IEEE Access
Multidisciplinary | Rapid Review | Open Access Journal

Received January 30, 2020, accepted February 16, 2020, date of publication February 24, 2020, date of current version March 10, 2020.
Digital Object Identifier 10.1109/ACCESS.2020.2974936

New Constructions of Short Length Binary Locally Repairable Codes

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This work was supported by the National Natural Science Foundation of China under Grant 11801564 and Grant 11901579.

ABSTRACT In this short paper, our main objective is to construct binary locally repairable codes (LRCs) with good properties. Two constructions of LRCs with short lengthes are proposed. The first one is that define a u-linearly independent set (u-LIS) of an LRC with disjoint repair groups (DRGs) and enlarge it into another one with bigger size to construct new LRCs. The second is puncturing check matrices of known codes to construct new LRCs. As an application, many new binary LRCs are constructed from distance optimal linear codes, which are also locality optimal according to the Cadambe-Mazumdar (C-M) bound.

INDEX TERMS Cadambe-Mazumdar bound, distance optimal linear code, locally repairable code.

I. INTRODUCTION

Locally repairable codes (LRCs) are designed for distributed storage systems to improve the repair efficiency. Since the pioneer work of Gopalan et al. in [1], LRC has been intensively researching in recent years. For a q-ary linear code $\mathcal{C} = [n, k, d]_q$, if for any $\mathbf{c} = (c_1, c_2, \ldots, c_n) \in \mathcal{C}$, the i-th code symbol c_i can be recovered by accessing no more than r other code symbols, c_i is said to have *locality* r. The code $\mathcal{C}$ is said to have *locality* r if all its symbols have locality at most r. A code $\mathcal{C} = [n, k, d]_q$ with locality r is denoted as $\mathcal{C} = [n, k, d; r]_q$ in [2]. If $q = 2$, $[n, k, d]_q$ and $[n, k, d; r]_q$ are denoted as $[n, k, d]$ and $[n, k, d; r]$ for short, respectively.

In [1], Gopalan *et al.* proposed an upper bound for an $[n, k, d; r]_q$ code named as Singleton-like bound:

$$d \leq n - k + 2 - \lceil \frac{k}{r} \rceil. \quad (1)$$

This bound is not tight over small fields [3], [4], especially over the binary field [5]. A bound taking field size into consideration was presented in [6], which is called Cadambe-Mazumdar (C-M) bound. This bound says that an $[n, k, d; r]_q$ code satisfies

$$k \leq k_{cm} = min_{t \in \mathbb{Z}^+}\{tr + k_{opt}^{(q)}(n - t(r+1), d)\}, \quad (2)$$

where $k_{opt}^{(q)}(n, d)$ is the largest possible dimension of a code of length n, for given field size q and minimum distance d.

Binary LRCs receive much more attentions, since they are easily implemented for no multiplications are needed in encoding, decoding and repair, see [2], [3], [5] and [7]–[19].

The associate editor coordinating the review of this manuscript and approving it for publication was Jin Sha.

Binary LRCs with small locality and meeting the C-M bound have been constructed by using anti-codes [7]. References [8]–[10] proposed binary LRCs for specific parameters by using cyclic codes. Authors of [2]–[3], [5] and [11]–[18] discussed constructions of binary LRCs from classical codes and obtained many codes meeting the C-M bound. Most of the binary LRCs given in this work have relative low rate or small distance, for detail please see [19]. So we focus on constructions of binary LRCs with relative higher rate and minimum distance $d \geq 6$, and discuss the optimality of obtained LRCs in terms of the C-M bound.

II. PRELIMINARIES

This section introduces basic concepts and some results on LRCs [1], [23], [24]. First, we give some notations which will be used later.

(i) Let F_2^n be the n-dimensional space over the binary field $F_2 = \{0, 1\}$. All codes, matrices and vectors in the rest of this paper are over F_2.

(ii) Let $[n] = \{1, 2, \ldots, n\}$. Denote $\mathbf{1}_n$ and $\mathbf{0}_n$ as the all-one and all-zero row vectors, their transposes are denoted as $\mathbf{1}_n^T$ and $\mathbf{0}_n^T$, respectively. Denote an $m \times n$ matrix $\mathbf{A}$ as $\mathbf{A}_{m,n}$.

For an $[n, k, d]$ code $\mathcal{C}$, a matrix $\mathbf{G}_{k,n}$ whose rows form a basis of $\mathcal{C}$ is called a generator matrix of $\mathcal{C}$. The dual code of $\mathcal{C}$ is defined as $\mathcal{C}^{\perp} = \{\mathbf{x} \in F_2^n | \mathbf{x} \cdot \mathbf{c} = 0 \text{ for all } \mathbf{c} \in \mathcal{C}\}$. We call a generator matrix H of $\mathcal{C}^{\perp}$ as a parity check matrix of $\mathcal{C}$ [22], [23]. If $\mathcal{C} = [n, k, d]$ and there is no $\mathcal{C}' = [n, k, d+1]$, $\mathcal{C}$ is called a distance optimal (d-optimal) code. For details on parameters of d-optimal binary codes, please see [21]. If $\mathcal{C} = [n, k, d; r]$ meets the C-M bound, it is called an r-optimal code.

Let $H = H_{m,n}$ be a parity check matrix of $\mathcal{C}$. For $i \in [n]$ and $j \in [m]$, if there is a row h_{j_i} in H and the i-th coordinate of h_{j_i} is nonzero, we say that the i-th coordinate is covered by h_{j_i}. If $H = \begin{pmatrix} H_L \\ H_G \end{pmatrix}$ and each $i \in [n]$ is covered by some rows of H_L, then we say H_L covers $[n]$ and the rows of H_L are called locality rows in [14]. The locality of $\mathcal{C} = [n, k, d]$ can be judged by its parity check matrix as follows:

Proposition 1 [7]: Let $H = \begin{pmatrix} h_1 \\ h_2 \\ \cdots \\ h_{n-k} \end{pmatrix}$ be a parity check matrix of $\mathcal{C} = [n, k, d]$. For each $i \in [n]$, there is a row h_{j_i} of H with weight at most $r + 1$ and the i-th coordinate is covered by h_{j_i}, then $\mathcal{C}$ has locality r.

Proposition 2 [14]: Let $H = \begin{pmatrix} H_L \\ H_G \end{pmatrix}$ be a parity check matrix of $\mathcal{C} = [n, k, d]$. If H_L covers $[n]$ and the weight of each locality row is at most $r + 1$, then $\mathcal{C}$ has locality r.

If $n = l(r + 1)$, $\mathcal{C} = [n, k, d; r]$ and $H = \begin{pmatrix} H_L \\ H_G \end{pmatrix}$ is its parity check matrix, where

$$H_L = \begin{pmatrix} \mathbf{1}_{r+1} & \mathbf{0}_{r+1} & \cdots & \mathbf{0}_{r+1} \\ \mathbf{0}_{r+1} & \mathbf{1}_{r+1} & \cdots & \mathbf{0}_{r+1} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{0}_{r+1} & \mathbf{0}_{r+1} & \cdots & \mathbf{1}_{r+1} \end{pmatrix},$$

then $\mathcal{C}$ is called an LRC with disjoint repair groups (DRGs).

In [18], a method of constructing an even distance LRC from an odd distance LRC was presented.

Proposition 3 [18]: Let d be odd and $\mathcal{C}_0 = [n, k, d; r_0]$. If the maximal weight of codewords of $\mathcal{C}_0^{\perp}$ is w_{max} and $w_{max} < n$, then there is a $\mathcal{C} = [n + 1, k, d + 1; r]$ where $r \le max\{n - w_{max}, r_0\}$.

To develop our discussion, we need a definition.

Definition 1 [20]: Let $M = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a set of m-dimensional column vectors. If any u vectors in M are linearly independent, then M is called as u-linearly independent set (u-LIS).

Let $u < m < n$. Given a u-LIS $M = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of m-dimensional column vectors, one can obtain an $m \times n$ matrix $H = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. If the rank of H is m, then an $[n, k, d] = [n, n - m, u + 1]$ code with parity check matrix H can be obtained. In this work, we only take care of u-LIS M which can give an $[n, k, d] = [n, n - m, u + 1]$ code, and use α_i to denote binary column vector.

III. CONSTRUCTIONS OF LRCs

In this section, we will give two constructions for LRCs.

A. NEW LRCs FROM LRCs WITH DRGs

In this subsection, we always assume $n = l(r + 1)$, $\mathcal{C} = [n, k, d; r]$ is an LRC with DRGs and its parity check matrix is $H = \begin{pmatrix} H_L \\ H_G \end{pmatrix} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. If $M = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ can be enlarged into $(d - 1)$-linearly independent set M', new LRCs can be obtained as follows.

Construction 1: Let $\mathcal{C} = [n, k, d; r]$ be an LRC with DRGs, whose parity check matrix is given above. Suppose $M = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ can be enlarged into a $(d - 1)$-LIS $M' = M \cup \{\alpha_{n+1}, \ldots, \alpha_{n+s}\}$. If $A = \begin{pmatrix} A_{l,s} \\ B \end{pmatrix} = (\alpha_{n+1}, \ldots, \alpha_{n+s})$, $A_{l,s}$ covers $[s]$ and its maximal row weight is w, then there exists an $[n + s, k + s, d; r + w]$ LRC.

Proof: Let $H' = [H\ A]$ and $H'_L = [H_L\ A_{l,s}]$. As $A_{l,s}$ covers $[s]$, one can see H'_L covers $[n + s]$. Since the maximal row weight of $A_{l,s}$ is w, it is obvious that the maximal row weight of H'_L is $r + w + 1$. Thus the code with parity check matrix $H' = [H\ A]$ is an $[n + s, k + s, d; r + w]$ LRC.

Example 1: An [18, 6, 8; 2] LRC with DRGs was given in [2]. This code has parity check matrix $H_{12,18} = \begin{pmatrix} H_L \\ H_G \end{pmatrix} = (\alpha_1, \alpha_2, \ldots, \alpha_{18})$. Using computer, we can enlarge $M = \{\alpha_1, \alpha_2, \ldots, \alpha_{18}\}$ into a 7-LIS $M' = M \cup \{\alpha_{19}, \ldots, \alpha_{24}\}$. The matrix $H_{12,24} = (H_{12,18} \mid A) = (\alpha_1, \ldots, \alpha_{18}, \ldots, \alpha_{24})$ is

$$H_{12,24} = \begin{pmatrix} H_L & A_{6,6} \\ H_G & O \end{pmatrix} = \left(\begin{array}{c|c} 111000000000000000 & 111110 \\ 000111000000000000 & 111101 \\ 000000111000000000 & 111011 \\ 000000000111000000 & 110111 \\ 000000000000111000 & 101111 \\ 000000000000000111 & 011111 \\ \hline 101000000011110011 & 000000 \\ 011101000011000101 & 000000 \\ 011011101000000110 & 000000 \\ 011011011110011101 & 000000 \\ 000011011000110011 & 000000 \\ 000000011011101101 & 000000 \end{array}\right).$$

From $H_{12,24}$ and its sub-matrices $H_{12,18+j} = (\alpha_1, \alpha_2, \ldots, \alpha_{19}, \ldots, \alpha_{18+j}\}$ for $1 \le j \le 6$, we can obtain six LRCs with parameters [19, 7, 8; 3], [20, 8, 8; 4], [21, 9, 8; 5], [22, 10, 8; 6], [23, 11, 8; 7] and [24, 12, 8; 7]. Localities of these LRCs are determined by the first six rows of their parity check matrices and can be calculated by hand.

Example 2: From [2] we can obtain a [24,11,8;3] LRC with DRGs, whose parity check matrix is $H_{13,24} = \begin{pmatrix} H_L \\ H_G \end{pmatrix} = (\alpha_1, \alpha_2, \ldots, \alpha_{24})$, where

$$H_G = \begin{pmatrix} 010100000000001101100011 \\ 001101010000001100000101 \\ 001100110101000000000110 \\ 001100110011011000110101 \\ 000000110011000001100011 \\ 000000000011001101010101 \\ 100010001000100010001000 \end{pmatrix}.$$

Using computer, we can enlarge $M = \{\alpha_1, \alpha_2, \ldots, \alpha_{24}\}$ into two 7-LIS $M' = M \cup \{\alpha'_{25}\}$ and $M'' = M \cup \{\alpha''_{25}\}$, where $(\alpha'_{25})^T = (1, 1, 1, 1, 1, 0, \cdots, 0)$ and $(\alpha''_{25})^T = (1, 1, 1, 1, 1, 1, 0, \cdots, 0, 1)$. $H'_{13,25} = (\alpha_1, \ldots, \alpha_{24}, \alpha'_{25})$ parity checks a [25, 12, 8; 4] LRC whose weight distribution is $w(z) = 1 + 503z^8 + 256z^{10} + \cdots + 128z^{18} + z^{24}$. $H''_{13,25} =$

$(\alpha_1, \alpha_2, \ldots, \alpha_{24}, \alpha''_{25})$ parity checks a [25, 12, 8; 4] LRC with weight distribution $w(z) = 1+375z^8+384z^9+\cdots+384z^{17}+z^{24}$. Localities of these LRCs are determined by the first six rows of their parity check matrices.

The seven LRCs obtained in this subsection are optimal LRCs according to the C-M bound. Now we check the optimality of the code [25, 12, 8; 4]. Suppose $t = 2$ and $r = 4$, by [21], then one can derive $tr + k_{opt}^{(2)}(25 - t(r+1), 8) = 8 + k_{opt}^{(2)}(15, 8) = 12$, hence a [25, 12, 8; 4] LRC achieves the C-M bound. Similarly, we can prove that the other six LRCs given in example 1 also achieve the C-M bound.

B. LRCs CONSTRUCTED BY DUAL PUNCTURING KNOWN CODES

In [16], a construction for LRCs was presented via puncturing anti-codes from generator matrices of Simplex codes, some optimal LRCs with small localities and low rates were derived. Using block-puncturing methods on generator matrices of Simplex codes, authors of [24] investigated minimum distance, locality, availability, joint information locality, and joint information availability of related LRCs. Many good LRCs concerning these properties were presented, yet these LRCs are low rate codes. Recently, in [18], we discussed construction of high rate LRCs by puncturing on check matrices of d-optimal codes. For a given parity check matrix H, by deleting a column with maximal weight each time, we obtained some good LRCs with length $n \leq 24$ and $n - k \leq 12$. However, the method in [18] often fail to get r-optimal LRC when $n - k > 12$. Thus we need to try new method for constructing optimal LRCs.

In this subsection, we assume $\mathcal{C} = [n, k, d; r]$ with parity check matrix $H = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. For $S = \{j_1, j_2, \ldots, j_s\} \subset [n]$ with $s =| S |\leq r - 1$, delete the columns $\alpha_{j_1}, \alpha_{j_2}, \cdots, \alpha_{j_s}$ from H and denote the result matrix as $H_{\bar{S}}$. The code with parity check matrix $H_{\bar{S}}$ is called the *dual punctured code* of $\mathcal{C}$ by S, and its locality is denoted as $r_{\bar{S}}$. From this, one can derive Construction 2 following.

Construction 2: Let $r \geq 3$ and $1 \leq s \leq r - 1$. If $\mathcal{C} = [n, k, d; r]$ is an LRC with parity check matrix $H = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, then there is an $[n - s, k - s, d; r_s]$ LRC, where $r_s = min\{r_{\bar{S}} \mid S \subset \text{[n] and } | S |= s\}$.

Proof: Let $H = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a parity check matrix of $\mathcal{C}$. For a set $S = \{j_1, j_2, \ldots, j_s\} \subset [n]$ with $s =| S |\leq r - 1$, delete the columns $\alpha_{j_1}, \alpha_{j_2}, \cdots, \alpha_{j_s}$ from H and denote the result matrix as $H_{\bar{S}}$. If rank($H_{\bar{S}}$)$= n - k$, using computer, we can obtain locality $r_{\bar{S}}$ of the code with parity check matrix $H_{\bar{S}}$, otherwise choose another subset of $[n]$ of size s and do the same thing. In this way, we can find an $S \subset [n]$ such that its dual punctured code has locality $r_s = min\{r_{\bar{S}} \mid S \subset \text{[n] and } | S |= s\}$ since the total number of such subset is $\binom{n}{s}$.

Using this construction, we can construct many new LRCs from distance optimal codes whose localities are calculated by computer according to the three proposition given in Section II.

TABLE 1. Localities of **$[n, n-12, 6; r]$** for **$32 \leq n \leq 48$**.

n	32	33	34	35	36	37	38	39	40
r	6	7	7	8	9	9	10	10	11
n	41	42	43	44	45	46	47	48	
r	12	12	13	13	14	15	16	17	

Example 3: From a [33, 23, 5; 11] LRC given by [17], according to proposition 3, one can construct a [34, 23, 6; 11] code with parity check matrix $H_{11,34}$, where

$$H_{11,34} = \begin{pmatrix} 1011101100110000000100010000010001 \\ 0011000100101000010110000100011010 \\ 0001000000100101110110000100101100 0 \\ 0000010111000001001001110101010000 \\ 0000101000011000010010001011110100 \\ 0110110100000000000100101100001101 0 \\ 0000000001101110100001100000111010 \\ 0000000010110011100000010110011001 \\ 0101000000000001001110000101001011 1 \\ 0001101000000011100000101010011100 0 \\ 0010010110001011010000000001011001 \end{pmatrix}.$$

Puncturing on $H_{11,34}$, one can construct LRCs $[n, k, 6; r] = [n, n-11, 6; r]$ for $25 \leq n \leq 34$. The localities of these LRCs are $r = 5, 6, 6, 7, 7, 8, 9, 9, 10, 11$ respectively.

Example 4: A [48, 36, 6; 17] code can be constructed from a [47, 36, 5; 17] LRC given in [17], this [48, 36, 6; 17] code has a parity check matrix $H_{12,48}$, where as shown at the top of the next page.

By puncturing on $H_{12,48}$, one can obtain $[n, k, 6; r] = [n, n - 12, 6; r]$ LRCs for $32 \leq n \leq 48$. The localities of these LRCs are listed in Table 1.

Example 5: (1) A [27, 10, 9] code and a [31, 13, 9] code can be obtained from [21], their parity check matrices are $H_{17,27}$ and $H_{18,31}$ respectively, where

$$H_{17,27} = \begin{pmatrix} 100000001000000000010000011 \\ 000000000010001000001010001 \\ 000000110000000000000001101 \\ 011000000000000110000000001 \\ 000100010000001000100000000 1 \\ 000010010000000000000100011 \\ 000001001000000000100010001 \\ 000000000011010100000000000 1 \\ 000000000100001000000100101 \\ 100000100000010100000000000 1 \\ 100100000000000000000110001 \\ 000000000000100001001000100 1 \\ 110000000000000000101000001 \\ 000100001010000001000000000 1 \\ 000010100000000011000000001 \\ 000000000100010000001010001 0 \\ 000001100000000000000001101 0 \end{pmatrix},$$

$$H_{18,31} = \begin{pmatrix} 1000000010100000000000000100001010 \\ 0000000000000101000101010100010000 \\ 0101000000010100000000010100000000 \\ 0000000101000000000100110001000 \\ 0010001000000010000001110000000000 \\ 0000000000011001110000100000000 \\ 1000110100000000000010010000000000 \\ 0000000110000000000100010010010 0 \\ 0000000000000000000001111000011 \\ 0000001010000000100010110000000 \\ 1000000100010000000000110100000 \\ 0000001001100010001000100000000 \\ 0110000011000000000100100000000 \\ 1001001001000000000001100000000 \\ 0000010001000000100000010101000 0 \\ 0100000001100000000000110000010 0 \\ 0010000010000010100000001000000 10 \\ 0000010100000000001000011001010 0 \end{pmatrix}.$$

It is not difficult to check that $H_{17,27}$ gives a [27, 10, 9; 4] LRC by computer, and similarly $H_{18,31}$ parity checks a [31, 13, 9; 5] LRC. Puncturing on $H_{17,27}$, one can derive LRCs [26, 9, 9; 3] and [25, 8, 9; 3]. LRCs [30, 12, 9; 4] and [29, 11, 9; 4] can be obtained by puncturing on $H_{18,31}$.

(2) According to Proposition 3, from the above six codes and weight distributions of their dual codes, one can derive LRCs [26, 8, 10; 3], [27, 9, 10; 3], [28, 10, 10; 4]; [30, 11, 10; 4], [31, 12, 10; 4] and [32, 13, 10; 5]. Further, puncturing on parity check matrix of the [30, 11, 10; 4] code, we can get a [29, 10, 10; 3] LRC.

Example 6: (1) A [31, 11, 11] LRC can be obtained from [21], whose parity check matrix $H_{20,31}$ is:

$$H_{20,31} = \begin{pmatrix} 1000010100000000000000000001011 \\ 0000000001000000100010010001 01 \\ 0000000011000101000100000000001 \\ 0110100000000010000000000100001 \\ 0001000000001000101010000000001 \\ 0000000000000000100101100100001 \\ 1000011000010000100000000000001 \\ 0000000110001000010000000000101 \\ 0000100001000000000000011000011 \\ 0000000100110000000010000000101 \\ 0010010000000001001000010000001 \\ 0101000000000100010000000010001 \\ 0000010010000100010000100000001 \\ 0100000000100100000000000001101 \\ 0101001000000000000010001000001 \\ 0000001000100100010100000000001 \\ 0010001000000000000000001001011 \\ 0010000000100001000000100000011 \\ 0001000001000000000101000000011 \\ 1000000010000000001010000000 1001 \end{pmatrix}.$$

It is not difficult to check $H_{20,31}$ gives a [31, 11, 11; 5] LRC by computer. Puncturing on $H_{20,31}$, one can derive [30, 10, 11; 4], [29, 9, 11; 3] and [28, 8, 11; 3] LRCs.

(2) According to Proposition 3, from the above four codes and weight distributions of their dual codes, one can derive [29, 8, 12; 3], [30, 9, 12; 3], [31, 10, 12; 4] and [32, 11, 12; 5] LRCs.

TABLE 2. Comparisons of LRCs.

Our LRCs	LRCs in [10]
[32, 21, 6; 9] Example 3	[32, 21, 6; 11] Table II
[31, 20, 6; 9] Example 3	[31, 20, 6; 10] Table II
[30, 19, 6; 8] Example 3	[30, 16, 6; 14] Table IV
[21, 9, 8; 5] Example 1	[21, 3, 8; 6] Table IV
[28, 10, 10; 4] Example 5	[28, 9, 8; 6] Table IV
[30, 9, 12; 3] Example 6	[30, 8, 8; 14] Table IV
[32, 10, 12; 3] Example 6	[32, 10, 8; 15] Table IV

(3) A [33, 11, 12] code can be obtained from [21], its parity check matrix is $H_{22,33}$, where

$$H_{22,33} = \begin{pmatrix} 000000000000000000000100100110001 \\ 001000000001000000000110000000100 \\ 000010000000100000000100010000010 \\ 000000100000000010000101000001000 \\ 110000000000001000000110000000000 \\ 000001000000000100100010000100000 0 \\ 100000000110000000000010000100000 0 \\ 000010010000000000100101000000000 0 \\ 000000000000001010101100000000000 \\ 010100000000000000000100001000001 \\ 100000001000000000000010000000110 0 \\ 000010000000001010000001000000000 01 \\ 000000000000001000011000000101000 \\ 001000000000000110000100000000001 0 \\ 000001000100000000000011100000000 00 \\ 000000000000001000000110000001001 \\ 101010000000000000000101000000000 00 \\ 000000000000100000000100011000100 \\ 000000000000100010001100000010000 \\ 000000000101001000000010000000001 0 \\ 000000010001000100000100010000000 0 \\ 010000001000000000000010000000101 0 \end{pmatrix}.$$

It is not difficult to check $H_{22,33}$ gives a [33, 11, 12; 4] LRC by computer. By puncturing on $H_{22,33}$, one can derive a [32, 10, 12; 3] LRC.

We have obtained 50 LRCs in this subsection, and we can show that all these LRCs meet the C-M bound by hand.

Summarizing all the examples in these section, we have constructed 57 LRCs, according to [21], all these codes are d-optimal codes. These results on d-optimal and r-optimal LRCs are shown as follows.

Results: There are d-optimal and r-optimal LRCs with parameters:

(1) [19, 7, 8; 3], [20, 8, 8; 4], [21, 9, 8; 5], [22, 10, 8; 6], [23, 11, 8; 7], [24, 12, 8; 7] and [25, 12, 8; 4].

(2) [25, 14, 6; 5], [26, 15, 6; 6], [27, 16, 6; 6], [28, 17, 6; 7], [29, 18, 6; 7], [30, 19, 6; 8], [31, 20, 6; 9], [32, 21, 6; 9], [33, 22, 6; 10], [34, 23, 6; 11].

(3) [32, 20, 6; 6], [33, 21, 6; 7], [34, 22, 6; 7], [35, 23, 6; 8], [36, 24, 6; 9], [37, 25, 6; 9], [38, 26, 6; 10], [39, 27, 6; 10], [40, 28, 6; 11], [41, 29, 6; 12], [42, 30, 6; 12], [43, 31, 6; 13], [44, 32, 6; 13], [45, 33, 6; 14], [46, 34, 6; 15], [47, 35, 6; 16], [48, 36, 6; 17].

$$
H_{12,48} = \begin{pmatrix}
110001001101000010000101010101000011101000001001 \\
000010101000111101000010100011000000000010011111 \\
000000110000100000011001111010111010000111000001 \\
010101011110010010100100000000010110000010110001 \\
001010010110101010011000101000000010010010100011 \\
010010011100001001101100100010000011010001001001 \\
010000111001100011001010101000000000100101010011 \\
001000101111000100000101100100101110000001100001 \\
100101010101001000100110100001000101000000010111 \\
011000001100001000000110011110100000000101011101 \\
101000010110000001011011010000010000101001011001 \\
010010100100000000011010101111000010101000010101
\end{pmatrix}.
$$

(4) [25, 8, 9; 3], [26, 9, 9; 3], [27, 10, 9; 4], [29, 11, 9; 4], [30, 12, 9; 4], [31, 13, 9; 5].

(5) [26, 8, 10; 3], [27, 9, 10; 3], [28, 10, 10; 4], [29, 10, 10; 3], [30, 11, 10; 4], [31, 12, 10; 4], [32, 13, 10; 5].

(6) [28, 8, 11; 3], [29, 9, 11; 3], [30, 10, 11; 4], [31, 11, 11; 5], [29, 8, 12; 3], [30, 9, 12; 3], [31, 10, 12; 4], [32, 11, 12; 5], [32, 10, 12; 3], [33, 11, 12; 4].

Remark: One can check by hand that an [18, 8, 4; 1] in [3] achieves both the Singleton-like bound and the C-M bound, yet it is not d-optimal code.

IV. CONCLUSION

In this paper, we have developed two constructions of LRCs. Using these methods, we can construct 57 new binary LRCs which are distance optimal codes and can achieve the C-M bound, hence they are also locality optimal codes. In fact, the approaches presented in this work can also be used to construct non-binary LRCs, we hope this question would attract much attention.

Table 2 lists some LRCs we constructed, which are all better than codes in Tables 2 and 4 of [10].

We only focus on localities of LRCs in our proposed two constructions. When considering availability and joint information localities of LRCs [24], we will make further study on these properties of LRCs in the future.

ACKNOWLEDGMENT

The authors are very grateful to the reviewers and the Associate Editor Dr. Y. Karaca for their detailed comments and suggestions that much improved the presentation and quality of this article.

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