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Sparse Array Design Exploiting the Augmented Conjugate Correlation Statistics for DOA Estimation

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ABSTRACT With the augmentation of the non-zero conjugate correlation statistics of some modulating signals, enlarged degrees of freedom (DOFs) can be achieved with conventional sparse arrays as proposed by many literatures. But little research has been carried out on sparse array design with the augmented statistics. In this paper, a novel sparse array geometry which exploits both the correlation and conjugate correlation statistics is proposed. It stems from the prototype nested array with transposed subarrays. The sensor locations are determined systematically, and the closed-form expression for the achieved aperture under given number of physical sensors is derived. In our scheme, a new hole-free virtual array called the sum difference coarray is constituted with the proposed sparse array, in which all the lags are consecutive. Due to the output of the coarray is obtained by averaging the repeated lags generated by the addition and subtraction of the physical sensor location, the hole-free structure of the coarray enables all the lags engaged in averaging and makes the output of the coarray more stable. Moreover, the coarray has larger aperture than existing sparse arrays, so that bringing more degrees of freedom. In the end, simulations are conducted to demonstrate the superior performance of the proposed scheme over other existing structures.

INDEX TERMS Conjugate correlation statistics, DOA estimation, hole-free virtual array, sparse arrays, sum difference coarray.

I. INTRODUCTION

Direction of arrival (DOA) estimation plays a significant role in the field of array signal processing, which aims at retrieving the direction information of sources from the received signals. It arises in many practical scenarios such as radar, modern communications and internet of things [1]–[6]. Conventional DOA estimators such as multiple signal classification (MUSIC) [7] and estimation of parameters by rotational invariant techniques (ESPRIT) [8] are mainly built on the uniform linear arrays (ULAs), where the inter-sensor spacing is less than or equal to half wavelength. For a N-sensor ULA, these DOA estimators can resolve at most $N-1$ uncorrelated sources.

Another family of DOA estimators are intensive investigated for underdetermined system, which aim at resolving more sources than sensors. These estimators are designed for sparse arrays, where the sensors are placed nonuniformly. And high degrees of freedom (DOFs) can be achieved by the underlying difference coarrays of sparse arrays. The traditional sparse arrays include minimum redundancy arrays [9] and the minimum hole arrays [10], [11], who have no closed-form expressions for their array geometries and the achieved virtual array aperture. The sensor location can only be determined by exhaustive searching. Recently, the development of sparse arrays such as the nested arrays [12] and the coprime arrays [13], [14] have generated a new wave of interest in the field of array signal processing. They have a more concise and flexible geometry for sparse array configuration, and there are analytical expressions for the achieved apertures. These sparse arrays can identify $O(N^2)$ uncorrelated sources using N physical sensors and they lead to better spatial resolution than ULAs with the same number of sensors. Many extensions of the nested arrays and the coprime arrays have been proposed, which can be divided into two categories. One is for reducing the mutual coupling between

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sensors by redistributing the dense subarray of the nested array, such as the super nested arrays [15] and the generalized nested arrays [16]. The other is for increasing the available DOFs. For instance, [17] has proposed generalized coprime arrays with variable inter-sensor spacing and inter-subarray spacing. And improved nested arrays composed of two ULAs and an additional sensor have been proposed in [18]. Moreover, the augmented nested arrays have been proposed in [19], who have both enhanced DOFs and reduced mutual coupling.

All of the aforementioned sparse arrays aim at constructing the difference coarray with received signals' covariance matrix. However, as indicated in [12] that both the sum coarray and the difference coarray can be utilized to construct the virtual array, where the location of the virtual sensors is obtained by the addition and the subtraction of the physical sensor location. Generally, two approaches are feasible to form such extended coarray. One is the active or transmit/receive sensing scheme such as the colocate multiple-input multiple-output (MIMO) radar, where the detection performance can be enhanced due to the enlarged coarray [20], [21]. The other is utilizing the specific non-zero conjugate correlation statistics such as the pseudo covariance or the conjugate cyclic correlation, which characterize two kinds of modulating signals, viz the noncircular signals and the cyclostationary signals. Some literatures have reported that by exploiting this statistic property, the virtual array aperture is extended by constructing the additional sum coarray with the pseudo covariance of noncircular signals [22] or with the conjugate cyclic correlation of cyclostationary signals [23], [24].

Two obvious issues are emerged in the aforementioned methods [22]–[24]. The additional sum coarrays are usually inconsecutive and irregular distributed, owning to that the majority of sparse arrays are designed to form consecutive difference coarrays regardless of the structure of the sum coarrays. Thus, the achieved DOFs provided by these sparse arrays are limited. The other issue is that the sum and the difference coarrays are constructed respectively, and some virtual sensors are contained in both coarrays at the same time. These repeated virtual sensors provide the same information, thus bringing redundancy to the augmented covariance matrix. Even though the repeated virtual sensors are removed in [25], [26], and the separated coarrays are merged into one array called sum difference coarray (SDCA), [25] is still built on the prototype coprime arrays, where its SDCA provide limited DOFs increasement. [26] redistributes the sensors in [25] to reduce the mutual coupling, but only comparative DOFs is achieved. Therefore, it is necessary to develop the SDCA-based sparse array, where the virtual sensors in both coarrays can be integrated in one virtual array with larger aperture. Some empirical results have been proposed, a customized 6-sensor sparse array in [27] provide the largest SDCA aperture among the 6-sensor arrays. And an adjacent coprime array (ADCA) in [28] can generate $N^2/2 + 3N + 1$ virtual sensors within the SDCA using N physical sensors. In addition, [29] has proposed a systematical methodology for designing a SDCA-based sparse array, which announced to have the largest SDCA aperture of N^2+N-1 with N sensors. However, there is no general way to design similar arrays with other than 6 sensors for [27], and the virtual sensors are not consecutive within the SDCA of the sparse array in [29], only the consecutive part of the SDCA can be utilized in the subspace-based DOA estimator.

Due to dramatic increasement of DOFs can be achieved when utilizing the augmented conjugated correlated statistics in sparse array design, the research of SDCA based sparse array design is promising. However, the aforementioned sparse arrays [25]–[29] suffer from their respective deficiencies, and no satisfactory DOFs increasement is achieve. Hence, there is still substantial potential to form a larger SDCA with fewer physical sensors. In this paper, we propose a novel SDCA-based sparse array called refined nested array with displaced subarrays (RNADS), which has at most $N^2 + 3N - 1$ virtual sensors with N physical sensors. In order to constitute a hole-free SDCA, we firstly transpose the ULAs of the nested array. The resultant array is called nested array with displaced subarrays (NADS), whose first subarray is ULA with D units inter-sensor spacing and the second subarray is a dense ULA. Then, by systematically analyzing the structure of the NADS, the inter-sensor spacing and the spacing between two subarrays are assigned elaborately, so that a coarray with consecutive lags can be formed. In order to make the sparse array more economical, we remove an inessential sensor, which is proved to be at a fixed position and make no change to the coarray when it is removed, and obtain the final version. The expressions for the achieved aperture and its maximum are derived. It should be emphasized that the RNADS has larger SDCA aperture than existing sparse arrays. The output of the coarray is obtained by averaging the repeated lags, thus the hole-free structure enables all the lags engaged in averaging and makes the coarray output more stable. Simulation results clearly demonstrate the superior performance of the proposed scheme over other existing structures in terms of the estimation accuracy, the number of resolvable sources and the angular resolution.

The remaining part of this paper is organized as follows. The data model and the underdetermined DOA estimation method for sparse array are briefly introduced in Section II. Then, in Section III, we propose the systematic methodology for designing RNADS and provide relevant proofs associated with the sparse array. Section IV demonstrate the numerical results of the proposed scheme in different scenarios with other existing sparse array structures. The conclusion is summarized in Section V.

Notations: Vectors, matrices and sets are denoted by lowercase bold letters, uppercase bold letters and uppercase outline letters, respectively. The superscripts $*$, T and H denote the complex conjugate, the transpose and the complex conjugate transpose, respectively. In addition, $\mathbb{E}(\cdot)$ is used to represent the expectation operation. The addition between a set and a scalar is represented as $\mathbb{A} \pm c = \{a \pm c | a \in \mathbb{A}\}$. Provided with two sets $\mathbb{A}$ and $\mathbb{B}$, the difference set between $\mathbb{A}$ and $\mathbb{B}$

are given by $diff(\mathbb{A}, \mathbb{B}) = \{a - b | a \in \mathbb{A}, b \in \mathbb{B}\}$ and their summation set is $sum(\mathbb{A}, \mathbb{B}) = \{a + b | a \in \mathbb{A}, b \in \mathbb{B}\}$.

II. PRELIMINARIES

The sources can be noncircular signals or cyclostationary signals. The way to construct the coarrays with cyclostationary signals can be found in [24]. In this paper, we assume K narrowband far-field noncircular signals $s_k(t)(k = 1, \dots K)$ impinge on a sparse linear array of N sensors, whose locations can be expressed by $\{p_1, p_2, \dots p_N\}d$. d is the unit spacing, which is equal to half wavelength $\lambda/2$, λ is the wavelength of the received signals. The integer set $\mathbb{S} = \{p_1, p_2, \dots p_N\}$ is usually called the location set for convenience. With the angle of the arrived kth source being θ_k, the observed signal $x_n(t)$ at the nth sensor is given by

$$x_n(t) = \sum_{k=1}^{K} a_n(\theta_k) s_k(t) + n(t) \tag{1}$$

where

$$a_n(\theta_k) = e^{-j2\pi p_n d sin\theta_k / \lambda} \tag{2}$$

$n(t)$ denotes the noise at the nth sensor, which is Gaussian distributed and uncorrelated with the sources. Assembling the signals of the whole N sensors gives the received vector as

$$\mathbf{x}(t) = \sum_{k=1}^{K} \mathbf{a}(\theta_k) s_k(t) + \mathbf{n}(t) \tag{3}$$

The covariance matrix $\mathbf{R_{xx}}$ of the received signal $\mathbf{x}$ can be written as

$$\mathbf{R_{xx}} = \mathbb{E}[\mathbf{x}(t)\mathbf{x}^H(t)] = \sum_{k=1}^{K} \mathbf{a}(\theta_k)\mathbf{a}^H(\theta_k)\sigma_k^2 + \sigma_n^2 \mathbf{I} \tag{4}$$

where σ_k^2 and σ_i^2 are the powers of the kth signal and the noise, respectively. The correlation between the observed signals of the mth and nth sensor can be expressed as

$$[\mathbf{R_{xx}}]_{m,n} = \sum_{k=1}^{K} e^{-j2\pi(p_m - p_n) d sin\theta_k/\lambda} \sigma_k^2 \tag{5}$$

It can be seen that (5) is similar with (1), which can be regarded as K real value signals with amplitude σ_k^2 $(k = 1 \dots K)$ from directions of θ_k $(k = 1 \dots K)$ impinge on a virtual sensor located at $\{p_m - p_n\}$. Then, the vectorization of $\mathbf{R_{xx}}$ can be regarded as a single snapshot of a virtual array called the difference coarray, whose location set are obtained by the subtraction of the physical sensor location as

$$diff(\mathbb{S}, \mathbb{S}) = \{p_m - p_n | p_m, p_n \in \mathbb{S}\} \tag{6}$$

The difference set of (6) acts as the virtual sensor locations, which can be utilized to form the difference coarray. Majority of the sparse array design concentrates on how to distribute the physical sensors to form a difference coarray which has consecutive segment of virtual sensors. And the covariance plays a fundamental role in constructing the difference coarray. Besides the covariance, the non-zero conjugate correlation statistics which is called the pseudo covariance for noncircular signals or the conjugate cyclic correlation for cyclostationary signals, can provide additional information that can be exploited to form new coarray structures.

Due to the noise is uncorrelated with the sources, the pseudo covariance for noncircular signals can be expressed as

$$\mathbf{R_{xx^*}} = \mathbb{E}[\mathbf{x}(t)\mathbf{x}^T(t)] = \sum_{k=1}^{K} \mathbf{a}(\theta_k)\mathbf{a}^T(\theta_k) g_k \tag{7}$$

where $g_k = \mathbb{E}[s_k(t)s_k^T(t)]$. And the conjugate correlation between the observed signals of the mth and the nth sensor can be written as

$$[\mathbf{R_{xx^*}}]_{m,n} = \sum_{k=1}^{K} e^{-j2\pi(p_m + p_n) d sin\theta_k/\lambda} g_k \tag{8}$$

(8) can be considered as a snapshot of a virtual sensor located at $\{p_m + p_n\}$. So we vectorize the pseudo covariance, the result can be regarded as one snapshot of a virtual array called sum coarray, whose location set is obtained by the addition of the physical sensor location and forms the following positive segment of sum coarray

$$sum(\mathbb{S}, \mathbb{S}) = \{p_m + p_n | p_m, p_n \in \mathbb{S}\} \tag{9}$$

Take the conjugate of covariance $\mathbf{R}_{\mathbf{xx}}^*$ and the conjugate of pseudo covariance $\mathbf{R}_{\mathbf{xx}^*}^*$ into account, their values can also be regarded as a snapshot of a virtual array. The vectorization of $\mathbf{R}_{\mathbf{xx}}^*$ forms the same difference set as (6). For $\mathbf{R}_{\mathbf{xx}^*}^*$, its vectorization forms the negative segment of sum set as

$$sum(-\mathbb{S}, -\mathbb{S}) = \{-p_m - p_n | p_m, p_n \in \mathbb{S}\} \tag{10}$$

Integrating these three location sets, we obtain the SDCA with the following location set

$$\mathbb{U} = diff(\mathbb{S}, \mathbb{S}) \cup sum(\mathbb{S}, \mathbb{S}) \cup sum(-\mathbb{S}, -\mathbb{S}) \tag{11}$$

The resultant coarray contain the lags from $\{-2p_N, \dots 2p_N\}$, where p_N denotes the location of the Nth sensor in the physical array. It should be noticed that the lags may not be consecutive. In order to form an enlarged coarray, we propose a novel sparse array which has consecutive SDCA in Section III.

For a underdetermined DOA estimator, we firstly define the extended observation vector $\mathbf{y}(t)$ as

$$\mathbf{y}(t) = [\mathbf{x}^T(t), \mathbf{x}^H(t)]^T \tag{12}$$

Then, we evaluate the augmented covariance corresponding to the extended observation vector as

$$\mathbf{R_y} = \mathbb{E}[\mathbf{y}(t)\mathbf{y}^H(t)] \tag{13}$$

After that, we vectorize the $\mathbf{R_y}$ followed by removing the repeated elements in the coarray and sorting the remaining elements in ascending order, the single snapshot received

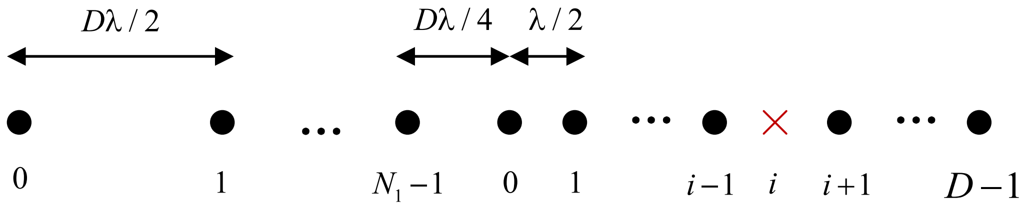

FIGURE 1. The array configuration of RNADS.

model of SDCA is formulated. The spatial smoothing [12] or the direct augmented method [30] can be utilized to recover the rank of the one snapshot covariance. Finally, the subspace-based DOA estimating algorithms can be performed.

III. PROPOSED SPARSE ARRAY

With the augmented conjugate correlation statistics, we propose a methodology to design a sparse array by exploring its SDCA structure. The initial version of the proposed array is termed as the nested array with displaced subarrays (NADS), which is composed by two ULAs with different inter-sensor spacings. The first ULA specified by integer set $\mathbb{S}_1$ with N_1 sensors has an inter-sensor spacing of D units, i.e., $D\lambda/2$ and the second subarray is a dense ULA with D sensors, which is $D/2$ units away from the last sensor of the first ULA. Then in the concept of sum difference coarray we can always find an inessential sensor in the second ULA, who has a fixed position i and can be removed without changing the coarray structure. Hence, we remove the inessential sensor to make the desired array more economical, and the resultant second subarray is specified by integer set $\mathbb{S}_2$. Finally, we obtain the sparse array termed as refined nested array with displaced subarrays (RNADS), whose location set $\mathbb{S} = \{\mathbb{S}_1, \mathbb{S}_2\}$ is given as

$$\mathbb{S}_1 = \{l | l = mD, 0 \leq m \leq N_1 - 1\} \quad (14)$$
$$\mathbb{S}_2 = \{l | l = (N_1 - 1/2)D + n, 0 \leq n \leq D - 1, n \neq i\} \quad (15)$$

The array geometry of the proposed sparse array are shown in Fig. 1. The black dots represent the physical sensors and the red cross represents the inessential sensor which is removed from the array.

Theorem 1 describes the assignment of parameters associated with RNADS that forms a consecutive SDCA.

Theorem 1: Consider a N-sensor RNADS consisting of two subarrays, one is an ULA with D units inter-sensor spacing, and the other is a $D-1$ sensor nonuniform linear array (NLA) generated by removing an inessential sensor located at i of a dense ULA with D sensors. To achieve a hole-free SDCA with large aperture, D should be an even number satisfying $D \geq 6$, the NLA should be $D/2$ units away from the last sensor of the first subarray, and the location of the inessential sensor should be $i = N_1 D$, where $N_1 = N - D + 1$ is the sensor number of the first subarray.

Before proving Theorem 1 we first provide a proposition to investigate the SDCA structure of the NADS before removing the inessential sensor from it.

Proposition 1: Consider a N-sensor NADS consisting of a ULA with D units inter-sensor spacing and a dense ULA with D sensors, represented by ULA-I and ULA-II, respectively. ULA-II is $D/2$ units away from the last sensor of ULA-I and D is an even number. Then the SDCA is a hole-free consecutive virtual array ranging from $-(2N - 2D + 1)D + 2$ to $(2N - 2D + 1)D - 2$.

Proof: Due to ULA-II is $D/2$ units away from the last sensor of ULA-I, D must be an even number to ensure that the inter-sensor spacings of the NADS are integer multiple of half-wavelength. Let integer sets $\mathbb{S}_1$ represent the sensor location of ULA-I, and $\mathbb{S}_3$ be the sensor location of ULA-II with the following expression

$$\mathbb{S}_3 = \{l | l = (N_1 - 1/2)D + n, 0 \leq n \leq D - 1\} \quad (16)$$

N_1 is the sensor number of ULA-I. For the difference coarray, it can be seen that the difference set of NADS is consisted of the self-difference and the cross-difference of the integer sets $\mathbb{S}_1$ and $\mathbb{S}_3$. The self-difference sets are evaluated as

$$diff(\mathbb{S}_1, \mathbb{S}_1) = \{0, \pm D, \ldots \pm (N_1 - 1)D\} \quad (17)$$
$$diff(\mathbb{S}_3, \mathbb{S}_3) = \{0, \pm 1, \ldots \pm (D - 1)\} \quad (18)$$

And the cross-difference set between $\mathbb{S}_3$ and $\mathbb{S}_1$ is

$$\begin{aligned} diff(\mathbb{S}_3, \mathbb{S}_1) &= \{\mathbb{S}_3 - 0\} \ldots \cup \{\mathbb{S}_3 - (N_1 - 1)D\} \\ &= \{0.5D, \ldots (N_1 + 0.5)D - 1\} \end{aligned} \quad (19)$$

Due to $diff(\mathbb{S}_1, \mathbb{S}_3)$ is the mirrored negative segment of $diff(\mathbb{S}_3, \mathbb{S}_1)$, we have

$$diff(\mathbb{S}_1, \mathbb{S}_3) = \{-(N_1 + 0.5)D + 1, \ldots - 0.5D\} \quad (20)$$

Integrating these four parts of difference sets, the difference coarray turns out to be a consecutive array ranging from $-(N_1 + 0.5)D + 1$ to $(N_1 + 0.5)D - 1$.

As to the sum coarray, the summation set is also made up of the self-summations and the cross-summation, which are evaluated as follows

$$sum(\mathbb{S}_1, \mathbb{S}_1) = \{0, D, \ldots 2(N_1 - 1)D\} \quad (21)$$
$$sum(\mathbb{S}_3, \mathbb{S}_3) = \{(2N_1 - 1)D, \ldots (2N_1 + 1)D - 2\} \quad (22)$$
$$\begin{aligned} sum(\mathbb{S}_3, \mathbb{S}_1) &= \{\mathbb{S}_3 + 0\} \ldots \cup \{\mathbb{S}_3 + (N_1 - 1)D\} \\ &= \{(N_1 - 0.5)D, \ldots (2N_1 - 0.5)D - 1\} \end{aligned} \quad (23)$$

Due to $sum(\mathbb{S}_3, \mathbb{S}_1) = sum(\mathbb{S}_1, \mathbb{S}_3)$, these four parts of summation sets takes up the segment of

$$\{(N_1 - 0.5)D, \ldots (2N_1 + 1)D - 2\}.$$

Furthermore, the summation set $sum(-\mathbb{S}, -\mathbb{S})$ is the mirrored negative segment of $sum(\mathbb{S}, \mathbb{S})$. Integrating both the difference set and the summation set, we can summarize that the SDCA is consecutive and rangs from $-(2N_1 + 1)D + 2$ to $(2N_1 + 1)D - 2$. N_1 is the sensor number of the ULA-I which can be substituted by $N - D$ and then we complete the proof of Proposition 1.

In the following, we introduce the definition called inessential sensor and demonstrate the influence of removing the inessential sensor to the sum difference coarray of NADS.

Lemma 1: The N-sensor RNADS array and the $N+1$-sensor NADS array with the same parameter D process the same SDCA, and the location of the inessential sensor in the NADS is $i = N_1D$, where N_1 is the sensor number of the first subarray of NADS.

Proof: we use the definition inessential sensor proposed in [31] with modification. Here a sensor is said to be inessential if the SDCA remains unchanged when it is removed from the array. In order to prove Lemma 1, we shall verify the location elements generated by subtraction operation and addition operation involving the inessential sensor can be replaced by other elements existing in the location set of SDCA. For the difference coarray, the removed sensor leads to the missing of some values when evaluating $diff(\mathbb{S}_2, \mathbb{S}_1)$, $diff(\mathbb{S}_1, \mathbb{S}_2)$ and $diff(\mathbb{S}_2, \mathbb{S}_2)$. It is evident that

$$diff(\mathbb{S}_2, \mathbb{S}_2) = \{0, \pm 1, \ldots \pm (D - 1)\}$$

which is still equal to (18) when the inessential sensor is existed in the array. The missing values in $diff(\mathbb{S}_1, \mathbb{S}_2)$ and $diff(\mathbb{S}_2, \mathbb{S}_1)$ are

$$\{N_1D - \mathbb{S}_1\} \cup \{\mathbb{S}_1 - N_1D\} = \{\pm D, \pm 2D, \ldots \pm N_1D\}$$

Among them, the elements $\{\pm D, \pm 2D, \ldots \pm (N_1 - 1)D\}$ can be replaced by $diff(\mathbb{S}_1, \mathbb{S}_1)$, the rest two holes $\{\pm N_1D\}$ are existed in the difference coarray, but they can be filled by the addition of the physical sensor location pairs $\{D, (N_1 - 1)D\}$ and $\{-D, -(N_1 - 1)D\}$. The removed sensor also makes no change to the positive segment of sum coarray due to

$$sum(\mathbb{S}_2, \mathbb{S}_2) = \{(2N_1 - 1)D, \ldots (2N_1 + 1)D - 2\}$$

which is coincident with (22). And the missing values in the positive segment of cross-summation set are

$$\{N_1D + \mathbb{S}_1\} = \{N_1D, (N_1 + 1)D, \ldots (2N_1 - 1)D\}$$

Among the missing elements, $\{N_1D, (N_1 + 1)D, \ldots (2N_1 - 2)D\}$ can be replaced by $sum(\mathbb{S}_1, \mathbb{S}_1)$, the remaining missing element $\{(2N_1 - 1)D\}$ can be replaced by the self-summation of element $\{(N_1 - 0.5)D\}$, which is the location of the first sensor of ULA-II. The missing values in the negative segment of sum coarray can also be replaced likewise, due to the negative segment is symmetric with the positive segment at the zero point. Thus, removing the inessential sensor located at $i = N_1D$ make no change to the SDCA, and the SDCA of RNADS is also ranging from $-2(N_1 + 0.5)D + 2$ to $2(N_1 + 0.5)D - 2$. For a N-sensor RNADS array, the physical sensor number of ULA-I is $N_1 = N - D + 1$, so in our scheme the aperture expression of SDCA is $4(N - D + 1.5)D - 3$.

Next, we study the parameter character of the RNADS and provide the following corollary.

Corollary 1: The inter-sensor spacing of the ULA-I within the RNADS array should be an even number satisfying $D \geq 6$, and the minimal sensor number of RNADS is 6.

Proof: As we know that the RNADS is obtained by removing an inessential sensor of the ULA-II in NADS. And in order to form a hole-free SDCA, the inessential sensor can not be the second from the bottom of ULA-II. Because the element $\{(2N_1 + 1)D - 3\}$ in the location set of SDCA can only be generated by the summation operation of the physical sensor location pair $\{(N_1 + 0.5)D - 2, (N_1 + 0.5)D - 1\}$, which are the locations of the last two sensor of NADS. Thus, the location of the inessential sensor must satisfy the following condition

$$N_1D < (N_1 + 0.5)D - 2 \tag{24}$$

It can be inferred from the above inequation that D satisfies

$$D > 4 \tag{25}$$

As D must be an even number explained in Proposition 1, the minimal even number for D is 6, and when $D = 6$ the minimal size of RNADS is composed of one sensor ULA-I and $D - 1 = 5$ sensor NLA, so the minimal sensor number of RNADS is 6.

Together with Proposition 1, Lemma 1 and Corollary 1, we complete the proof of Theorem 1 about how to construct a sparse array with hole-free SDCA. Thus, if we know the sensor number as well as the inter-sensor spacing D, we can determine the physical configuration of RNADS and the coarray aperture with the expression $4(N - D + 1.5)D - 3$.

With the explicit analysis of the RNADS and its corresponding coarray structures, an example is provided for better illustration. Consider a 10-sensor RNADS with parameter $D = 8$. According to Theorem 1, the first subarray has $N_1 = N - D + 1 = 3$ sensors with the location set of $\{0, 8, 16\}$, the second subarray with $D - 1 = 7$ sensors ranges from $(N_1 - 0.5)D = 20$ to $(N_1 + 0.5)D - 1 = 27$, the location of the inessential sensor is $N_1D = 24$, thus the second subarray has the location set as $\{20, 21, 22, 23, 25, 26, 27\}$. The physical sensors and its corresponding coarrays are illustrated in Fig. 2. As the analysis results, the difference coarray is ranging from $-(N_1 + 0.5)D + 1 = -27$ to $(N_1 + 0.5)D - 1 = 27$ with two holes located at $\pm N_1D = \pm 24$, and the sum coarray has two consecutive segments from $(N_1 - 0.5)D = 20$ to $(2N_1 + 1)D - 2 = 54$ and its mirrored negative segment. Combining these two coarrays, the SDCA presents to be a consecutive array with aperture $4(N - D + 1.5)D - 3 = 109$. The structures of coarrays in Fig. 2 verify the above analysis and manifest the capability to form a hole-free SDCA of our scheme.

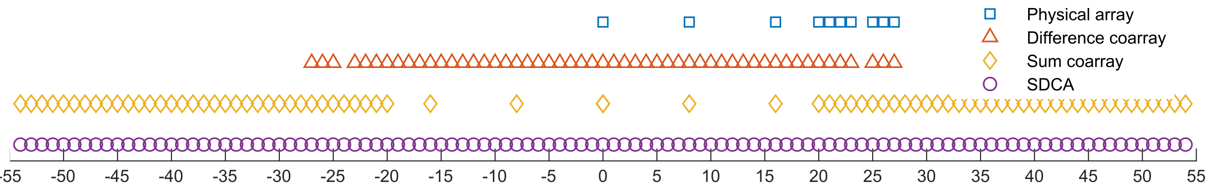

FIGURE 2. **An example of the RNADS with 10 physical sensors.**

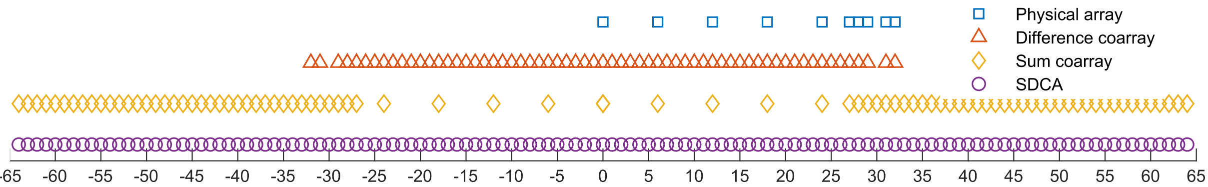

FIGURE 3. **An example of the 10 sensor optimum RNADS.**

The sparse array RNADS and the achieved aperture can be uniquely determined once the parameter D is assigned. But it is still unknown about how to choose D to maximize the virtual array aperture under given number of sensors. In the following, we propose Theorem 2 to show the way to choose D and the explicit expression for the maximum aperture.

Theorem 2: For a N-sensor RNADS array, the aperture of SDCA is maximum when the inter-sensor spacing of the first subarray satisfying

$$D = \frac{N}{2} + \frac{3}{4} \tag{26}$$

and the maximum is $N^2 + 3N - 1$.

Proof: How to choose D to maximize the aperture of SDCA leads to the following optimizing problem

$$max \;\; 4(N - D + 1.5)D - 3 \tag{27}$$

The quadratic maximizing problem can be easily solved with the following result

$$D = \frac{N}{2} + \frac{3}{4} \tag{28}$$

And the maximum is $N^2 + 3N - 0.75 \approx N^2 + 3N - 1$. In practice, D is chosen as the approximated even number satisfying $D \geq 6$. Hence the achieved aperture maybe slightly smaller than $N^2 + 3N - 1$, but it still can be explicitly determined by $4(N - D + 1.5)D - 3$.

Here, we reconsider the array geometry of the 10-sensor RNADS. For an optimum RNADS, we firstly determine the inter-sensor spacing for the first subarray with $D = N/2 + 3/4 = 5.75$, so we choose $D = 6$ that satisfies the criteria specified in Corollary 1. Then the first subarray has $N_1 = N - D + 1 = 5$ sensors located at $\{0,6,12,18,24\}$, the second subarray is ranging from $(N_1 - 0.5)D = 27$ to $(N_1 + 0.5)D - 1 = 32$, after removing the inessential sensor at $N_1 D = 30$, the sensors of the second subarray locate at $\{27, 28, 29, 31, 32\}$. Finally, the corresponding SDCA is a hole-free array ranging from -64 to 64 with the aperture of 129, which is equal to the theoretical result $N^2 + 3N - 1 = 129$. Fig. 3 shows the physical array and its corresponding coarrays, which verify the analysis results. The optimum RNADS has 20 more lags than that of RNADS with $D = 8$. Compared to the method in [29] which announce to have the largest SCDA aperture of $N^2 + N - 1$, our method also has 20 more lags when $N = 10$.

TABLE 1. **Comparison of the number of lags in the SDCA between proposed array and other related arrays.**

Number of Sensors	ADCA	NADiS (Optimized)	RNADS ($D = 6$)	RNADS ($D = 8$)	RNADS ($D = 10$)
10	81	109	**129**	109	57
12	109	155	**177**	173	137
14	141	209	225	**237**	217
16	177	271	273	**301**	297
18	217	341	321	365	**377**
20	261	419	369	429	**457**

In this section, we compare the SDCA aperture of the proposed arrays with $D = 6, 8, 10$ and two recently proposed sparse arrays named ADCA [28], and NADiS [29], which also aim to construct the SDCA utilizing both the correlation and the conjugate correlation statistics. Table 1 shows their respective coarray aperture where the numbers of physical sensors are chosen to be 10 to 20. It can be seen that the ADCA with analytical aperture expression $N^2/2 + 3N + 1$ have obviously shorter aperture than the NADiS and the RNADS with optimal sensor distribution. The coarray aperture of RNADS with optimized D are in bold font, which are chosen to be even numbers approximated to $N/2 + 3/4$ according to Theorem 2. And the coarray aperture of RNADS with optimized D is shown to be $2N$ larger than that of the NADiS when $D = 10, 14, 18$ and $2N - 2$ larger than that of the NADiS when $D = 12, 16, 20$.

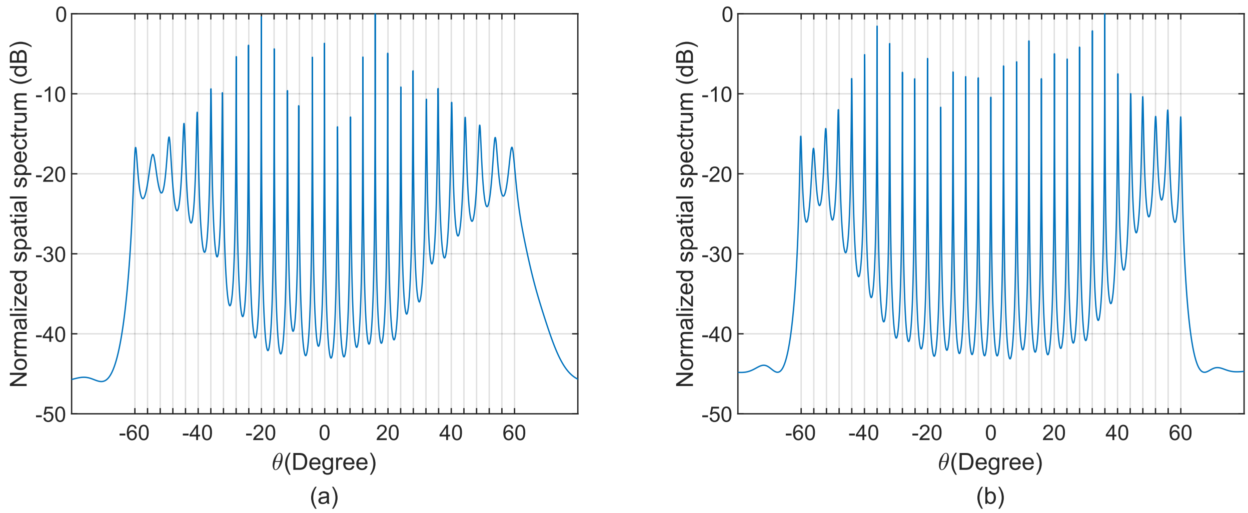

FIGURE 4. The spatial spectrum for the NADiS (a) and the RNADS (b) with 31 sources.

IV. SIMULATION RESULTS

In this section, simulations are performed to study the performance of the proposed array structure in terms of DOA estimation and compared with other existing schemes, including the ADCA, NADiS and the initial version of the proposed array called NADS described in Proposition 1. The compressed sensing based methods and the subspace-based DOA estimation methods can be utilized in the underdetermined estimation problem. For the subspace-based method, the DOFs can be obtained by $(L-1)/2$ when the number of lags within a coarray is L. In this paper, we adopt direct augmented approach [30] followed by spectrum-searching MUSIC method with 8-sensor sparse arrays in the first experiment, which demonstrates the spatial spectrum under multiple sources scenario. In the following experiments involving evaluating the mean square error (MSE) of the estimation, the sensor number increases to 10. And the Root MUSIC approach is adopted, which can directly obtain the estimates of DOA with low computational complexity.

A. SPATIAL SPECTRUM

In the first simulation, we study the performance in terms of spatial spectrum of the proposed array. The 8-sensor sparse array geometries are specified as

$$\begin{aligned} \mathbb{S}_{\text{ADCA}} &= \{0, 4, 5, 8, 10, 12, 15, 16\}, \\ \mathbb{S}_{\text{NADiS}} &= \{0, 1, 2, 3, 4, 13, 22, 31\}, \\ \mathbb{S}_{\text{NADS}} &= \{0, 6, 9, 10, 11, 12, 13, 14\}, \\ \mathbb{S}_{\text{RNADS}} &= \{0, 6, 12, 15, 16, 17, 19, 20\}, \quad D = 6, \end{aligned}$$

According to these array geometries, we know that the RNADS contains 81 consecutive lags and its DOFs are 40, the DOFs for both the ADCA and the NADS are 28, and the NADiS contains 71 consecutive lags so that its DOFs are 35. We assume $K = 31$ uncorrelated baseband BPSK sources, who belong to noncircular signals, impinge on these arrays. These sources are distributed uniformly between -60° to 60° with the step of 4°. The SNR is chosen to be 20dB and the number of snapshot is 1000. The search interval for the MUSIC method is fixed to 0.01°. We ignore the performance of ADCA and NADS, due to the number of sources is beyond their DOFs. Fig. 4 (a) and Fig. 4 (b) demonstrate the spatial spectrums of the NADiS and the RNADS, respectively. As seen in Fig. 4 (a), the NADiS fails to resolve the sources from $\pm 56^\circ$ and $\pm 52^\circ$, even though its DOFs are larger than the number of sources. The reason can be attributed to the lack of stability of the NADiS, which is further tested in the following subsection. In Fig. 4 (b), there are 31 distinguishable peaks, which match with the ideal peaks marked by ticks on the θ axis. It is indicated that the proposed scheme achieves superior detection performance due to its larger DOFs and the stability induced by its hole-free coarray.

B. ESTIMATION ACCURACY

In this subsection, we examine the estimation accuracy of these sparse arrays in terms of MSE. The curve of CRAMER RAO bound (CRLB) for DOA estimation using sum difference coarray is also shown to evaluate the estimation performance, whose detailed derivation can be found in [29]. Due to it is represented in the form of normalized DOA $\bar{\theta} = \sin(\theta)/2$, our MSE is also evaluated with the normalized DOA as

$$MSE = \frac{1}{IK} \sum_{i=1}^{I} \sum_{k=1}^{K} (\hat{\theta}_{i,k} - \bar{\theta}_k)^2 \tag{29}$$

where K is the number of sources, I denotes the number of Monte Carlo trials, which is chosen to be 1000 in each scenario. The sensor number increases to 10, and the array geographies are given as

$$\begin{aligned} \mathbb{S}_{\text{ADCA}} &= \{0, 5, 6, 10, 12, 15, 18, 20, 24, 25\}, \\ \mathbb{S}_{\text{NADiS}} &= \{0, 1, 2, 3, 4, 5, 16, 27, 38, 49\}, \end{aligned}$$

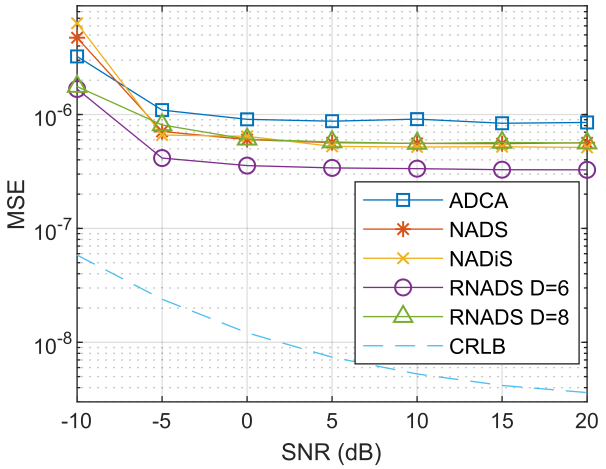

FIGURE 5. MSE results versus input SNR.

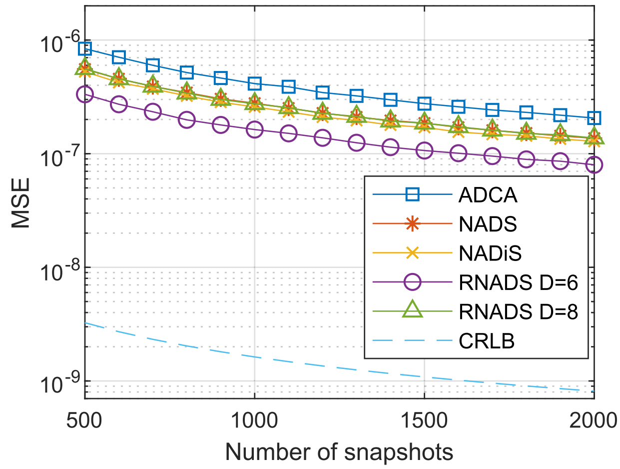

FIGURE 6. MSE results versus number of snapshots.

$$\mathbb{S}_{\text{NADS}} = \{0, 6, 12, 18, 21, 22, 23, 24, 25, 26\},$$
$$\mathbb{S}_{\text{RNADS}} = \{0, 6, 12, 18, 24, 27, 28, 29, 31, 32\}, \quad D = 6$$
$$\mathbb{S}_{\text{RNADS}} = \{0, 8, 16, 20, 21, 22, 23, 25, 26, 27\}, \quad D = 8$$

The curves between the MSE versus the input SNR, the number of snapshots and the number of sources are drawn in different scenarios. Firstly, we assume 27 uncorrelated baseband BPSK sources distributed uniformly between $-60°$ to $60°$ impinge on these sparse arrays. And the angles range from -1.0472 to 1.0472 in the normalized DOA domain. Fig. 5 shows the curves of MSE results versus input SNR, where the snapshot number is fixed to 500. The RNADS with $D = 6$ achieves the lowest MSE for the whole input SNR range, because it possesses the largest coarray aperture including 129 consecutive lags. Besides the large coarray aperture, the stability also contributes to the low MSE. In our scheme, all the lags within the SDCA are consecutive and engaged in averaging, which makes the estimates of the coarray output more accurate and stable. The consecutive lags for the RNADS with $D = 8$, the NADiS and the NADS with $D = 6$ are 109, 109 and 105, respectively. The close coarray aperture make them share similar MSE. And the ADCA with the smallest consecutive lags of 81 has a larger MSE than the other sparse arrays. It is worth noting that there is a big gap between the MSE and the CRLB even at high SNR. This is because the equivalent received signals of the virtual sensors cannot further improve the estimation accuracy as the authentic received signals of physical sensors, even though they can be utilized to improve the number of resolvable sources. Fig. 6 depicts the comparison of MSE results versus the number of snapshots, where the SNR is fixed to 20dB. It can be see a similar trend and the proposed RNADS with $D = 6$ achieves the lowest MSE.

Then we study the estimation accuracy in terms of the MSE as a function of the number of sources, where SNR is selected as 0dB and the number of snapshots is fixed to 1000. The number of sources increases from 25 to 40. It can be seen from Fig. 7 that the RNADS with $D = 6$ possesses the lowest MSE

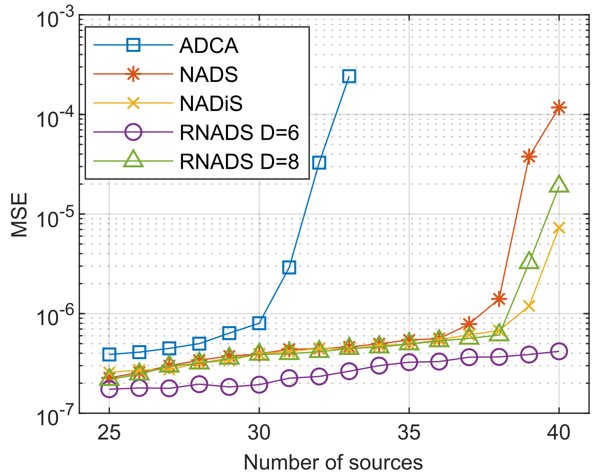

FIGURE 7. MSE results versus number of sources.

for the whole range. Due to the uptight DOFs, the ADCA begin to fail when the number of sources is larger than 30. The MSE of the RNADS with $D = 8$ and the NADiS are smaller than that of the NADS when the number of sources is larger than 36 due to they have 2 more DOFs than the NADS. But they fail to estimate the sources' DOA when the number of sources is larger than 38. Our scheme with $D = 6$ can resolve even 40 sources with moderate MSE.

C. ANGULAR RESOLUTION

The angular resolution is examined in the last subsection. Two sources from directions $\theta_1 = 30°$ and $\theta_2 = 30° + \Delta\theta$ impinge on these sparse arrays. We define that two sources are correctly resolved when the estimated DOAs $\hat{\theta}_1$ and $\hat{\theta}_2$ satisfy $|\hat{\theta}_1 - \theta_1| < \Delta\theta/2$ and $|\hat{\theta}_2 - \theta_2| < \Delta\theta/2$ at the same time. The SNR is 0dB and the number of snapshots is 1000. $\Delta\theta$ varies from $0.1°$ to $0.8°$. The angular resolution probability versus the angular interval is shown in Fig. 8. All of these sparse arrays have 100 percent resolution probability when the angular interval is bigger than $0.65°$. The resolution probability of the NADiS is smaller than that of the RNADS

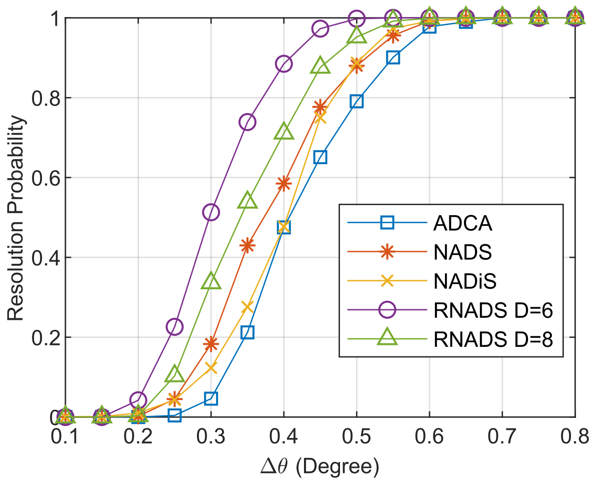

FIGURE 8. **Resolution probability versus the angular interval.**

with $D = 8$ and the NADS, even its DOFs are larger than that of the NADS. The reason is that the NADiS scheme is less stable than others. Beside the 109 consecutive lags, 20 lags out of the consecutive part are ignored and not engaged in averaging the estimate of the repeated virtual sensors, thus it leads to less stable estimate. Our scheme with $D = 6$ achieves higher angular resolution probability than the other sparse arrays, which can still be attributed to its large coarray aperture and its stability.

V. CONCLUSION

In this letter, a novel sparse array called RNADS has been proposed, which can utilize both the signals' correlation statistics and the conjugate correlation statistics to construct the coarray. The resultant sum difference coarray is hole-free and contain at most $N^2 + 3N - 1$ lags with N physical sensors. The explicit parameter assignment associated with the RNADS is provided, the analytical expressions of the coarray aperture and its maximum are derived. Simulation results demonstrate the efficiency of the proposed scheme and the advantage over other recently proposed structures in DOA estimation.

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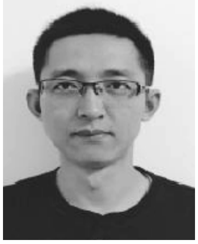

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