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Some Progress on Quantum Error Correction for Discrete and Continuous Error Models

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ABSTRACT Quantum computing has increasingly gained attention for decades since it can surpass classical computing in various aspects. A crucial issue of quantum computing is how to protect information from noise interference such that the error rate during quantum information processing can be limited within an acceptable bound. The technique to address this issue is quantum error correction (QEC). Developing QEC technologies needs to define noise as formal error models and design QEC approaches for these error models. Discrete error models and continuous error models are two kinds of definitions of noise. The former assumes noise occurs independently and can be discretized into a set of basic errors. The former also has a tool, named quantum operations, to describe a specific error model. The latter describes that noise is continuous in time by using differential equations. In this paper, we categorize QEC approaches into three types according to the different error models: discrete error models, specific error models, and continuous error models. We also analyze the state-of-the-art QEC approaches and discuss some future directions. Furthermore, we propose the perturbed error models and their possible definitions, aiming to find the effect of the perturbation during quantum information processing.

INDEX TERMS Quantum computing, quantum error correction, quantum information processing, discrete error models, continuous error models, quantum operations.

I. INTRODUCTION

Quantum computing has provided great potentiality compared to classical computing. For instance, Shor's algorithm [1] is nearly exponentially faster than the most efficient classical factoring algorithm. Therefore, it is significant to ensure the accuracy of *quantum information processing*. Otherwise, any computing would be unreliable. The theory developed for protecting fragile information against noise is *quantum error correction (QEC)*. QEC is one of the foundations to build large-scale and fault-tolerant quantum computers. With the development of quantum theory and practice, building quantum circuits and experimenting with QEC approaches are achievable on platforms such as IBM Q Experience.[1]

The common model of quantum information processing in quantum computers includes three main steps, encoding information, transmitting through channels and decoding information, as shown in Figure 1. Information in quantum computers is stored as *quantum states* of quantum bits (*qubits*). Since noise can occur anytime and anywhere, QEC approaches should protect information all along the processing way. In encoding procedures, QEC approaches add redundancy to the initial information to resist possible noise in advance. The noise during transmission should be formed into proper error models, discretely or continuously. In decoding procedures, QEC approaches need to correct errors and perform encoding procedures backward, where error correction includes error detection and recovery. In general, designing QEC approaches should consider the following two key points [16].

- Low Error Correction Cost. The ancilla qubits needed and other resources to protect information shall be as few as possible.

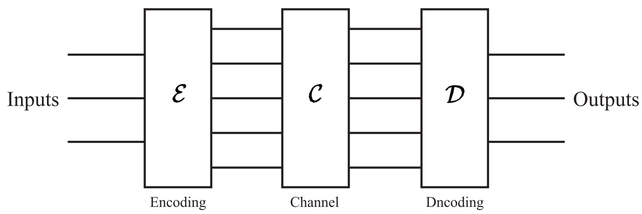

FIGURE 1. Quantum information processing.

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[1] http://www.research.ibm.com/ibm-q/

- High Error Correction Accuracy. Error correction shall be found to accurately recover the initial information;

Since Wootters and Zurek [2] addressed that a quantum state can not be cloned in 1982, duplicating quantum states as classical error correction schemes do is impossible. The difficulty of developing QEC remained unsolved until the late 90s. When noise is discretized into basic error sets, named the *discrete error models*, Shor [3] successfully invented a *quantum code* to protect one-qubit information against one arbitrary error with nine qubits. Later, Calderbank and Shor [4] and Steane [5] each constructed quantum codes based on a special class of classical linear codes. This method is named the CSS construction and quantum codes constructed by it can be described using the stabilizer formalism [6] uniformly. It was then presented in [7] that the minimal number of qubits to protect one-qubit information against one arbitrary error is five. Various advanced quantum codes with lower error correction cost were further constructed in the 21st century, such as entanglement-assisted quantum codes [8] and nonadditive quantum codes [9].

Quantum codes focus on correcting arbitrary errors during transmission. However, such generic approaches cost a large number of ancilla qubits. What if the noisy channel to transmit information is known at first? It turns out correcting errors in an amplitude damping channel only needs four qubits [12] while correcting arbitrary errors needs five qubits at least. A specific error model can be described using a *quantum operation*, which is a general tool to describe the changes of quantum states. QEC approaches based on convex optimization problems [13], named *optimization-based approaches*, can find the optimal error correction procedures for specific error models, thus have lower error correction cost. *Operator quantum error correction* [14] aims to find the "error-free" spaces for specific error models, in which the information encoded is immune to noise. These approaches have lower error correction cost than quantum codes since they only correct targeted errors while maintaining accuracy.

The discrete error models that above approaches focus on are idealized formalisms, which assume errors occur discretely and independently on qubits. More practical error models consider errors occurring continuous in time, i.e., the *continuous error models*. Paz and Zurek [10] first proposed the *continuous-time QEC (CTQEC)* that described both errors and error correction procedures continuous in time by using differential equations. The implementation of continuous error correction procedures in small time intervals is an essential issue of this approach, including directly acting on the information (the direct CTQEC) or using ancilla qubits (the indirect CTQEC). When the encoding procedures are chosen as quantum codes and CTQEC is chosen for error correction, Hsu and Brun [11] proved that the error correction cost of CTQEC is comparable to pure quantum codes. The error correction accuracy of CTQEC is related to the error rate and the error correction rate.

We attempt to put forward our analysis for QEC approaches from a more comprehensive perspective,

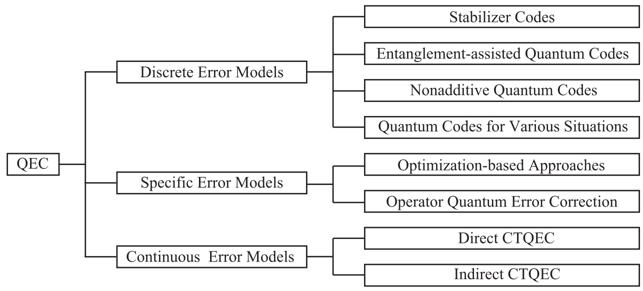

FIGURE 2. **QEC approaches for different error models.**

as shown in Figure 2, while previous work only compared the performance of stabilizer codes and nonadditive quantum codes [15]. Note that the basic error sets and quantum operations are two kinds of descriptions for discrete error models. We categorize QEC approaches according to the different error models. Each type has their respective scope of application and makes progress hand in hand. For discrete error models described using the basic error sets, quantum codes are easy to implement and abundant methods can be "transplanted" from the classical code theory. For specific error models described using quantum operations, optimization-based approaches and operator quantum error correction provide better performance for these errors compared to generic quantum codes. For continuous error models, CTQEC can correct errors all along the processing way, which fits the real situations better. Further, we will discuss the case of perturbed errors while current work supposes noise is constant. To what extent the information is perturbed in *perturbed error models* remains to be explored. It is no doubt that more efforts are needed towards more advanced QEC approaches.

The rest of the paper is organized as follows. Section II enumerates some necessary concepts for QEC. Various advanced quantum codes for discrete error models are summarized in Section III. Section IV introduces optimization-based approaches and operator quantum error correction for specific error models. Section V reviews CTQEC for continuous error models. Section VI discusses some future QEC directions and presents the perturbed error models. Finally, Section VII concludes the paper.

II. PRELIMINARIES

This section briefly introduce some necessary quantum mechanics concepts, error models, quantum codes and fidelity. More concrete details about quantum computing can be referred to in [16].

A. HILBERT SPACE

A *Hilbert space* $\mathcal{H}$ is a complex vector space equipped with an inner product that is also a complete metric space. The dimension of $\mathcal{H}$ is denoted as $dim(\mathcal{H})$. A mapping $A : \mathcal{H} \to \mathcal{H}'$, where $\mathcal{H}$ and $\mathcal{H}'$ are two Hilbert spaces, is called a *linear operator* from $\mathcal{H}$ to $\mathcal{H}'$ if $A(\sum_i \lambda_i \mathbf{v}_i) = \sum_i \lambda_i A(\mathbf{v}_i)$, $\forall \mathbf{v}_i \in \mathcal{H}$, $\forall \lambda_i \in \mathbb{C}$. The set of all linear operators from $\mathcal{H}$ to $\mathcal{H}'$ is denoted as $\mathcal{L}(\mathcal{H}, \mathcal{H}')$ and $\mathcal{L}(\mathcal{H})$ is the shorthand for $\mathcal{L}(\mathcal{H}, \mathcal{H})$. Linear operators can be represented using matrices explicitly. The *trace* of a matrix, abbreviated as $tr(\cdot)$, is the sum of its diagonal elements.

B. QUANTUM STATE AND QUANTUM SYSTEM

The concepts of quantum mechanics are commonly introduced in a two-dimensional complex vector space $\mathbb{C}^2$. An orthonormal basis of space $\mathbb{C}^2$ can be $|0\rangle = (1, 0)^\intercal$ and $|1\rangle = (0, 1)^\intercal$, where $|\cdot\rangle$ is the *Dirac notation*. The basis can be used to denote an arbitrary *quantum state* $|\psi\rangle$ of a qubit on $\mathbb{C}^2$, i.e., $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$.

A *quantum system* consists of n qubits. A *pure* quantum state of a quantum system is described using a vector in $(\mathbb{C}^2)^{\otimes n}$. A *mixed* quantum state of a quantum system is described using a *density matrix* ρ, where $\rho = \sum_j p_j|\psi_j\rangle\langle\psi_j|$ and p_j is the probability that the quantum system is in quantum state $|\psi_j\rangle$. Specifically, ρ is pure when the probability of $|\psi\rangle$ is 1, i.e., $\rho = |\psi\rangle\langle\psi|$. The set of all density operators on a Hilbert space $\mathcal{H}$ is denoted by $\mathcal{D}(\mathcal{H})$.

A *closed quantum system* indicates that qubits are free from the environment interference. An *open quantum system* indicates that the principal quantum system is coupled to the environment, thus noise exists. Let $\rho^{\mathscr{R}\mathscr{Q}}$ be a quantum state of a composite system $\mathscr{R}\mathscr{Q}$. The *partial trace* over a subsystem $\mathscr{R}$ returns the density matrix of subsystem $\mathscr{Q}$

$$\rho^{\mathscr{Q}} = tr_{\mathscr{R}}(\rho^{\mathscr{R}\mathscr{Q}}) = \sum_k (\langle k^{\mathscr{R}}| \otimes I^{\mathscr{Q}})\rho^{\mathscr{R}\mathscr{Q}}(|k^{\mathscr{R}}\rangle \otimes I^{\mathscr{Q}}),$$

where $\{|k^{\mathscr{R}}\rangle\}$ is a set of basis of subsystem $\mathscr{R}$, $I^{\mathscr{Q}}$ is a d-dimensional identity matrix and $d = dim(\mathcal{H}^{\mathscr{Q}})$.

C. QUANTUM STATE TRANSFORMATION

The evolution of a quantum system is described by the transformation of its quantum state, and the transformation of a quantum state is described using operators. Consider an operator A. If $A = A^\dagger$, A is Hermitian. If $AB - BA = 0$ ($AB + BA = 0$), A and B are *commuting (anti-commuting)* with each other, denoted as $[A, B] = 0$ ($\{A, B\} = 0$). In a closed quantum system, the evolution of a quantum state is described using *unitary operators*, where a unitary operator U satisfies $U^\dagger U = I$ (the dimension of an identity matrix is omitted in trivial cases). Four frequently-used unitary operators are listed in Definition 1.

*Definition 1: **Pauli matrices** are defined as*

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$
$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

A more general tool to describe the evolution of quantum states in both closed quantum systems and open quantum systems is a map $\mathcal{E} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ named the *quantum operation*, which acts on a quantum state $\rho \in \mathcal{L}(\mathcal{H})$ as $\mathcal{E}(\rho)$. The operator-sum representation is an explicit form to represent quantum operations, written as $\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger$, where $\{E_k\}$ is a set of linear operators named as operator elements.

Measurements $\{M_m\}$ are specific quantum operations that each M_m acting on a quantum system in quantum state $|\psi\rangle$ returns a corresponding outcome $|\psi_m\rangle$ with probability p_m, where $p_m = \langle\psi|M_m^\dagger M_m|\psi\rangle$ and $|\psi_m\rangle = M_m|\psi\rangle/\sqrt{p_m}$. *POVM (positive operator-valued measure) measurements* $\{P_m\}$ concern more about the probability $p_m = \langle\psi|P_m|\psi\rangle$ rather than the corresponding outcome quantum state, where each P_m is named the POVM operator. *Projective measurements* $\{P_m\}$ are a kind of POVM measurements with $P_m = |m\rangle\langle m|$, where P_m is named the *projector* and $\{|m\rangle\}$ is a set of orthogonal basis.

D. ERROR MODELS

Errors come from the interference with the environment in open quantum systems. The discrete error models assume that quantum errors can be discretized to linear combinations of Pauli matrices and occur on different qubits independently. It is sufficient to correct arbitrary errors by only correct Pauli matrices, as shown in Theorem 10.2 in [16]. An error can be written as $W \otimes W \cdots \otimes W$, where $W \in \{I, X, Y, Z\}$. An alternate expression, $W_1 W_2 \cdots W_n$, omits tensor product for brevity, where the subscript means a Pauli matrix X, Y or Z occurs on the i-th qubit. I_i is not included since it does not cause any error on the i-th qubit. For instance, an error $X \otimes I \otimes Z$ is equal to $X_1 Z_3$. The *weight* of an error is the number of places that are not I.

The set of Pauli matrices occur on n qubits can also be denoted by X, Z and multiplicative factors uniformly since $Y = iXZ$. Let $\mathbf{a} = (a_1, a_2, \cdots, a_n) \in \{0, 1\}^n$, $\mathbf{b} = (b_1, b_2, \cdots, b_n) \in \{0, 1\}^n$. An error E is given by

$$E = i^c X_{\mathbf{a}} Z_{\mathbf{b}}, \quad c \in \{0, 1, 2, 3\}. \tag{1}$$

The weight of E is given by

$$w(E) = \#\{j | 1 \leq j \leq n, a_j \vee b_j = 1\}, \tag{2}$$

where # is the number of elements in a set.

A quantum operation $\mathcal{E}$ with operator elements $\{E_k\}$ is a general tool to describe a specific error model since it captures the discrete changes of quantum states. Operator elements can be recognized as error operators that are linear combinations of Pauli matrices. In addition, we often use $\mathcal{E}$ to represent a *noisy channel*. For instance, the amplitude damping channel $\mathcal{E}_a$ has operator elements $\{E_1, E_2\}$, where

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-r} \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & \sqrt{r} \\ 0 & 0 \end{bmatrix}.$$

The *continuous error models* are derived from real situations using differential equations, where the evolution of open quantum systems is continuous in time. The explicit form is introduced in Section V.

E. QUANTUM CODES

QEC approaches originate from quantum codes, thus it is necessary to bring up some basic knowledge about the code theory.

*Definition 2: Each K-dimensional subspace $\mathcal{C}$ of Hilbert space $(\mathbb{C}^2)^{\otimes n}$ is an (n, K) **quantum code**. If $K = 2^k$, $\mathcal{C}$ is an $[n, k]$ quantum code.*

Consider a quantum operation $\mathcal{E}$ that describes quantum errors. An *$\mathcal{E}$-quantum error correcting code* is a quantum code that has a recovery operation for $\mathcal{E}$ satisfying the error correction condition in Theorem 1. Generally, the notation $\mathcal{E}$ will be omitted since a quantum error correcting code can correct arbitrary errors.

Theorem 1 (Knill-Laflamme Condition [17]): Let $\mathcal{E}$ be a quantum operation with operator elements $\{E_k\}$ and $\mathcal{C}$ a quantum code. A recovery operation $\mathcal{R}$ correcting $\mathcal{E}$ on $\mathcal{C}$ exists if and only if $\langle m|E_a^\dagger E_b|n\rangle = \alpha_{ab}\delta_{mn}$ for all orthogonal $|m\rangle, |n\rangle \in \mathcal{C}$, where $E_a, E_b \in \{E_k\}$, α_{ab} are complex numbers for a Hermitian matrix α and $\delta_{mn} = \langle m|n\rangle$.

A quantum error correcting code $\mathcal{C}$ with code distance d can correct errors of weight $\lfloor (d-1)/2 \rfloor$ at most. In such cases a quantum code $\mathcal{C}$ is denoted as an (n, K, d) or $[n, k, d]$ quantum error correcting code. We introduce a bound with parameters n, k, d on the ability of quantum error correcting codes to correct errors.

Lemma 1 (Quantum Singleton Bound [16]): Let $\mathcal{C}$ be an $[n, k, d]$ quantum error correcting code. The quantum Singleton bound states that $n - k \geq 2(d-1)$.

Quantum error correcting codes that achieve the quantum Singleton bound equality are named the *maximum distance separable (MDS) codes*.

1) STABILIZER CODES

Stabilizer codes are an important class of quantum codes and can be constructed from classical dual-containing linear codes [18]. We present a brief introduction here since many quantum codes is constructed based on the stabilizer formalism. Stabilizer codes explored so far with good parameters are listed in [19].

The group theory is the key to the stabilizer formalism. We define a *Pauli group* $\mathcal{P}_n$ on n qubits that consists of all n-fold tensor products of Pauli matrices, together with multiplicative factors $\pm 1, \pm i$. Consider a group $\mathcal{S}$ and a set of elements $g_1, \cdots, g_l \in \mathcal{S}$. $g_1, \cdots, g_l$ are the *generators* of $\mathcal{S}$, denoted as $\mathcal{S} = \langle g_1, \cdots, g_l\rangle$, if each element in $\mathcal{S}$ can be written as a product of elements from $g_1, \cdots, g_l$.

A vector space $\mathcal{V}$ is stabilized by group $\mathcal{S}$ if $\forall |\psi\rangle \in \mathcal{V}$, $\forall g \in \mathcal{S}$, $g|\psi\rangle = |\psi\rangle$ holds, where $\mathcal{S}$ is named the *stabilizer* of $\mathcal{V}$. It can be seen that a stabilizer $\mathcal{S}$ has properties $-I \notin \mathcal{S}$ and $[g_i, g_j] = 0, \forall g_i, g_j \in \mathcal{S}$, i.e., $\mathcal{S}$ is an *abelian group*. The *normalizer* of stabilizer $\mathcal{S}$ in $\mathcal{P}_n$, denoted as $N(\mathcal{S})$, is a set of operators U such that $UgU^\dagger = g, \forall g \in \mathcal{S}$.

*Definition 3: Let $\mathcal{S}$ be an abelian subgroup of $\mathcal{P}_n$. An $[n, k]$ **stabilizer code** is a 2^k-dimensional subspace of Hilbert space $(\mathbb{C}^2)^{\otimes n}$ that is stabilized by $\mathcal{S}$.*

The error correction procedure with a stabilizer code includes error detection and recovery. Error detection is done by measuring the eigenvalues of the stabilizer generators, which are initially all $+1$. The effect of an correctable error E acting on the stabilizer code is equivalent to acting on the stabilizer generators, i.e., for a quantum state $|\psi\rangle$, $E|\psi\rangle = Eg_i|\psi\rangle = Eg_iE^\dagger E|\psi\rangle, \forall g_i \in \mathcal{S}$. Some eigenvalues of the generators will turn to -1 after being affected by errors. For instance, the stabilizer generator of a quantum state $|0\rangle$ is Z with an eigenvalue being $+1$. An error X turns the state into $|1\rangle$, which is stabilized by Z with an eigenvalue being -1. The outcomes of measuring the generators after errors are *error syndromes*. Then, recovery operations are chosen conditioned on the error syndromes.

Example 1: The $[5, 1, 3]$ code encodes one-qubit information in five qubits. According to the quantum Singleton bound, it is known as the minimal size of a quantum error correcting code to correct one arbitrary error. The stabilizer $\mathcal{S}$ of the $[5, 1, 3]$ code has generators $g_1 = X_1Z_2Z_3X_4$, $g_2 = X_2Z_3Z_4X_5$, $g_3 = X_1X_3Z_4Z_5$ and $g_4 = Z_1X_2X_4Z_5$. If error X_1 occurs, generator g_4 will turn to $X_1(Z_1X_2X_4Z_5)X_1^\dagger = -Z_1X_2X_4Z_5$. g_1, g_2, g_3 remain unchanged. Thus, the error syndromes are $+1, +1, +1, -1$. The recovery operation is chosen by applying X_1.

F. FIDELITY

The fidelity is a common tool to measure the difference between quantum states. It will be used in optimization-based approaches in Section IV.

*Definition 4: The **fidelity** of quantum states ρ and σ is*

$$F(\rho, \sigma) = tr\sqrt{\rho^{1/2}\sigma\rho^{1/2}}. \quad (3)$$

Specifically, the fidelity between a pure state $|\psi\rangle$ and an arbitrary state σ can be calculated as

$$F(|\psi\rangle, \sigma) = \sqrt{\langle\psi|\sigma|\psi\rangle}. \quad (4)$$

The *channel fidelity* to measure a noisy channel $\mathcal{E}$ preserving a quantum state can be defined as the *minimal fidelity* since the initial state may be unknown and we should consider the worst-case quantum systems. The channel fidelity of mixed initial states is guaranteed to be larger than pure initial states by the joint concavity of fidelity. Thus, the minimal fidelity is defined over pure states $|\psi\rangle$,

$$F_{min}(|\psi\rangle, \mathcal{E}) = \min_{|\psi\rangle} F(|\psi\rangle, \mathcal{E}(|\psi\rangle\langle\psi|)). \quad (5)$$

When the average-case quantum systems are considered, i.e., a system is in quantum state ρ_j with probability p_j, The channel fidelity is defined as the *ensemble average fidelity*, $F_{ave}(\rho, \mathcal{E}) = \sum_{p_j} F(\rho_j, \mathcal{E}(\rho_j))^2$. The *entanglement fidelity* is the lower bound of the ensemble average fidelity and has an explicit form for calculation. Thus, the entanglement fidelity is often applied for the average-case quantum systems. The entanglement fidelity measures to what extent a noisy channel $\mathcal{E}$ preserving the entanglement between systems when consider the average-case of quantum systems. Let ρ be a quantum state of system $\mathscr{Q}$. A reference system $\mathscr{R}$ is introduced such that ρ corresponds to a pure state $|\psi\rangle^{\mathscr{R}\mathscr{Q}}$ in $\mathcal{H}^{\mathscr{R}} \otimes \mathcal{H}^{\mathscr{Q}}$ with $\rho = tr_{\mathscr{R}}(|\psi\rangle^{\mathscr{R}\mathscr{Q}\mathscr{R}\mathscr{Q}}\langle\psi|)$. The entanglement fidelity is defined as

$$F_{ent}(\rho, \mathcal{E}) = F(|\psi\rangle^{\mathscr{R}\mathscr{Q}}, (I^{\mathscr{R}} \otimes \mathcal{E})(|\psi\rangle^{\mathscr{R}\mathscr{Q}\mathscr{R}\mathscr{Q}}\langle\psi|))^2. \quad (6)$$

Let $\mathcal{E}$ be a noisy channel with operator elements $\{E_k\}$. The entanglement fidelity has an explicit form with $|\cdot|$ being the complex norm,

$$F_{ent}(\rho, \mathcal{E}) = \sum_k |tr(\rho E_k)|^2. \quad (7)$$

III. ADVANCED QUANTUM CODES

Quantum codes are designed for discrete error models that are described using Pauli matrices and stabilizer codes have grounded the development of quantum codes. Then improved approaches emerge to pursue lower error correction cost and higher error correction accuracy. This section presents some advanced quantum codes with different error correction abilities. Entanglement-assisted quantum codes can be constructed from arbitrary classical linear codes using pre-shared entanglement. Nonadditive quantum codes, analogous to classical nonlinear codes, can encode more information than stabilizer codes with the same number of qubits. Nonbinary quantum codes, asymmetric quantum error correcting codes, quantum burst error correcting codes, quantum convolutional codes, concatenated quantum codes and topological quantum error correcting codes consider modified discrete error models that suit for various real situations. For instance, in practice, errors occur predominantly in adjacent places rather than random places assumed by discrete error models.

A. ENTANGLEMENT-ASSISTED QUANTUM CODES

Entanglement-assisted quantum codes, first designed for quantum teleportation, use unlimited entanglement that is prepared in advance and shared by the sender and the receiver separately. The first example of entanglement-assisted quantum codes [20] protects one-qubit information against one arbitrary error with two pairs of maximally entangled states $|\Psi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. The quantum state $|\Psi\rangle$ of two qubits is named an *ebit*. The optimal stabilizer code for one arbitrary error needs four ancilla qubits. Since experiments showed that pre-shared entanglement is a weaker resource than transmitting ancilla qubits [21], entanglement-assisted quantum codes can have better performance than stabilizer codes using the same number of qubits.

We introduce the formal definition of entanglement-assisted quantum codes in the following.

*Definition 5: Let $\mathscr{A}$ and $\mathscr{B}$ share c ebits, where $\mathscr{A}$ has another $n-c$ qubits and $\mathscr{B}$ holds its c qubits noiselessly. An $(n, K; c)$ **entanglement-assisted quantum code** $\mathcal{C}$ is a K-dimensional subspace of $(\mathbb{C}^2)^{\otimes(n+c)}$ such that*

$$tr_{\mathscr{A}}(|\psi\rangle\langle\psi|) = 1/2^c I^{\mathscr{B}}, \quad \forall|\psi\rangle \in \mathcal{C}. \quad (8)$$

If $K = 2^k$, $\mathcal{C}$ is an $[n, k; c]$ entanglement-assisted quantum code.

The definition is further explained here. Information is encoded into Hilbert space $(\mathbb{C}^2)^{\otimes(n+c)}$ on $n+c$ qubits since $\mathscr{A}$ and $\mathscr{B}$ share c ebits. Equation (8) ensures that the encoding procedures only perform on $\mathscr{A}$ and $\mathscr{B}$ is noiseless. If $\mathcal{C}$ corrects errors of weight $\lfloor(d-1)/2\rfloor$ at most, $\mathcal{C}$ is an $(n, K, d; c)$ or $[n, k, d; c]$ entanglement-assisted quantum error correcting code.

The error correction procedure using an entanglement-assisted quantum code works as follows: a) a quantum state $|\psi\rangle$ is encoded with $n-k-c$ ancilla qubits prepared in state $|0\rangle$ and c ebits $|\Phi\rangle$, i.e. $|\psi'\rangle = |\psi\rangle|0\rangle^{\otimes(n-k-c)}|\Phi\rangle^{\otimes c}$; b) $\mathscr{A}$ encodes its n qubits and transmits them through a noisy channel, $\mathscr{B}$ holds its c qubits noiselessly; c) $\mathscr{B}$ decodes n qubits from $\mathscr{A}$ using its c qubits, obtains error syndromes by measurements, and corrects errors.

The construction of an entanglement-assisted quantum code can be generalized from the stabilizer codes. The stabilizer codes are constructed from classical dual-containing linear codes where the corresponding stabilizer generators are commuting with one another. However, entanglement-assisted quantum codes do not require the dual-containing property of classical linear codes since the non-commuting generators can be embedded into larger commuting generators using ebits.

Among a certain ancilla qubits, applying more ebits to the stabilizer codes can optimize the code distance [22]. The larger the code distance, the higher the error correction accuracy. Thus, researchers began to explore the situation with a maximal number of ebits, i.e., all ancilla qubits are ebits, and codes with such parameters are named the *maximal entanglement-assisted quantum codes*. The duality in the stabilizer formalism of entanglement-assisted quantum codes can led to a linear programming bound that explore the code distance [23]. A table of the upper and lower bounds on the code distance of the maximal entanglement-assisted quantum codes with $n \leq 15$ was also presented. Constructions from quaternary zero radical codes improved the bounds of entanglement-assisted quantum codes to $n \leq 20$ [24]. The bounds were further progressed by generalizing linear programming bounds to cases that apply ebits to non-stabilizer codes [25].

Another direction to construct entanglement-assisted quantum codes with good error correction abilities is to find entanglement-assisted quantum MDS codes that achieve the equality of entanglement-assisted quantum Singleton bound, i.e., $K \leq 2^{n+c-2(d-1)}$. Entanglement-assisted quantum MDS codes can be constructed directly from classical MDS codes [26], constacyclic codes [8] and so on. The up-to-date families of entanglement-assisted quantum MDS codes were summarized in [27].

The performance of entanglement-assisted quantum codes we concern about other than error correction abilities is the transmission rate and the complexity of encoding circuits. The transmission rate is defined to be k/n or net rate $(k-c)/n$. In general, the net rate can be negative since the number of pre-shared ebits is unlimited. The complexity of encoding circuits of entanglement-assisted quantum codes is $\mathcal{O}(n(n-k+c)/\log n)$ [28].

The idealized quantum teleportation process assumes $\mathscr{B}$ holds its c qubits noiselessly. However, this cannot be assured in real situations. The authors in [29] protected c qubits held by $\mathscr{B}$ with a stabilizer code. They also found that given the same number of qubits $n+c$, an $[n,k,d;c]$ entanglement-assisted quantum code has better performance than an $[n+c,k,d]$ stabilizer code, which validates that pre-shared entanglement is a weaker resource.

Entanglement-assisted quantum codes can also be used against the time evolution of quantum states in quantum computers, where $\mathscr{A}$ and $\mathscr{B}$ do not separate spatially. These codes are named the *catalytic QEC codes* [21]. The transmission rate of a catalytic QEC code should be positive since ebits can not be prepared in advance with an arbitrary amount as in quantum teleportation. Further, qubits held by $\mathscr{B}$ shall be sent through a noiseless channel, i.e., $(\mathcal{E}\otimes I^{\otimes c})(\rho)$, given ρ the encoded information and $\mathcal{E}$ the noisy channel.

Exploring the optimal entanglement-assisted quantum codes is a hot topic since a) they can be constructed from arbitrary classical linear codes; b) entanglement is a strictly weaker resource than quantum communication. The entanglement-assisted method can also be applied to improve quantum codes other than stabilizer codes, such as nonadditive quantum codes that will be introduced in the following subsection. We notice that the protection of c qubits held by $\mathscr{B}$ in catalytic QEC codes draws our attention to "error-free" spaces in Section IV. It is also interesting to study the trade-offs of catalytic QEC codes between the maximal number of pre-shared ebits and the cost of protecting these ebits, which reflects the error correction abilities of entanglement-assisted quantum codes in quantum computing.

B. NONADDITIVE QUANTUM CODES

Analogous to classical nonlinear codes, nonadditive quantum codes can encode more information than stabilizer codes with the same number of qubits. Two equivalent constructions were presented in the same year. A *union stabilizer code* [30] is a nonadditive quantum code based on stabilizer codes. A *codeword stabilized quantum code* [31] is constructed from so-called graph states [32], where a graph state corresponding to an $[n,0,d]$ stabilizer code integrates a finite abelian group with a graph. Here we introduce nonadditive quantum codes using the union stabilizer code structure.

A stabilizer code $\mathcal{C}$ with stabilizer $\mathcal{S}$ is a joint $+1$ eigenspace of all elements in $\mathcal{S}$, whose dimension is 2^k. It can be noticed that the $n-k$ generators in $\mathcal{S}$ having eigenvalues $+1$ or -1 yield a decomposition of $(\mathbb{C}^2)^{\otimes n}$ into 2^{n-k} mutually orthogonal subspaces labeled by the eigenvalues of the $n-k$ generators. Based on this decomposition, union stabilizer codes can be constructed from several mutually orthogonal stabilizer codes so that the dimension to encode information is larger than pure stabilizer codes.

Definition 6: Let $\mathcal{C}_0$ be an $[n,k]$ stabilizer code with stabilizer $\mathcal{S}_0$ and $\mathcal{T}_0 = \{t_1,\cdots,t_s\}$ a subset of the coset representatives of $N(\mathcal{S}_0)$ in $\mathcal{P}_n$. A ***union stabilizer code*** *$\mathcal{C}$ is defined as $\mathcal{C}=\bigoplus_{t_i\in\mathcal{T}_0} t_i\mathcal{C}_0$.*

The dimension K of $\mathcal{C}$ is $s\cdot 2^k$. The code distance d of $\mathcal{C}$ is the minimal code distance of $t_i\mathcal{C}_0$, $\forall 1\leq i\leq s$. The parameters of a nonadditive quantum code are denoted as (n,K,d).

It has been noticed that nonadditive quantum codes can encode more information than stabilizer codes using the same number of qubits. We may wonder how many errors can be corrected with nonadditive quantum codes? The first example of nonadditive quantum codes is the $(5,6,2)$ code in [33]. Inspired by the optimal $[5,1,3]$ stabilizer code $\mathcal{C}_0$ that corrects one arbitrary error, the authors successfully constructed the union stabilizer code with 6 dimensions. Since codes with code distance 2 can detect one arbitrary error or correct one *erasure*, where an erasure means the location of the error is known, it is reasonable to explore codes with code distance 2. Then, an infinite family of $(2m+1, 3\cdot 2^{2m-3}, 2)$ code was presented in [34]. The $(9,12,3)$ code [35] formulated by graph states was first found to correct one arbitrary error. The authors presented the $(10,24,3)$ code [36] later through a graphical approach.

We have seen that nonadditive quantum codes outperform stabilizer codes with the same number of qubits and code distance. The next question is how to find more nonadditive quantum codes with better parameters? As shown in [31], a key point to construct nonadditive quantum codes is to find classical nonlinear codes that can correct desired errors. This problem is equivalent to finding the maximum clique of an induced graph, which is known to be an NP-complete problem. Three structure theorems [37] were proposed to reduce the search space when finding the maximum clique. However, this approach is restrained by its algorithm complexity and thus is hard to find nonadditive quantum codes with $n>11$. Yu *et al.* [9] found two infinite families of nonadditive single-error-correcting codes that combined some stabilizer codes with two known nonadditive quantum codes.

Although nonadditive quantum codes can encode more information than stabilizer codes, their constructions lack a regular method and the encoding and decoding circuits are complicated. We need more approaches to construct nonadditive quantum codes with larger code distance. For instance, the entanglement-assisted method was taken into consideration and a $(7,4,5;4)$ code was presented [38]. Constructions of union stabilizer codes and codeword stabilized quantum codes have provided perspectives for searching nonadditive quantum codes, yet pursuing better parameters are limited by search algorithms. Particular methods such as constructions from known nonadditive quantum codes or entanglement-assisted methods also present some examples. However, a more generic construction of nonadditive quantum codes remains unknown.

C. QUANTUM CODES FOR VARIOUS SITUATIONS

The aforementioned quantum codes are great improvements compared to pure stabilizer codes. This subsection lists some quantum codes that are suitable for various situations.

1) NONBINARY QUANTUM CODES

Nonbinary quantum codes [39] were developed right after the binary stabilizer codes that generalized the code space from 2-dimensional Hilbert space $(\mathbb{C}^2)^{\otimes n}$ to $(\mathbb{C}^q)^{\otimes n}$. Nonbinary quantum codes can complete the quantum code field mathematically and can be applied to fault-tolerant quantum computing.

Definition 7: A ***nonbinary quantum code*** *$\mathcal{C}$ is a subspace of $(\mathbb{C}^q)^{\otimes n}$ and a basis of $\mathcal{C}$ is denoted as*

$$\{|c_1c_2\cdots c_n\rangle : (c_1, c_2, \cdots, c_n) \in (\mathbb{F}_q)^{\otimes n}\},$$

where $q = p^m$ is the power of an odd prime number p and $(\mathbb{F}_q)^{\otimes n}$ is the q-ary finite field on n dimensions.

The code bounds such as Hamming bound and quantum Singleton bound for high-dimensional cases were analyzed in [40]. Various quantum codes can be generalized to the nonbinary cases, such as nonbinary quantum cyclic codes constructed using graph method [41], nonbinary codeword stabilized quantum codes with nonadditive properties [42], or nonbinary entanglement-assisted quantum codes [43].

2) ASYMMETRIC QUANTUM ERROR CORRECTING CODES

In standard discrete error models Pauli matrices occur with the same probability. In real situations, however, the probability of phase-flip errors (Z) is much greater than bit-flip errors (X). Thus, *asymmetric quantum error correcting codes* [44] are more flexible for such cases. The probability that error X and error Z occur is measured by the corresponding error weight w_X and w_Z. Let $\{E_k\}$ be a set of errors on n qubits, where $E_k = i^c X_{\mathbf{a}} Z_{\mathbf{b}}$, $c \in \{0, 1, 2, 3\}$, $\mathbf{a} = (a_1, a_2, \cdots, a_n) \in \{0, 1\}^n$, $\mathbf{b} = (b_1, b_2, \cdots, b_n) \in \{0, 1\}^n$, then

$$\begin{aligned} w_X(E_k) &= \#\{j|1 \leq j \leq n, a_j \neq 0\}, \\ w_Z(E_k) &= \#\{j|1 \leq j \leq n, b_j \neq 0\}. \end{aligned} \tag{9}$$

Definition 8: Let $\mathcal{C}$ be an (n, K) quantum code, $\{E_k\}$ a set of errors on n qubits, where $w_X(E_k) \leq d_x - 1$, $w_Z(E_k) \leq d_z - 1$. $\mathcal{C}$ is an $(n, K, d_z/d_x)$ ***asymmetric quantum error correcting code*** *if it corrects $\{E_k\}$.*

The properties and construction approaches of asymmetric quantum error correcting codes were presented in [45]. Bounds of asymmetric quantum error correcting codes were analyzed in [46]. Some known asymmetric quantum error correcting codes were constructed from nonadditive quantum codes [47] and nonbinary quantum codes [48].

3) QUANTUM BURST ERROR CORRECTING CODES

Another kind of modified discrete error models aims for errors occurring predominantly in adjacent positions since entanglement in quantum circuits mainly exists among local qubits. The approach to correct such errors rather than errors occurring in random places is the *quantum burst error correcting code* [49]. The burst length $bl(\cdot)$ of an error E is counted by the number of places that are nonidentity consecutively.

Definition 9: Let $\mathcal{C}$ be an (n, K) quantum code, $\{E_k\}$ a set of errors on n qubits, where $bl(E_k) < l$. $\mathcal{C}$ is an (n, K) ***quantum burst error correcting code*** *if it corrects $\{E_k\}$.*

Research problems are similar to asymmetric quantum error correcting codes. Some known quantum burst error correcting codes were constructed from cyclic codes and quantum tensor product codes [50].

4) QUANTUM CONVOLUTIONAL CODES

Analogous to classical cases, initial information can be prepared in streams rather than blocks. The quantum version for such cases is the *quantum convolutional codes* [51]. Quantum codes in the above sections are all block codes. One of the main benefits of quantum convolutional codes is that information can be transmitted in small pieces at any time while block codes shall prepare a fixed amount of information in advance. The encoding procedure of quantum convolutional codes is related to the former information in the stream while block codes encode information independently from block to block. Thus quantum convolutional codes have higher error correction accuracy.

5) CONCATENATED QUANTUM CODES

Concatenated quantum codes concatenate block codes [52] or quantum convolutional codes with block codes [53] to combine quantum codes with different error correction abilities. The $[9, 1, 3]$ Shor code is a common concatenated quantum code that applies three-qubit phase-flip code as the inner code and three-qubit bit-flip code as the outer code. The concatenated quantum code can successfully correct one arbitrary error. Concatenating $[5, 1, 3]$ code multiple times can achieve higher fidelity [54]. Concatenating an asymmetric quantum error correcting code with a stabilizer code for an amplitude damping channel has a larger code distance than the best-known stabilizer code [55].

6) TOPOLOGICAL QUANTUM ERROR CORRECTING CODES

All the above quantum codes assume that the encoding and decoding procedures are noiseless. However, implementing these procedures requires adding quantum gates, which brings noise naturally. Thus, topological quantum error correcting codes [56] that apply the stabilizer formalism locally on topological structures were proposed to be fault-tolerant. These codes are promising in building large-scale and fault-tolerant quantum computers.

IV. QEC APPROACHES FOR ERRORS DESCRIBED USING QUANTUM OPERATIONS

Quantum codes are powerful QEC approaches as they can correct arbitrary errors. However, such generic approaches can be inefficient when correcting specific error models, which are described using quantum operations.

For instance, if a noisy channel is known to be the amplitude damping channel in advance, there exists a four-qubit encoding operation [12] while the optimal generic [5, 1, 3] quantum code needs five qubits. This section first presents optimization-based approaches to obtain optimal QEC operations for specific error models that have highest recovery accuracy. The *operator quantum error correction (OQEC)* is then presented to protect information into "error-free" spaces such that the information can be immune to specific error models.

A. OPTIMIZATION-BASED APPROACHES

Optimization-based approaches use numerical methods to obtain optimal encoding and recovery operations for specific error models. It is done by maximizing the channel fidelity of the initial information and a noisy channel. This problem can be transformed to optimization problems.

1) OPTIMIZATION PROBLEM TRANSFORMATION

Suppose a quantum state ρ is transmitted through a noisy channel $\mathcal{E}$ with a fixed encoding operation $\mathcal{U}$. To find the optimal recovery operation $\mathcal{R}$, the objective function Γ_f based on the fidelity f is

$$\Gamma_f = \arg\max_{\mathcal{R}} f(\rho, \mathcal{R} \circ \mathcal{E} \circ \mathcal{U}), \tag{10}$$

where arg refers to $\mathcal{R}$ such that Γ achieves the maximum and $f(\cdot)$ refers to the minimal fidelity F_{min} or the entanglement fidelity F_{ent} as defined in equations (5) and (6).

To transform equation (10) into a convex optimization problem, we need to overcome that $\mathcal{E}$ has a variety of elements in operator-sum representations, which is inconvenient during the calculation. Fortunately, there is a unique operator for $\mathcal{E}$, the *Choi matrix*. Before the definition of Choi matrix, we introduce a representation that denotes an $N \times N$ density matrix ρ into a single column vector $|\rho\rangle\rangle$ of dimension N^2. It is done by stacking columns of the matrix in right-to-left order on top of one another.

Let $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be the set of operators from Hilbert space $\mathcal{H}_1$ to $\mathcal{H}_2$ and $C = \sum_{ij} c_{ij}|i\rangle\langle j|$ an operator in $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, where $\{|i\rangle\}$ and $\{|j\rangle\}$ are bases for $\mathcal{H}_2$ and $\mathcal{H}_1$ respectively, $c_{ij} = \langle i|C|j\rangle$. Operator C is defined in ket form as a vector on Hilbert space $\mathcal{H}_2 \otimes \mathcal{H}_1$,

$$|C\rangle\rangle = \sum_{ij} c_{ij}|j\rangle \otimes |i\rangle. \tag{11}$$

This yields three useful relations: a) $\langle\langle C_1|C_2\rangle\rangle = tr(C_1^\dagger C_2)$, b) $tr_{\mathcal{H}_1}(|C_1\rangle\rangle\langle\langle C_2|) = C_1 C_2^\dagger \in \mathcal{L}(\mathcal{H}_2)$, c) $(C_1 \otimes C_2)|C_3\rangle\rangle = |C_2 C_3 C_1^\mathsf{T}\rangle\rangle$, where $C_1, C_2, C_3 \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$.

Definition 10: Let $\mathcal{E} : \mathcal{L}(\mathcal{H}_1) \to \mathcal{L}(\mathcal{H}_2)$ be a quantum operation with operator elements $\{E_k\}$. The ***Choi matrix*** *is calculated from all operator elements $\{E_k\}$ of $\mathcal{E}$ as*

$$X_\mathcal{E} = \sum_k |E_k\rangle\rangle\langle\langle E_k|, \tag{12}$$

where $X_\mathcal{E} \geq 0$ and $tr_{\mathcal{H}_2} X_\mathcal{E} = I$.

Let ρ be a quantum state on $\mathcal{H}_1$, $\mathcal{E}(\rho)$ can be rewritten using the Choi matrix as

$$\begin{aligned} \mathcal{E}(\rho) &= \sum_k E_k \rho E_k^\dagger \\ &= \sum_k tr_{\mathcal{H}_1} |E_k \rho\rangle\rangle\langle\langle E_k| \\ &= \sum_k tr_{\mathcal{H}_1} [(\rho^\mathsf{T} \otimes I)|E_k\rangle\rangle\langle\langle E_k|] \\ &= tr_{\mathcal{H}_1}[(\rho^\mathsf{T} \otimes I) X_\mathcal{E}]. \end{aligned} \tag{13}$$

A quantum state ρ' on $\mathcal{H}_2$ that $\mathcal{E}(\rho)$ returns can be written as an one-sided matrix operation $|\rho'\rangle\rangle = (\sum_k E_k^* \otimes E_k)|\rho\rangle\rangle$. Since $tr(\rho E_k) = tr(|E_k\rangle\rangle\langle\langle\rho|) = \langle\langle\rho|E_k\rangle\rangle$, we can rewrite the entanglement fidelity as

$$\begin{aligned} F_{ent}(\rho, \mathcal{E}) &= \sum_k \langle\langle\rho|E_k\rangle\rangle\langle\langle E_k|\rho\rangle\rangle \\ &= \langle\langle\rho|X_\mathcal{E}|\rho\rangle\rangle. \end{aligned} \tag{14}$$

Inserting equation (14) into (10), where the encoding operation $\mathcal{U} : \mathcal{L}(\mathcal{H}_1) \to \mathcal{L}(\mathcal{H}_2)$, the noisy channel $\mathcal{E} : \mathcal{L}(\mathcal{H}_2) \to \mathcal{L}(\mathcal{H}_2)$ and the recovery operation $\mathcal{R} : \mathcal{L}(\mathcal{H}_2) \to \mathcal{L}(\mathcal{H}_1)$ have operator elements $\{U_l\}$, $\{E_m\}$ and $\{R_n\}$ respectively, we successfully transform the entanglement fidelity into a convex optimization problem,

$$\begin{aligned} \Gamma_{F_{ent}} &= \arg\max_{\mathcal{R}} \sum_{mnl} \langle\langle\rho|R_n E_m U_l\rangle\rangle\langle\langle R_n E_m U_l|\rho\rangle\rangle \\ \Gamma_{F_{ent}} &= \arg\max_{\mathcal{R}} \sum_{mnl} \langle\langle\rho U_l^\dagger E_m^\dagger|R_n\rangle\rangle\langle\langle R_n|\rho U_l^\dagger E_m^\dagger\rangle\rangle \\ \Gamma_{F_{ent}} &= \arg\max_{\mathcal{R}} \sum_{lm} \langle\langle\rho U_l^\dagger E_m^\dagger|X_\mathcal{R}|\rho U_l^\dagger E_m^\dagger\rangle\rangle \\ &\quad \text{s.t. } X_\mathcal{R} \geq 0, \quad tr_{\mathcal{H}_1}(X_\mathcal{R}) = I. \end{aligned} \tag{15}$$

2) MAXIMIZING THE ENTANGLEMENT FIDELITY

Finding the optimal encoding operations and recovery operations with entanglement fidelity being the objective function was firstly solved by the power method [62]. The process works as follows: a) fixes a random encoding operation and optimizes the entanglement fidelity over recovery operations; b) fixes the recovery operation and optimizes the entanglement fidelity over encoding operations; c) proceeds iteratively until the entanglement fidelity is convergent to a threshold. The explicit algorithm was proposed in [63] that applied methods for solving the semidefinite programming (SDP) problems.

It can be seen from equation (15) that the objective function is linear in recovery operations with constraints $X_\mathcal{R} \geq 0$. However, the dimension of the optimization problem grows exponentially with the number of qubits and the SDP problem is a resource-intensive operation, thus it is hard to apply on large cases [64]. Another difficulty states that the obtained recovery operations are not constrained by the structure of quantum operations, thus do not have intuitive physical forms. Defining a recovery operation based on the projector and a unitary operator can solve this problem, though this will lead to suboptimal results [65]. In terms of eigenanalysis, the proposed greedy algorithm constructed a channel-adapted recovery operation by successively choosing the syndrome subspace that maximizes the entanglement fidelity. The performance of obtained recovery operations was verified on the amplitude damping channel, which is superior to quantum codes [66].

Moreover, machine learning techniques were considered in [67] to optimize the entanglement fidelity. Recovery operations are learned from tremendous known quantum states and actual noisy channels.

3) MAXIMIZING THE MINIMAL FIDELITY

The entanglement fidelity considers the initial quantum states in an ensemble notion, providing a relatively comprehensive evaluation of the QEC performance. However, a quantum computer should be able to protect the information in arbitrary quantum states. Thus, worst-case quantum systems shall be considered, which are gauged by the minimal fidelity.

The recovery operation obtained from maximizing the minimal fidelity was proposed in [68], which is a non-convex optimization problem. Therefore, the authors relaxed the initial quantum states into a larger Hilbert space to convert the problem to a typical convex optimization problem and obtained suboptimal recovery operations. Later, they presented an exact solution in [69] when the initial information is encoded on one qubit. In this case, the problem can be perfectly converted into the SDP problem based on the sum of squares (SOS) characterization. When the initial information is encoded on multi-qubits, they relaxed the problem such that the SOS characterization holds. The obtained recovery operations are suboptimal.

Alternatively, maximizing the minimal fidelity can consider recovery operations with different forms, i.e., the *transpose channel* [70]. The authors proved that a recovery operation and a transpose channel are the same maps. Another approach in [71] defined a *complementary channel*, which cast the problem of finding optimal recovery operations to a dual optimization problem of finding minimal noise caused by the environment. Although both alternated approaches still obtain the approximate recovery operations, they simplified the solving process.

4) MINIMIZING THE CHANNEL NONIDEALITY

The channel nonideality [72] is an indirect approach to maximize the channel fidelity, which measures the "distance" between an actual channel $\mathcal{R} \circ \mathcal{E} \circ \mathcal{U}$ and an ideal channel I, where the encoding operation $\mathcal{U}$, the noisy channel $\mathcal{E}$ and the recovery operation $\mathcal{R}$ have operator elements $\{U_l\}$, $\{E_m\}$ and $\{R_n\}$ respectively. Given the encoding operation $\mathcal{U}$ and the noisy channel $\mathcal{E}$ in advance, we aim to find the corresponding optimal recovery operations. From the Knill-Laflamme condition we obtain $R_n E_m U_l = \alpha_{lmn} I$, indicating that the recovery operation $\mathcal{R}$ for the noisy channel $\mathcal{E}$ is perfect. Thus, the objective function Γ_c to minimize the channel nonideality based on the Hilbert-Schmidt norm is

$$\Gamma_c = \arg\min_{\{R_n\}} \sum_{lmn} \|R_n E_m U_l - \alpha_{lmn} I\|^2, \tag{16}$$

where arg refers to the set of operator elements $\{R_n\}$ of the recovery operation $\mathcal{R}$.

A unique operator of $\mathcal{R}$ can be obtained by stacking the operator elements on the right of one another. Transforming the objective function Γ_c will also lead to convex optimization problems [73]. This approach can further be applied to entanglement-assisted QEC cases [74].

The advantage of optimization-based approaches is that the obtained optimal encoding and recovery operations can involve fewer ancilla qubits. As the cloud-based quantum computers from IBM and Rigetti opening to the public, optimization-based approaches in [73] was demonstrated in [75] in 2019. The channel-adapted recovery operations were decomposed into implementable quantum gates at the expense of losing recovery accuracy. Thus the realization of optimization-based approaches is a trade-off between low error correction cost and high error correction accuracy.

B. OPERATOR QUANTUM ERROR CORRECTION

A significant application of the OQEC [76] is to find the "error-free" spaces for specific error models such that the information encoded in these spaces is immune to errors. A *decoherence-free subspace* is a kind of "error-free" space in which the information is affected by the noise unitarily and can be recovered easily. A noiseless subsystem is another kind of "error-free" space that requires the noise acting on it to be identity. The advantage of the OQEC is that it "hides" information into the "error-free" spaces and thus do not need error correction after transmission.

The formal definition of "error-free" spaces is given below.

Definition 11: Let $\mathcal{H}$ be a Hilbert space with decomposition

$$\mathcal{H} = \bigoplus_i (\mathcal{H}^{\mathscr{A}_i} \otimes \mathcal{H}^{\mathscr{B}_i}) \oplus \mathcal{K}, \tag{17}$$

$\mathcal{E}$ be a noisy channel on $\mathcal{H}^{\mathscr{A}_i} \otimes \mathcal{H}^{\mathscr{B}_i}$. $\mathcal{H}^{\mathscr{B}_i}$ is a ***noiseless subsystem*** *if $\forall \rho^{\mathscr{A}_i} \in \mathcal{D}(\mathcal{H}^{\mathscr{A}_i})$, $\forall \rho^{\mathscr{B}_i} \in \mathcal{D}(\mathcal{H}^{\mathscr{B}_i})$, $\exists \sigma^{\mathscr{A}_i} \in \mathcal{D}(\mathcal{H}^{\mathscr{A}_i})$, s.t.,*

$$\mathcal{E}(\rho^{\mathscr{A}_i} \otimes \rho^{\mathscr{B}_i}) = \sigma^{\mathscr{A}_i} \otimes \rho^{\mathscr{B}_i}. \tag{18}$$

When $dim(\mathcal{H}^{\mathscr{A}_i}) = 1$, $\mathcal{H}^{\mathscr{B}_i}$ is a ***decoherence-free subspace****.*

Subsystem $\mathcal{H}^{\mathscr{B}_i}$ can be named as the *subsystem codes*. The self-correcting property was shown explicitly without external QEC approaches. The stabilizer formalism of subsystem codes requires us to transform some stabilizer generators to *gauge operators*, which are designed to leave the encoded quantum states unaffected. For instance, the Bacon-Shor code [77] is a subsystem code transformed from the Shor code. Although reducing the number of stabilizer generators will lead to fewer error syndromes, gauge operators can make up for this by preserving quantum states passively. Thus, subsystem codes and stabilizer codes are equivalent in code distance with the same number of total qubits.

A necessary and sufficient error correction condition for the OQEC can be developed from the Knill-Laflamme condition [78]. Let P be the projector onto $\mathcal{H}^{\mathscr{A}_i} \otimes \mathcal{H}^{\mathscr{B}_i}$ and $\mathcal{E}$ a noisy channel with operator elements $\{E_k\}$. The OQEC is feasible when a noisy channel $\mathcal{E}$ satisfies

$$PE_aE_b^{\dagger}P = U_{ab}^{\mathscr{A}_i} \otimes I^{\mathscr{B}_i}, \forall E_a, E_b \in \{E_k\}, \tag{19}$$

where $U_{ab}^{\mathscr{A}_i}$ is an operator on $\mathcal{H}^{\mathscr{A}_i}$ and $I^{\mathscr{B}_i}$ is an identity matrix on $\mathcal{H}^{\mathscr{B}_i}$. Thus, the OQEC is suitable for specific error models with symmetry properties on subsystems $\mathscr{B}_i$.

The key issue of the OQEC is to find the "error-free" spaces for a specific error model. Consider a *unital* noisy channel $\mathcal{E}$ with operator elements $\{E_k\}$, where unital means $\mathcal{E}(I) = I$, the "error-free" property states that $\mathcal{E}(\rho) = \rho$. Thus, $[\rho, E_k] = 0$ for all k. Choi and Kribs [79] proposed a method to find $\mathcal{E}$-invariant subspaces using C^*-algebra. The algebraic structure induces a decomposition of the corresponding Hilbert space. Then noiseless subsystems are obtained. The explicit algorithm was developed in [80]. The authors in [81] proposed a numerical method to block-diagonalize C^*-algebra with Wedderburn decomposition, which is inspired by semidefinite programming.

The OQEC suits for specific error models with symmetry properties. The information encoded into the "error-free" spaces does not need error correction procedures. Up to now, methods to find decoherence-free subspaces or noiseless subsystems mainly aim for unital noisy channels while general noise may be non-unital.

V. CONTINUOUS-TIME QUANTUM ERROR CORRECTION

In real situations, the noise of an open quantum system comes from continuous interaction with the environment. Thus the continuous error models are presented to describe the evolution of open quantum systems continuously in time. To correct errors in these models, the continuous-time quantum error correction (CTQEC) is needed other than quantum codes. The advantages of CTQEC are that a) it corrects continuous errors, which fit the real situations better; b) it can correct errors during transmission rather than after transmission. In this section, we introduce the continuous error model and the error correction procedures for this model.

Continuous errors are described by the evolution of open quantum systems, which is governed by the Schrödinger equation

$$\frac{d\rho^{\mathscr{O}}(t)}{dt} = -\frac{i}{\hbar}[H, \rho^{\mathscr{O}}(t)], \tag{20}$$

where $\rho^{\mathscr{O}}(t)$ is the state of an open quantum system at time t, $\hbar$ is the Planck constant and operator H is named the *Hamiltonian* of the system. The solution of equation (20) returns $\rho^{\mathscr{O}}(t) = e^{-iHt/\hbar}\rho^{\mathscr{O}}(0)e^{-iHt/\hbar}$.

Most of the time we only want to track down the states of the principal quantum system without the environment. Let $\rho(t)$ be the state of the principal quantum system. The evolution of the principle system can be described using a master equation in the Lindblad form [57], $d\rho(t)/dt = \mathcal{L}(\rho(t))$. The Lindblad master equation is modeled as Markovian. The equation consists of the Hamiltonian H_s of the principle system, a set of Lindblad operators $\{L_j\}$ and the corresponding error rate λ_j, written as

$$\begin{aligned}\mathcal{L}(\rho(t)) = &-i[H_s, \rho(t)] \\ &+ \frac{1}{2}\sum_j \lambda_j(2L_j\rho(t)L_j^{\dagger} - L_j^{\dagger}L_j\rho(t) - \rho(t)L_j^{\dagger}L_j).\end{aligned} \tag{21}$$

CTQEC [10] described that both errors and error correction procedures are continuous in time. A state $\rho(t)$ undergoes the error correction procedures during a time step dt is

$$\rho(t + dt) \rightarrow (1 - \kappa dt)\rho(t) + \kappa dt\mathcal{R}(\rho(t)), \tag{22}$$

where κ is the error correction rate and $\mathcal{R}(\cdot)$ is the error correction operation that consists of error detection and recovery. In the limit of $dt \rightarrow 0$, the continuous error correction procedure $\mathcal{Q}(\cdot)$ is

$$\mathcal{Q}(\rho(t)) = \kappa(\mathcal{R}(\rho(t)) - \rho(t)) \tag{23}$$

Then the full master equation for the evolution of a principal quantum system subject to Markovian noise plus the error correction procedure is

$$\frac{d\rho(t)}{dt} = \mathcal{L}(\rho(t)) + \mathcal{Q}(\rho(t)). \tag{24}$$

Example 2: Consider a one-qubit principal quantum system with initial quantum state $\rho(0) = |0\rangle\langle 0|$ and bit-flip errors as noise, whose continuous error model is

$$\mathcal{L}(\rho(t)) = \lambda(X\rho(t)X - \rho(t)). \tag{25}$$

In any moment during CTQEC the state can be represented by the fidelity $\mathcal{F}(t) = F(\rho(0), \rho(t))$ as

$$\rho(t) = \mathcal{F}(t)|0\rangle\langle 0| + (1 - \mathcal{F}(t))|1\rangle\langle 1|, \mathcal{F}(t) \in [0, 1], \tag{26}$$

The error correction procedure is

$$\mathcal{Q}(\rho(t)) = \kappa(|0\rangle\langle 0|\rho(t)|0\rangle\langle 0| + |0\rangle\langle 1|\rho(t)|1\rangle\langle 0| - \rho(t)). \tag{27}$$

Applying equations (25) and (27) into (24), the evolution of the system is thus

$$\begin{aligned}\frac{d\rho(t)}{dt} = &[\lambda + \kappa - (2\lambda + \kappa)\mathcal{F}(t)]|0\rangle\langle 0| \\ &+ [(2\lambda + \kappa)\mathcal{F}(t) - \lambda - \kappa]|1\rangle\langle 1|.\end{aligned} \tag{28}$$

An alternative way to describe the evolution of the system can be obtained from equation (26) as

$$\frac{d\rho(t)}{dt} = \frac{d\mathcal{F}(t)}{dt}|0\rangle\langle 0| - \frac{d\mathcal{F}(t)}{dt}|1\rangle\langle 1|. \tag{29}$$

It is obviously from equations (28) and (29) that

$$\frac{d\mathcal{F}(t)}{dt} = \lambda + \kappa - (2\lambda + \kappa)\mathcal{F}(t) \tag{30}$$

with solution $\mathcal{F}(t) = (1-\theta)e^{-(2\lambda+\kappa)t} + \theta$, *where* $\theta = 1 - 1/(2+r)$ *and* $r = \kappa/\lambda$ *is the ratio between the error correction rate and error rate. We see that the fidelity* $\mathcal{F}(t)$ *is confined above its asymptotic value* θ *and* θ *can be made arbitrarily close to* 1 *for sufficiently large* r.

CTQEC for continuous error models applies quantum codes in the encoding procedures and considers continuous error correction procedures. In real situations, however, it is impossible to implement error correction procedures continuously during infinite small time. Thus *weak* measurements are needed that only cause small changes to a quantum state and obtain little information. It is known that each POVM operator can be decomposed into a sequence of weak measurements [58]. A weak measurement operator can be written as $P_i = p_i(I + W)$, where $0 \le p_i \le 1$ and $\|W\| \ll 1$. The following example describes how to implement CTQEC with weak measurements.

Example 3: Consider the quantum system in Example 2 and a sequence of weak measurement $\{P_1, P_2\}$, *where*

$$P_1 = \frac{I+\epsilon X}{2}, \quad P_2 = \frac{I-\epsilon X}{2}, \quad \epsilon \ll 1. \tag{31}$$

POVM operators are defined as $P_1 = M_1^\dagger M_1$, $P_2 = M_2^\dagger M_2$, *where*

$$M_1 = \sqrt{\frac{I+\epsilon X}{2}}, \quad M_2 = \sqrt{\frac{I-\epsilon X}{2}}. \tag{32}$$

Based on the outcomes of P_1, P_2, *the corresponding weak recovery operators* R_1, R_2 *are given by,*

$$R_1 = \frac{I+i\tau Y}{\sqrt{1+\tau^2}}, \quad R_2 = \frac{1-i\tau Y}{\sqrt{1+\tau^2}}, \quad \tau \ll 1. \tag{33}$$

Then the evolution of quantum state $\rho(t)$ *during the error correction procedure* $\mathcal{Q}(\rho(t))$ *is*

$$\mathcal{Q}(\rho(t)) = \frac{d\rho(t)}{dt} = R_1 M_1 \rho(t) M_1 R_1^\dagger + R_2 M_2 \rho(t) M_2 R_2^\dagger. \tag{34}$$

Weak measurements in the above example directly act on the quantum state and weak recovery operations only condition on the most recent measurements, thus historical measurement outcomes are discarded. We name this approach the *direct CTQEC*. The *indirect CTQEC* implements weak measurements on the stabilizer generators [59]. For a quantum state in Example 2, its stabilizer generator is Z, the corresponding weak measurements are $\{P_1', P_2'\}$, where

$$P_1' = \frac{I+\epsilon' Z}{2}, \quad P_2' = \frac{I-\epsilon' Z}{2}, \quad \epsilon' \ll 1. \tag{35}$$

Weak recovery operators are applied based on the outcomes of measuring stabilizer generators. The evolution of the principle quantum system subject to the indirect CTQEC is described using a stochastic differential equation. Weak recovery operators can also be applied based on the difference between outcomes from two weak measurements [60].

Weak measurement implementation can be described on the Bloch sphere explicitly. Consider Example 2 again. The initial quantum state is on the z-axis. Bit-flip errors will transform it towards the center of the sphere, i.e., the maximally mixed state. Weak measurements on quantum states cause a rotation around the y-axis. Thus the corresponding weak recovery operators in equation (33) are also designed with a rotation around y-axis. Weak measurements on stabilizer Z cause a rotation around the x-axis and weak recovery operators are designed as weak X operators.

Physical operations such as weak measurements make a discretization for interactions between the principal system and the environment. More implementation details can be found in [61]. It is proved that implementing CTQEC with weak measurements needs $n - k + 1$ ancilla qubits when the encoding procedure is chosen as an $[n, k]$ stabilizer code. Thus, the error correction cost of CTQEC is comparable to quantum codes and the error correction accuracy is higher than quantum codes with a high error correction rate.

VI. DISCUSSIONS AND FUTURE DIRECTIONS

The above sections have reviewed the state-of-the-art work of QEC. In the beginning, quantum codes are designed for discrete error models that are described using Pauli matrices. Then Optimization-based approaches and the OQEC are introduced that had better performance for specific error models that are described using quantum operations. Further, CTQEC is proposed to correct errors continuously. In this section, we first summarize the features of each approach and discuss some future directions. We then propose some likely definitions of the perturbed error models, aiming to explore the relation between the perturbation on an error model and information preservation.

Table 1 summarizes how these approaches work in each step during quantum information processing and their respective features. Each approach has different error correction ability and it is up to the researchers to choose what they interest most for further study.

A. ADVANCED QUANTUM CODES

The development of quantum codes is promising since the digital error correction is easy to implement and it is closely related to the classical code theory. There remains certain work to complete the quantum code theory, such as a) constructing quantum codes with better parameters from classical codes using the CSS construction, b) combining known quantum codes that have different error correction abilities, c) finding more code structures beyond the scope of known quantum codes.

Entanglement-assisted quantum codes and nonadditive quantum codes are two main approaches that expand the scope of pure stabilizer codes. The former uses pre-shared entanglement so that quantum codes can be constructed from arbitrary classical linear codes. The latter, analogous to classical nonlinear codes, aims to encode more information than stabilizer codes with the same number of qubits. Various quantum codes such as asymmetric quantum error correcting codes are proposed to fit the real situations in discrete form.

TABLE 1. Summarization of QEC approaches.

Approaches	Encoding	Error Models	Decoding	Features
Stabilizer codes	Quantum codes	Discrete error models described using Pauli matrices	Error syndrome measurements and recovery operations	Can be constructed from classical dual-containing linear codes
Entanglement-assisted quantum codes				Can be constructed from arbitrary classical linear codes using entanglement
Nonadditive quantum codes				Can encode more information than stabilizer codes with the same number of qubits, which is analogous to classical nonlinear codes
Quantum codes for various situations				Have respective scope of application (includes fault-tolerance, asymmetric errors and burst errors)
Optimization-based Approaches	Quantum codes/Optimal encoding operations	Specific error models described using quantum operations	Optimal recovery operations	Numerically return the optimal operations, which are not directly implementable quantum gates though
OQEC	"Error-free" spaces		None	Hides information passively rather than actively corrects errors
Direct CTQEC	Quantum codes/Optimal encoding operations	Continuous error models	Weak measurements and weak recovery operations	Directly applies weak measurements on information
Indirect CTQEC				Applies weak measurements on the stabilizer generators

In the future, combining entanglement with known quantum codes can lead to better code parameters. A more generic construction of nonadditive quantum codes also remains an open issue. Besides, more code structures can be presented to fit the real situations discretely. Further, a table of best-known stabilizer codes [19] can be kept updated by constructions from various classical dual-containing linear codes.

B. QEC APPROACHES FOR ERRORS DESCRIBED USING QUANTUM OPERATIONS

As pointed out in [15], the performance of a QEC approach for different errors models varies and there should be a matching between a noisy channel and a QEC approach. Thus, approaches for specific error models that are described using quantum operations were proposed. Optimization-based approaches are numerical methods that find optimal encoding and recovery operations given the initial information and a noisy channel. However, the obtained optimal encoding and recovery operations are not unitary and cannot be directly implemented through quantum gates. Thus, future work shall solve it while maintaining the near-optimal quantum error correction performance.

The OQEC encodes information into the "error-free" spaces for specific error models such that the information is preserved without active error correction procedures. Thus this approach is widely applied in QEC. To apply the OQEC, the symmetry property of error models is particularly important. If a perturbation breaks the symmetry of an error model, can these "error-free" spaces still work? This question leads to our perturbed error models in subsection VI-D.

C. CTQEC

CTQEC is a theoretically optimal choice for QEC when the error correction rate is high enough. However, continuous error correction procedures still need to be implemented with discrete physical operations, i.e., weak measurements and weak operations, in small time intervals. The direct CTQEC applies weak measurements on information directly, which changes the information itself. Since all former measurement outcomes are discarded, weak recovery operators can only condition on the most recent weak measurements. The indirect CTQEC applies weak measurements on the stabilizer generators. Weak recovery operators depend on the outcomes of the last weak measurements. Physicists will keep pursuing wiser feedback choices for recovery, aiming towards more accurate error correction. Further, exploring whether CTQEC can be fault-tolerant may have wider applications in quantum computing.

D. PERTURBED ERROR MODELS

There has been abundant work for discrete or continuous error models. Further, our main concern is the perturbed error models. One of the main applications of perturbed error models is exploring to what extent perturbed errors will affect the information in the "error-free" spaces. The "error-free" spaces correspond to error models with symmetry property and the perturbation on these models will break the symmetry. Thus, it is meaningful to obtain a relation between the perturbation and information preservation and design QEC approaches to correct perturbed errors.

Previous work considered the following two kinds of perturbations. Continuous errors are described by the time evolution of an open quantum system, which is generated by a Hamiltonian H. Perturbation H^ϵ, which is added to the initial system Hamiltonian as $H' = H + \epsilon H^\epsilon$, on decoherence-free subspaces have been proved in [82] that there is no term proportional to the perturbation parameter ϵ when measuring the distance between before-perturbation fidelity and after-perturbation fidelity.

While the Hamiltonian describes errors over the whole open quantum system, the Lindblad operators focuses on the errors that only act on the principal quantum system without the environment. Perturbations L_j^ϵ are discussed using Lindblad operators in equation (21) as $L_j' = L_j + \epsilon L_j^\epsilon$ or $L_j = L_j^\epsilon L_j L_j^{\epsilon\dagger}$ [14].

Other than the Hamiltonian and the Lindblad operators, the quantum operation is also an powerful tool to describe errors. Thus it is worthy to discuss the perturbation using quantum operations. We raise the question: Let $\mathcal{E}$ be an error model with operator elements $\{E_i\}$. What kind of perturbation is worthy discussed? Analogous to the Lindblad operators, we may consider $\{E_i\}$ is perturbed by $\{E_i + M_i\}$, where M_i is the perturbation matrix, $\|M_i - I\|_p \leq \epsilon$, and $\|\cdot\|_p$ is a suitable norm. The perturbed error model is denoted as $\mathcal{E}'$. Applying the completeness relation, we can obtain

$$\sum_i (E_i + M_i)^\dagger (E_i + M_i) = I$$
$$\sum_i (E_i^\dagger M_i + M_i^\dagger E_i + M_i^\dagger M_i) = \mathbf{0}. \quad (36)$$

The evolution of a quantum state ρ going through $\mathcal{E}'$ is

$$\begin{aligned}\mathcal{E}'(\rho) &= \sum_i (E_i + M_i)\rho(E_i + M_i)^\dagger \\ &= \mathcal{E}(\rho) + C, \end{aligned} \quad (37)$$

where $C = \sum_i (E_i \rho M_i^\dagger + M_i \rho E_i^\dagger + M_i \rho M_i^\dagger)$.

Information preservation can be calculated by the fidelity of before-perturbation quantum state and after-perturbation quantum state $F(\mathcal{E}(\rho), \mathcal{E}'(\rho))$. When the before-perturbation quantum state $\rho = |\psi\rangle\langle\psi|$ is pure, the fidelity is

$$F(\mathcal{E}(\rho), \mathcal{E}'(\rho)) = \sqrt{1 + \langle\psi|C|\psi\rangle}. \quad (38)$$

A second way to perform the perturbation is a perturbed error model $\mathcal{E}''$ with operator elements $\{M_i E_i(t)\}$. Applying the completeness relation, we can obtain

$$\sum_i E_i^\dagger M_i^\dagger M_i E_i = I \quad (39)$$

The evolution of a quantum state ρ going through $\mathcal{E}''$ is

$$\mathcal{E}''(\rho) = \sum_i M_i E_i \rho E_i^\dagger M_i^\dagger. \quad (40)$$

The fidelity with pure before-perturbation quantum state is

$$F(\mathcal{E}(\rho), \mathcal{E}''(\rho)) = \sum_k \langle\psi|M_i E_i|\psi\rangle. \quad (41)$$

A third way to perform the perturbation is inspired by the quantum process tomography. The vector space of a single qubit has dimension $d = 2$ with 2×2 density matrices. The vector space of n qubits has dimension $d = 2^n$ with $d \times d$ density matrices. Thus the operator elements on a d-dimensional vector space have a basis $\{\tilde{E}_i\}$, $1 \leqslant i \leqslant d^2$ and each E_i can be written as

$$E_i = \sum_j w_{ij} \tilde{E}_j, \quad (42)$$

where complex numbers w_{ij} are elements of matrix W. Then a perturbation matrix T_ϵ acting on W will lead E_i to

$$E_i' = \sum_j v_{ij} \tilde{E}_j, \quad (43)$$

where complex numbers v_{ij} are elements of matrix $T_\epsilon W$. The completeness relation states that

$$\sum_i \sum_j v_{ij}^* v_{ij} \tilde{E}_j^\dagger \tilde{E}_j = I. \quad (44)$$

The difficulty of analyzing such perturbations is that the number of variants involved could be $2d^4 + d^2$.

There are other perturbed error models to be discovered. We aim to find out which ones can deduce intuitive relations between the perturbation and information preservation and further propose approaches to correct these perturbed errors.

VII. CONCLUSION

In this paper, we analyze some progress on QEC approaches according to the different error models. We attempt to explain how these approaches work and then discuss their respective features. Quantum codes for discrete error models are promising since they only focus on correcting Pauli matrices. Optimization-based approaches and the OQEC can achieve better performance for specific error models that are described using quantum operations. CTQEC for continuous error models corrects errors all along the processing way, where errors and error correction procedures are described using different equations. We also discuss some challenging open issues. With the development of quantum computing, it is necessary to design QEC approaches with lower error correction cost and higher error correction accuracy.

We also propose likely definitions of the perturbed error models and discuss the evolution of quantum states caused by the perturbation. The perturbed error models are described using quantum operations. In the future, we will validate if these definitions can deduce intuitive relations between the perturbation and information preservation. Further, we hope to find the corresponding error correction procedures for the perturbation. The perturbed error models may open up new possibilities towards robust QEC approaches.

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