$$

From (30)–(33) it follows that

$$A_1 \hat{u} - \frac{\partial}{\partial x} \left(k(u) \frac{\partial u}{\partial x} \right) (\bar{x}, \hat{t}) = O(\tilde{h}^2 + \tau). \tag{35}$$

Using the Taylor series expansion

$$\begin{aligned} u_x &= u'(\bar{x}) + \frac{h_+ + 2h}{6} u''(\bar{x}) + O(\tilde{h}^2), \\ u_{\bar{x}} &= u'(\bar{x}) - \frac{2h_+ + h}{6} u''(\bar{x}) + O(\tilde{h}^2), \\ a_+(u) &= k(\bar{x}) + \frac{h_+ + 2h}{6} k'(\bar{x}) + O(\tilde{h}^2), \\ a(u) &= k(\bar{x}) - \frac{2h_+ + h}{6} k'(\bar{x}) + O(\tilde{h}^2), \end{aligned}$$

we conclude that

$$a_+(u) u_x = (ku')(\bar{x}) + \frac{h_+ + 2h}{6}(ku')'(\bar{x}) + O(\tilde{h}^2),$$

$$a(u) u_{\bar{x}} = (ku')(\bar{x}) - \frac{2h_+ + h}{6}(ku')'(\bar{x}) + O(\tilde{h}^2).$$

Since

$$r^+(u) + r^-(u) = r(u),$$

$$b^+(u) + b^-(u) = \frac{1}{3} \left(\frac{r}{k}(u_-) + \frac{r}{k}(u) + \frac{r}{k}(u_+) \right)$$

$$= \frac{r}{k}(\bar{x}) + O(\tilde{h}^2),$$

then

$$b^+(u) a_+(u) u_x + b^-(u) a(u) u_{\bar{x}} = (ru')(\bar{x}) + R(u) (ku')'(\bar{x}) + O(\tilde{h}^2). \quad (36)$$

Using (36) we get

$$b^+(u) a_+(u) \hat{u}_x + b^-(u) a(u) \hat{u}_{\bar{x}} = \left(r(u) \frac{\partial u}{\partial x} \right) (\bar{x}, \hat{t}) + R(u) \frac{\partial}{\partial x} \left(k(u) \frac{\partial u}{\partial x} \right) (\bar{x}, \hat{t}) + O(\tilde{h}^2 + \tau). \quad (37)$$

Finally, from (34)–(35), (37) we find out that the approximation error is of second order in space

$$\psi(\bar{x}, \hat{t}) = -u_{t(\beta_1\beta_2)} + \kappa(u) A_1 \hat{u} + b^+(u) a_+(u) \hat{u}_x + b^-(u) a(u) \hat{u}_{\bar{x}} - \bar{d} \hat{u}_{(\beta_5\beta_6)} + \varphi = \frac{R^2(u)}{1 + R(u)} \frac{\partial}{\partial x} \left(k(u) \frac{\partial u}{\partial x} \right) (\bar{x}, \hat{t}) + O(\tilde{h}^2 + \tau).$$

Hence, order of spatial approximation of the difference scheme (29) is two and order of temporal approximation is one.

C. MONOTONICITY, TWO-SIDED AND A PRIORI ESTIMATES

We write the difference scheme (29) in the canonical form (22)–(23) with coefficients defined as follows

$$A_i^n = -\beta_{2i} + 0.5\kappa_i^n \tau [(k_{(\beta_1\beta_2)}(y^n) + k(y_{i-1}^n)) / (\tilde{h}_i h_i) - \beta_{4i} k_{\tilde{x}\hat{x},i}(y^n)] - \tau b_i^-(y^n) a_i^n / h_i - \tau \bar{d}_i \beta_{6i},$$

$$B_i^n = -\beta_{1i} + 0.5\kappa_i^n \tau [(k_{(\beta_1\beta_2)}(y^n) + k(y_{i+1}^n)) / (\tilde{h}_i h_{i+1}) - \beta_{3i} k_{\tilde{x}\hat{x},i}(y^n)] + \tau b_i^+(y^n) a_{i+1}^n / h_{i+1} - \tau \bar{d}_i \beta_{5i},$$

$$C_i^n = 1 + \tau \bar{d}_i + A_i^n + B_i^n, \quad F_i^n = y_{(\beta_1\beta_2)}^n + \tau \varphi_i^{n+1},$$

$$\varphi_i^{n+1} = f(\bar{x}_i, t_{n+1}), \quad \bar{d}_i = q(\bar{x}_i), \quad \bar{x}_i = x_i + \tilde{h}_i.$$

Let the following inequality be fulfilled:

$$\tau \geq \frac{(1 + 0.5\tilde{h}c_0) \|h_+^2 - h^2\|_C}{6k_1},$$

$$\tilde{h} = \max_{1 \leq i \leq N} h_i, \quad c_0 = \max_{u \in \bar{D}_u} \frac{|r(u)|}{k(u)}, \quad (38)$$

Theorem 4: Let the conditions (38) be met. Then the finite-difference scheme (29) is monotone, its solution belongs to the value interval of exact solution $y \in \bar{D}_u$ and the following two-sided estimates hold at any point $(x, t_n) \in \omega$

$$m_1^n \leq y(x, t_n) \leq m_2^n, \quad (39)$$

where

$$m_1^n = \sup_{\lambda > 0} \min \left\{ 0, \min_{\omega_{t_n}} e^{\lambda(t_n-t)} \{ \mu_1(t), \mu_2(t), u_0(x) \}, \min_{\omega_{t_n}} \frac{\tau f(x, t) e^{\lambda(t_n-t)}}{e^{\lambda\tau} (1 + \tau q) - 1} \right\},$$

$$m_2^n = \inf_{\lambda > 0} \max \left\{ 0, \max_{\omega_{t_n}} e^{\lambda(t_n-t)} \{ \mu_1(t), \mu_2(t), u_0(x) \}, \max_{\omega_{t_n}} \frac{\tau f(x, t) e^{\lambda(t_n-t)}}{e^{\lambda\tau} (1 + \tau q) - 1} \right\}.$$

Proof: To prove the upper bound (39), we consider an auxiliary function $z = z(x, t_n) = y(x, t_n) e^{-\lambda t_n}$, where $\lambda > 0$ is a parameter. Let (x^0, t^0) be the maximum point of z in the grid domain ω_{t_τ} and $z^0 = z(x^0, t^0)$. There are only the following possibilities:

1. The maximum z^0 is non-positive;
2. The point (x^0, t^0) is on the boundary of the grid domain ω_{t_τ} ;
3. The maximum z^0 is positive, and the point (x^0, t^0) is the interior point of the grid domain ω_{t_τ} .

In case (3.) at the maximum point (x^0, t^0) the following relations are fulfilled:

$$C_{(*)i}^n z_i^{n+1} = A_{(*)i}^n z_{i-1}^{n+1} + B_{(*)i}^n z_{i+1}^{n+1} + K_i^n z_{(\beta_1, \beta_2)i}^n + F_{(*)i}^n, \quad i = 1, 2, \dots, N - 1,$$

where

$$A_{(*)i}^n = e^{\lambda\tau} A_i^n, \quad B_{(*)i}^n = e^{\lambda\tau} B_i^n, \quad K_i^n = 1,$$

$$C_{(*)i}^n = e^{\lambda\tau} (1 + \tau q_i) + A_{(*)i}^n + B_{(*)i}^n, \quad F_{(*)i}^n = \tau f_i^{n+1} e^{-\lambda t_n},$$

$$D_{(*)i}^n = C_{(*)i}^n - A_{(*)i}^n - B_{(*)i}^n - K_i^n = e^{\lambda\tau} (1 + \tau q_i) - 1 > 0$$

for all $\lambda\tau > 0$.

Now we need to find a condition so that $y_i^n \in \bar{D}_u$ for all i, n . If $n = 0$, it is obvious that $y_i^0 = u_0(x_i) \in \bar{D}_u$. Assume that, for any arbitrary n , $y_i^n \in \bar{D}_u$ is also true for all i . From this assumption for the case of $\tilde{h} > 0, k_{\tilde{x}\hat{x}} > 0$ (we do not consider trivial cases of $\tilde{h} = 0$ and $k_{\tilde{x}\hat{x}} = 0$) we get the concrete values of the weights

$$\beta_1 = \tilde{h}/h_+ > 0, \quad \beta_2 = \beta_3 = \beta_5 = 0,$$

$$\beta_4 = \beta_6 = -\tilde{h}/h < 0.$$

The expressions $k_{(\beta_1\beta_2)}(y)$, $-\beta_4 k_{\tilde{x}\hat{x}}(y)$, $-\tau \bar{d} \beta_6$ are converted to the form

$$k_{(\beta_1\beta_2)}(y) = \frac{\tilde{h}}{h_+} k(y_+) + \left(1 - \frac{\tilde{h}}{h_+} \right) k(y)$$

$$= \frac{\tilde{h}}{h_+} k(y_+) + \frac{2h_+ + h}{3h_+} k(y) > 0,$$

$$-\beta_4 k_{\tilde{x}\tilde{x}}(y) = \frac{\tilde{h}}{h} k_{\tilde{x}\tilde{x}}(y) > 0, \quad -\tau \bar{d} \beta_6 = \tau \bar{d} \frac{\tilde{h}}{h} \geq 0.$$

It follows that $A > 0$. Discarding the positive term in the expression for the coefficient B with $b_i^+(y^n)$, we obtain

$$B > -\frac{\tilde{h}}{h_+} + 0.5\tau \frac{\kappa \left(\left(1 + \frac{\tilde{h}}{h_+}\right) k(y_+) + \left(1 - \frac{\tilde{h}}{h_+}\right) k(y) \right)}{\hbar h_+}.$$

Since $|\tilde{h}/h_+| < 1$, then $\left(1 + \frac{\tilde{h}}{h_+}\right) k_+ + \left(1 - \frac{\tilde{h}}{h_+}\right) k_- \geq 2k_1$. On the other hand, we have

$$\begin{aligned} \min_{u \in \bar{D}_u} \kappa &= \min_{u \in \bar{D}_u} (1 + R)^{-1} = \left(1 + \max_{u \in \bar{D}_u} R\right)^{-1}, \\ \max_{u \in \bar{D}_u} R &= \max_{u \in \bar{D}_u} \left[b^+(u)(h_+ + 2h)/6 - b^-(u)(2h_+ + h)/6 \right] \\ &\leq 0.5\bar{h}c_0, \\ \bar{h} &= \max_{1 \leq i \leq N} h_i, \quad c_0 = \max_{u \in \bar{D}_u} \frac{|r(u(x_i, t_n))|}{k(u(x_i, t_n))}. \end{aligned}$$

It follows that

$$B > -\frac{\tilde{h}}{h_+} + \tau \frac{(1 + 0.5\bar{h}c_0)^{-1} k_1}{\hbar h_+}.$$

From the last inequality we conclude that $B_i^n > 0$ at $\tau \geq (1 + 0.5\bar{h}c_0) |h_{i+1}^2 - h_i^2| / (6k_1)$. It is similar, we can find out conditions for other cases.

Therefore, the inequality (38) also assures that the positivity of the coefficients (6)–(7) is fulfilled (i.e. the difference scheme (29) is monotone). According to Lemma 1 on the base of the estimate (8) for arbitrary $t = t_n \in \omega_\tau$ and all $i = 0, 1, \dots, N$, we have

$$z_i^{n+1} \leq \max_{\omega_{n+1}} \frac{\tau f(x, t) e^{-\lambda t}}{e^{\lambda \tau} (1 + \tau q) - 1}.$$

Therefore, in all above cases the function z will satisfy the following estimate

$$z(x, t_{n+1}) \leq \max \left\{ 0, \max_{\omega_{n+1}} e^{-\lambda t} \{ \mu_1(t), \mu_2(t), u_0(x) \}, \max_{\omega_{n+1}} \frac{\tau f(x, t) e^{-\lambda t}}{e^{\lambda \tau} (1 + \tau q) - 1} \right\}$$

and for the original function y , we acquire the upper bound (39). Analogous arguments for the minimum point hold the lower estimate (39). As

$$\frac{\tau}{e^{\lambda \tau} (1 + \tau q) - 1} \leq \frac{1}{\lambda + q} \quad \text{for all } \lambda, \tau > 0,$$

we see that the estimates (19) and (39) imply the inequalities

$$m_1 \leq m_1^n, \quad m_2^n \leq m_2,$$

i.e., $y_i^{n+1} \in [m_1, m_2]$, $i = \overline{1, N-1}$. The proof is completed by induction.

Remark 6: Inequalities

$$m_1 \leq m_1^n, \quad m_2^n \leq m_2,$$

show that difference estimates are consistent with the estimates for the solution of the differential problem.

Basing on the maximum principle in a standard way [36], we can acquire the a priori estimate in the C -norm:

Theorem 5: Let the condition (38) be fulfilled. Then for the solution of the difference problem (29) the following a priori estimate holds

$$\|y^n\|_{\bar{C}} \leq \max \left\{ \max_{t \in \bar{\omega}_\tau} \{ |\mu_1(t)|, |\mu_2(t)| \}, \|u_0\|_{\bar{C}} \right\} + T \max_{t \in \bar{\omega}_\tau} \|f(t)\|_C, \quad n = \overline{0, N_0}. \quad (40)$$

Proof: According to Corollary 1, we can conclude that $\|y^{n+1}\|_{\bar{C}} \leq \max \left\{ |\mu_1^{n+1}|, |\mu_2^{n+1}|, \left\| \frac{F^n}{D^n} \right\|_C \right\}$. Since the variable weighting factors $\beta_1, \beta_2 \geq 0$ are non-negative, then

$$\begin{aligned} \left\| \frac{F^n}{D^n} \right\|_C &\leq \left\| \frac{y_{(\beta_1 \beta_2)}^n}{1 + \tau \bar{d}} \right\|_C + \tau \left\| \frac{\varphi^{n+1}}{1 + \tau \bar{d}} \right\|_C \\ &\leq \|y_{(\beta_1 \beta_2)}^n\|_{\bar{C}} + \tau \|\varphi^{n+1}\|_C \leq \|y^n\|_{\bar{C}} + \tau \|\varphi^{n+1}\|_C. \end{aligned}$$

Substituting this estimate in the last inequality, we find the chain of relations

$$\begin{aligned} \|y^{n+1}\|_{\bar{C}} &\leq \max \left\{ |\mu_1^{n+1}|, |\mu_2^{n+1}|, \|y^n\|_{\bar{C}} + \tau \|\varphi^{n+1}\|_C \right\} \\ &\leq \max \left\{ |\mu_1^{n+1}|, |\mu_2^{n+1}|, \max \{ |\mu_1^n|, |\mu_2^n| \} \right. \\ &\quad \left. + \tau \|\varphi^{n+1}\|_C, \|y^{n-1}\|_{\bar{C}} + \tau (\|\varphi^n\|_C + \|\varphi^{n+1}\|_C) \right\} \leq \dots \\ &\leq \max \left\{ \max_{1 \leq k \leq n+1} \left\{ |\mu_1^k|, |\mu_2^k| \right\} + \sum_{k=0}^n \tau \|\varphi^{k+1}\|_C, \|y^0\|_{\bar{C}} + \sum_{k=0}^n \tau \|\varphi^{k+1}\|_C \right\}. \quad (41) \end{aligned}$$

Given that

$$\begin{aligned} \max_{1 \leq k \leq n+1} \left\{ |\mu_1^k|, |\mu_2^k| \right\} &\leq \max_{t \in \bar{\omega}_\tau} \{ |\mu_1(t)|, |\mu_2(t)| \}, \\ \sum_{k=0}^n \tau \|\varphi^{k+1}\|_C &\leq t_{n+1} \max_{t \in \bar{\omega}_\tau} \|f(t)\|_C, \end{aligned}$$

from (41) we obtain the required relation (40). \square

D. NUMERICAL TEST

We consider the problem (13)–(17) with a given input data $l_1 = 0, l_2 = \pi, T = 1, k(u) = u^2, r(u) = u, q(x) = 1 + x^2$ and the exact solution $u(x, t) = e^t(\sin x + 1)$. The starting non-uniform spatial nodes are shown in Table 1. To increase the number of the nodes, we use the following law

$$x_{2i} = (0, 375 + r) x_{i+1} + (0, 625 - r) x_i,$$

where $r \in [0; 0, 25)$ – is a random variable following the normal distribution.

For illustration of the efficiency of the new algorithms on non-uniform grids the error of the method in maximum form

$$\|z\|_C = \|y - u\|_C = \max_{(x,t) \in \omega} |y(x, t) - u(x, t)|,$$

TABLE 1. Starting non-uniform spatial nodes (N = 10).

x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
0	0,3	0,5	1	1,4	1,6	2	2,5	2,8	2,9	π

TABLE 2. Numerical results on non-uniform spatial grids for problem (13)–(17).

$N; h_{max}; \tau = 0, 025$	10; 0,5; τ	20; 0,27; $\tau/4$	40; 0,175; $\tau/16$
$\ z\ _C$	0,0955	0,0236	0,00614

of difference scheme (29) approximating the given problem are shown in Table 2.

The numerical experiments illustrate the increasing of the accuracy of the new scheme and for scheme (29) the order of accuracy $O(h^2 + \tau)$ is reached on coarse grids.

VI. ANOTHER APPROACH FOR CONSTRUCTION OF MONOTONE SECOND-ORDER FINITE-DIFFERENCE SCHEMES ON NON-UNIFORM GRIDS

To simplify evaluation, we consider the linear Gamma equation with the convective transport being non-divergent [44]

$$\begin{aligned}
 Lu &= (k(x)u') + r(x)u' - d(x)u = -f(x), \\
 k(x) &\geq k_1 > 0, \quad d(x) \geq d_1 > 0, \quad l_1 < x < l_2, \\
 u(l_1) &= \mu_1, \quad u(l_2) = \mu_2.
 \end{aligned} \tag{42}$$

We can construct similar schemes by using the identities (28) and $ku' = 0.5((ku)' + ku' - k'u)$. Suppose that $b(x) = r(x)/k(x)$ and $L_1v = v'' + bv'$ we rewrite the equation (42) in the form

$$Lu = 0.5L_1(ku) + 0.5kL_1u - 0.5uL_1k - du = -f$$

We change the differential operator L_1 on the grid $\hat{\omega}_h$ by the following second order difference operator L_{1h} :

$$L_{1h}v = \kappa v_{\bar{x}\bar{x}} + b^+ v_x + b^- v_{\bar{x}} = L_1v(\bar{x}) + O(h^2), \tag{43}$$

On the grid $\hat{\omega}_h$ we change the differential operator L by L_h

$$\begin{aligned}
 L_hu &= 0.5L_{1h}(ku) + 0.5k_{(\beta_1\beta_2)}L_{1h}u \\
 &\quad - 0.5u_{(\bar{\beta}_3\bar{\beta}_4)}L_{1h}k - \bar{d}u_{(\beta_5\beta_6)},
 \end{aligned} \tag{44}$$

where

$$\begin{aligned}
 b^+ &= \frac{r^+}{k}(\bar{x}) \geq 0, \quad b^- = \frac{r^-}{k}(\bar{x}) \leq 0, \\
 r^\pm &= 0.5(r \pm |r|), \quad \kappa = \frac{1}{1+R}, \\
 R &= \frac{h_+ + 2h}{6}b^+ - \frac{2h_+ + h}{6}b^- \geq 0, \quad \bar{d} = d(\bar{x}).
 \end{aligned}$$

The variable in space weights $\bar{\beta}_3, \bar{\beta}_4, \beta_5, \beta_6$ are defined by fulfilment of the condition (32) in the following way

$$\begin{aligned}
 \bar{\beta}_3 &= 0.5 \left(\tilde{h}L_{1h}k - \left| \tilde{h}L_{1h}k \right| \right) / (h_+L_{1h}k), \\
 \bar{\beta}_4 &= -0.5 \left(\tilde{h}L_{1h}k + \left| \tilde{h}L_{1h}k \right| \right) / (hL_{1h}k), \\
 \beta_5 &= 0.5 \left(\tilde{h} - \left| \tilde{h} \right| \right) / h_+, \quad \beta_6 = -0.5 \left(\tilde{h} + \left| \tilde{h} \right| \right) / h.
 \end{aligned}$$

Therefore, the difference scheme

$$L_hy = -\varphi, \quad \varphi = f(\bar{x}), \quad y_0 = \mu_1, \quad y_N = \mu_2, \tag{45}$$

can be used for approximating differential problem (42) on arbitrary non-uniform grid with the second order approximation. For uniform grid $R = 0.5h|r|/k$, the difference scheme (45) will degenerate to be the well-known monotone scheme of second-order approximation [39]. The difference scheme (45) can be rewritten as the following canonical form (5)

$$\begin{aligned}
 A_i y_{i-1} - C_i y_i + B_i y_{i+1} &= -F_i, \quad i = 1, 2, \dots, N-1, \\
 y_0 &= \mu_1, \quad y_N = \mu_2.
 \end{aligned}$$

with coefficients

$$\begin{aligned}
 A_i &= 0.5 \left[(k_{(\beta_1\beta_2)} + k_{i-1}) (\kappa_i - b^- \bar{h}_i) / (\bar{h}_i h_i) \right. \\
 &\quad \left. - \bar{\beta}_{4i} L_{1h,i} k \right] - \beta_{6i} \bar{d}_i, \\
 B_i &= 0.5 \left[(k_{(\beta_1\beta_2)} + k_{i+1}) (\kappa_i + b^+ \bar{h}_i) / (\bar{h}_i h_{i+1}) \right. \\
 &\quad \left. - \bar{\beta}_{3i} L_{1h,i} k \right] - \beta_{5i} \bar{d}_i, \\
 C_i &= \bar{d}_i + A_i + B_i, \quad F_i = f(\bar{x}_i).
 \end{aligned} \tag{46}$$

It is obvious that $A_i > 0, B_i > 0, D_i = \bar{d}_i > 0$. Hence, for arbitrary non-uniform grid refinement, the coefficients (46) of the difference scheme (45) will fulfill the conditions (6)–(7) (unconditional monotonicity). By Lemma 1 we acquire two-sided estimates for the difference scheme solution (45) for $i = \bar{0}, \bar{N}$

$$\begin{aligned}
 y_i &\geq \min \left\{ \mu_1, \mu_2, \min_{1 \leq i \leq N-1} (\bar{f}_i / \bar{d}_i) \right\}, \\
 y_i &\leq \max \left\{ \mu_1, \mu_2, \max_{1 \leq i \leq N-1} (\bar{f}_i / \bar{d}_i) \right\}.
 \end{aligned}$$

Moreover, according to Corollary 1, the difference scheme (45) is stable by right-hand side and by the boundary conditions. For the solution, we have the following a priori estimate

$$\|y\|_{\bar{C}} \leq \max \{ |\mu_1|, |\mu_2|, \|\bar{f}_i / \bar{d}_i\|_{\bar{C}} \}. \tag{47}$$

Substituting $y = z + u$ in the equation (45), we get the problem for the method error

$$L_h z = -\psi, \quad \psi = L_h u + \varphi, \quad z|_{\gamma_h} = 0. \tag{48}$$

It is obvious that $\psi = O(\bar{h}^2)$, $\bar{h} = \max_{1 \leq i \leq N} h_i$. Because for the problem (48) all conditions (6)–(7) of the maximum principle are fulfilled, from (47) we can find out that $\|z\|_{\bar{C}} \leq \|\psi\|_{\bar{C}} \leq c\bar{h}^2$, i.e. the difference scheme (45) will converge to the exact solution with the order convergence being two.

It is similar, we can use formulas (33), (34), (43), (44) to construct the usual six-point stencil of monotone second-order approximation difference scheme for the Gamma equation (13)–(15) on the non-uniform grid $\omega = \hat{\omega}_h \times \omega_\tau$ with the

support of the change $g(x, u) = r(x)/k(u)$ and the operator $\Lambda_h v = v'' + g'v', v = v(u)$

$$\begin{aligned}
 y_{t(\beta_1\beta_2)} &= 0.5 \left[\Lambda_h(k(y)\hat{y}) + k_{(\beta_1\beta_2)}(y) \Lambda_h \hat{y} \right. \\
 &\quad \left. - \hat{y}_{(\tilde{\beta}_3\tilde{\beta}_4)} \Lambda_h k(y) \right] \\
 &\quad - \tilde{d}_{\hat{y}_{(\beta_5\beta_6)}} + \varphi, y_i^0 = u_0(x_i), \\
 y_0^{n+1} &= \mu_1(t_{n+1}), \quad y_N^{n+1} = \mu_2(t_{n+1}), \quad (49)
 \end{aligned}$$

where

$$\begin{aligned}
 \Lambda_h v &= \bar{\kappa} v_{\bar{x}\bar{x}} + g^+ v_x + g^- v_{\bar{x}}, \quad \Lambda_h \hat{v} = \bar{\kappa} \hat{v}_{\bar{x}\bar{x}} + g^+ \hat{v}_x + g^- \hat{v}_{\bar{x}}, \\
 \tilde{\beta}_3 &= 0.5 \left(\tilde{h} \Lambda_h k - \left| \tilde{h} \Lambda_h k \right| \right) / (h_+ \Lambda_h k), \\
 \tilde{\beta}_4 &= -0.5 \left(\tilde{h} \Lambda_h k + \left| \tilde{h} \Lambda_h k \right| \right) / (h \Lambda_h k), \\
 g^+ &= \bar{r}^+(y) \bar{k}(y) \geq 0, \quad g^- = \bar{r}^-(u) \bar{k}(y) \leq 0, \\
 \bar{r}^\pm &= 0.5 (\bar{r} \pm |\bar{r}|), \\
 \bar{k}(y) &= [1/k(y_-) + 1/k(y) + 1/k(y_+)]/3, \\
 \bar{\kappa} &= (1 + \bar{R})^{-1}, \quad \bar{R} = \frac{h_+ + 2h}{6} g^+ - \frac{2h_+ + h}{6} g^- \geq 0, \\
 \tilde{d} &= q(\bar{x}), \quad \varphi = f(\bar{x}, \hat{t}), \quad \bar{x} = x + \tilde{h}.
 \end{aligned}$$

It is easy to show, that the difference scheme (49) approximates the original differential problem with second order on arbitrary non-uniform spatial grid, and the inequality (38) guarantees the fulfilment of the positivity condition for the coefficients (6)–(7) (i.e., the difference scheme (49) is monotone), and for a difference solution the two-sided estimates of the form (39) are valid.

VII. IMPLICATIONS

A great attention of many scholars to mathematical models utilised for pricing complex financial instruments, in recent years, has suggested the interesting research area of financial mathematics in which Black-Scholes model (BS) is a prominent example. Indeed, there is a large number of studies on this equation in the solution and application context with the employment of diverse numerical schemes like explicit difference schemes [13], finite difference scheme [45], multivariate padé approximation scheme [46] and Cauchy Euler method [32]. Nevertheless, applications of monotone finite difference scheme of second-order approximation for pricing models in finance are still very few. Therefore, to fill this research gap in mathematical financial, this paper adopts this kind of scheme for Gamma equation which is transformed from the non-linear BS equation for option price in an effort to derive the approximation solution of the nonlinear parabolic equations (for the second derivative of the option price). Specifically, in the problems of unlimited nonlinearity (or nonlinearity of unlimited growth), the basic properties of the differential problem may not be preserved when converting from a problem for a differential equation to the corresponding one for a difference equation. For instance, the approximation solution's values may not belong in a certain neighborhood of the exact solution's values.

Hence, two-side estimates for difference solution need to be proved to attain the approximation solution that belongs to some neighborhood of the exact solution (i.e. difference solution $y \in \bar{D}_u$, then condition (18) is satisfied, and it follows that equation (16) is uniformly parabolic). Based on the technique of O.A Ladyzhenskava, the findings of this paper show that a two-side estimate for the difference solution, which is completely consistent with a similar solution estimate of the differential problem, was obtained. The empirical results are expected to be generalised to construct monotone finite difference schemes for two-dimensional quasilinear parabolic BS equations. These issues will be clarified in future work. In addition, this application of this method to solve BS pricing models can be reproduced for other mathematical models in finance such as Kolmogorov equations, Vasicek pricing equations.

VIII. BLACK-SCHOLES EQUATION FROM THE PERSPECTIVE OF ELECTRICAL AND ELECTRONICS ENGINEERING

The proposed difference schemes can be used for solving the Black-Scholes equation (by supporting the Gamma equation). The Black-Scholes equation plays an important role not only in financial engineering but also in electrical and electronics engineering. In this section, we discuss on two well-known equations that have many important applications in electrical and electronics engineering: the heat equation and the Schrodinger equation. They can be transformed into the Black-Scholes equation. The equations are widely used in studying quantum computing, light-emitting diodes, microprocessor, radiological imaging [48], microscopy imaging [49], image restoration (e.g., based on Perona-Malik model [50]), etc.

A. TRANSFORMATION OF THE HEAT EQUATION TO THE BLACK-SCHOLES EQUATION

Let consider the following heat equation:

$$\begin{aligned}
 \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \quad x \in (-\infty, +\infty), \quad 0 \leq t \leq \frac{\sigma^2}{2} T, \\
 u(x, 0) &= e^{-\alpha x} v(x, 0) = e^{-\alpha x} f(e^x), \quad x \in (-\infty, +\infty).
 \end{aligned}$$

Supposing

$$\alpha = \frac{\sigma^2 - 2r}{2\sigma^2}, \quad \beta = -\left(\frac{\sigma^2 + 2r}{2\sigma^2} \right)^2,$$

and we consider that

$$v(x, t) = e^{\alpha x + \beta t} u(x, t) := \phi u.$$

Computing the partial derivatives:

$$\begin{aligned}
 \frac{\partial v}{\partial t} &= \beta \phi u + \phi \frac{\partial u}{\partial t}, \\
 \frac{\partial v}{\partial x} &= \alpha \phi u + \phi \frac{\partial u}{\partial x}, \\
 \frac{\partial^2 v}{\partial x^2} &= \alpha^2 \phi u + 2\alpha \phi \frac{\partial u}{\partial x} + \phi \frac{\partial^2 u}{\partial x^2}.
 \end{aligned}$$

Hence, we acquired the following Black-Scholes equation:

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \left(\frac{2r}{\sigma^2} - 1 \right) \frac{\partial v}{\partial x} - \frac{2r}{\sigma^2} v,$$

$$x \in (-\infty, +\infty), \quad t \in \left[0, \frac{\sigma^2}{2} T \right],$$

$$v(x, 0) = V(e^x, T) = f(e^x), \quad x \in (-\infty, +\infty).$$

B. TRANSFORMATION OF THE SCHRÖDINGER EQUATION TO THE BLACK-SCHOLES EQUATION

We consider the following Schrodinger equation:

$$i\hbar \frac{\partial}{\partial t} \psi(x, t) = \hat{H} \psi(x, t),$$

where

$$\hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x, t),$$

i – the imaginary unit, $\hbar = \frac{h}{2\pi}$, h – the Planck const, $V(x, t)$ – a potential of an electric field. We only consider the free Schrodinger equation, i.e., $V(x, t) = 0$. Then, we set:

$$\tau := it, \quad \hbar := 1, \quad m := \frac{1}{\sigma^2}, \quad x := \ln S,$$

$$\psi(x, t) := e^{-\left(\frac{1}{\sigma^2} \left(\frac{\sigma^2}{2} - r \right) x + \frac{1}{2\sigma^2} \left(\frac{\sigma^2}{2} + r \right) t \right)} C(x, t).$$

Then we acquired the Black-Scholes equation as follows:

$$\frac{\partial C(S, \tau)}{\partial \tau} = -\frac{\sigma}{2} S^2 \frac{\partial^2 C(S, \tau)}{\partial S^2} - rS \frac{\partial C(S, \tau)}{\partial S} + rC(S, \tau).$$

As we can see that the Black-Scholes equation, as well as the Gamma equation, has a close relation to equations of mathematical physics such as the heat equation and the Schrodinger equation. In other words, the proposed difference schemes can be applied for solving problems of the field of electrical and electronics engineering.

IX. CONCLUSION

Based on regularization principle, we developed monotone finite-difference schemes with second-order of local approximation on both uniform and nonuniform grids for the initial boundary value problem for the Gamma equation. Moreover, two-side estimates of the difference solution are presented. Such estimates not only provide a manner to prove the non-negativity of the exact solution, but it is also helpful to find out sufficient conditions based on the input data if the nonlinear problem is parabolic. Consequently, a priori estimate of the approximate solution in the grid norm C depending on the initial and boundary conditions only is proved.

REFERENCES

- [1] G. Arora and G. S. Bhatia, "Radial basis function methods to solve partial differential equations arising in financial applications—A review," *Nonlinear Stud.*, vol. 24, no. 1, pp. 15–25, 2017.
- [2] A. A. E. F. Saib, D. Y. Tangman, and M. Bhuruth, "A new radial basis functions method for pricing American options under Merton's jump-diffusion model," *Int. J. Comput. Math.*, vol. 89, no. 9, pp. 1164–1185, Jun. 2012.
- [3] R. Brummelhuis and R. T. L. Chan, "A radial basis function scheme for option pricing in exponential Lévy models," *Appl. Math. Finance*, vol. 21, no. 3, pp. 238–269, 2014.
- [4] S. Haq and M. Hussain, "Selection of shape parameter in radial basis functions for solution of time-fractional Black–Scholes models," *Appl. Math. Comput.*, vol. 335, pp. 248–263, Oct. 2018.
- [5] J. Ankudinova and M. Ehrhardt, "On the numerical solution of nonlinear Black–Scholes equations," *Comput. Math. Appl.*, vol. 56, pp. 799–812, Aug. 2008.
- [6] R. Company, A. L. González, and L. Jódar, "Numerical solution of modified Black–Scholes equation pricing stock options with discrete dividend," *Math. Comput. Model.*, vol. 44, nos. 11–12, pp. 1058–1068, Dec. 2006.
- [7] R. Company, L. Jódar, G. Rubio, and R. J. Villanueva, "Explicit solution of Black–Scholes option pricing mathematical models with an impulsive payoff function," *Math. Comput. Model.*, vol. 45, nos. 1–2, pp. 80–92, 2007.
- [8] S. Haq, M. Hussain, and A. Ghafoor, "A computational study of variable coefficients fractional advection–diffusion–reaction equations via implicit meshless spectral algorithm," *Eng. Comput.*, pp. 1–21, May 2019, doi: 10.1007/s00366-019-00760-x.
- [9] V. Gülkaç, "Comparative study analytic and numerical methods for solving non-linear Black–Scholes equation with European call option," *Int. J. Sci. Innov. Math. Res.*, vol. 3, no. 2, pp. 66–78, 2015.
- [10] D. Y. Tangman, A. Gopaul, and M. Bhuruth, "Numerical pricing of options using high-order compact finite difference schemes," *J. Comput. Appl. Math.*, vol. 218, no. 2, pp. 270–280, Sep. 2008.
- [11] R. Company, E. Navarro, J. R. Pintos, and E. Ponsoda, "Numerical solution of linear and nonlinear Black–Scholes option pricing equations," *Comput. Math. Appl.*, vol. 56, no. 3, pp. 813–821, Aug. 2008.
- [12] S. K. Bhowmik, "Fast and efficient numerical methods for an extended Black–Scholes model," *Comput. Math. Appl.*, vol. 67, no. 3, pp. 636–654, Feb. 2014.
- [13] M. N. Anwar and L. S. Andallah, "A study on numerical solution of Black–Scholes model," *J. Math. Finance*, vol. 8, no. 2, pp. 372–381, 2018.
- [14] G. Barles and H. M. Soner, "Option pricing with transaction costs and a nonlinear Black–Scholes equation," *Finance Stochastics*, vol. 2, no. 4, pp. 369–397, Aug. 1998.
- [15] F. Black and M. Scholes, "The pricing of options and corporate liabilities," *J. Political Economy*, vol. 81, no. 3, pp. 637–654, May 1973.
- [16] R. Company, L. Jódar, E. Ponsoda, and C. Ballester, "Numerical analysis and simulation of option pricing problems modeling illiquid markets," *Comput. Math. Appl.*, vol. 59, no. 8, pp. 2964–2975, Apr. 2010.
- [17] E. Dremkova and M. Ehrhardt, "A high-order compact method for nonlinear Black–Scholes option pricing equations of american options," *Int. J. Comput. Math.*, vol. 88, no. 13, pp. 2782–2797, Sep. 2011.
- [18] I. Faragó and R. Horváth, "Discrete maximum principle and adequate discretizations of linear parabolic problems," *SIAM J. Sci. Comput.*, vol. 28, no. 6, pp. 2313–2336, Jan. 2006.
- [19] I. Faragó, R. Horváth, and S. Korotov, "Discrete maximum principles for FE solutions of nonstationary diffusion–reaction problems with mixed boundary conditions," *Numer. Methods Partial Differ. Equ.*, vol. 27, no. 3, pp. 702–720, Oct. 2009.
- [20] I. Farago, J. Karatson, and S. Korotov, "Discrete maximum principles for nonlinear parabolic PDE systems," *IMA J. Numer. Anal.*, vol. 32, no. 4, pp. 1541–1573, Feb. 2012.
- [21] R. Frey, *Market Illiquidity as a Source of Model Risk in Dynamic Hedging in Model Risk*, R. Gibson, Ed. London, U.K.: RISK Publications, 2000.
- [22] A. Friedman, *Partial Differential Equations of Parabolic Type*. Englewood Cliffs, NJ, USA: Prentice-Hall, 1964.
- [23] F. R. Gantmakher, "Theory matrices," (in Russian), Nauka, Moscow, Russia, Tech. Rep., 1967. [Online]. Available: https://books.google.com.vn/books?hl=en&lr=&id=ePFtMw9v92sC&oi=fnd&pg=PR3&dq=theory+matrices&ots=sldiqE8NG4&sig=5cFznHlpPT8F78306Jfk4DSShg&redir_esc=y#v=onepage&q=theory%20matrices&f=false
- [24] S. Godunov and V. Ryabenkii, "Difference schemes," (in Russian), Nauka, Moscow, Russia, Tech. Rep., 1977. [Online]. Available: https://scholar.google.com/scholar_lookup?title=Raznostnye%20skhemy&publication_year=1977&author=Godunov%20S.K.&author=Ryabenkii%20V.S
- [25] O. Hyong-Chol, J. J. Jo, and J. S. Kim, "General properties of solutions to inhomogeneous Black–Scholes equations with discontinuous maturity payoffs," *J. Differ. Equ.*, vol. 260, no. 4, pp. 3151–3172, 2016.
- [26] M. Jandačka and D. Ševčovič, "On the risk-adjusted pricing methodology-based valuation of vanilla options and explanation of the volatility smile," *J. Appl. Math.*, vol. 2005, no. 3, pp. 235–258, 2005.

- [27] M. N. Koleva and L. G. Vulkov, "A second-order positivity preserving numerical method for gamma equation," *Appl. Math. Comput.*, vol. 220, pp. 722–734, Sep. 2013.
- [28] O. A. Ladyzhenskaya, "Solution of the first boundary problem in the large for quasi-linear parabolic equations," *Trudy Moskovskogo Matematicheskogo Obshchestva*, vol. 7, pp. 149–177, 1958.
- [29] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Ural'tseva, "Lineinye i kvazilineinye uravneniya parabolicheskogo tipa (Linear and Quasilinear Equations of Parabolic Type)," (in Russian), Nauka, Moscow, Russia, Tech. Rep., 1967. [Online]. Available: https://scholar.google.com/scholar?hl=en&as_sdt=0%2C5&q=Lineinye+i+kvazilineinye+uravneniya+parabolicheskogo+tipa+%28linear+and+quasilinear+equations+of+parabolic+type%29%2C+moscow%2C+1967&btnG=
- [30] L. M. Hieu, T. T. H. Hanh, and D. N. H. Thanh, "Finite-difference method for the Gamma equation on non-uniform grids," *Vietnam J. Sci., Technol. Eng.*, vol. 61, no. 4, pp. 3–8, 2019, doi: [10.31276/VJSTE.61\(4\).03-08](https://doi.org/10.31276/VJSTE.61(4).03-08).
- [31] D. Malafei, "Economical monotone difference schemes for multidimensional convection-diffusion problems on non-uniform grids," (in Russian), *Doklady NAS Belarus*, vol. 44, no. 4, pp. 21–25, 2000.
- [32] J. Manafian and M. Paknezhad, "Analytical solutions for the Black–Scholes equation," *Appl. Appl. Math., Int. J.*, vol. 12, no. 2, pp. 843–852, 2017.
- [33] P. Matus, F. Gaspar, L. M. Hieu, and V. T. K. Tuyen, "Monotone difference schemes for weakly coupled elliptic and parabolic systems," *Comput. Methods Appl. Math.*, vol. 17, no. 2, pp. 287–298, Jan. 2017.
- [34] P. Matus, L. M. Hieu, and L. G. Vulkov, "Maximum principle for finite-difference schemes with non sign-constant input data," (in Russian), *Doklady Nat. Acad. Sci. Belarus*, vol. 59, no. 5, pp. 13–17, 2015.
- [35] P. Matus, D. Poliakov, and L. M. Hieu, "On the consistent two-side estimates for the solutions of quasilinear convection–diffusion equations and their approximations on non-uniform grids," *J. Comput. Appl. Math.*, vol. 340, pp. 571–581, Oct. 2018.
- [36] P. Matus, L. M. Hieu, and L. G. Vulkov, "Analysis of second order difference schemes on non-uniform grids for quasilinear parabolic equations," *J. Comput. Appl. Math.*, vol. 310, pp. 186–199, Jan. 2017.
- [37] P. Matus, V. T. K. Tuyen, and F. Gaspar, "Monotone difference schemes for linear parabolic equations with mixed boundary conditions," (in Russian), *Doklady Nat. Acad. Sci. Belarus*, vol. 58, no. 5, pp. 18–22, 2014.
- [38] P. N. Vabishchevich, A. A. Samarskii, and P. P. Matus, "Second-order accurate finite-difference schemes on nonuniform grids," *Comput. Math. Math. Phys.*, vol. 38, no. 3, pp. 399–410, 1998.
- [39] A. A. Samarskii, *The Theory of Difference Schemes*. New York, NY, USA: Marcel Dekker, 2001.
- [40] A. Samarskii, V. Mazhukin, and P. Matus, "Difference schemes on non-uniform grids for two-dimensional parabolic equations," *Differents Uravneniya*, vol. 34, no. 2, pp. 269–280, In Russian, 1998.
- [41] A. Samarskii, P. Vabishchevich, and P. Matus, "Difference schemes of increased order of accuracy on non-uniform grids," (in Russian), *Differents Uravneniya*, vol. 32, no. 2, pp. 265–274, 1996.
- [42] L. M. Hieu, T. T. H. Hanh, and D. N. H. Thanh, "A finite-difference scheme for initial boundary value problem of the gamma equation in the pricing of financial derivatives," *J. Math. Comput. Sci.*, vol. 20, no. 4, pp. 283–291, Feb. 2020, doi: [10.22436/jmcs.020.04.02](https://doi.org/10.22436/jmcs.020.04.02).
- [43] A. A. Samarskii, P. P. Matus, and V. G. Rychagov, "High-order accurate monotone difference schemes on nonuniform grids for convection–diffusion problems," *Mat. Model.*, vol. 9, no. 2, pp. 95–96, 1997.
- [44] A. A. Samarskii and P. N. Vabishchevich, "Numerical methods for solution of convection–diffusion problems," (in Russian), Editorial YRSS, Moscow, Russia, Tech. Rep., 1999.
- [45] M. K. S. Uddin, M. Ahmed, and S. K. Bhowmik, "A note on numerical solution of a linear Black–Scholes model," *GANIT, J. Bangladesh Math. Soc.*, vol. 33, pp. 103–115, Jan. 2014.
- [46] N. Özdemir and M. Yavuz, "Numerical solution of fractional Black–Scholes equation by using the multivariate Padé approximation," *Acta Phys. Polonica A*, vol. 132, no. 3, pp. 1050–1053, 2017.
- [47] V. A. Zhuk, "Monotone difference scheme for parabolic and elliptic equations with lower order derivatives on nonuniform grids," *Vestsi NAN Bel., Ser. Fiz.-Mat.*, vol. 2, pp. 12–17, 1998.
- [48] D. Thanh, P. Surya, and L. M. Hieu, "A review on CT and X-ray images denoising methods," *Informatica*, vol. 43, no. 2, pp. 151–159 Jun. 2019, doi: [10.31449/inf.v43i2.2179](https://doi.org/10.31449/inf.v43i2.2179).
- [49] D. N. H. Thanh and S. D. Dvoenko, "A denoising of biomedical images," *Int. Arch. Photograph., Remote Sens. Spatial Inf. Sci.*, vols. XL-5/W6, pp. 73–78, May 2015, doi: [10.5194/isprsarchives-XL-5-W6-73-2015](https://doi.org/10.5194/isprsarchives-XL-5-W6-73-2015).
- [50] V. B. S. Prasath, D. N. H. Thanh, and N. H. Hai, "On selecting the appropriate scale in image selective smoothing by nonlinear diffusion," in *Proc. IEEE 7th Int. Conf. Commun. Electron. (ICCE)*, Hue, Vietnam, Jul. 2018, pp. 267–272, doi: [10.1109/CCE.2018.8465764](https://doi.org/10.1109/CCE.2018.8465764).
- [51] L. M. Hieu, D. N. H. Thanh, and V. B. S. Prasath, "Second order monotone difference schemes with approximation on non-uniform grids for two-dimensional quasilinear parabolic convection-diffusion equations," *Vestnik St Petersburg Univ.-Math.*, to be published.
- [52] A. Al-Arabi, A. Correia, D. Naiff, G. Jardim, and Y. Saporito, "Solving nonlinear and high-dimensional partial differential equations via deep learning," 2018, *arXiv:1811.08782*. [Online]. Available: <http://arxiv.org/abs/1811.08782>
- [53] A. Quarteroni, A. Manzoni, and F. Negri, *Reduced Basis Methods for Partial Differential Equations*. Cham, Switzerland: Springer, 2016.
- [54] S. Markolefas, "Standard Galerkin formulation with high order langrange finite elements for option markets pricing," *Appl. Math. Comput.*, vol. 195, pp. 707–720, 2008.
- [55] D. Jeong, M. Yoo, C. Yoo, and J. Kim, "A hybrid Monte Carlo and finite difference method for option pricing," *Comput. Econ.*, vol. 53, no. 1, pp. 111–124, 2019, doi: [10.1007/s10614-017-9730-4](https://doi.org/10.1007/s10614-017-9730-4).



LE MINH HIEU received the bachelor's degree, the M.Sc. degree in applied mathematics, and the Ph.D. degree in computational mathematics from Belarusian State University, in 2009, 2010, and 2018, respectively.

He is currently an Assistant Professor with the University of Economics–The University of Danang, Vietnam. He has more than 30 works on international peer-reviewed journals and conference proceedings. His research interests are finite difference schemes, nonlinear PDEs, image processing, machine learning, financial mathematics, and computational mathematics. In 2018, he was awarded a certificate for excellent researchers by the Danang People's Committee.



TRUONG THI HIEU HANH was born in Danang, Vietnam. She received the B.S. degree in international trade from the University of Economics–The University of Danang, in 2014, and the M.B.A. degree in finance programme from the Bangor Business School, Bangor University, U.K., in 2017.

Since 2018, she has been researching and teaching in the University of Economics–The University of Danang. Her research interests are about international trade, ecommerce, foreign direct investment, and issues on international financial market and financial mathematics.



DANG NGOC HOANG THANH received the bachelor's degree and the M.Sc. degree in applied mathematics from Belarusian State University, in 2008 and 2009, respectively, and the Ph.D. degree in computer science from Tula State University, Russia, in 2016.

He was a Lecturer with the Department of Information Technology, Hue College of Industry, Vietnam. He is currently an Assistant Professor with the Department of Information Technology, School of Business Information Technology, University of Economics Ho Chi Minh City, Vietnam. He has more than 50 works on international peer-reviewed journals and conference proceedings, five book chapters, one book, and one European Patent. His research interests are image processing, computer vision, machine learning, data classification, computational mathematics, fuzzy mathematics, and optimization. He is a member of scientific organization INSTICC, Portugal, ACM, USA, and IAENG, Taiwan. He is also a member of international conferences committee, such as the IEEE ICCE 2018, Vietnam, IWBBIO, Spain, the IEEE ICIEV, USA, the IEEE ICEEE, Turkey, ICIEE, Japan, ICOTCA, Australia, and ICMTEL, U.K.