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RMPC for Uncertain Nonlinear Systems With Non-Additive Dynamic Disturbances and Noisy Measurements

SAMI EL-FERIK
Systems Engineering Department, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia
e-mail: selferik@kfupm.edu.sa

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ABSTRACT In this paper, we present a robust, model-predictive control scheme for the general class of uncertain and constrained discrete-time nonlinear systems subject to noisy measurements. The relationships between the system's dynamics, uncertainties, disturbances and the measurement noise are nonlinear and not necessarily additive. In particular, the disturbance is the output of an uncertain system with an unknown input. This study serves the threefold ultimate objective of ensuring robust satisfaction of the state constraints, recursive feasibility and stability. To satisfy state constraints, the proposed algorithms adopt a constraints tightening approach using the restricted constraint sets computed online. Several bounds on the prediction level and rate are derived and the size of the terminal region is maximized using polytopic linear differential inclusions (PLDI). An explicit bound on the maximum allowable disturbance for recursive feasibility is also derived based on optimization of the one-step ahead controllable set to the terminal region. The disturbance and uncertainties are non-vanishing and therefore only Input-to-state practical stability (ISpS) can be ensured. A simulation example demonstrates the efficacy of the mathematical framework and algorithms developed in this work.

INDEX TERMS Nonlinear MPC, robust MPC, constraint tightening, convex optimization.

I. INTRODUCTION

Model predictive control (MPC) is a moving horizon control approach that has been recognized as the most used control strategy for systems under inputs and state constraints. MPC is a model-based control and can be called either linear or nonlinear depending on the nature of the model used in the prediction of the system's dynamics. In both linear MPC and nonlinear MPC (NMPC) schemes, the closed loop system is nonlinear due to the presence of constraints. Linear MPC has become a preferred control strategy in many industrial applications [24]. It is worthwhile noting that despite the fact that industrial systems are inherently nonlinear, several features, including the ability to handle constraints and balance competing control objectives, contributed to the widespread use of MPC strategy. However, there is a growing need to operate the systems closer to the admissible operating conditions for maximum efficiency and profit. Linear models are therefore no longer adequate to satisfy requirements that are more rigorous and the use of more representative nonlinear system models is necessary. In practice, due to modeling errors all real-world systems suffer from system uncertainties, exogenous disturbances and measurement noise.

The idea behind the model predictive control strategy is very well known, mature, and can be summarized as follows. Based on the system's information collected at time t, MPC solves an online open-loop constrained optimal control problem over a predetermined finite-horizon at every sampling instant. At each instant, the controller uses the system's model to predict the dynamic of the real system under a set of control actions up to a predetermined horizon N_p. The control is optimized over a horizon N_c and only the first optimal control action is implemented. These steps in the control strategy repeat themselves over time at every single sampling period and because the prediction and optimization are done open loop, the stability and performance of the closed-loop behavior can differ from what is expected. To overcome this weakness, MPC applies the optimal control law obtained for only a single sampling period and repeat the open loop optimization

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problem all over again. The addition of constraints in the optimal control problem (OCP) and dynamic uncertainties considerably affect the stability and performance of MPC. The different robustness aspects expected from MPC algorithms are defined as follows [34]:

1) robust feasibility: the guarantee to, both, satisfy the constraints at each sampling instance despite the uncertainty and meet the terminal conditions. In other words,robust feasibility if the answer for the following question: starting from an initially feasible solution, can the algorithm guarantee the existence of a feasible control input sequence that constitutes the solution of the finite horizon optimal control problem at all subsequent time instances?
2) robust stability: the guarantee that the system remains stable despite uncertainties, constraints and disturbances. Thus, it is how to guarantee the stability of the system in closed loop based on the open-loop finite horizon optimal problem solution [34].
3) Robust (closed-loop) performance: the guarantee that the required performance specifications of the system in closed loop system are met regardless of the uncertainties and disturbances.

These three robustness aspects may influence each other. Study [7] states that MPC has zero-robustness to uncertainties if the optimal control problem (OCP) is constrained. Likewise, feasibility is not ensured by the nominal cost constrained minimization for future state vectors under any disturbance realization [4]. Therefore, if the system is uncertain then stability as well as feasibility may be lost when using nominal MPC making it necessary to include some knowledge about the uncertainty in the optimization problem [29]. Three main techniques exist for Robust MPC (RMPC) in the literature. The first is Min-Max optimization technique. In this approach (see for example [11], [18], [31]), the constrained finite horizon open loop OCP is solved at each time instant while considering the worst possible realizations of the uncertainty for any possible disturbances. This results in adopting a pessimistic control actions applied repeatedly. Beside, being a computationally taxing approach [4], it can also generate a high unjustified operational cost and cannot be implemented in real world applications ([25], [34]). The second technique can also be classified as an open loop approach and is originally introduced by [4]. The MPC (or nominal MPC) algorithm uses the nominal model in the optimization process but subject to tightened state constraints to guarantee the original constraints satisfaction for the uncertain system. Thus, if the perturbed system satisfies these tightened or constricted constraints then it will automatically satisfy the original constraints. The tightening of the constraints will increase with time due to the increase of the predicted uncertainty. This is also a conservative approach and leads to a large spread of trajectories but it is more computationally efficient and can be used for large systems (see for example [15], [16], [29]). The third technique is Tube-based MPC. It is a closed loop approach [26] used to ensure that the deviation of the actual state from the nominal state is smaller than the one obtained by open-loop nominal MPC. In Tube-MPC, the controller is composed of two terms:

- the nominal control input, which is the online solution of nominal finite horizon control problem when applying nominal MPC and
- an additive state feedback law which is computed off-line and guarantees that the real trajectory of the closed-loop system will be within an envelope for all possible trajectories called hyper-tube [27].

The center of the envelope is the nominal trajectory [29]. Several successful studies using Tube-MPC have been reported in various literature (see [6], [12], [32], [33], [9], [13], [21]). More recently, [38] addressed RMPC for continuous systems. In [1] and [41], tube-based RMPC for time varying system is presented. There are 508 publications since 2018, some with successful application of tube-based MPC (see [21], [36], [22]). Tube-based MPC includes constraint tightening in the MPC optimization problem and a simple constraint tightening formulation has been proposed in [12]. Most of the available literature either for nominal-MPC or Tube-based MPC have considered different constraint tightening approaches for systems with additive uncertainties ([12], [17]). In [30], the study addressed uncertainties which decay as function of the system states. In [28], an output feedback model predictive controller (MPC) with the integration of an extended state observer (ESO) is proposed for hydraulic systems. The experimental application shows an enhancement of the robusteness.

A. CONTRIBUTION

In this paper, we extend the work initially developed in [35] and consider the system to be affected by an external dynamic disturbance having its own exo-system dynamic model. Both states and disturbance measurements needed for the control action and also to initialize the prediction phase are considered imperfect and noisy. The robust MPC design takes into account the bounds of the disturbance, the measurement noise, and the dynamic uncertainty. Therefore, in the case of this work, the tightened constraints sets are more general. Asymptotic stability for MPC strategies can be proven in case of additive and vanishing disturbance (decaying with state) (see [23]), however, only ISpS can be guaranteed in case the uncertainties are non-vanishing [18]. Likewise, in this work, the uncertainties are also not decaying with the state and hence only ISpS can be considered. Therefore, Input-to-state practical stability (ISpS) framework is used to prove the stability of the proposed algorithm. Moreover, stability can also be guaranteed by imposing a terminal cost and terminal constraint set [24], and the stability margin of the system is established depending on the size of this terminal set [3]. Several approaches can be used to maximize the size of the terminal set (see for example [3], [42]). To determine recursive feasibility, one needs to find the size of the l-step controllable sets to the terminal region, $\mathcal{C}_l(X_f)$. A conservative set-based estimation of $\mathcal{C}_1(X_f)$ by iteratively computing convex inner approximations of $\mathcal{C}_1(X_f)$ is presented in [29], however, the sensitivity of the size of $\mathcal{C}_1(\cdot)$ with respect to the disturbance has not been taken into account. Another set-based approach [40] considers Minkowski differences of a collection of polytopes. This approach is applied to determine the one-step ahead robust invariant sets in the case of polytopic additive uncertainties. A similar approach solved as a mixed-integer feasibility program is presented in [10]. Set-based techniques are recognized to be conservative and provide only inner approximations. In this work, linear differential inclusions (LDI) approach is adopted taking into account the sensitivity of the size of $\mathcal{C}_1(\cdot)$ with respect to the disturbance. In addition, the paper provides the following contributions to the literature.

i. When using Lipschitz bounds, the literature mainly considers either dynamic uncertainty assuming perfect measurements [30] or measurement noise with perfect model [8]. The present work is an attempt to develop a unified framework to address both types of uncertainties. The non-additive disturbance is generated by an uncertain exo-system with an unknown input.
ii. The ISpS framework is extended to take into account the different types of uncertainties considered here.
iii. The prediction error grows exponentially. New upper and lower bounds for the size and rate of the prediction error value and rate are derived. In addition, constraint tightening for robust satisfaction of original constraint set in the presence of this variety of uncertainties is also a new development.
iv. A newly developed theorem guarantees recursive feasibility based on the size of the one-step ahead controllable set to the terminal region.
v. The RMPC algorithm is composed of offline and online procedures where two new algorithms for constraint tightening and online optimization are presented.

B. ORGANIZATION

The paper is organized as follows. Some preliminary definitions and a general ISpS result (Theorem 1) are introduced in Section II-A. The problem is stated in Section III-A. Subsequently, in Section III-B we derive explicit expressions bounding either the error level or its growth rate along the prediction horizon. In addition, we utilize the bounds to tighten the constraints. Then, conditions on recursive feasibility (Section and practical stability III-C) are derived. The terminal constraint set and its associated control law are optimized using PLDIs in Section III-D. The robust one-step controllability set to the terminal region is calculated in Section III-D, which is extended in Section III-E to find the robust output feasible set. Finally simulation results are presented in Section IV. The paper is concluded in Section V with recommendations for future work.

II. PROBLEM FORMULATION

A. NOTATION

Let $\mathbb{R}, \mathbb{R}_{\geq 0}, \mathbb{Z}, \mathbb{Z}_{\geq 0}$ denote real, non-negative real, integer and non-negative integer sets of numbers, respectively. For a set $A \subseteq \mathbb{R}^n$, the point to set distance from $\zeta \in \mathbb{R}$ to A is denoted by $\text{dist}(\zeta, A) \triangleq \inf\{|\ \eta - \zeta\ |, \eta \in A\}$, and if A is a closed set, its boundary is denoted by ∂A. The difference between two sets $A, B \subseteq \mathbb{R}^n$ is denoted by $A \backslash B \triangleq \{x : x \in A, x \notin B\}$.The Pontryagin (or Minkowski) difference between these sets is defined as $A \sim B \triangleq \{x \in \mathbb{R}^n : x + y \in A, \forall y \in B\}$, while the Minkowski sum of these two sets is defined as $A \oplus B \triangleq \{x + y \in \mathbb{R}^n\, \forall x \in A,\ y \in B\}$. Let L_2 Euclidean norm be denoted by $|\cdot|$. For a discrete-time series $\phi = [\phi_0, \phi_1, \ldots\ldots]^T$, we define $\|\phi\| \triangleq \sup_{l\geq 0}\{|\phi_l|\}$ and $\|\phi_{[t]}\| \triangleq \sup_{0\leq l\leq t}\{|\phi_l|\}$. We also use class $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$ comparison functions [37].

B. SYSTEM NOTATION AND PRELIMINARIES

Consider the discrete-time nonlinear system

$$x_{t+1} = f(x_t, u_t) \tag{1}$$

with $f(0, 0) = 0$, where $x_t \in \mathbb{R}^n$ is the state, and $u_t \in \mathbb{R}^p$ is the control input. Given a signal x, let $x_{t,t+N}$ denote the discrete-time realization of x_t within the period t to $t + N$. $\mathcal{I} : \mathbb{R} \to \mathbb{R}$ is the identity function, γ_1 and γ_2 by $\gamma_1 \circ \gamma_2$ is functional composition of γ_1 and γ_2. α^{-1} is the inverse function of α.

Definition 1: If $x_t \in \Xi, \forall t > t_0$ whenever $x_{t_0} \in \Xi$ and bounded input $w_t \in W$, then Ξ is called a Robust Positively Invariant (RPI) set.

Moreover,

Definition 2: if Ξ is compact, RPI and contains the origin as an interior point, the system (1) is said to be regionally Input-to-State practically Stable (ISpS) in Ξ for $x_0 \in \Xi$ and $w \in W$, if there exists $\mathcal{KL}$-function β, $\mathcal{K}$-function γ and constant $c > 0$ such that

$$|x_t| \leq \beta\left(|x_0|, t\right) + \gamma\left(||w||\right) + c \tag{2}$$

If $c \equiv 0$, then the system is said to be regionally Input-to-State Stable (ISS) in Ξ [37].

Definition 3: Function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is an ISpS Lyapunov function in Ξ, if for suitable functions $\alpha_{1,2,3}$, $\sigma_3 \in \mathcal{K}_\infty$, $\sigma_{1,2} \in \mathcal{K}$ and constants $\bar{c}, \bar{\bar{c}} > 0$, there exists a compact and RPI set Ξ and another set $\Omega \subset \Xi$ with origin as an interior-point (Ω is also RPI), such that the following conditions hold,

$$V(x_t, w_t) \geq \alpha_1(|x_t|), \quad \forall x_t \in \Xi \tag{3}$$

$$\begin{aligned} &V\left(f\left(x_t, w_t\right), w_{t+1}\right) - V\left(x_t, w_t\right) \\ &\leq -\alpha_2\left(|x_t|\right) + \sigma_1\left(|w_t|\right) + \sigma_2\left(|w_{t+1}|\right) + \bar{c}, \quad \forall x_t \in \Xi \end{aligned} \tag{4}$$

$$V\left(x_t, w_t\right) \leq \alpha_3\left(|x_t|\right) + \sigma_3\left(|w_t|\right) + \bar{\bar{c}}, \quad \forall x_t \in \Omega \tag{5}$$

ISS implies ISpS, but converse is not true, since an ISS system with 0−input, i.e. $w_k = 0, \forall k \geq 0$ implies asymptotic stability to the origin, while for an ISpS system, 0−input implies asymptotic stability to a compact set (ball of radius c) containing the origin. In this paper, the stability analysis will demonstrate that according to the proposed control approach, closed-loop dynamics is ISpS, not ISS, due to uncertainty resulting from data compression. We also state an important result in regional input-to-state practical stability.

Theorem 1 (Ref. [5]): If system $x_{t+1} = f(x_t, w_t)$ admits an ISpS-Lyapunov function in Ξ, then it is regionally ISpS and satisfies condition (2), with $\beta(r,s) \triangleq \alpha_1{}^{-1}(3\hat{\beta}(3\alpha_3(r), s))$, $\gamma(s) \triangleq \alpha_1{}^{-1}(3(\hat{\gamma}(3\sum_{i=1}^{3}\sigma_i(s)) + \hat{\beta}(3\sigma_3(s), 0)))$ and $c \triangleq \alpha_1{}^{-1}(3(\hat{\beta}(3(\bar{\bar{c}} + d), 0) + \alpha_1{}^{-1}\hat{\gamma}(\mu(3\bar{\bar{c}})) + \alpha_1{}^{-1}\hat{\gamma}(3\bar{c}))$, where μ, $\hat{\gamma} \in \mathcal{K}_\infty$ while $\hat{\beta} \in \mathcal{KL}$ for some $d > 0$.

III. ROBUST MODEL PREDICTIVE CONTROL

A. SETUP

The paper addresses NMPC of the following nonlinear discrete-time system

$$x_{t+1} = f(x_t, u_t, w_t),$$
$$w_{t+1} = g(w_t, \phi_t), \tag{6}$$

where states x_t, controls u_t and disturbance w_t belong to the following constrained convex sets: $x_t \in X \subset \mathbb{R}^n$, $u_t \in U \subset \mathbb{R}^m$, $w_t \in W \subset \mathbb{R}^p$, bounded by some maximum and minimum values. The dynamic of the system $f(.,.,.)$ and the dynamic of the disturbance are both uncertain. In addition, the states and disturbance are assumed to have both inaccurate estimates $\hat{x}_t$ and $\hat{w}_t$ respectively such that

$$\hat{x}_t = x_t + \eta_{x_t}, \quad \hat{x}_{t+1} = x_{t+1} + \eta_{x_{t+1}},$$
$$\hat{w}_t = w_t + \eta_{w_t}, \quad \hat{w}_{t+1} = w_{t+1} + \eta_{w_{t+1}}, \tag{7}$$

$\eta_{x_t} \in H_x \subset \mathbb{R}^n$ and $\eta_{w_t} \in H_w \subset \mathbb{R}^p$ are bounded random noise affecting the measurements of x_t and w_t respectively. ϕ is an unknown input vector. Nominal states and disturbance are denoted by $\tilde{x}$ and $\tilde{w}$, for which we have the following nominal models used for prediction.

$$\tilde{x}_{t+1} = \tilde{f}(\tilde{x}_t, u_t, \tilde{w}_t), \quad \tilde{w}_{t+1} = \tilde{g}(\tilde{w}_t), \tag{8}$$

Let

$$\Delta_x(x,u,w) = f(x_t, u_t, w_t) - \tilde{f}(x_t, u_t, w_t) \tag{9}$$

$$\Delta_w(w) = g(w_t, \phi_t) - \tilde{g}(w_t) \tag{10}$$

Therefore, equation 6 can be re-written as

$$\hat{x}_{t+1} = \tilde{f}(x_t, u_t, w_t) + \Delta_x(x,u,w) + \eta_{x_{t+1}},$$
$$\hat{w}_{t+1} = \tilde{g}(w_t) + \Delta_w(w) + \eta_{w_{t+1}}. \tag{11}$$

In the remaining of the paper we will use $\hat{x}_t$ and x_t as well as $\hat{w}_t$ and w_t interchangeably as x and w are accessible only through measurements. The cost function $J_t(x, u, w, N_c, N_p, k_f)$ is defined by

$$J_t(x,u,w,N_c,N_p,k_f) = \sum_{l=t+1}^{t+Nc-1}[h_l(x_l,u_l) + q_l(x_l,w_l)] + \sum_{l=t+Nc}^{t+Np-1}\left[h_l(x_l,k_f(x_l)) + q_l(x_l,w_l)\right] + h_f(x_{t+N_p}), \tag{12}$$

where N_p and N_c are prediction and control horizons. Cost function (12) consists of transition cost h_l, terminal cost h_f and robustness cost q_l. Control sequence $u_{t,t+N_p}$ consists of two parts, $u_{t,t+N_c-1}$ and $u_{t+N_c,t+N_p-1}$. The latter part is generated by a terminal control law $u_l = k_f(\tilde{x}_l)$ for $l \geq t + N_c$, while the former is finite horizon optimal control $u_{t,t+N_p}$ which is the solution of the OCP 1.

Problem 1: At every instant $t \geq 0$, given prediction and control horizons $N_p, N_c \in \mathbb{Z}_{\geq 0}$, auxiliary control $k_f(\tilde{x}) : \mathbb{R}^n \to \mathbb{R}$, state $\tilde{x}_t$ and disturbance $\tilde{w}_t$ measurements, find the optimal control sequence $u^0_{t,t+N_c-1}$, which minimizes the finite horizon cost (12)

$$u^0_{t,t+N_c-1} = \arg\min_{u_{t,t+N_c-1} \in U^{N_c}} J_t(\tilde{x}_{t,t+N_p}, \tilde{w}_{t,t+N_p}, u_{t,t+N_c-1}, N_c, N_p), \tag{13}$$

subject to nominal plant model and nominal exo-system model (8)*, tightened constraints $\tilde{x} \in \tilde{X}_{t,t+Np}$, and terminal state $\tilde{x}_{t+Np}$ is constrained to an invariant terminal set $X_f \in \tilde{X}_{t+Nc}$, i.e.*

$$\tilde{x}_{t+l} \in X_f, \quad \forall l = N_C, \ldots, N_P \tag{14}$$

The loop is closed by implementing only the first element of $u^0_{t,t+N_c-1}$ at each instant, such that the NLMPC implicit control law becomes

$$\Theta_t(\tilde{x}, \tilde{w}) = u^0_t(\tilde{x}_{t,t+N_p}, \tilde{w}_{t,tN_p}, N_p, N_c) \tag{15}$$

In the remaining of this paper,

- section III-B will go through al the details mentioned in Step 2 in Algortithm 1.
- Guarantees for recursive feasibility will be detailed in section III-C
- Determination of $\mathcal{C}_1(X_f)$ in Step 4 will covered in section III-D.
- Determination of X_{MPC} in Step 5 will covered in section III-E.
- Robust Stability of Algorithm 1 will be discussed in I.

B. CONSTRAINT TIGHTENING

The system under consideration contains several possible sources of uncertainty which is more representative to the real word than those in the literature (e.g. [17] and [30]). To ensure stability, an envelop similar to a growing tube will be determined. The dynamic bounds take into accounts all these type of uncertainties and unlike [12], they do not decay exponentially. For that, a bound on the growth on uncertainty will be determined first. Due to uncertainty, the constraint sets x and w are 'larger' than the constraint sets for $\tilde{x}$ and $\tilde{w}$, such that $\tilde{x}_t \in \tilde{X}_t \subset X$, $\tilde{w}_t \in \tilde{W}_t$. Moreover, the tightened constraint sets have the contractive property such that $\tilde{X}_{t+l} \subseteq \tilde{X}_{t+l+1}$ for $l \geq 2$. Since uncertainty grows with horizon length, one needs to assume that measurements and transition uncertainties are bounded (see Assumption 1). The standard MPC assumptions of initial feasibility [29] is also assumed.

Assumption 1:

i. Each prediction starts from the present state $\tilde{x}_{t|t} = \hat{x}_t = x_t + \eta_{x_t}$ and $\tilde{w}_{t|t} = \hat{w}_t = w_t + \eta_{w_t}$. where $\eta_{x_t} \triangleq |x_t - \tilde{x}_{t|t}| \leq \bar{\eta}_x$ and $\eta_{w_t} \triangleq |w_t - \tilde{w}_{t|t}| \leq \bar{\eta}_w$,
ii. The transition uncertainties $|\Delta_x(x, u, w)| \triangleq |\tilde{f}(x, u, w) - f(x, u, w)| \leq \bar{\Delta}_x$ and $|\Delta_w(w)| \triangleq |\tilde{g}(w) - g(w, \phi)| \leq \bar{\Delta}_w(w)$.
iii. The nominal maps are locally Lipschitz with constants L_{fx}, L_{fu}, L_{fw} and L_{gw} with respect to x, u and w, and $\tilde{f}(0, 0, 0) = 0$, $\tilde{g}(0) = 0$.
iv. A compact *robust output feasible set* $X_{MPC} \subseteq X$, defined as the set of initial states for which optimal control problem (Problem 1) is feasible, exists.

Lemma 1: Under Assumption 1, the l-step ahead prediction errors $\hat{\rho}_{x_{t+l|t}} \triangleq |x_{t+l} - \tilde{x}_{t+l|t}|$ *and* $\hat{\rho}_{w_{t+l|t}} \triangleq |w_{t+l} - \tilde{w}_{t+l|t}|$; $l = 1, \ldots, N_p$ *are bounded by*

$$
\begin{aligned}
\hat{\rho}_{w_{t+l|t}} &\leq \bar{\rho}_{w_{t+l|t}} \\
\bar{\rho}_{w_{t+l|t}} &= \bar{\eta}_w \frac{\left(L_{gw}^{l+1} - 1\right)}{\left(L_{gw} - 1\right)} + \bar{\Delta}_w \frac{\left(L_{gw}^{l} - 1\right)}{\left(L_{gw} - 1\right)}, \\
\hat{\rho}_{x_{t+l|l}} &\leq \hat{\rho}_{x_{t+l|l}} \\
\bar{\rho}_{x_{t+l|l}} &= \bar{\eta}_x \frac{L_{fx}^{l+1} - 1}{L_{fx} - 1} + \bar{\Delta}_x \frac{L_{fx}^{l} - 1}{L_{fx} - 1} \\
&\quad + \bar{\eta}_w \frac{L_{fw}}{L_{gw} - 1} \left(L_{gw} \frac{L_{fx}^l - L_{gw}^l}{L_{fx} - L_{gw}} - \frac{L_{fx}^l - 1}{L_{fx} - 1} \right) \\
&\quad + \bar{\Delta}_w \frac{L_{fw}}{L_{gw} - 1} \left(\frac{L_{fx}^l - L_{gw}^l}{L_{fx} - L_{gw}} - \frac{L_{fx}^l - 1}{L_{fx} - 1} \right), \quad (16)
\end{aligned}
$$

for $l = 0, \ldots, N_P$, and $L_{fx}, L_{gw} \neq 1$ and $L_{fx} \neq L_{gw}$.

Proof: From Assumption 1, for $l = 1$

$$
\begin{aligned}
|w_{t+1} - \tilde{w}_{t+1|t}| &= \left|\tilde{g}(w_t) + \Delta_w(w) + \eta_{w_{t+1}} - \tilde{g}\left(\tilde{w}_{t|t}\right)\right|, \\
&\leq \left|\tilde{g}\left(w_t\right) - \tilde{g}\left(\tilde{w}_{t|t}\right)\right| + \bar{\Delta}_w + \bar{\eta}_w, \\
&\leq L_{gw}\left|w_t - \tilde{w}_{t|t}\right| + \bar{\Delta}_w + \bar{\eta}_w, \\
&\leq \left(L_{gw} + 1\right)\bar{\eta}_w + \bar{\Delta}_w.
\end{aligned}
$$

Using similar arguments, $|w_{t+l} - \tilde{w}_{t+l|t}|$ can be written for $l \geq 1$

$$
|w_{t+l} - \tilde{w}_{t+l|t}| \leq \bar{\eta}_w \left(\sum_{k=1}^{k=l+1} L_{gw}^{k-1} \right) + \bar{\Delta}_w \left(\sum_{k=1}^{k=l} L_{gw}^{k-1} \right),
$$

$$
|w_{t+l} - \tilde{w}_{t+l|t}| \leq \bar{\eta}_w \frac{\left(L_{gw}^{l+1} - 1\right)}{\left(L_{gw} - 1\right)} + \bar{\Delta}_w \frac{\left(L_{gw}^{l} - 1\right)}{\left(L_{gw} - 1\right)}.
$$

This proves the first part of (16). For the states, at $l = 1$,

$$
\begin{aligned}
|x_{t+1} - \tilde{x}_{t+1|t}| &= |\tilde{f}(x_t, u_t, w_t) + \Delta_x(x, u, w) + \eta_{x_{t+1}} \\
&\quad - \tilde{f}\left(\tilde{x}_t, u_t, \tilde{w}_t\right)| \\
&\leq |\tilde{f}\left(x_t, u_t, w_t\right) - \tilde{f}\left(\tilde{x}_{t|t}, u_t, \tilde{w}_{t|t}\right)| \\
&\quad + \bar{\Delta}_x + \bar{\eta}_x \\
&\leq \left(L_{fx} + 1\right)\bar{\eta}_x + L_{fw}\bar{\eta}_w + \bar{\Delta}_x.
\end{aligned}
$$

When $l = 2$,

$$
\begin{aligned}
|x_{t+2} - \tilde{x}_{t+2|t}| &\leq L_{fx}|x_{t+1} - \tilde{x}_{t+1|t}| \\
&\quad + L_{fw}|w_{t+1} - \tilde{w}_{t+1|t}| \\
&\quad + \bar{\eta}_x + \bar{\Delta}_x \\
&\leq \left(L_{fx}^2 + L_{fx} + 1\right)\bar{\eta}_x \\
&\quad + \bar{\Delta}_x\left(L_{fx} + 1\right) \\
&\quad + \bar{\eta}_w L_{fw}\left(L_{fx} + L_{gw} + 1\right) + L_{fw}\bar{e}_w
\end{aligned}
$$

Using similar derivation, *l-step ahead* prediction error can be written as

$$
\begin{aligned}
|x_{t+l} - \tilde{x}_{t+l|t}| &\leq \bar{\eta}_x \sum_{k=0}^{k=l} L_{fx}^k + \bar{\Delta}_x \sum_{k=0}^{k=l-1} L_{fx}^k \\
&\quad + \bar{\eta}_w L_{fw} L_{fx}^{l-1} \sum_{k=0}^{k=l-1} \frac{\left(\sum_{j=0}^{j=k} L_{gw}^j\right)}{L_{fx}^k} \\
&\quad + \bar{\Delta}_w L_{fw} L_{fx}^{l-1} \sum_{k=0}^{k=l-1} \frac{\left(\sum_{j=0}^{j=k-1} L_{gw}^j\right)}{L_{fx}^k} \quad (17)
\end{aligned}
$$

Using nested geometric series' sum will lead to the second part of (16). □

In Lemma 1, L_{fx} and L_{gw} are assumed $\neq 1$ and $L_{fx} \neq L_{gw}$. However, the result for the case where $L_{fx}, L_{gw} = 1$ and $L_{fx} = L_{gw}$ is straight forward. Bounds (16) are conservative due to the use of the Lipschitz constants and methods suggested in [17] to reduce this conservatism can be used. A lower bound on rate of convergence of the prediction error can be established in the following claim.

Claim 1: There exists a terminal control $u_t = k_f(\tilde{x}_l) \in U$, $l = N_C, \ldots, N_P - 1$ *such that the solution* X_f *of the nominal plant dynamic given in* (8) *is robust positively invariant (RPI) i.e.* $x_l \in X_f$ *and* $\tilde{x}_l \in X_f$, $\forall l = t+N_c+1, \ldots, t+N_p$ *for any* $\tilde{x}_{t+N_c} \in X_f$, *such that,*

i. *The rate of convergence* $\delta_{\rho_{l+1}} \triangleq |\tilde{f}(\tilde{x}_{t+l|t+1}, k_f(\tilde{x}_{t+l|t+1}, 0)) - \tilde{x}_{t+l+1|t}|$ *of nominal state under control* $k_f(\tilde{x})$ *is* *lower bounded by*

$$\delta_{\rho_{l+1}} \geq \bar{\delta}_{\rho_{l+1}}$$
$$\bar{\delta}_{\rho_{l+1}} = \bar{\eta}_x L_{fx}^{l+1} + L_{fx}^{l}\left(\bar{\Delta}_x + (\bar{\eta}_w + \bar{\Delta}_w)\frac{L_{fw}}{L_{gw}-1}\right) + \left(\bar{\eta}_w L_{gw} + \bar{\Delta}_w\right)\frac{L_{fw}}{L_{gw}-1} \times \left(\frac{L_{fx}^{l}(L_{fx}-1) - L_{gw}^{l}(L_{gw}-1)}{L_{fx}-L_{gw}}\right) \quad (18)$$

for $l = N_c - 1 \ldots N_p - 2$

ii. *There exists* $a \in \mathbb{Z}_{\geq 0}$ *and* $0 \leq Q_f \in \mathbb{R}^{n\times n}$ *such that*

$$\tilde{x}^T Q_f \tilde{x} \leq a, \quad \forall \tilde{x} \in X_f \quad (19)$$

Proof:

i. Using the definition of the prediction error $\hat{\rho}_{t+l|t} = x_{t+l} - \tilde{x}_{t+l|t}$. $\delta_{\rho_{l+1}} \triangleq |\tilde{f}\left(\tilde{x}_{t+l|t+1}, k_f(\tilde{x}_{t+l|t+1}, \tilde{w}_{t+l|t+1})\right) - \tilde{x}_{t+l+1|t}|$ can be written as $\delta_{\rho_{l+1}} \triangleq |\tilde{x}_{t+L+1|t+1} - \tilde{x}_{t+l+1|t}|$. Adding and subtracting x_{t+l+1} leads to $\delta_{\rho_{l+1}} \geq \bar{\rho}_{x_{t+l+1|t}} - \bar{\rho}_{x_{t+l+1|t+1}}$. Using Lemma 1, the proof is complete.

ii. The proof for part ii) and the existence of $k_f(\tilde{x}_l) \in U$, $l = N_C, \ldots, N_P - 1$ is typical and based on linear theory analysis using a lineraized version of the nonlinear model. Similar claim have been made in the litterature (see for example [29] or [35]). It is omitted in this paper for the benefit of the reader and a better management of the paper length.

□

By taking into account the established prediction uncertainty bounds and study their ipact on the constrained FHOCP, it is possible to guarantee the admissibility of the state/output dynamic behavior of the actual system.

Theorem 2: With nominal constraints X and W on the system (6), *let the* `tightened constraints` *for nominal model* (8) *be given by*

$$\tilde{X}_{t+l} \triangleq X \sim \mathcal{B}^n\left(\bar{\rho}_{x_{t+l}}\right), \quad (20)$$

for $l = 0, \ldots, N_P$, *and* $\bar{\rho}_x$ *and* $\bar{\rho}_w$ *are prediction error bounds defined in* (16). $\mathcal{B}^n(\bar{\rho})$ *is an n-ball centered at the origin and with a radius* ($\bar{\rho}$. *Then, any admissible control sequence* $[u_{t,t+Nc-1}, k_f(\tilde{x}_t + Nc, t + Np - 1)]$ *which is feasible with respect to tightened constraints* (20), *guarantees the satisfaction of original constraints, i.e.* $x_{t+l} \in X$ *for* $w_{t+l} \in W$, $l = 0, \ldots, N_P$ *and* $x_t \in X_{MPC}$.

Proof: To prove the theorem, one needs to proceed in two steps:

1) Let $\eta'_x < \bar{\eta}_x$, $\eta'_w < \bar{\eta}_w$, $\Delta'_x(x,u,w) \leq \bar{\Delta}'_x < \bar{\Delta}_x$ and $\Delta'_w(w) \leq \bar{\Delta}'_w < \bar{\Delta}_w$. Then, following the same procedure as in proof of Lemma 1, one can find that $|x_{t+l} - \tilde{x}_{t+l}| \leq \rho'_{x_{t+l}} \triangleq \bar{\eta}'_x \frac{L_{fx}^{l+1}-1}{L_{fx}-1} + \bar{\Delta}'_x \frac{L_{fx}^{l}-1}{L_{fx}-1} + \bar{\eta}'_w \frac{L_{fw}}{L_{gw}-1}\left(L_{gw}\frac{L_{fx}^{l}-L_{gw}^{l}}{L_{fx}-L_{gw}} - \frac{L_{fx}^{l}-1}{L_{fx}-1}\right) + \bar{\Delta}'_w \frac{L_{fw}}{L_{gw}-1}\left(\frac{L_{fx}^{l}-L_{gw}^{l}}{L_{fx}-L_{gw}} - \frac{L_{fx}^{l}-1}{L_{fx}-1}\right)$ Since $\tilde{x}_{t+l} \in \tilde{X}_{t+l}$, therefore $x_{t+l} \in \tilde{X}_{t+l} \oplus \mathcal{B}^n\left(\rho'_{x_{t+l}}\right)$. Comparing with (16) and using (17), $\rho'_{x_{t+l}} < \bar{\rho}_{x_{t+l}}$ and hence $\mathcal{B}^n\left(\rho'_{x_{t+l}}\right) \subset \mathcal{B}^n\left(\bar{\rho}_{x_{t+l}}\right)$. This leads to $x_{t+l} \in \tilde{X}_{t+l} \oplus \mathcal{B}^n\left(\bar{\rho}_{x_{t+l}}\right)$. On the other hand, based on (20), one can state that $X = \tilde{X}_{t+l} \oplus \mathcal{B}^n\left(\bar{\rho}_{x_{t+l}}\right)$ and $x_{t+l} \in X$, for $l = 0, \ldots, N_C$.

2) This part is similar to what was reported in the literature (see for example [29]). The sektech of the proof is as follows: Consider the prediction $\tilde{x}_{t+l|t}$ and $\tilde{x}_{t+l|t+1}$ made using the control sequence $u_{t,t+N_c-1|t} = k_f(\tilde{x}_{t,t+N_c-1|t})$ and $u_{t,t+N_c-1|t} = k_f(\tilde{x}_{t,t+N_c-1|t+1})$ respectively and with the initial conditions $\tilde{x}_{t|t} = \hat{x}_t = x_t + \eta_t$, $\tilde{x}_{t+1|t+1} = \tilde{f}(\tilde{x}_{t|t}, k_f(\tilde{x}_{t|t}, \tilde{w}_{t|t}) = x_t + \eta_t$ for $l = N_c \ldots N_p - 1$. Assuming $\tilde{x}_{t+l|t} \in X \sim \mathcal{B}^n(\bar{\rho}_{t+l|t})$. Let $\xi = \hat{x}_{t+l|t+1} - \hat{x}_{t+l|t} + \epsilon$ where $\epsilon \in \mathcal{B}^n(\bar{\rho}_{t+l|t+1})$. Using (16), $|\xi| \leq \delta_{\rho_{t+l|t}} + \rho_{t+l|t+1}$ and in vue of (18), $|\xi| \leq \rho_{t+l|t}$. This leads to $\hat{x}_{t+l|t+1} \in X \sim \mathcal{B}^n(\bar{\rho}_{t+l|t+1})$. □

□

C. RECURSIVE FEASIBILITY OF NMPC ALGORITHM

This section addresses the important issue of recursive feasibility. Let the one-step controllability set towards the terminal constraint set X_f be defined as $\mathcal{C}_1(X_f, \tilde{X}_{t+Nc}) \triangleq \left\{\tilde{x}_t \in \tilde{X}_{t+Nc} : \tilde{f}(\tilde{x}_t, u_t, \tilde{w}_t) \in X_f, \ \ u_t \in U, \tilde{w}_t \in \tilde{W}_t\right\}$. Let us also define the minimum size of this set as $\bar{d} \triangleq \mathrm{dist}(\tilde{X}_{t+Nc} \backslash \mathcal{C}_1(X_f, \tilde{X}_{t+Nc}), X_f)$.

Theorem 3: Under Assumption 1, terminal control (Claim 1) and tightened constraints (Theorem 2), given the feasibility of initial state $\tilde{x}_t \in X_{MPC}$, *Problem 1 is recursively feasible, if the minimum size of* $\mathcal{C}_1(X_f)$ *upper bounds the uncertainties as follows*

$$\left(\begin{array}{l} L_{fx}^{Nc-1}\left(\left(L_{fx}+1\right)\bar{\eta}_x + L_{fw}\bar{\eta}_w + \bar{\Delta}_x\right) + \\ L_{fw}\frac{L_{fx}^{Nc-1}-L_{gw}^{Nc-1}}{L_{fx}-L_{gw}}\left(\left(L_{gw}+1\right)\bar{\xi}_w + \bar{\Delta}_w\right)\end{array}\right) \leq \bar{d} \quad (21)$$

Proof: The sketch of the proof is presented here. First, one needs to show that starting at a time instant t from an initially feasible state $\tilde{x}_t \in X_{MPC}$ and a feasible control input u_t, a feasible control input solution of the FHOCP (1) for $t+1$ can be found. This will lead to the determination of the control input sequence for the entire horizon. Assume that at t, a feasible control $u^0_{t,t+N_c-1|t}$ exists. Let $u'_{t+1,t+N_c|t+1} = \mathtt{col}\,[u^0_{t+1,t+Nc-1|t}, u'_{t+N_c|t}]$ be a possible feasible control sequence for $t+1, \ldots, t+N_c$. The objective is to find (a) a feasible $u'_{t+1,t+N_c-1|t+1}$ for the progressive tightened constraints (20) for $t+1, \ldots, t+N_c-1$, and (b) $u'_{t+N_c|t+1}$ at $t+N_c$.

(a) Given $\tilde{x}_{t|t}$ and $\tilde{w}_{t|t}$ at t, assume the existence of a feasible control $u^0_{t,t+Nc-1|t}$ with state and disturbance predictions $\tilde{x}_{t,t+Nc|t}$ and $\tilde{w}_{t,t+Nc|t}$ respectively. At $t+1$, new predictions generated using $u'_{t+1,t+Nc-1|t+1} = u^0_{t+1,t+Nc-1|t}$ are $\tilde{x}_{t+1,t+Nc|t+1}$ and $\tilde{w}_{t+l,t+Nc|t+1}$. Using Assumption 1, $\tilde{x}_{t+l|t+1} - \tilde{x}_{t+l|t} \leq L_{fx}|\tilde{x}_{t+l-1|t+1} - \tilde{x}_{t+l-1|t}| + L_{fw}|\tilde{w}_{t+l-1|t+1} - \tilde{w}_{t+l-1|t}|$. By induction, it follows that

$$\tilde{x}_{t+l|t+1}-\tilde{x}_{t+l|t} \le L_{fx}^{l-1}\left|\tilde{x}_{t+1|t+1}-\tilde{x}_{t+1|t}\right| + L_{fw}\frac{L_{fx}^{l-1}-L_{gw}^{l-1}}{L_{fx}-L_{gw}}\left|\tilde{w}_{t+1|t+1}-\tilde{w}_{t+1|t}\right|, \quad (22)$$

for $l = 1,\ldots,N_c-1$. Let $\delta_{\tilde{x}_1} \triangleq |\tilde{x}_{t+1|t+1}-\tilde{x}_{t+1|t}|$ and $\delta_{\tilde{w}_1} \triangleq |\tilde{w}_{t+1|t+1}-\tilde{w}_{t+1|t}|$. Now, based on the assumption that $\tilde{x}_{t+l|t} \in \tilde{X}_{t+l|t} \triangleq X \sim \mathcal{B}^n(\bar{\rho}_{x_{t+l|t}})$, and using (22), one gets $\tilde{x}_{t+l|t+1} \in X \sim \mathcal{B}^n\left(\bar{\rho}_{x_{t+l|t}}\right) \oplus \mathcal{B}^n\left(L_{fx}^{l-1}|\delta_{\tilde{x}_1}| + L_{fw}\frac{L_{fx}^{l-1}-L_{gw}^{l-1}}{L_{fx}-L_{gw}}|\delta_{\tilde{w}_1}|\right)$. On the other hand, $\tilde{x}_{t+l,t+Nc|t+1} \in \tilde{X}_{t+l|t+1} \triangleq X \sim \mathcal{B}^n(\bar{\rho}_{x_{t+l|t+1}})$ requires that $\bar{\rho}_{x_{t+l|t+1}} \le \bar{\rho}_{x_{t+l|t}} - L_{fx}^{l-1}|\delta_{\tilde{x}_1}| + L_{fw}\frac{L_{fx}^{l-1}-L_{gw}^{l-1}}{L_{fx}-L_{gw}}|\delta_{\tilde{w}_1}|$. In view of $\bar{\rho}_{x_{t+l|t+1}} = \bar{\rho}_{x_{t+l-1|t}}$, it turns out that $\bar{\rho}_{x_{t+l|t}} - \bar{\rho}_{x_{t+l-1|t}} \ge L_{fx}^{l-1}|\delta_{\tilde{x}_1}| + L_{fw}\frac{L_{fx}^{l-1}-L_{gw}^{l-1}}{L_{fx}-L_{gw}}|\delta_{\tilde{w}_1}|$. Using triangle inequality leads to $\bar{\rho}_{x_{t+l|t}} - \bar{\rho}_{x_{t+l-1|t}} \ge L_{fx}^{l-1}\left|\tilde{x}_{t+1|t+1}-x_{t+1}\right| + L_{fw}\frac{L_{fx}^{l-1}-L_{gw}^{l-1}}{L_{fx}-L_{gw}}\left|\tilde{w}_{t+1|t+1}-w_{t+1}\right|$. Knowing that $0 \le \left|\tilde{x}_{t+1|t+1}-x_{t+1}\right|$ and $0 \le \left|\tilde{w}_{t+1|t+1}-w_{t+1}\right|$, one gets $\bar{\rho}_{x_{t+l|t}} \ge \bar{\rho}_{x_{t+l-1|t}}$, and hence, $\tilde{x}_{t+l|t+1} \in X \sim \mathcal{B}^n(\bar{\rho}_{x_{t+l-1|t}}) \subseteq X \sim \mathcal{B}^n(\bar{\rho}_{x_{t+l|t+1}}) \overset{\delta}{=} \tilde{X}_{t+l|t+1}$ for $l = 1,\ldots,N_c$.

(b) To prove $\tilde{x}_{t+Nc+1|t+1} \in X_f \subseteq \tilde{X}_{t+Nc+1|t+Nc+1}$: inequality (22) is written as $\tilde{x}_{t+Nc|t+1} - \tilde{x}_{t+Nc|t} \le L_{fx}^{Nc-1}\left(\eta_{x_{t+1}}+\eta_{x_t}\right) + L_{fw}\frac{L_{fx}^{Nc-1}-L_{gw}^{Nc-1}}{L_{fx}-L_{gw}}\left(\eta_{w_{t+1}}+\eta_{w_t}\right)$. The upper bounds on the accumulated prediction errors are given by (16), therefore one has $\tilde{x}_{t+Nc|t+1} - \tilde{x}_{t+Nc|t} \le L_{fx}^{Nc-1}\left|(L_{fx}+1)\bar{\eta}_x + L_{fw}\bar{\eta}_w + \bar{\Delta}_x\right| + L_{fw}\frac{L_{fx}^{Nc-1}-L_{gw}^{Nc-1}}{L_{fx}-L_{gw}}\left|(L_{gw}+1)\bar{\eta}_w + \bar{\Delta}_w\right|$. In view of (21), it follows that $\tilde{x}_{t+Nc|t+1} - \tilde{x}_{t+Nc|t} \le \bar{d}$. Using $\tilde{x}_{t+Nc|t} \in X_f$, leads to $\tilde{x}_{t+Nc|t+1} \in X_f \oplus \mathcal{B}^n(\bar{d}) \triangleq \mathcal{C}_1(X_f, X)$. □

D. DETERMINATION OF $\mathcal{C}_1(X_f)$

In this section, a method is presented to determine the minimum size $\bar{d}$ of $\mathcal{C}_1(X_f)$ as given by (21). To find $\bar{d}$, the topology of $\mathcal{C}_1(X_f)$ should be known. The proposed approach uses min-max optimization to find the estimate of $\mathcal{C}_1(X_f)$. Let $\partial(X_f)$ and $\partial\left(\mathcal{C}_1(X_f)\right)$ be the boundaries of X_f and $\mathcal{C}_1(X_f)$ respectively.

Problem 2: Given target set X_f and tightened constraints (20), if the boundary of X_f is discretized appropriately into $\bar{N}$ point $\tilde{x}_f^i \in \partial(X_f)$ for $i = 1,\ldots,\bar{N}$, then $\mathcal{C}_1(X_f)$ is the solution of the following $\bar{N}$ min-max OCPs

$$\tilde{x}_{c_1}^i = \max_{\tilde{w}}\left(\min_u\left(-\log\left(\tilde{x}_{c_1}^i Q_f \tilde{x}_{c_1}^i\right)\right)\right) \quad (23)$$

for $i = 1,\ldots,\bar{N}$, subject to

$$\tilde{x}_f^i = \tilde{f}(\tilde{x}_{c_1}^i, u, \tilde{w}) \quad (24)$$

$$1 - \tilde{x}_{c_1}^i Q_f \tilde{x}_{c_1}^i \le 0 \quad (25)$$

$$\tilde{x}_{c_1}^i \in \tilde{X}_{Nc-1}, \quad u \in U, \ \tilde{w} \in \tilde{W}_{Nc-1} \quad (26)$$

Let $\partial\left(\mathcal{C}_1(X_f)\right)$ denote the boundary of $\mathcal{C}_1(X_f)$, it follows that $\partial\left(\mathcal{C}_1(X_f)\right) = \left\{\tilde{x}_{c_1}^i, \ \forall i = 1,\ldots,\bar{N}\right\}$.

Remark 1:
- *The cost functional (23) is convex by design. However, the overall OCP may not be convex due to the presence of the nonlinear constraints (24)(25). To overcome this challenge and mainly avoid local minima, a prior knowledge of an adequate initial guess is required to converge to a feasible solution. (25) ensures that is outside X_f.*
- *while computationally expensive, the determination of X_f runs offline, therefore the complexity in terms of time and memory is not of great concern.*

E. DETERMINATION OF X_{MPC}

In this section, the entire feasibility region of the MPC algorithm will be presented. The following propositions are needed and some of them will be stated without proof as they are straightforward.

Proposition 1: $X_f \subset \mathcal{C}_1(X_f)$.

Proof: X_f is RPI if and only if $X_f \subset \mathcal{C}_1(X_f)$ (see for example [19]). and since X_f is RPI, therefore Proposition 1 follows. □ □

Proposition 2: $\mathcal{C}_1(X_f)$ is contained in the robust output feasible set X_{MPC}, i.e. $\mathcal{C}_1(X_f) \subseteq \mathcal{C}_1(X_{MPC})$

Proof: According to [10], $\Omega \subseteq \bar{\Omega} \implies \mathcal{C}_1(\Omega) \subseteq \mathcal{C}_1(\bar{\Omega})$. It follows that $X_f \subset X_{MPC}$, and hence $\mathcal{C}_1(X_f) \subseteq \mathcal{C}_1(X_{MPC})$. □

Proposition 3: $\mathcal{C}_1(X_f)$ is a finite union of polyhedra.

Proof: Given X_f is convex (RPI) and $X_f \subset \mathcal{C}_1(X_f)$ according to Proposition 1, $\mathcal{C}_1(X_f)$ is compact and is the union of intersecting polyhedra. □

Proposition 4: $\mathcal{C}_1(X_f)$ contains the boundary of the terminal set, $\partial(X_f)$, i.e. $\partial(X_f) \subset \mathcal{C}_1(X_f)$.

The following main theorem can now be stated

Theorem 4: Given the terminal set X_f, the tightened constraints (20) and the control $u \in U$, X_{MPC} is obtained by recursively applying the one-step controllable set operator $\mathcal{C}_1(\cdot)$ to solve the OCP 2 as follows

$$X_{MPC} = \bigcup_{l=2}^{l=Nc} \mathcal{C}_1\left(\mathcal{C}_{l-1}(X_f)\right) \cup \mathcal{C}_1\left(X_f\right) \cup X_f \quad (27)$$

Proof: One can see that $\mathcal{C}_2(X_f) = \mathcal{C}_1\left(\mathcal{C}_1(X_f)\right)$ and $\mathcal{C}_3(X_f) = \mathcal{C}_1\left(\mathcal{C}_2(X_f)\right)$ which can be written as $\mathcal{C}_1(X_f) = \mathcal{C}_1\left(\mathcal{C}_1\left(\mathcal{C}_1(X_f)\right)\right)$. Inference leads to $X_{MPC} = \mathcal{C}_1\left(\mathcal{C}_{N_C-1}(X_f)\right) = \ldots = \mathcal{C}_1\left(\mathcal{C}_1\left(\ldots\mathcal{C}_1(X_f)\right)\right)$. Similarly, from Proposition 4, it follows that $\partial(X_f) \subset \mathcal{C}_1(X_f)$. Hence, $\partial\left(\mathcal{C}_{N_C-1}(X_f)\right) \subset \mathcal{C}_1\left(\mathcal{C}_{N_C-1}(X_f)\right)$ and $\partial\left(\mathcal{C}_{N_C-2}(X_f)\right) \subset \mathcal{C}_1\left(\mathcal{C}_{N_C-2}(X_f)\right),\ldots,\partial(X_f) \subset \mathcal{C}_1(X_f)$. But, since $X_{MPC} = \mathcal{C}_1\left(\mathcal{C}_{N_C-1}(X_f)\right)$, it follows that $X_{MPC} = \mathcal{C}_1\left(\mathcal{C}_{N_C-1}(X_f)\right) \cup \mathcal{C}_1\left(\mathcal{C}_{N_C-2}(X_f)\right) \cup \ldots \mathcal{C}_1(X_f) \cup X_f$. □

While computationally demanding, all these calculations are also offline.

F. ROBUST STABILITY

In this work, the uncertainties η_x and η_w are non-vanishing and only ISpS can be proven as explained previously. In order to state the main theorm characterizing ISpS of the proposed control strategy, the following assumptions and properties are introduced.

Assumption 2 (Cost Lipschitz Continuity): i. The cost functions $k_f(\cdot)$, $h(\cdot)$, $q(\cdot)$ and $h_f(\cdot)$ are nonlinear, continuous and locally Lipschitz with constants L_{kf}, L_{hx}, L_{hu}, L_{qx}, L_{qw} and L_{hf}.

ii. $|k_f(\tilde{x})| \leq L_{k_f}|\tilde{x}|$, for $\tilde{x} \in \tilde{X}_{f_t}$

iii. $|h(\tilde{x}, u)| \leq L_{hx}|\tilde{x}| + L_{hu}|u|$, for $\tilde{x} \in \tilde{X}_t$ and $u \in U$

iv. $0 \leq |q(\tilde{x}, \tilde{w})| \leq L_{qx}|\tilde{x}| + L_{qw}|\tilde{w}|$, for $x \in \tilde{X}_t$ and $w \in \tilde{W}_t$

v. $|h_f(\tilde{x})| \leq L_{hf}|\tilde{x}|$ for $\tilde{x} \in \tilde{X}_{f_t}$

vi. $\alpha_1(|\tilde{x}_t|) \leq h(\tilde{x}_t, u_t)$, for $\tilde{x} \in \tilde{X}_t$.

vii. $\alpha_{1,f}(|\tilde{x}_t|) \leq h_f(\tilde{x}_t) \leq \alpha_{2,f}(|\tilde{x}_t|)$, for all $\tilde{x}_t \in \tilde{X}_t$,

Assumption 3 (Technical): Based on the definition of ISpS Lyapunov functions given in Section II-A, the following holds,

(i). $\alpha_1(s) = \alpha_2(s) \triangleq \min h(s, 0)$

(ii). $\alpha_3(s) \triangleq \alpha_{2,f}(L_{fx}^{N_p} s) + c_1\, s$

(iii). $\sigma_1(s) \triangleq (c_2 + c_3 + c_4 + c_5) L_{fw}\left(L_{gw} s\right)$

(iv). $\sigma_2(s) \triangleq \sigma_1\left(L_{gw}{}^{-1} s\right) + \psi(s)$

(v). $\sigma_3(s) \triangleq L_{qw}\left(L_{gw} - 1\right)^{-1}\left(\left(L_{gw}\right)^{N_p} - 1\right) s$

(vi). $\bar{c} \triangleq c_6 c_7 + L_{hu}\left|u_{\max} - u_{\min}\right|$

(vii) $\bar{\bar{c}} = 0$,

where, $c_1 = \left(L_{hx} + L_{hu}L_{kf} + L_{qx}\right)\frac{(L_{fx})^{N_p}-1}{L_{fx}-1}$, $c_2 = \frac{L_{qx}+L_{hx}}{L_{fx}-L_{gw}}\left(\frac{L_{fx}^{Np-1}-1}{L_{fx}-1} - \frac{L_{gw}^{Np-1}-1}{L_{gw}-1}\right)$, $c_3 = \frac{L_{gw}^{Np-1}-1}{L_{gw}-1}\frac{L_{qw}}{L_{fw}}$, $c_4 = \frac{L_{hu}L_{kf}}{L_{fx}-L_{gw}}\left(\frac{L_{fx}^{Np-1}-L_{fx}^{Nc}}{L_{fx}-1} - \frac{L_{gw}^{Np-1}-L_{gw}^{Nc}}{L_{gw}-1}\right)$, $c_5 = \frac{L_{fx}^{Np-1}-L_{gw}^{Np-1}}{L_{fx}-L_{gw}}$, $c_6 = \bar{\eta}_x + L_{fx}\bar{\eta}_x + L_{fw}\bar{\eta}_w + \bar{\Delta}_x$ and $c_7 = \frac{L_{fx}^{Np-1}-1}{L_{fx}-1}\left(L_{qx} + L_{hx}\right) + L_{hu}L_{kf}\frac{L_{fx}^{Np-1}-L_{fx}^{Nc}}{L_{fx}-1} + L_{fx}^{Np-1}$ The following Theorem presents the main stability result.The proof follows the standard approach used in the literature

*Theorem 5: Let $\tilde{X}_f \subset \tilde{X}$, $k_f(x)$ as defined in Claim **??**, and $\psi \in \mathcal{K}$. In vue of Assumptions 1-2 and under NMPC optimal control* (15) *and cost* (12)*, if the following condition holds for all $\tilde{x} \in X_f$ and $\tilde{w} \in \tilde{W}$,*

$$h_f\left(\tilde{f}\left(\tilde{x}, k_f(\tilde{x})\right)\right) - h_f\left(\tilde{x}\right) \leq -h(\tilde{x}, k_f(\tilde{x})) - q(\tilde{x}, \tilde{w}) + \psi\left(|\tilde{w}|\right), \tag{28}$$

then $V_t \triangleq J_t(\tilde{x}_{t,t+N_p}, u^0_{t,t+N_p}, \tilde{w}_{t,t+N_p})$ is an ISpS Lyapunov function and the nominal system (8) *is ISpS for all initial states within $X_{MPC} \subseteq X$.*

Proof: From (12),it holds that $V_{t|t} = h(\tilde{x}_{t|t}, u^o_{t|t}) + q(\tilde{x}_{t|t}, \tilde{w}_{t|t}) + \sum_{l=t+1}^{t+Nc-1}\left[h(\tilde{x}_{l|t}, u^o_{l|t}) + q(\tilde{x}_{l|t}, \tilde{w}_{l|t})\right] + \sum_{l=t+Nc}^{t+Np-1}\left[h(\tilde{x}_{l|t}, u^o_{l|t}) + q(\tilde{x}_{l|t}, \tilde{w}_{l|t})\right] + h_f(\tilde{x}_{t+Np|t})$. Using (Assumption 2), it turns out that $\alpha_1(|\tilde{x}_{t|t}|) \leq V_{t|t}$ for $\tilde{x}_{t|t} \in \tilde{X}_{t|t} \supseteq X, \tilde{w}_{t|t} \in \tilde{W}_{t|t} \subseteq W$. The input sequence $\tilde{u}_{t,t+N_C-1|t} = [k_f(\tilde{x}_{t|t}), \ldots, k_f(\tilde{x}_{t+N_C-1|t})]^T$ is feasible for any $\tilde{x}_{t|t} \in X_f$. From Assumptions 1-2, one gets $V_{t|t} \leq \alpha_{2f}\left(\left(L_{fx}\right)^{N_p}|\tilde{x}_{t|t}|\right) + \sum_{l=t}^{t+N_p-1}\left[c_8 L_{fx}^{l-t}\left|\tilde{x}_{t|t}\right| + L_{qw}L_{gw}^{l-t}\left|\tilde{w}^i_{t|t}\right|\right]$ for $c_8 = L_{hx} + L_{hu}L_{kf} + L_{qx}$. Summing the geometric series, we obtain $V_t|t \leq \alpha_3(|\tilde{x}_{t|t}|) + \sigma_3(|\tilde{w}_{t|t}|) + \bar{\bar{c}}$, with α_3, σ_3 and $\bar{\bar{c}}$ defined in Lemma 3. In vue of Theorem 3, using the optimal control $u^0_{t,t+Nc-1|t}$ and $\tilde{x}_{t|t} \in \tilde{X}_t$, it turns out that at least one feasible control $u'_{t+1,t+Nc|t+1} = [u^o_{t+1,t+N^i_c-1|t}, u'_{t+Nc|t+1}]^T$ at $t+1$, where $u'_{t+Nc|t+1} \in U$ such that $\tilde{x}_{t+Nc+1|t+1} \in X_f$ any $x_t \in X^i_{MPC}$. it follows that $V_{t+1|t+1} \leq \sum_{l=t+1}^{t+Nc-1}\left[h(\tilde{x}_{l|t+1}, u^o_{l|t}) + q(\tilde{x}_{l|t+1}, \tilde{w}_{l|t+1})\right] + h(\tilde{x}_{t+Nc|t+1}, u'_{t+Nc|t+1}) + q(\tilde{x}_{t+Nc|t+1}, \tilde{w}_{t+Nc|t+1}) + \sum_{l=t+Nc+1}^{t+Np}\left[h(\tilde{x}_{l|t+1}, k_f(\tilde{x}_{l|t+1})) + q(x_{l|t+1}, \tilde{w}_{l|t+1})\right] + h_f(\tilde{f}(\tilde{x}_{t+Np|t+1}, k_f(\tilde{x}_{t+Np|t+1})))$. This will lead to

$$\begin{aligned}
&V_{t+1|t+1} - V_{t|t} \\
&\leq -h(\tilde{x}_{t|t}, u^o_{t|t}) - q(\tilde{x}_{t|t}, \tilde{w}_{t|t}) \\
&\quad + h(\tilde{x}_{t+Nc|t+1}, u'_{t+Nc|t+1}) - h\left(\tilde{x}_{t+Nc|t}, k_f(\tilde{x}_{t+Nc|t})\right) \\
&\quad + q(\tilde{x}_{t+Nc|t+1}, \tilde{w}_{t+Nc|t+1}) - q(\tilde{x}_{t+Nc|t}, \tilde{w}_{t+Nc|t}) \\
&\quad + \sum_{l=t+Nc+1}^{t+Np-1}\left[\begin{array}{l} h(\tilde{x}_{l|t+1}, k_f(\tilde{x}_{l|t+1})) - h\left(\tilde{x}_{l|t}, k_f(\tilde{x}_{l|t})\right) \\ +q(x_{l|t+1}, \tilde{w}_{l|t+1}) - q\left(\tilde{x}_{l|t}, \tilde{w}_{l|t}\right) \end{array}\right] \\
&\quad + h(\tilde{x}_{t+Np|t+1}, k_f(\tilde{x}_{t+Np|t+1})) + q(\tilde{x}_{t+Np|t+1}, \tilde{w}_{t+Np|t+1}) \\
&\quad + h_f\left(\tilde{f}\left(\tilde{x}_{t+Np|t+1}, k_f(\tilde{x}_{t+Np|t+1})\right)\right) - h_f\left(\tilde{x}_{t+Np|t}\right) \\
&\quad + \sum_{l=t+1}^{t+Nc-1}[h(\tilde{x}_{l|t+1}, u^o_{t|t}) - h(\tilde{x}_{l|t}, u^o_{l|t}) + q(\tilde{x}_{l|t+1}, \tilde{w}_{l|t+1}) \\
&\quad - q(\tilde{x}_{l|t}, \tilde{w}_{l|t})]
\end{aligned} \tag{29}$$

To find the upper bounds of the different terms in (29), one proceeds as follows. From (22) for $l = 1, \ldots, N_C - 1$, it follows that $\left|h(\tilde{x}_{t+l|t+1}, u^o_l) - h(\tilde{x}_{l|t}, u^o_l)\right| \leq c_6 L_{hx} L_{fx}^{l-1} + L_{hx}L_{fw}\frac{L_{fx}^{l-1}-L_{gw}^{l-1}}{L_{fx}-L_{gw}}\left(\left|\tilde{w}_{t+1|t+1}\right| + L_{fx}\left|\tilde{w}_{t|t}\right|\right)$. Similarly,

$$\begin{aligned}
&\left|q(\tilde{x}_{t+l|t+1}, \tilde{w}_{t+l|t+1}) - q(\tilde{x}_{t+lt}, \tilde{w}_{t+l|t})\right| \leq \\
&+L_{qw}L_{gw}^{l-1}\left(L_{gw}\left|\tilde{w}_{t|t}\right| + \left|\tilde{w}_{t+1|t+1}\right|\right) + c_6 L_{qx} L_{fx}^{l-1} \\
&+L_{qx}L_{fw}\frac{L_{fx}^{l-1} - L_{gw}^{l-1}}{L_{fx} - L_{gw}}\left(L_{gw}\left|\tilde{w}_{t|t}\right| + \left|\tilde{w}_{t+1|t+1}\right|\right)
\end{aligned}$$

$l = 1, \ldots, N_P - 1$. Similarly, for $l = N_C + 1, \ldots, N_P - 1$

$$\begin{aligned}
&\left|h(\tilde{x}_{l|t+1}, k_f(\tilde{x}_{l|t+1})) - h\left(\tilde{x}_{l|t}, k_f(\tilde{x}_{l|t})\right)\right| \leq c_6 c_9 L_{fx}^{l-1} \\
&+c_9 L_{fw}\frac{L_{fx}^{l-1} - L_{gw}^{l-1}}{L_{fx} - L_{gw}}\left(L_{gw}\left|\tilde{w}_{t|t}\right| + \left|\tilde{w}_{t+1|t+1}\right|\right)
\end{aligned}$$

for $c_9 = L_{hx} + L_{hu}L_{kf}$. Similarly, at N_C

$$\begin{aligned}
&\left|h(\tilde{x}_{t+Nc|t+1}, u'_{t+Nc|t+1}) - h\left(\tilde{x}_{t+Nc|t}, k_f(\tilde{x}_{t+Nc|t})\right)\right| \\
&\leq c_6 L_{hx} L_{fx}^{Nc-1} + L_{hu}\left|u_{\max} - u_{\min}\right| \\
&+L_{hx}L_{fw}\frac{L_{fx}^{Nc-1} - L_{gw}^{Nc-1}}{L_{fx} - L_{gw}}\left(L_{gw}\left|\tilde{w}_{t|t}\right| + \left|\tilde{w}_{t+1|t+1}\right|\right)
\end{aligned}$$

and finally, $h_f\left(\tilde{x}_{t+Np|t+1}\right) - h_f\left(\tilde{x}_{t+Np|t}\right) \leq c_6 L_{fx}^{Np-1} + L_{fw}\frac{L_{fx}^{Np-1}-L_{gw}^{Np-1}}{L_{fx}-L_{gw}}\left(L_{gw}\left|\tilde{w}_{t|t}\right| + \left|\tilde{w}_{t+1|t+1}\right|\right)$. Using (29), and in vue of (28), $V_{t+1|t+1} - V_{t|t}$ be writen as $V_{t+1|t+1} - V_{t|t} \leq -\alpha_2(|\tilde{x}_t|) + \sigma_1(|\tilde{w}_t|) + \sigma_2(|\tilde{w}_{t+1}|) + \bar{c}$ for $\tilde{x}_t \in X_{MPC}, \forall \tilde{w}_t \in \tilde{W}_t, \tilde{w}_{t+1} \in \tilde{W}_{t+1}$, with α_2, σ_1, σ_2, and $\bar{c}$ are given in Lemma 3. Therefore, according to Theorem 1

$$\left|\tilde{x}_{t+l|t+l}\right| \leq \tilde{\beta}\left(\left|\tilde{x}_{t|t}\right|, l\right) + \tilde{\gamma}\left(\left\|\tilde{w}_{[t+l|t+l]}\right\|\right) + \tilde{c} \tag{30}$$

□

Stability of the uncertain system is always a concern. The following theorem establishes the conditions that allow to conclude about the stability of the uncertain system based on the ISpS of the nominal model.

Theorem 6: If the nominal system (8) *with tightened constraint set* (20) *is ISpS under the RH control law* (15), *then the uncertain system* (6) *is also ISpS.*

Proof: In vue of Assumption 1, $\left|x_{t+l} - \tilde{x}_{t+l|t+l}\right| \leq \bar{\eta}_x$, hence $\left|\tilde{x}_{t+l|t+l}\right| \geq |x_{t+l}| - |\bar{\eta}_x|$. Using (30), one gets $|x_{t+l}| \leq \tilde{\beta}\left(\left|\tilde{x}_{t|t}\right|, l\right) + \tilde{\gamma}\left(\left\|\tilde{w}_{[t+l|t+l]}\right\|\right) + \bar{c} + \bar{\eta}_x$. Similarly, $\left|\tilde{x}_{t|t}\right| \leq |x_t| + \bar{\eta}_x$, therefore from the proof of Theorem 5, $|x_{t+l}| \leq \tilde{\beta}\left(|x_t| + \bar{\eta}_x, l\right) + \tilde{\gamma}\left(\left\|\tilde{w}_{[t+l|t+l]}\right\|\right) + \bar{c} + \bar{\eta}_x \leq \tilde{\beta}\left(2|x_t|, l\right) + \tilde{\gamma}\left(\left\|\tilde{w}_{[t+l|t+l]}\right\|\right) + \bar{c} + \bar{\eta}_x + \tilde{\beta}\left(2\bar{\eta}_x, 0\right)$. On the other hand $\left\|\tilde{w}_{t,t+l|t+l}\right\| \leq \left\|w_{t,t+l}\right\| + \bar{\eta}_w$, hence $|x_{t+l}| \leq \tilde{\beta}\left(2|x_t|, l\right) + \tilde{\gamma}\left(\left\|w_{t,t+l}\right\| + \bar{\eta}\right) + \bar{c} + \bar{e}ta_x + \tilde{\beta}\left(2\bar{\eta}_x, 0\right) \leq \tilde{\beta}\left(2|x_t|, l\right) + \tilde{\gamma}\left(2\left\|w_{t,t+l}\right\|\right) + \bar{c} + \bar{\xi}_x + \tilde{\beta}\left(2\bar{\eta}_x, 0\right) + \tilde{\gamma}\left(2\bar{e}ta_w\right)$. This leads to

$$|x_{t+l}| \leq \beta\left(|x_t|, l\right) + \gamma\left(\left\|w_{t,t+l}\right\|\right) + c \tag{31}$$

with $\beta(r,s) \triangleq \tilde{\beta}(2r, s)$, $\gamma(s) = \tilde{\gamma}(2s)$ and $c \triangleq \bar{c} + \bar{\eta}_x + \tilde{\beta}\left(2\bar{\eta}_x, 0\right) + \tilde{\gamma}\left(2\bar{\xi}_w\right)$. Consequently, using (31), one can conclude that the perturbed system (6) under RH control law (15) is ISpS. □

The following algoritm 1 summarizes the robust nonlinear model predictive control approach presented in this paper. Two classes of optimization problems are solved: offline and online.

Algorithm 1 Robust NMPC

1: **Define** nominal model $\tilde{f}(\tilde{x}, u, w)$, the constraint sets X, U and W, nominal cost (12) and error bounds as in Assumption 1.
2: **Determine** the tightned constraint sets using Algorithm .
3: **Compute** optimal terminal set X_f and terminal control k_f.
4: **Compute** One-step controllability set $\mathcal{C}_1(X_f)$.
5: **Compute** Robust output feasibility set X_{MPC} using Algorithm
6: **Initialization** at $t, l = 0$
7: **while** Target state is not reached **do**
8: **Compute** state $\tilde{x}_{t+l}$ and disturbance $\tilde{w}_{t+l}$
9: **Determine** $u^0_{t+1,t+l+Nc}$ solution the FHOCP 1
10: **Check** feasibility of the optimal solution.
11: **Implement** first the control action, u^0_t
12: **end while**

IV. SIMULATION EXAMPLE

The numerical example of a simple nonlinear oscillator conisdered in this paper has been studied in the literature and will be used here with an important modification to illustrate the main concepts proposed in this paper. The simulation work used the semi-definite programming packages SDPT3-4.0 [39] and PENBMI [14], running on the optimization toolbox YALMIP [20]. The second order uncertain nonlinear oscillator system dynamic of interest has been modified to include a non-additive nonlinear disturbance and is given by

$$\begin{aligned} x_{1_{t+1}} &= x_{1_t} + a_1\left[-x_{2_t} + a_2\left(1 + x_{1_t}\right)u_t\right]\left(1 + w_{1_t}\right) + w_{1_t} \\ x_{2_{t+1}} &= x_{2_t} + a_3\left[x_{1_t} + a_4\left(1 - 4x_{2_t}\right)u_t\right]\left(1 + w_{2_t}\right) + w_{2_t} \end{aligned} \tag{32}$$

where the uncertain parmeters $a_1 = 0.051$, $a_2 = 0.4902$, $a_3 = 0.049$ and $a_4 = 0.5102$. The external input w is driven by the following exosystem

$$\begin{aligned} w_{1_{t+1}} &= 10^{-3} w_{1_t} + \phi_{1_t} \\ w_{2_{t+1}} &= 10^{-3} w_{2_t} + \phi_{2_t} \end{aligned}$$

where $|\phi_t| \leq 10^{-4}$ is a random noise. The constraints are $U = \{u \in \mathbb{R} : -2 \leq u \leq 2\}$, $X = \{x \in \mathbb{R} : -0.125 \leq x_1 \leq 0.125,\ -0.125 \leq x_2 \leq 0.125\}$ and $W = \left\{w_{1,2} \in \mathbb{R} : -5 \times 10^{-3} \leq w_{1,2} \leq 5 \times 10^{-3}\right\}$. The nominal model is given as

$$\begin{aligned} \tilde{x}_{1_{t+1}} &= \tilde{x}_{1_t} + \tilde{a}_1\left[-\tilde{x}_{2_t} + \tilde{a}_2\left(1 + \tilde{x}_{1_t}\right)u_t\right]\left(1 + \tilde{w}_{1_t}\right) + \tilde{w}_{1_t} \\ \tilde{x}_{2_{t+1}} &= \tilde{x}_{2_t} + \tilde{a}_3\left[\tilde{x}_{1_t} + \tilde{a}_4\left(1 - 4\tilde{x}_{2_t}\right)u_t\right]\left(1 + \tilde{w}_{2_t}\right) + \tilde{w}_{2_t} \end{aligned} \tag{33}$$

with the nominal parameters $\tilde{a}_1 = 0.05$, $\tilde{a}_2 = 0.5$, $\tilde{a}_3 = 0.05$ and $\tilde{a}_4 = 0.5$. On the other hand, the nominal dynamic model of the exosystem causing the disturbance is

$$\begin{aligned} \tilde{w}_{1_{t+1}} &= 1.0110^{-3}\tilde{w}_{1_t} \\ \tilde{w}_{2_{t+1}} &= 1.0110^{-3}\tilde{w}_{2_t} \end{aligned}$$

The random measurement uncertainties are not known but bounded by $|x_t - \hat{x}_{t|t}| \leq \bar{\eta}_x = 10^{-3}$ and $|w_t - \hat{w}_{t|t}| \leq \bar{\eta}_w = 10^{-5}$. Let the cost functional be given by (**??**), with $Q = 0.1 \times I_2$, $R = 1$, $S = 10^{-3} I_2$, $N_P = 12$ and $N_c = 5$. The lipschitz constants are $L_{fx} = 1.189$, $L_{fu} = 0.0801$, $L_{fw} = 1.4142$, $L_{gw} = 1.01 \times 10^{-3}$, $\alpha_1(|r|) = \alpha_2(|r|) = 0.1|r|^2$, $L_{hx} = 0.06325$, $L_{hu} = 2$, $L_{qx} = 6.325 \times 10^{-4}$, $L_{qw} = 1.414 \times 10^{-5}$, $\bar{\Delta}_x = 6.237 \times 10^{-4}$ and $\bar{\Delta}_w = 5 \times 10^{-8}$. The shrunk constraint sets are determined using with (20). The exponential growth of the uncertainty, leads to tighter constraint sets which may lead in turn to unfeasibal solution of OCP. Therefore, as it is expected and known in the literature, the maximum horizon which can be selected and still ensures feasibility is limited. To determine the set $M \subset Z$ within which the OCP for maximum terminal region is feasible, the algorithm starts from Z and makes progressive

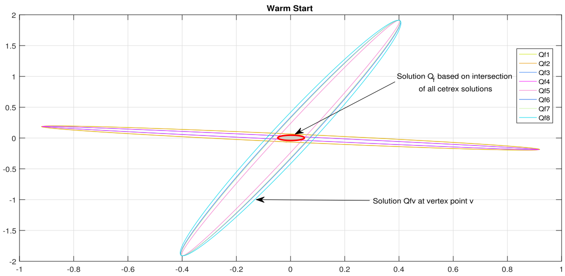

FIGURE 1. **Warm start ellipsoids and the result of computing the optimum terminal region.**

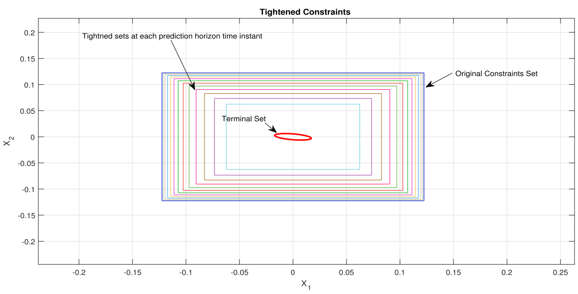

FIGURE 2. **Tightened constraint sets along the prediction horizon and the optimal terminal region.**

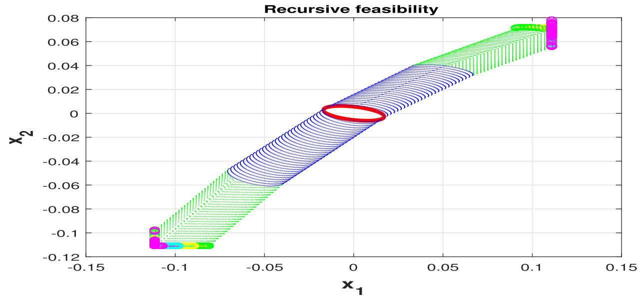

FIGURE 3. **Feasibility sets for Example 32.**

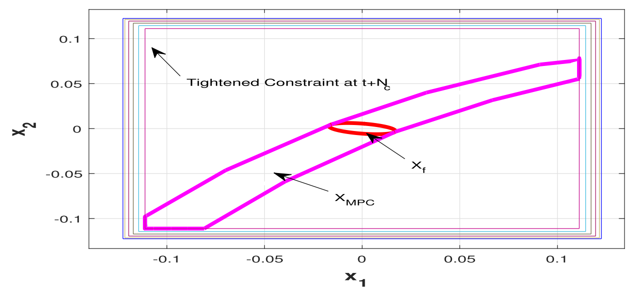

FIGURE 4. **Feasible set X_{MPC} with tightened constraints $r\hat{h}o_{t+NC} = \bar{\rho}_{t+Nc}$.**

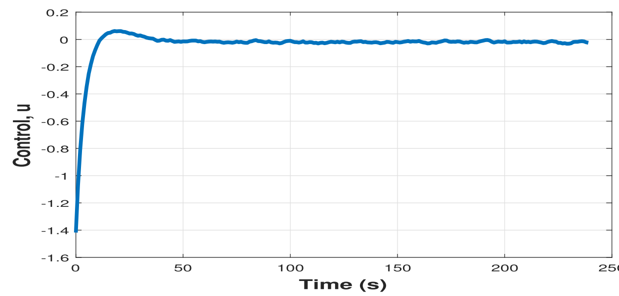

FIGURE 5. **Optimal control input.**

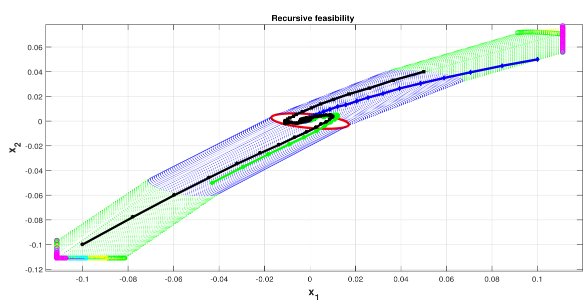

FIGURE 6. **State trajectory solved with algorithm 1 using different initial conditions.**

tightening of the constraints. Therefore, set $M \subset Z$ is selected as as $-2 \leq u \leq 2$, $-0.1125 \leq \tilde{x}_1 \leq 0.1125$ and $-0.1125 \leq \tilde{x}_{(2)} \leq 0.1125$. For that, the normalized set value is ($\hat{v} = 6$)

$$M = \left\{ \left[\tilde{x}_1 \; \tilde{x}_2 \; u \right]^T \in \mathbb{R}^{n+m} : \bar{c}_v \tilde{x} + \bar{d}_v u \leq 1 \right\}$$

with $\bar{c}_1 = \bar{c}_3 = [-8.89\ 0]$, $\bar{c}_2 = \bar{c}_4 = [0\ 8.89]$, $\bar{c}_5 = c_6 = [0\ 0]$, $\bar{d}_{1,2,3,4} = 0$, $\bar{d}_5 = -0.5$ and $\bar{d}_5 = 0.5$. The linearized nominal system is $\tilde{x}_{t+1} = A_v \tilde{x}_t + B_v u$, with $A_v = \begin{bmatrix} 1 + 0.025u_t & -0.05 \\ 0.05 & 1 - 0.1u_t \end{bmatrix}$ and $B_v = 0.025\left[1 + \tilde{x}_{1,t} \;\; 1 - 4\tilde{x}_{2,t} \right]^T$. There are $\bar{v} = 8$ vertices and $\tilde{S}$ has to be selected such that $-q(\tilde{x}, \tilde{w}) > +\psi(|\tilde{w}|) \leq \tilde{x}^T \tilde{S} \tilde{x}$. So, let $\tilde{S} = 10 \cdot S$. Figure 4 illustrates the results where $Q_f^\infty = \begin{bmatrix} 342.04 & 69.16 \\ 69.16 & 670.08 \end{bmatrix}$ and $K^\infty = \left[-6.52 \; -10.45 \right]^T$.

Starting from the warm start results, the optimum terminal region is computed for $a = 1$ as indicated in figure . Warm starting generally helps to improve the determination of a larger optimum terminal region. Figure 2 shows the evolution of the tightned constraints sets during the the prediction horizon. The exponential growth rate of the uncertainty depends on the size of the bounds elative to the disturbance, dynamic uncertainty and noise level.

$$Q_f = \begin{bmatrix} 6222.7 & 631.1 \\ 631.1 & 3709.7 \end{bmatrix}, \quad K = \left[-11.66 \; -36.33 \right]$$

Determination of $\mathcal{C}_1(X_f)$ leads to $\tilde{x}_c^i \in \partial\left(\mathcal{C}_1(X_f)\right)$ for $i = 1, \ldots, \bar{N}$.The minimum size of $\mathcal{C}_1(X_f)$ is $\bar{d} = \min_i \left(\left| \tilde{x}_{c_1}^i - \tilde{x}_f^i \right| \right), \forall i = 1, \ldots, \bar{N} = 6.18 \times 10^{-2}$. The left side of inequality (21) gives $1.34 \times 10^{-3} \leq \bar{d}$. Figure 3 displays the shape and evolution of the feasibility sets within the state space based on an intial feasible solution in X_{MPC}. Recursive feasibility of OCP 1 is guaranteed. In Figure 4, one can also observe that boundary of X_{MPC} starts from the boundaries of the tightened state constraint set $\partial\tilde{X}_{t+Nc}$. This is

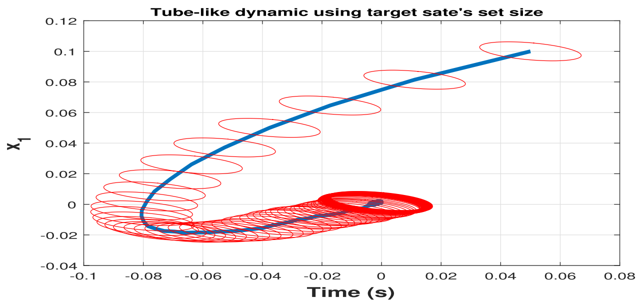

FIGURE 7. Tube-based dynamic based on optimize final set.

because the uncertainty bound is set at $\bar{\rho}_{t+Nc}$. The generated optimal control signal can be seen in Figure 5. The online part ofAlgorithm 1 uses `fmincon` package of Matlab [2]. Several initial conditions has been tried as can be seen in Figure 7 shows state trajectories to the origin. Using the terminal set size one can observe the equivalent Tube-based dynamic

V. CONCLUSION

This paper addressed the problem of robust model predictive control of uncertain nonlinear system. Several sources of uncertainties have been considered and algorithms that can facilitate the implementation has been presented. Some of these algorithms are offline, while others are online. They can be used as a guideline to solve constraint tightening, terminal region optimization, feasibility determination and online optimization. The analysis confirmed several observations seen in the literature especially those related to tightened constraints and uncertainty growth over the prediction horizon. To maximize the terminal region and increase robustness, polytopic linear difference inclusions (PLDI) based on LMI formulation has been used. Only practical stability (ISpS) can be ensured for this type of system and the limit of disturbances' levels that can be handled by the controller are bounded by the size of the one-step controllability set to the terminal constraint region. Simulations results confirmed the theoretical analysis and showed performance of the proposed approach.

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SAMI EL-FERIK received the B.Sc. degree in electrical engineering from Laval University, Quebec, QC, Canada, and the M.S. and Ph.D. degrees in electrical and computer engineering from Ecole Polytechnique, University of Montreal, Montreal, QC.

His Ph.D. work, on flexible manufacturing systems modeling and control, was co-supervised with mechanical engineering. After completion of his Ph.D. and postdoctoral positions, he was worked as a Staff Control Analyst with the Research and Development Center of Systems, Controls, and Accessories, Pratt and Whitney Canada. He is currently an Associate Professor in control and instrumentation with the Department of Systems Engineering, King Fahd University of Petroleum and Minerals (KFUPM). His research interests are in sensing, monitoring, and control with strong multidisciplinary research and applications. His research contributions are in control of drug administration, process control and control loop performance-monitoring, control of systems with delays, modeling and control of stochastic systems, analysis of network stability, condition monitoring, and condition-based maintenance.