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Queues With the Dropping Function and Non-Poisson Arrivals

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ABSTRACT We deal with the single-server queueing system, in which an arriving job (packet, customer) is not allowed to the queue with the probability depending on the queue size. Such a rejected job is lost and never returns to the queue. The study is motivated, but not limited to, active queue management in Internet routers. The exponential service times and general interarrival times are assumed, what makes the model to be a generalization of classic G/M/1 and G/M/1/N queueing models. Firstly, a replacement for the $\rho < 1$ stability condition, which is too excessive in the considered system, is proven. Then, several popular performance characteristics are derived, including the distribution of the queue size, waiting time, workload and the time to reach a given level, as well as the loss ratio. Finally, numerical examples are presented, demonstrating the impact of the standard deviation of the interarrival time on the system performance, as well as the performance of the system for different parameterizations of the dropping function.

INDEX TERMS G/M/1 queue, G/M/1/N queue, dropping function, active queue management, infinite buffer, stability condition, queue size, workload, loss ratio.

I. INTRODUCTION

When exploiting a queueing system, we often want to improve its performance. For instance, we may want to reduce the average queue size. Obviously, this goal can be accomplished by manipulating the arrival or service rate, as a higher service rate or a lower arrival rate makes the queue shorter on average. Several examples of systems with variable service or arrival rate have been studied in the queueing literature, see e.g. [1]–[4].

However, in many real queueing systems, the arrival stream and the service time are difficult to manipulate, or cannot be manipulated at all. The performance of such systems can be adjusted in other way – by rejecting some arriving jobs. One of the most popular schemes of this type is based on the dropping function. Namely, an arriving job is not allowed to the queue (dropped) randomly, with the probability being a function of the queue size at the arrival epoch of this job. A dropped job is lost and never returns to the queue.

The practical usage of the dropping function started a quarter of century ago, when a simple linear dropping function was used in management of queues of packets in Internet routers, [5]. Since then, several different dropping functions have been proposed for packet queueing, including the doubly linear function, [6], the exponential function, [7], the quadratic function, [8], the cubic function, [9], and the newest mixture of cubic and linear functions, [10]. The benefits of usage of the dropping function in routers are well understood - they include not only the reduction of the queue size and queueing delay, but also mitigation of the interflow synchronization, improvement of the interflow fairness, and other.

The active queue management in Internet routers is still the most important area of application of queues with the dropping function. However, other applications have been proposed to date. For instance, in some call centers, it is perhaps better to reject the caller immediately when the queue is long, rather than to keep him waiting for a long time.

It should be stressed, that the queueing system with the dropping functions have universal sense and great potential of applicability. First of all, the system generalizes classic infinite-buffer and finite-buffer queues. Namely, if the dropping function is always 0, we obtain the classic single-server queue with the infinite buffer. If the dropping function is equal to 1 for queue sizes above N, and 0 otherwise, we get the classic queue with the finite buffer of size N. The potential of applicability of the dropping function follows from that fact, that it can be used to obtain an arbitrary value of arbitrary

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performance characteristic, at least in the range limited by the infinite buffer-model. For instance, it is a simple matter to obtain an arbitrary average queue size or loss ratio. In fact, much more than that can be achieved, including optimization of an objective function which depends on a few performance characteristics (see e.g. [11]).

In this paper, we deal with the single-server queue with the dropping function, infinite buffer, general distribution of the interarrival time and exponential distribution of the service time. In other words, we deal with the G/M/1 model with the addition of the dropping function. As argued above, such model incorporates the classic G/M/1 and G/M/1/N models, and the G/M/1/N model with the dropping function, i.e. such that the dropping function is 1 for queue sizes above some threshold. In the considered model, the dropping function can have an arbitrary form. The general distribution of the interarrival time enables modeling of arrival streams of varied characteristics, for instance having a large, or a very small variance of the interarrival time. This is in contrast with Poisson-arrival models, in which this variance is just $\frac{1}{\lambda^2}$ and cannot be altered.

When studying an infinite-buffer queueing system, it is obligatory to ask about its stability. Unfortunately, the classic stability condition, $\rho < 1$, where ρ is the load of the system (see Section III), is not very useful when the dropping function is used. Obviously, the condition holds true, but becomes far too excessive, as there are many useful and stable systems with the dropping function and $\rho \geq 1$. For this reason, an easy to use and much more general condition sufficient for the system stability is proven in Section IV.

Having settled the system stability, we can analyze its performance using different characteristics. Several popular characteristics are derived in this paper, including the steady-state distributions of the queue size, waiting time and workload, as well as the loss ratio. These steady-state characteristics are accompanied by two transient ones: the distribution of the time to reach a given level by the queue size, and, closely related, the probability that in a time interval of length t the queue size does not reach a given level.

The theorems and formulas are illustrated via numerical examples. We first focus on the possibility of modeling a large standard deviation of the interarrival time distribution. In particular, the impact of this deviation on all derived performance characteristics is depicted by using gamma-distributed interarrival time as an example. Secondly, we use a parameter-dependent class of dropping functions to demonstrate possibilities of altering performance characteristics of the system.

The remaining part of the paper is structured in the following manner. In Section II, the related work is recalled. In Section III, a formal description of the queueing model and the notation used in the paper are introduced. In Section IV, a condition sufficient for the system stability is presented in Theorem 1. Then, in Section V, several mentioned above performance characteristics are derived and gathered in Theorem 2. In Section VI, numerical examples are shown and discussed. Finally, conclusions and some suggestions of future work are presented in Section 7.

II. RELATED WORK

To the best of the author's knowledge, the results presented herein are new. The performance of queues with the dropping function have been studied before either with the Poisson arrival stream, or under the finite-buffer assumption.

Analytical studies of queues with finite buffers and Poisson arrivals began with [12], where an approximate analysis of the system with the linear dropping function was carried out. An exact solution of the model with batch Poisson arrivals, the exponential service and the general dropping function, was given in [13]. The steady state and transient analysis of the M/G/1/N model with the general dropping function was performed in [14] and [15], respectively. In [16], a system with Poisson arrivals and multiple service channels was studied.

Queues with finite buffers, general interarrival times and exponential service times were studied in [17] and [18]. Namely, in [17] an approximate solution in the steady state was presented, while in [18], the transient analysis was carried out.

Finally, in [19], [20] finite-buffer systems with the arrival process governed by a modulating Markov chain, were analyzed. Namely, [19] dealt with the Markov-modulated Poisson process, while [20] – with the batch Markovian arrival process.

The infinite-buffer queue with the dropping function has been studied so far in [21] and [11], with Poisson arrivals only. In both of these papers, a stability analysis was performed first, then followed by derivations of the most important performance characteristics and numerical examples. In particular, in [21] the M/M/1 model with the dropping function was studied, while in [11] – the M/G/1 model. The presented herein study on the G/M/1 model is complementary to those papers. The main stability condition, (8), is common in all the mentioned models. Unfortunately, a separate proof has to be given in the G/M/1 case, as the previous proofs rely on the memoryless property of the interarrival time, which is not present herein.

The G/M/1 queue without the dropping function is well known and discussed in most queueing theory textbooks, e.g. [22]–[24].

III. QUEUEING MODEL AND NOTATION

In what follows, $\mathbb{P}$ denotes probability, $\mathbb{E}$ denotes the average value of a random variable or probability distribution, while $\mathbb{D}$ – its standard deviation.

We consider herein the G/M/1 queueing model with the dropping function. Namely, jobs arrive according to the general renewal process with the interarrival time distribution $G(t)$. The service time distribution function is $1 - e^{-\mu t}$ and the standard independence assumptions are made. The waiting room (buffer) is infinite, thus the queue size can be arbitrarily large.

Additionally, every arriving job may not be allowed to the queue with the probability $d(n)$, where n is the queue size upon arrival of this job (including the service position, is occupied). The dropping function, $d(n)$, $n = 0, 1, 2, \ldots$, is not further specified, except from the fact that $d(0) < 1$. (The latter requirement eliminates trivial, always empty systems). A job not allowed to the queue leaves the system and never returns.

The queueing discipline is FIFO, but it matters only when the waiting time distribution is computed. For the remaining characteristics, it can be LIFO or any other, non-preemptive discipline. This is due to the fact, that the waiting time is computed from the point of view of a job, thus it matters where in the queue it is placed upon arrival. The remaining characteristics are functions of the number of jobs in the queue only, thus the jobs can be arbitrary ordered, without affecting those characteristics.

It is assumed that $G(t)$ is not a lattice distribution and

$$\mathbb{E}G = \int_0^\infty t dG(t) < \infty. \tag{1}$$

The standard deviation of the interarrival time distribution is denoted by $\mathbb{D}G$, the arrival intensity is:

$$\lambda = \frac{1}{\mathbb{E}G}, \tag{2}$$

while the load of the system:

$$\rho = \frac{\lambda}{\mu}. \tag{3}$$

In derivations, we will be using the Laplace-Stieltjes transform of the interarrival time distribution:

$$g(s) = \int_0^\infty e^{-st} dG(t), \quad s \geq 0. \tag{4}$$

Finally, $X(t)$ denotes the queue size at time t, including the service position, if occupied. We adopt the convention that $X(t)$ is left-continuous, i.e. $X(t) = X(t-)$.

IV. STABILITY

We say that a queueing system is stable if and only if for every $n = 0, 1, \ldots$, the following limit exists:

$$\lim_{t\to\infty} \mathbb{P}(X(t) = n) = p_n, \tag{5}$$

and numbers p_n fulfill:

$$\sum_{n=0}^{\infty} p_n = 1. \tag{6}$$

Theorem 1: If

$$c = \limsup_{n\to\infty} \rho(1 - d(n)) < 1, \tag{7}$$

then the system is stable.

Proof: The proof consists of two parts. The first part is devoted to proving that the limit in (5) exists for every n. It exploits the renewal theory and ends after (12) is obtained. The second part is devoted mainly to proving that m occurring in (12) is finite. From this, (6) is easily obtained.

If (7) holds, then there exists a natural number, N, such that:

$$\rho(1 - d(n)) < \frac{c+1}{2} < 1, \quad n > N. \tag{8}$$

Without loss of generality, we may assume that $t = 0$ is an arrival time and $X(0+) = N + 1$. Let τ_k, $k = 0, 1, 2, \ldots$, denote a sequence of arrival times in which the queue size increases from N to $N + 1$. Formally, we have:

$$\begin{aligned} \tau_0 &= 0, \\ \tau_k &= \inf\{t > \tau_{k-1} : X(t) = N, X(t+) = N + 1\}, \quad k \geq 1. \end{aligned} \tag{9}$$

Let us define random variables T_k's as follows:

$$T_k = \tau_k - \tau_{k-1}, \quad k = 1, 2, \ldots. \tag{10}$$

From the memoryless property of the exponential distribution, it follows that each τ_k is a regeneration point of the queue size process. This means that the evolution of the queue size from time τ_k is exactly the same, as from time τ_{k-1}. Therefore, variables T_k are independent and identically distributed, while moments τ_k constitute a renewal process with the renewal function:

$$H(u) = \sum_{k=0}^{\infty} F^{k*}(u),$$

where

$$F(x) = \mathbb{P}(T_k < x) = \mathbb{P}(T_1 < x),$$

$$F^{0*}(u) = 1, \quad F^{k*}(u) = \int_0^u F^{(k-1)*}(u-v)dF(v), \ k \geq 1.$$

Then we have:

$$\begin{aligned} &\mathbb{P}(X(t) = n) \\ &= \sum_{k=0}^{\infty} \mathbb{P}(X(t) = n, \tau_k \leq t < \tau_{k+1}) \\ &= \sum_{k=0}^{\infty} \int_0^t \mathbb{P}(X(t) = n, \tau_k \leq t < \tau_{k+1} | \tau_k = u) dF^{k*}(u) \\ &= \sum_{k=0}^{\infty} \int_0^t \mathbb{P}(X(t-u)=n, \tau_1 > t-u | X(0+)=N+1) dF^{k*}(u) \\ &= \int_0^t \mathbb{P}(X(t-u) = n, \tau_1 > t-u | X(0+) = N+1) dH(u). \end{aligned} \tag{11}$$

The third to last formula is obtained from the second to last using the regeneration property of τ_k times. Namely, the queue size process between τ_k and τ_{k+1} is replaced by the same process between $\tau_0 = 0$ and τ_1.

Applying the key renewal theorem (see e.g. [23], p. 102) to (11) yields:

$$\begin{aligned} p_n &= \lim_{t\to\infty} \mathbb{P}(X(t) = n) \\ &= \frac{1}{m} \int_0^\infty \mathbb{P}(X(t) = n, \tau_1 > t | X(0+) = N+1) dt, \ n \geq 0, \end{aligned} \tag{12}$$

where

$$m = \mathbb{E}(\tau_1 | X(0+) = N+1). \tag{13}$$

In this way, we have proven that the limit in (5) exists for every n. Therefore, we are left with the task of proving (6). Firstly, we need to prove that:

$$m < \infty. \tag{14}$$

Otherwise, (12) would be 0 for every n.

To prove (14), we can use a G/M/1 system with the same, as previously, interarrival and service time distributions, but with a simplified dropping function, $\tilde{d}$. It is defined as follows:

$$\tilde{d}(n) = \begin{cases} q, & \text{if } n > N, \\ r, & \text{otherwise,} \end{cases} \tag{15}$$

$$q = \frac{2\lambda - c\mu - \mu}{2\lambda}, \quad r = \max\{d(n) : n = 0, 1, \ldots, N\}. \tag{16}$$

Every characteristic of the new system will be denoted with tilde, e.g. $\tilde{X}(t)$, $\tilde{m}$, $\tilde{\tau}_k$, to distinguish it from the analogous characteristic of the original system. As previously, we assume that $t = 0$ is an arrival epoch and $\tilde{X}(0+) = N+1$. Moreover, without loss of generality, we may assume that $r < 1$. (In the case $r = 1$, we simply have the finite-buffer system, which is well known to be stable).

We will show now that:

$$\tilde{m} = \mathbb{E}(\tilde{\tau}_1 | X(0+) = N+1) < \infty. \tag{17}$$

Let $\tilde{\alpha}$ be the first time, when the queue size reaches the level N, i.e.:

$$\tilde{\alpha} = \inf\{t > 0 : \tilde{X}(t) = N\}. \tag{18}$$

If $\tilde{\eta}$ denotes the first arrival time after $\tilde{\alpha}$, then $\tilde{\tau}_1$ can be expressed as:

$$\tilde{\tau}_1 = \tilde{\eta} + \tilde{\theta}, \tag{19}$$

where

$$\tilde{\theta} = \tilde{\tau}_1 - \tilde{\eta}. \tag{20}$$

Hence to prove (17), it suffices to prove $\mathbb{E}\tilde{\eta} < \infty$ and $E\tilde{\theta} < \infty$.

To prove $\mathbb{E}\tilde{\eta} < \infty$, note that from (15) it follows that for queue sizes above N, the dropping probability is constant. Thus the arrival process is a thinned renewal process with thinning probability q. It is well known that the thinned renewal process is again a renewal process, with the interarrival time distribution $G_q(t)$ given by the Laplace-Stieltjes transform:

$$g_q(s) = \int_0^\infty e^{-st} dG_q(t) = \frac{qg(s)}{1 - g(s) + qg(s)}, \quad s \geq 0, \tag{21}$$

where $g(s)$ is the transform of $G(t)$ (see e.g. [25]). Therefore, for queue sizes above N, the new system operates in exactly the same way as the classic G/M/1 system without the dropping function, but with the interarrival time distribution given by (21) and the average value:

$$\mathbb{E}G_q = \frac{\mathbb{E}G}{1-q} = \frac{2}{\mu(c+1)} < \infty. \tag{22}$$

The load, ρ_q, of this G/M/1 system equals:

$$\rho_q = \frac{1}{\mu\, \mathbb{E}G_q} = \frac{c+1}{2} < 1. \tag{23}$$

Thus this classic system is underloaded and stable. It is easy to see that $\tilde{\eta}$ is simply the duration of the busy cycle of this classic G/M/1 system. Therefore, we can use some known results. In [22], p. 97, it is shown that under conditions (22) and (23), the distribution of $\tilde{\eta}$ is proper, and for its expected value we have:

$$\mathbb{E}\tilde{\eta} = \frac{1}{\rho_q y_0} < \infty, \tag{24}$$

where y_0 is the largest positive solution of the equation:

$$y = \mu - \mu g_q(y), \tag{25}$$

which is known to exist. This finishes the proof of $\mathbb{E}\tilde{\eta} < \infty$.

To show $E\tilde{\theta} < \infty$, we note that for queue sizes shorter than $N+1$, the dropping probability is constant again. Therefore, before the queue size reaches $N+1$, the arrival process in the system with $\tilde{d}$ is yet another thinned renewal process, with thinning probability r and the interarrival time distribution $G_r(t)$ given by the transform:

$$g_r(s) = \int_0^\infty e^{-st} dG_r(t) = \frac{rg(s)}{1 - g(s) + rg(s)}, \quad s \geq 0, \tag{26}$$

and the average value:

$$\mathbb{E}G_r = \frac{\mathbb{E}G}{1-r} < \infty. \tag{27}$$

Denote $\tilde{X}(\tilde{\eta}+) = K$. Obviously, $1 \leq K \leq N+1$. If $K = N+1$, then $\tilde{\eta} = \tilde{\tau}_1$ and $\tilde{\theta} = 0$, thus $E\tilde{\theta} < \infty$. Assume now $1 \leq K \leq N$. In this case, random variable $\tilde{\eta}$ is simply the time of growth of the queue size from the level K to $N+1$, in the classic G/M/1 system, with the interarrival distribution given by (26) and the service rate μ. Therefore, we can use known results again. Applying Theorem 6.2 of [26], after simple algebra we obtain:

$$\mathbb{E}\tilde{\theta} = \sum_{l=0}^{N} \left(\sum_{k=1}^{N+1} R_{N+1-k} z_{k,l} - \sum_{k=1}^{K} R_{K-k} z_{k,l} \right), \tag{28}$$

where

$$R_0 = 0, \quad R_1 = \frac{1}{g_r(\mu)}, \tag{29}$$

$$R_{k+1} = R_1 (R_k - \sum_{i=0}^{k} a_{i+1} R_{k-i}), \quad k \geq 1, \tag{30}$$

and

$$a_k = \int_0^\infty \frac{e^{-\mu t}(\mu t)^k}{k!} dG_r(t), \quad k \geq 0, \tag{31}$$

$$z_{k,l} = \begin{cases} \mathbb{E}G_r - \sum_{i=0}^{k-1} \int\limits_0^\infty \frac{e^{-\mu t}(\mu t)^i}{i!}(1-G_r(t))dt, \\ \quad \text{if } l=0, \\ \int\limits_0^\infty \frac{e^{-\mu t}(\mu t)^{k-l}}{(k-l)!}(1-G_r(t))dt, \\ \quad \text{if } 0<l\leq k, \\ 0, \\ \quad \text{otherwise.} \end{cases} \tag{32}$$

From (27), it follows that all integrals in (31) and (32) are finite for every k and l. Therefore (28) is finite, as a finite sum of finite summands. This completes the proof of $\mathbb{E}\tilde{\theta} < \infty$.

Now we can proceed to showing that $m < \infty$ in our main system, with the dropping function $d(n)$. From (15) and (16) it follows that:

$$\tilde{d}(n) \leq d(n), \quad n > N, \tag{33}$$

$$\tilde{d}(n) \geq d(n), \quad 0 \leq n \leq N, \tag{34}$$

From (33) and (34) we obtain:

$$\mathbb{E}\eta \leq \mathbb{E}\tilde{\eta} < \infty, \tag{35}$$

$$\mathbb{E}\theta \leq \mathbb{E}\tilde{\theta} < \infty, \tag{36}$$

respectively. Therefore, we have:

$$m = \mathbb{E}\eta + \mathbb{E}\theta < \infty. \tag{37}$$

This completes the proof of $m < \infty$ in the main system.

To prove (6), it suffices to notice that:

$$m = \int_0^\infty \mathbb{P}(\tau_1 > t | X(0+) = N+1)dt. \tag{38}$$

Now, combining (12) with (38) and (37) we get:

$$\sum_{n=0}^\infty p_n = \frac{1}{m}\sum_{n=0}^\infty \int_0^\infty \mathbb{P}(X(t)=n, \tau_1 > t|X(0+)=N+1)dt$$
$$= \frac{1}{m}\int_0^\infty \mathbb{P}(\tau_1 > t|X(0+) = N+1) = 1, \tag{39}$$

which completes the proof of Theorem 1. □

V. PERFORMANCE CHARACTERISTICS

In this section, a theorem on several important performance characteristics of the system will be proven. In the steady state, distributions of the queue size, workload and waiting time, with their average values, and the job loss ratio, will be derived. In the transient case, the probability that the queue size does not reach a given level in some interval, and the average time to reach a given level, will be found.

The loss ratio, L, is defined as the fraction of jobs rejected in a long time interval, i.e. as $t \to \infty$. It characterizes a system with losses, like the one studied herein. The loss ratio is especially important in networking, where the lost packets influence profoundly the communication quality. The loss ratio in systems without the dropping functions has been studied extensively using measurements (e.g. [27]–[29]) and mathematical modeling (e.g. [30], [31]). Finally, the probability that the queue size does not reach a given level can be especially useful when the dropping function is used, as one of the main reasons to use the dropping function is to keep the queue size low.

The system workload at time t is defined as the time that a job entering the queue at time t would spend in the queue, before service. It will be denoted by $v(t)$, while its distribution in the steady state by $V(x)$, i.e.:

$$V(x) = \lim_{t\to\infty} \mathbb{P}(v(t) < x). \tag{40}$$

The actual waiting time of the n-th arriving job will be denoted by w_n, while the distribution of the waiting time in the steady state by $W(x)$, namely:

$$W(x) = \lim_{n\to\infty} \mathbb{P}(w_n < x). \tag{41}$$

We adopt the convention, that rejected jobs are not counted in the distribution of the waiting time. To take into account rejected jobs, one has to simply add their fraction, L, with the waiting time 0.

The probability that the queue size does not reach the level M in a time interval of length t, if it starts from the level n, will be denoted by $Q_{n,M}(t)$. Formally, we have:

$$Q_{n,M}(t) = \mathbb{P}\big(R_{n,M} > t\big), \tag{42}$$

where

$$R_{n,M} = \inf\{t > 0 : X(t) = M | X(0+) = n\}, \quad 0 \leq n < M. \tag{43}$$

We may notice that $Q_{n,M}(t)$ is also the tail of the distribution of the time of hitting the level M, starting from the level n. Thus having $Q_{n,M}(t)$, we can easily compute the average value of the hitting time, $\mathbb{E}R_{n,M}$.

The function $Q_{n,M}(t)$ will be computed in terms of the Laplace transform:

$$q_{n,M}(s) = \int_0^\infty e^{-st}Q_{n,M}(t)dt, \quad s \geq 0. \tag{44}$$

Finally, let X_n denote the queue length just before the n-th arrival time. It is easy to see that X_n constitutes a Markov chain. Let $p_{i,j}$ denote the transition probability of chain X_n:

$$p_{i,j} = \mathbb{P}(X_{n+1} = j | X_n = i), \tag{45}$$

while β_n – its stationary distribution:

$$\beta_n = \lim_{k\to\infty} \mathbb{P}(X_k = n), \quad n = 0, 1, \ldots. \tag{46}$$

This distribution is not hard to find in a stable system – at the end of this section it will be shown how to do this effectively. For now, we assume that β_n is known.

Theorem 2: If the system is stable, then the loss ratio equals:

$$L = \sum_{n=0}^\infty \beta_n d(n), \tag{47}$$

the distribution of the queue size in the steady state:

$$p_n = \rho(1-d(n-1))\beta_{n-1}, \quad n = 1, 2, \ldots, \tag{48}$$

$$p_0 = 1 - \rho + \rho L, \tag{49}$$

the average queue size:

$$\mathbb{E}X = \rho \sum_{k=1}^{\infty} n\big(1-d(n-1)\big)\beta_{n-1}, \tag{50}$$

the distribution of the waiting time:

$$W(x) \begin{cases} 0, & \text{if } x \le 0, \\ 1 - \frac{1}{1-L}\sum_{n=1}^{\infty}\sum_{i=0}^{n-1}(1-d(n))\beta_n \frac{e^{-\mu x}(\mu x)^i}{i!}, & \text{if } x > 0, \end{cases} \tag{51}$$

the average waiting time:

$$\mathbb{E}W = \frac{\mathbb{E}X}{(1-L)\lambda} - \frac{1}{\mu}, \tag{52}$$

the distribution of the workload:

$$V(x) = \begin{cases} 0, & \text{if } x \le 0, \\ 1-\rho \sum_{n=1}^{\infty}\sum_{i=0}^{n-1}(1-d(n-1))\beta_{n-1} \frac{e^{-\mu x}(\mu x)^i}{i!}, & \text{if } x > 0, \end{cases} \tag{53}$$

and the average workload:

$$\mathbb{E}V = \frac{\mathbb{E}X}{\mu}. \tag{54}$$

Moreover, no matter if the system is stable or not, the transform of the probability that the queue size does not reach the level M in an interval of length t is:

$$q_M(s) = A_M^{-1}(s)\, 1\, \frac{1-g(s)}{s}, \tag{55}$$

where, (56)–(59), as shown at the bottom of this page, while the average time to reach the queue size M starting from the queue size m:

$$\mathbb{E}R_{n,M} = \lim_{s\to 0+} [A_M^{-1}(s)\, 1]_{n-1} \frac{1-g(s)}{s}, \quad n < M, \tag{60}$$

where $[]_n$ denotes the n-the element of a vector.

Proof: Formula (47) follows immediately from definitions of L, X_n and β_n.

To show (48), we can use the rate conservation law (see e.g. [24], p. 218). Namely, in a long time interval of length σ, there are $\sigma p_n \mu$ jumps of the process $X(t)$ from the level n to the level $n-1$, where $n \ge 1$. On the other hand, in this interval there are clearly $\sigma\lambda\beta_{n-1}(1-d(n-1))$ jumps of $X(t)$ from the level $n-1$ to n. From the rate conservation law we know that these numbers must be equal in a stable system. Therefore we have:

$$p_n\mu = \lambda\beta_{n-1}(1-d(n-1)), \quad n = 1, \ldots, \tag{61}$$

which completes the proof of (48).

To show (49), we can use (6) and (48). We have:

$$p_0 = 1 - \sum_{n=1}^{\infty} p_n = 1 - \rho\sum_{n=1}^{\infty}\beta_{n-1} + \rho\sum_{n=1}^{\infty} d(n-1)\beta_{n-1}, \tag{62}$$

which finishes the proof of (49).

Formula (50) follows directly from (48).

$$q_M(s) = [q_{0,M}(s), \ldots, q_{M-1,M}(s)], \tag{56}$$

$$1 = [1, \ldots, 1]^T, \quad A_M(s) = [a_{i,j}(s)]_{i=0\ldots M-1, j=0\ldots M-1}, \tag{57}$$

$$a_{i,j}(s) = \begin{cases} 1-\overline{b}_0(s)d(0), & \text{if } i=0, j=0, \\ -\overline{b}_0(s)(1-d(0)), & \text{if } i=0, j=1, \\ 1-\overline{b}_1(s)+\overline{b}_1(s)d(0)-b_0(s)d(1), & \text{if } i=1, j=1, \\ -\overline{b}_i(s)d(0), & \text{if } 1\le i\le M-1, j=0, \\ -b_0(s)(1-d(i)), & \text{if } 1\le i\le M-2, j=i+1, \\ -\overline{b}_i(s)+\overline{b}_i(s)d(0)-b_{i-1}(s)d(1), & \text{if } 2\le i\le M-1, j=1, \\ 1-b_1(s)+b_1(s)d(i-1)-b_0(s)d(i), & \text{if } 2\le i\le M-1, j=i, \\ -b_{i-j+1}(s)+b_{i-j+1}(s)d(j-1)-b_{i-j}(s)d(j), & \text{if } 2\le i\le M-1, 2\le j\le i-1, \\ 0, & \text{otherwise}, \end{cases}$$

$$b_k(s) = \int_0^{\infty} \frac{e^{-(\mu+s)u}(\mu u)^k}{k!} dG(u), \tag{58}$$

$$\overline{b}_k(s) = \sum_{i=k}^{\infty} b_i(s) = g(s) - \sum_{i=0}^{k-1} b_i(s), \tag{59}$$

To prove (51), let us derive the probability $\overline{W}(x)$, that the waiting time of an arbitrary accepted job exceeds some x, where $x > 0$. Firstly, the system cannot be empty upon the arrival of this job. (Otherwise its waiting time would be 0). Secondly, if there are $n \geq 1$ jobs in the system upon the arrival of the new job, its waiting time in the queue is a sum of n exponentially distributed service times. Due to the memoryless property of the exponential distribution, it does not matter that one of the jobs has been already partially completed. Summarizing these considerations, we obtain:

$$\overline{W}(x) = \frac{1}{1-L}\sum_{n=1}^{\infty}\beta_n\big(1-d(n)\big)\big(1-(1-e^{-\mu x})^{n*}\big), \quad (63)$$

where $n*$ denotes the n-fold convolution of a distribution function with itself. Note that the need for normalization $\frac{1}{1-L}$ follows from the fact, that we do not take rejected jobs into account when deriving $\overline{W}(x)$.

Now the proof of (51) can be easily completed using (63), the fact $W(x) = 1 - \overline{W}(x)$ and the relation:

$$(1-e^{-\mu x})^{n*} = 1 - \sum_{i=0}^{n-1}\frac{e^{-\mu x}(\mu x)^i}{i!}. \quad (64)$$

Formula (52) can be obtained directly from (51). Namely, we have:

$$\begin{aligned}\mathbb{E}W &= \int_0^\infty (1-W(x))dx \\ &= \frac{1}{1-L}\sum_{n=1}^{\infty}\sum_{i=0}^{n-1}\beta_n(1-d(n))\int_0^\infty \frac{e^{-\mu x}(\mu x)^i}{i!}dx \\ &= \frac{1}{(1-L)\mu}\sum_{n=1}^{\infty} n\beta_n(1-d(n)) \\ &= \frac{1}{(1-L)\mu}\Big(\sum_{n=0}^{\infty}(n+1)\beta_n(1-d(n)) \\ &\quad -\sum_{n=0}^{\infty}\beta_n(1-d(n))\Big) \\ &= \frac{1}{(1-L)\mu}\Big(\frac{\mathbb{E}X}{\rho}-1+L\Big) = \frac{\mathbb{E}X}{(1-L)\lambda}-\frac{1}{\mu}. \end{aligned} \quad (65)$$

Formula (53) can be proven in a similar way as (51), i.e. by deriving the probability $\overline{V}(x)$ that the amount of unfinished work at arbitrary time exceeds $x > 0$. Reasoning in the same way about the remaining service time and using the queue size distribution at arbitrary time, p_n, we obtain immediately:

$$\overline{V}(x) = \sum_{n=1}^{\infty} p_n\big(1-(1-e^{-\mu x})^{n*}\big), \quad (66)$$

Now (53) follows from (48), (64) and the fact $V(x) = 1 - \overline{V}(x)$.

To obtain (54), we can use simply the memoryless property of the exponential service time, which assures that if the queue size at time t is n, then the average workload at this time is n/μ, no matter if one job is already in progress at time t. (A direct integration of $1 - V(x)$ leads to the same result).

To show (55), we can first note that for $1 \leq n \leq M-1$:

$$\begin{aligned} &Q_{n,M}(t) \\ &= \int_0^t \sum_{k=0}^{n-1}\frac{(\mu u)^k e^{-\mu u}}{k!} d(n-k)Q_{n-k,M}(t-u)dG(u) \\ &\quad + \int_0^t \sum_{k=n}^{\infty}\frac{(\mu u)^k e^{-\mu u}}{k!} d(0)Q_{0,M}(t-u)dG(u) \\ &\quad + \int_0^t \sum_{k=0}^{n-1}\frac{(\mu u)^k e^{-\mu u}}{k!}\big(1-d(n-k)\big)Q_{n-k+1,M}(t-u)dG(u) \\ &\quad + \int_0^t \sum_{k=n}^{\infty}\frac{(\mu u)^k e^{-\mu u}}{k!}\big(1-d(0)\big)Q_{1,M}(t-u)dG(u) \\ &\quad +1-G(t), \end{aligned} \quad (67)$$

where $Q_{M,M}(t)$ is 0 by definition. Equation (67) is obtained using the formula of total probability, where the first two summands in (67) cover the case, in which there is an arrival in the interval $(0, t)$ and the arriving job is rejected. The third and the fourth summand cover the case, in which there is an arrival in $(0, t)$ and the arriving job is accepted. The last summand, $1 - G(t)$, covers the case in which there are no arrivals in $(0, t)$. In this case, obviously, $Q_{n,M}(t) = 1$. Reasoning in a similar way for $n = 0$ we obtain:

$$\begin{aligned} Q_{0,M}(t) &= \int_0^t d(0)Q_{0,M}(t-u)dG(u) \\ &\quad + \int_0^t \big(1-d(0)\big)Q_{1,M}(t-u)dG(u) + 1 - G(t). \end{aligned} \quad (68)$$

Applying the Laplace transform to (67) and (68) yields:

$$\begin{aligned} &q_{n,M}(s) \\ &= \sum_{k=0}^{n-1} d(n-k)q_{n-k,M}(s)b_k(s) \\ &\quad +d(0)q_{0,M}\overline{b}_n(s) + \sum_{k=0}^{n-1}\big(1-d(n-k)\big)q_{n-k+1,M}(s)b_k(s) \\ &\quad +\big(1-d(0)\big)q_{1,M}(s)\overline{b}_n(s) + \frac{1-g(s)}{s}, \quad 1 \leq n \leq M-1, \end{aligned} \quad (69)$$

and

$$\begin{aligned} q_{0,M}(s) &= d(0)q_{0,M}(s)g(s) + \big(1-d(0)\big)q_{1,M}(s)g(s) \\ &\quad + \frac{1-g(s)}{s}, \end{aligned} \quad (70)$$

respectively. Clearly, (69) and (70) constitute now a system of M linear equations. Collecting unknowns $q_{n,M}(s)$ on one side and using the matrix notation, this systems can be rewritten as:

$$A_M(s)q_M(s) = \mathbb{1}\,\frac{1-g(s)}{s}, \quad (71)$$

from which (55) is obtained immediately.

Finally, (60) follows from (55) and the fact that:

$$\mathbb{E}R_{n,M} = \int_0^\infty Q_{n,M}(t)dt = \lim_{s\to 0+} q_{n,M}(s). \quad (72)$$

This completes the proof of Theorem 2. □

In order to use Theorem 2 in practice, we have to compute the vector:

$$\beta = (\beta_0, \beta_1, \beta_2, \ldots), \quad (73)$$

which is the stationary vector for Markov chain X_n defined above. Fortunately, transition probabilities for this chain are rather easy to obtain. Namely, we have:

$$p_{i,j} = \begin{cases} (1-d(i))\overline{b}_{i+1}(0) + d(i)\overline{b}_i(0) \text{ if } i \geq 0, j=0, \\ (1-d(i))b_{i-j+1}(0) + d(i)b_{i-j}(0), \\ \qquad\qquad \text{if } i>0, 0<j<i+1, \\ (1-d(i))b_0(0), \text{ if } i\geq 0, j=i+1, \\ 0, \text{ if } i \geq 0, j>i+1, \end{cases} \quad (74)$$

where $b_k(0)$ and $\overline{b}_k(0)$ are given in (58) and (59), respectively. Note that $b_k(0)$ and $\overline{b}_k(0)$ are well known in queueing theory and are either easy to calculate symbolically (for several popular distributions) or suitable for numerical integration.

Now, having the transition matrix:

$$P = [p_{i,j}]_{i,j=0,1,\ldots}, \quad (75)$$

in a stable system β can be computed using the well-known equations:

$$\beta P = \beta, \quad (76)$$

$$\sum_{j=0}^{\infty} \beta_j = 1. \quad (77)$$

The system (76), (77) is infinite. Fortunately, it has been widely studied as a popular tool in the Markov chain theory, and is well understood now. In particular, in [32] it was shown, that the solution ${}_{(n)}\beta$ of the truncated system:

$${}_{(n)}\beta_{(n)}P = {}_{(n)}\beta, \quad (78)$$

converges elementwise to β as $n \to \infty$, where ${}_{(n)}P$ is the $n\times n$ northwest corner truncation of the matrix P. Thus we can calculate the vector β with a high precision using a solution of a finite linear system.

Finally, to obtain numbers from (55), a method for numerical inversion of the Laplace transform has to be used. Many such methods can be found in literature. In the following numerical examples, the Zakian method is used, [33].

VI. EXAMPLES

In the examples, the following gamma-distributed interarrival time is used:

$$G'(t) = \frac{a^a}{\Gamma(a)} t^{a-1} e^{-at}, \quad a > 0. \quad (79)$$

It can be easily verified that $\mathbb{E}G = 1$ and $\mathbb{D}G = 1/\sqrt{a}$. Thus manipulating a, we can manipulate the standard deviation of the interarrvial time, keeping its average value equal to 1. It is assumed that $\mu = 1$, which gives $\rho = 1$.

Finally, the following class of dropping functions is used:

$$d_r(n) = r \begin{cases} \frac{1}{2} e^{\frac{n-50}{10}}, & \text{if } n \leq 50, \\ 1 - \frac{1}{2} e^{\frac{50-n}{10}}, & \text{if } n > 50. \end{cases} \quad (80)$$

where $r \in (0, 1]$ is a parameter. A few functions from this class are depicted in Fig. 1.

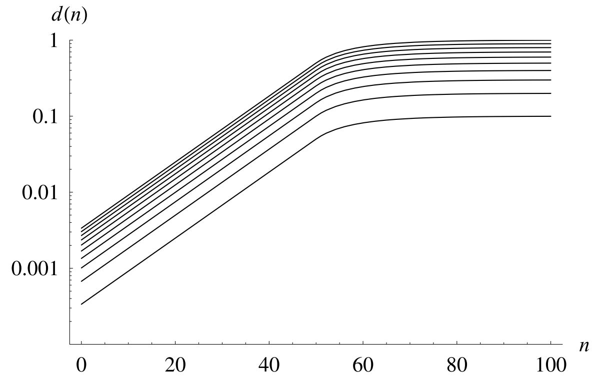

FIGURE 1. Dropping function $d_r(n)$ for $r = 0.1, 0.2, \ldots, 1$, counting from the bottom.

From Theorem 1 it follows immediately, that the system is stable for every $r \in (0, 1]$. It is also clear, that it would have been unstable without the dropping function.

A. DEPENDENCE ON THE STANDARD DEVIATION

In this set of examples, we check the influence of the standard deviation of the interarrival time on the performance of the system.

Firstly, the values of characteristics derived in the previous section are presented in Table 1, for three values of $\mathbb{D}G$ and the function $d_{0.5}$. In particular, results for $\mathbb{D}G = 1$ (i.e. Poisson arrivals) are shown in the second column, while in the next two, the standard deviation is multiplied twice by the factor of 5.

As we can see, $\mathbb{D}G$ has a profound impact on the queueing performance, even when ρ is unaltered, as in our case. The performance for Poisson arrivals is pretty good; we have a small queue size, even smaller its standard deviation, low loss ratio (2.7%), and a very long average time to reach length 50. Increasing $\mathbb{D}G$ 5 times, we observe a moderate increase in the queue size and its deviation (twice), but other characteristics deteriorate more visibly – the loss ratio increases 6 times (to 16.6%), while the average time to reach length 50 is 17 times shorter. Increasing $\mathbb{D}G$ 5 times once more, to 25, we can see that the queue size deteriorates rather severely (5 times), and its standard deviation even more (10 times). Other characteristics also get far worse.

Comparing the results for $\mathbb{D}G = 1$ with those for $\mathbb{D}G = 25$ we can see, how misleading is usage of a simplified traffic model (e.g. Poisson), in modelling of a real system, when the real deviation of the interarrival time is far from $1/\lambda$.

TABLE 1. Detailed performance characteristics for three different values of the standard deviation of the interarrvial time, $\mathbb{D}G$, and the dropping function $d_{0.5}$.

performance characteristic	$\mathbb{D}G = 1$	$\mathbb{D}G = 5$	$\mathbb{D}G = 25$
average queue size, $\mathbb{E}X$	19.521	28.707	141.45
std. dev. of the queue size, $\mathbb{D}X$	12.778	23.504	225.61
loss ratio, L	0.0274	0.1666	0.4527
average waiting time, $\mathbb{E}W$	19.070	33.444	257.45
average workload, $\mathbb{E}V$	19.521	28.707	141.45
average time to reach the length 50, $\mathbb{E}R_{0,50}$	3164.7	179.05	61.154
prob. the queue is below 50 in first 100s, $Q_{0,50}(100)$	0.9998	0.5806	0.1266

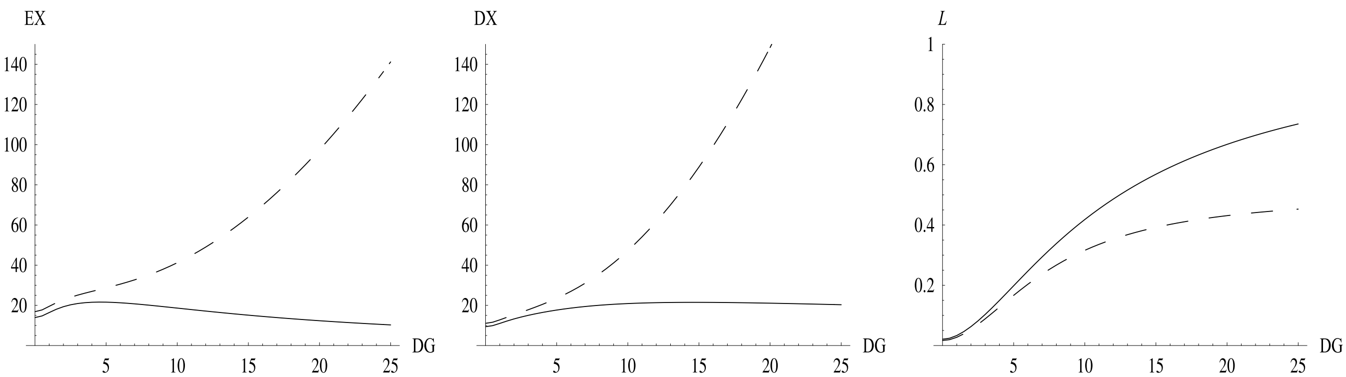

FIGURE 2. Performance characteristics versus the standard deviation of the interarrival time for the dropping function $d_{0.5}$ (dashed lines) and d_1 (continuous lines).

TABLE 2. Detailed performance characteristics for three different values of parameter r and $\mathbb{D}G = 4$.

performance characteristic	$r = 0.1$	$r = 0.5$	$r = 1$
average queue size, $\mathbb{E}X$	81.077	26.950	21.548
std. dev. of the queue size, $\mathbb{D}X$	77.185	20.491	16.471
loss ratio, L	0.0628	0.1276	0.1496
average waiting time, $\mathbb{E}W$	85.508	29.890	24.339
average workload, $\mathbb{E}V$	81.077	26.950	21.548
average time to reach the length 75, $\mathbb{E}R_{0,75}$	490.86	1909.2	$1.311 \cdot 10^{10}$
prob. the queue is below 75 in first 1000s, $Q_{0,75}(1000)$	0.1141	0.6036	0.99999987

In general, this effect was to be expected, as a strong dependence of the system performance on $\mathbb{D}G$ occurs also in systems without the dropping function. However, herein the detailed performance of the system depends also on the shape of the dropping function, in a complicated way. For instance, using a more aggressive dropping function, we can improve some poor characteristics, occurring due to the high $\mathbb{D}G$, at the cost of some other characteristics.

In Fig. 2, the average queue size, its standard deviation, and the loss ratio are depicted as functions of $\mathbb{D}G$, for the dropping function $d_{0.5}$ (dashed lines) and the function d_1 (continuous lines). In the case of $d_{0.5}$, the queue size grows quickly with $\mathbb{D}G$. The standard deviation grows even more quickly. In the case of the more aggressive d_1, the queue size decreases with $\mathbb{D}G$ from some point, what is a totally different behaviour. This is, however, at the cost of the increased number of losses, which is visible on the right hand side of Fig. 2.

B. DEPENDENCE ON THE DROPPING FUNCTION

In these calculations, we check the influence of the parameter r of the dropping function on the system performance. In every case, $\mathbb{D}G = 4$ is used.

In Tab. 2, various characteristics are presented for three values of r, i.e. $r = 0.1$ (mild dropping), $r = 0.5$ (moderate dropping) and $r = 1$ (aggressive dropping). Clearly, every performance characteristic gets improved as r grows, except for the loss ratio. Additionally, in Fig. 3 the average waiting time, the loss ratio and the probability that the queue does not reach size 75 in first 1000s, are depicted as functions of r.

We can draw at least two conclusions from these results. Firstly, manipulating r we can manipulate the value of each characteristic in a relatively wide range. Secondly, there are decreasing and increasing functions depicted in Fig. 3, thus we can clearly solve optimization problems with respect to r, using a cost function. Such cost function can have an arbitrary form and may depend on $\mathbb{E}W$, L, $Q_{n,M}(t)$, and other characteristics. For example, we may want to minimize the cost function:

$$C = a\,\mathbb{E}W + b\,L, \tag{81}$$

or

$$C = a\,L - b\,Q_{0,75}(1000), \tag{82}$$

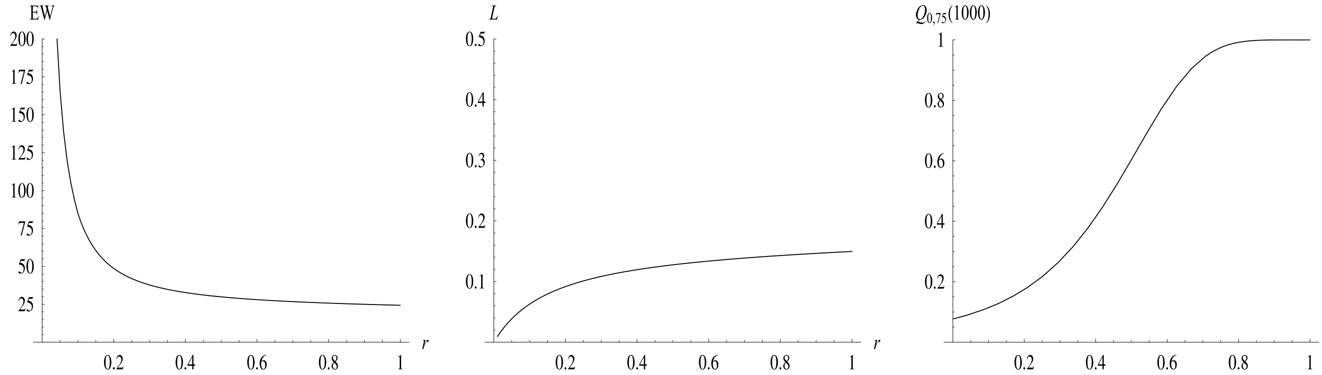

FIGURE 3. **Performance characteristics versus the parameter *r* of the dropping function.**

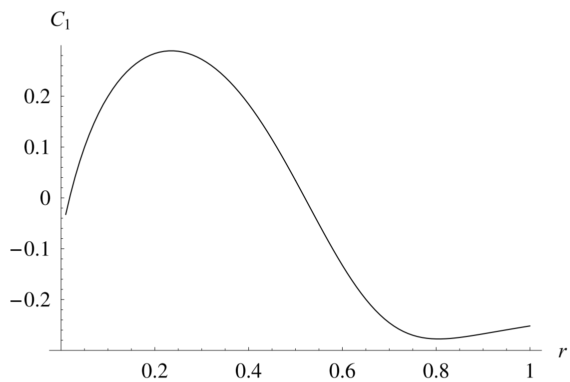

FIGURE 4. **The cost C_1 versus parameter *r*.**

or

$$C = \mathbb{E}W^a L^b, \tag{83}$$

with respect to r, or any other.

For instance, assume that we want to minimize the cost function:

$$C_1 = 5L - Q_{0,75}(1000). \tag{84}$$

This means, that we want to have the loss ratio as small as possible, and at the same time, the probability of not reaching the length 75 in 1000s as high as possible. Moreover, the former goal is 5 times more important than the latter.

In Fig. 4, the cost C_1 is presented as a function of r. It reaches a minimum for $r = 0.81$.

VII. CONCLUSION

In the paper, an analysis of the queue with the dropping function and non-Poisson arrivals has been carried out. A stability condition has been proven and accompanied with derivations of several steady-state and transient characteristics. Numerical examples, which illustrated the influence of the interarrival time variance on the system behaviour, as well as the possibility of adjusting the performance of the system by varying the dropping function, have been presented.

As the general form of the dropping function has been assumed, the considered model incorporates the classic G/M/1 and G/M/1/N models without the dropping function, and the G/M/1/N model with the dropping function.

In the future work, a few different research directions can be explored.

First of all, the G/G/1 model with the dropping function can be studied. As it does not have the convenient embedded Markov chain structure, an approximate analysis may be useful.

As for the G/M/1 queue with the dropping function, the analysis can be extended to the time-dependent distribution of the queue size, workload and the number of losses. Finally, an alternative stability condition can be searched for. For instance, in [21] it was proven that the M/M/1 system with the dropping function is stable if

$$\lim_{n\to\infty} \rho \prod_{i=1}^{n} \sqrt[n]{1 - d(i-1)} < 1, \tag{85}$$

or

$$\lim_{n\to\infty} n \frac{1 - \rho(1 - d(n))}{\rho(1 - d(n))} > 1. \tag{86}$$

It is not sure, whether condition (85) or (86) works in the case of the G/M/1 model as well.

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