IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received January 22, 2020, accepted February 17, 2020, date of publication February 21, 2020, date of current version March 3, 2020.

Digital Object Identifier 10.1109/ACCESS.2020.2975628

Hermite-Hadamard Type Inequalities via Exponentially (p, h)-Convex Functions

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This work was supported by the National University of Sciences and Technology (NUST), Islamabad, Pakistan.

ABSTRACT Here we introduce new class of exponentially convex function namely exponentially (p, h)-convex function. We find the Hermite-Hadamard type inequalities via exponentially (p, h)-convex functions. We extend the various familar results.

INDEX TERMS Hermite-Hadamard inequalities, (p, h)-convex function, exponentially (p, h)-convex function.

I. INTRODUCTION

Last couple of decades, the notion convex functions and its generalizations have become more familar because of marvelous nature. The Hermite-Hadamard inequality [5], [6] for a convex function $\zeta : \mathcal{Y} \to \mathbb{R}$ on an interval $\mathcal{Y}$ is

$$\zeta\left(\frac{y_1+y_2}{2}\right) \leq \frac{1}{y_2-y_1}\int_{y_1}^{y_2}\zeta(w)dw \leq \frac{\zeta(y_1)+\zeta(y_2)}{2}, \tag{I.1}$$

for all $y_1, y_2 \in \mathcal{Y}$ with $y_1 < y_2$. Selections of suitable functions one can use inequality (I.1) to derive other useful mean inequalities. For example see [1]–[4], [7]–[14]. Above inequalities are true in the reversed order if ζ is concave.

Definition 1 ([7]): Let $s \in (0, 1]$. A function $\zeta : \mathcal{Y} \subset [0, \infty) \to [0, \infty)$ is called s-convex in the second sense, if

$$\zeta(\kappa y_1 + (1-\kappa)y_2) \leq \kappa^s\zeta(y_1) + (1-\kappa)^s\zeta(y_2), \tag{I.2}$$

for all $y_1, y_2 \in \mathcal{Y}$ and $\kappa \in [0, 1]$.

Definition 2 ([17]): Let $h : \mathcal{J} \subseteq \mathbb{R} \to \mathbb{R}$ be a positive function. Consider an interval $\mathcal{Y} \subset (0, \infty)$, then a function $\zeta : \mathcal{Y} \to \mathbb{R}$ is called h-convex, if ζ is non-negative and

$$\zeta(\kappa y_1 + (1-\kappa)y_2) \leq h(\kappa)\zeta(y_1) + h(1-\kappa)\zeta(y_2), \tag{I.3}$$

for all $y_1, y_2 \in \mathcal{Y}$ and $\kappa \in [0, 1]$. If (I.3) is reversed then ζ is called h-concave.

Definition 3 ([9]): Consider an interval $\mathcal{Y} \subset (0, \infty)$ and $p \in \mathbb{R}\backslash\{0\}$. A function $\zeta : \mathcal{Y} \to \mathbb{R}$ is called p-convex, if

$$\zeta\left(\left[\kappa y_1^p + (1-\kappa)y_2^p\right]^{\frac{1}{p}}\right) \leq \kappa\zeta(y_1) + (1-\kappa)\zeta(y_2), \tag{I.4}$$

for all $y_1, y_2 \in \mathcal{Y}$ and $r \in [0, 1]$. If (I.4) is in reversed order then ζ is called p-concave.

Sarikaya *et. al.* [16] and Iscan [9] proved Hadmard inequalities for h- and p- convex functions, respectively.

Theorem 4 ([16]): Let $\zeta : \mathcal{Y} \to \mathbb{R}$ be h-convex function and $y_1, y_2 \in \mathcal{Y}$ with $y_1 < y_2$ and $\zeta \in L_1([y_1, y_2])$. Then

$$\frac{1}{2h\left(\frac{1}{2}\right)}\zeta\left(\frac{y_1+y_2}{2}\right) \leq \frac{1}{y_2-y_1}\int_{y_1}^{y_2}\zeta(w)dw \leq [\zeta(y_1)+\zeta(y_2)]\int_0^1 h(\kappa)d\kappa. \tag{I.5}$$

Theorem 5 ([9]): Consider an interval $\mathcal{Y} \subset (0, \infty)$, and $p \in \mathbb{R}\backslash\{0\}$. Let $\zeta : \mathcal{Y} \to \mathbb{R}$ is p-convex and $y_1, y_2 \in \mathcal{Y}$, $y_1 < y_2$. If $\zeta \in L_1([y_1, y_2])$, then we have

$$\zeta\left(\left[\frac{y_1^p+y_2^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{y_2^p-y_1^p}\int_{y_1}^{y_2}\frac{\zeta(w)}{w^{1-p}}dw \leq \frac{\zeta(y_1)+\zeta(y_2)}{2}. \tag{I.6}$$

Lemma 6 ([9]): Let $\zeta : \mathcal{Y} \to \mathbb{R}$ be a differentiable function on $\mathcal{Y}^\circ$, i.e., the interior of $\mathcal{Y}$, and $y_1, y_2 \in \mathcal{Y}, y_1 < y_2$, and $p \in \mathbb{R}\backslash\{0\}$. If $\zeta' \in L_1([y_1, y_2])$, then

$$\frac{\zeta(y_1)+\zeta(y_2)}{2} - \frac{p}{y_2^p-y_1^p}\int_{y_1}^{y_2}\frac{\zeta(w)}{w^{1-p}}dw = \frac{y_2^p-y_1^p}{2p}\int_0^1\frac{1-2\kappa}{[\kappa y_1^p+(1-\kappa)y_2^p]^{1-\frac{1}{p}}} \times\zeta'\left([\kappa y_1^p+(1-\kappa)y_2^p]^{\frac{1}{p}}\right)d\kappa. \tag{I.7}$$

The associate editor coordinating the review of this manuscript and approving it for publication was Yan-Jun Liu.

Fang and Shi [4] defined (p,h)-convex function and gave Hadamard inequalities.

Definition 7 ([4]): Let $h:\mathcal{J}\subseteq\mathbb{R}\to\mathbb{R}$ be a positive function. Let $\mathcal{Y}\subset(0,\infty)$ be an interval, and $p\in\mathbb{R}\backslash\{0\}$. A function $\zeta:\mathcal{Y}\to\mathbb{R}$ is called (p,h)-convex, if ζ is non-negative and

$$\zeta\left(\left[\kappa y_1^p+(1-\kappa)y_2^p\right]^{\frac{1}{p}}\right)\leq h(\kappa)\zeta(y_1)+h(1-\kappa)\zeta(y_2), \quad (I.8)$$

for all $y_1,y_2\in\mathcal{Y}$ and $\kappa\in[0,1]$. If (I.8) is in reversed order then ζ is called (p,h)-concave.

Theorem 8 ([4]): Let $\zeta:\mathcal{Y}\to\mathbb{R}$ be (p,h)-convex function and $y_1,y_2\in\mathcal{Y}$ with $y_1<y_2$ and $\zeta\in L_1([y_1,y_2])$. Then

$$\frac{1}{2h\left(\frac{1}{2}\right)}\zeta\left(\left[\frac{y_1^p+y_2^p}{2}\right]^{\frac{1}{p}}\right)\leq\frac{p}{y_2^p-y_1^p}\int_{y_1}^{y_2}\frac{\zeta(w)}{w^{1-p}}dw \leq[\zeta(y_1)+\zeta(y_2)]\int_0^1 h(\kappa)d\kappa. \quad (I.9)$$

Awan *et al.* [1] and Mehreen and Anwar [12] established new class of convex functions and gave several Hadamard's type inequalities.

Definition 9 ([1]): A function $\zeta:\mathcal{Y}\subseteq\mathbb{R}\to\mathbb{R}$ is called exponentially convex, if

$$\zeta(\kappa y_1+(1-\kappa)y_2)\leq\kappa\frac{\zeta(y_1)}{e^{\alpha y_1}}+(1-\kappa)\frac{\zeta(y_2)}{e^{\alpha y_2}}, \quad (I.10)$$

for all $y_1,y_2\in\mathcal{Y}$, $\kappa\in[0,1]$ and $\alpha\in\mathbb{R}$. If the inequality (I.10) is in reversed order then ζ is called exponentially concave.

Definition 10 ([12]): Let $s\in(0,1]$ and $\mathcal{Y}\subset[0,\infty)$ be an interval. A function $\zeta:\mathcal{Y}\to\mathbb{R}$ is called exponentially s-convex in the second sense, if

$$\zeta(\kappa y_1+(1-\kappa)y_2)\leq\kappa^s\frac{\zeta(y_1)}{e^{\alpha y_1}}+(1-\kappa)^s\frac{\zeta(y_2)}{e^{\alpha y_2}}, \quad (I.11)$$

for all $y_1,y_2\in\mathcal{Y}$ and $\kappa\in[0,1]$. If (I.11) is in reversed order then ζ is called exponentially s-concave.

Theorem 11 ([12]): Let $\zeta:\mathcal{Y}\subset[0,\infty)\to\mathbb{R}$ be an integrable exponentially s-convex function in the second sense on $\mathcal{Y}^\circ$. Then for $y_1,y_2\in\mathcal{Y}$ with $y_1<y_2$ and $\alpha\in\mathbb{R}$, we have

$$2^{s-1}\zeta\left(\frac{y_1+y_2}{2}\right)\leq\frac{1}{y_2-y_1}\int_{y_1}^{y_2}\frac{\zeta(w)}{e^{\alpha w}}dw \leq A_3(\kappa)\frac{\zeta(y_1)}{e^{\alpha y_1}}+A_4(\kappa)\frac{\zeta(y_2)}{e^{\alpha y_2}}, \quad (I.12)$$

where

$$A_3(\kappa)=\int_0^1\frac{\kappa^s d\kappa}{e^{\alpha(\kappa y_1+(1-\kappa)y_2)}},$$

and

$$A_4(\kappa)=\int_0^1\frac{(1-\kappa)^s d\kappa}{e^{\alpha(\kappa y_1+(1-\kappa)y_2)}}.$$

Definition 12 ([12]): Consider an interval $\mathcal{Y}\subset(0,\infty)=\mathbb{R}_+$ and $p\in\mathbb{R}\backslash\{0\}$. A function $\zeta:\mathcal{Y}\to\mathbb{R}$ is called exponentially p-convex, if

$$\zeta\left(\left[\kappa y_1^p+(1-\kappa)y_2^p\right]^{\frac{1}{p}}\right)\leq\kappa\frac{\zeta(y_1)}{e^{\alpha y_1}}+(1-\kappa)\frac{\zeta(y_2)}{e^{\alpha y_2}}, \quad (I.13)$$

for all $y_1,y_2\in\mathcal{Y}$, $\kappa\in[0,1]$ and $\alpha\in\mathbb{R}$. If the inequality (I.13) is in opposite direction then ζ is called exponentially p-concave.

Theorem 13 ([12]): Let $\zeta:\mathcal{Y}\to\mathbb{R}$ be an integrable exponentially p-convex function. Let $y_1,y_2\in\mathcal{Y}$ with $y_1<y_2$. Then for $\alpha\in\mathbb{R}$, we have

$$\zeta\left(\left[\frac{y_1^p+y_2^p}{2}\right]^{\frac{1}{p}}\right)\leq\frac{p}{y_2^p-y_1^p}\int_{y_1}^{y_2}\frac{\zeta(w)}{w^{1-p}e^{\alpha w}}dw \leq A_1(\kappa)\frac{\zeta(y_1)}{e^{\alpha y_1}}+A_2(\kappa)\frac{\zeta(y_2)}{e^{\alpha y_2}}, \quad (I.14)$$

where

$$A_1(\kappa)=\int_0^1\frac{\kappa d\kappa}{e^{\alpha(\kappa y_1^p+(1-\kappa)y_2^p)^{\frac{1}{p}}}},$$

and

$$A_2(\kappa)=\int_0^1\frac{(1-\kappa)d\kappa}{e^{\alpha(\kappa y_1^p+(1-\kappa)y_2^p)^{\frac{1}{p}}}}.$$

The Beta and Hypergeometric function are defined as:

$$\beta(y_1,y_2)=\int_0^1 w^{y_1-1}(1-w)^{y_2-1}dw,\ y_1,y_2>0,$$

and

$$ {}_2F_1(y_1,y_2;t;z) =\frac{1}{\beta(y_2,t-y_2)}\int_0^1 w^{y_2-1}(1-w)^{t-y_2-1}(1-zw)^{-y_1}dw,$$

$t>y_2>0$, $|z|<1$, respectively.

II. MAIN RESULTS

We define exponentially (p,h)-convex functions as:

Definition 14: Let $h:\mathcal{J}\subseteq\mathbb{R}\to\mathbb{R}$ be a positive function. Consider an interval $\mathcal{Y}\subset(0,\infty)=\mathbb{R}_+$ and $p\in\mathbb{R}\backslash\{0\}$. A function $\zeta:\mathcal{Y}\to\mathbb{R}$ is called exponentially (p,h)-convex, if

$$\zeta\left(\left[\kappa y_1^p+(1-\kappa)y_2^p\right]^{\frac{1}{p}}\right)\leq h(\kappa)\frac{\zeta(y_1)}{e^{\alpha y_1}}+h(1-\kappa)\frac{\zeta(y_2)}{e^{\alpha y_2}}, \quad (II.1)$$

for all $y_1,y_2\in\mathcal{Y}$, $\kappa\in[0,1]$ and $\alpha\in\mathbb{R}$. If the inequality (II.1) is in opposite direction then ζ is called exponentially (p,h)-concave.

Remark 15: In Definition 14,

(a) by taking $\alpha=0$, we attain inequality (I.8) of Definition 7.

(b) by taking $\alpha=0$ and $p=1$, we attain inequality (I.3) of Definition 2.

(c) by taking $\alpha = 0$, $h(\kappa) = k^s$ and $p = 1$, we attain inequality (I.2) of Definition 1.

(d) by taking $\alpha = 0$ and $h(\kappa) = k$, we attain inequality (I.4) of Definition 3.

(e) by taking $p = 1$ and $h(\kappa) = k$, we attain inequality (I.10) of Definition 9.

(f) by taking $p = 1$, we attain inequality (I.11) of Definition 10.

(g) by taking $h(k) = k$, we attain inequality (I.13) of Definition 12.

(h) by taking $h(\kappa) = \kappa$, $\alpha = 0$ and $p = 1$, we get the definition of convex function.

Throughout the section, we symbolize $\mathcal{Y} \subset (0, \infty) = \mathbb{R}_+$ for an interval and its interior as $\mathcal{Y}^\circ$, $p \in \mathbb{R}\backslash\{0\}$ and $h : \mathcal{J} \subseteq \mathbb{R} \to \mathbb{R}$ be a positive function.

Theorem 16: Let $\zeta : \mathcal{Y} \to \mathbb{R}$ be an integrable exponentially (p, h)-convex function. Let $y_1, y_2 \in \mathcal{Y}$ with $y_1 < y_2$. Then for $\alpha \in \mathbb{R}$, we have

$$\frac{1}{2h(\frac{1}{2})}\zeta\left(\left[\frac{y_1^p + y_2^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{y_2^p - y_1^p}\int_{y_1}^{y_2}\frac{\zeta(w)}{w^{1-p}e^{\alpha w}}dw \leq S_1\frac{\zeta(y_1)}{e^{\alpha y_1}} + S_2\frac{\zeta(y_2)}{e^{\alpha u_2}}, \quad \text{(II.2)}$$

where

$$S_1 = \int_0^1 \frac{h(\kappa)d\kappa}{e^{\alpha(\kappa y_1^p + (1-\kappa)y_2^p)^{\frac{1}{p}}}},$$

and

$$S_2 = \int_0^1 \frac{h(1-\kappa)d\kappa}{e^{\alpha(\kappa y_1^p + (1-\kappa)y_2^p)^{\frac{1}{p}}}}.$$

Proof 17: Since ζ is exponentially (p, h)-convex function, we have

$$\frac{1}{h(\frac{1}{2})}\zeta\left(\left[\frac{v_1^p + v_2^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{\zeta(v_1)}{e^{\alpha v_1}} + \frac{\zeta(v_2)}{e^{\alpha v_2}}. \quad \text{(II.3)}$$

Let $v_1^p = \kappa y_1^p + (1-\kappa)y_2^p$ and $v_2^p = (1-\kappa)y_1^p + \kappa y_2^p$, we get

$$\frac{1}{h(\frac{1}{2})}\zeta\left(\left[\frac{y_1^p + y_2^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{\zeta\left(\left[\kappa y_1^p + (1-\kappa)y_2^p\right]^{\frac{1}{p}}\right)}{e^{\alpha(\kappa y_1^p + (1-\kappa)y_2^p)^{\frac{1}{p}}}} + \frac{\zeta\left(\left[(1-\kappa)y_1^p + \kappa y_2^p\right]^{\frac{1}{p}}\right)}{e^{\alpha((1-\kappa)y_1^p + \kappa y_2^p)^{\frac{1}{p}}}}. \quad \text{(II.4)}$$

Integrating (II.4) with respect to $\kappa \in [0, 1]$ and using changes of variable, we get

$$\frac{1}{h(\frac{1}{2})}\zeta\left(\left[\frac{y_1^p + y_2^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{2p}{y_2^p - y_1^p}\int_{y_1}^{y_2}\frac{\zeta(w)}{w^{1-p}e^{\alpha w}}dw. \quad \text{(II.5)}$$

Hence the first inequality of (II.2) is proved. For second one, use exponentially (p, h)-convexity of ζ, we find

$$\frac{\zeta\left(\left[\kappa y_1^p + (1-\kappa)y_2^p\right]^{\frac{1}{p}}\right)}{e^{\alpha(\kappa y_1^p + (1-\kappa)y_2^p)^{\frac{1}{p}}}} \leq \frac{h(\kappa)\frac{\zeta(y_1)}{e^{\alpha y_1}} + h(1-\kappa)\frac{\zeta(y_2)}{e^{\alpha y_2}}}{e^{\alpha(\kappa y_1^p + (1-\kappa)y_2^p)^{\frac{1}{p}}}}. \quad \text{(II.6)}$$

Integrating with respect to $\kappa \in [0, 1]$, we get

$$\frac{p}{y_2^p - y_1^p}\int_{y_1}^{y_2}\frac{\zeta(w)}{w^{1-p}e^{\alpha w}}dw \leq \frac{\zeta(y_1)}{e^{\alpha y_1}}\int_0^1 \frac{h(\kappa)d\kappa}{e^{\alpha(\kappa y_1^p + (1-\kappa)y_2^p)^{\frac{1}{p}}}} + \frac{\zeta(y_2)}{e^{\alpha y_2}}\int_0^1 \frac{h(1-\kappa)d\kappa}{e^{\alpha(\kappa y_1^p + (1-\kappa)y_2^p)^{\frac{1}{p}}}}. \quad \text{(II.7)}$$

From (II.5) and (II.7), we get (II.2).

Remark 18: In Theorem 16,

(a) by taking $\alpha = 0$, we attain inequality (I.9) of Theorem 8.

(b) by taking $\alpha = 0$ and $p = 1$, we attain inequality (I.5) of Theorem 4.

(c) by taking $\alpha = 0$ and $h(\kappa) = \kappa$, we attain inequality (I.6) of Theorem 5.

(d) by taking $h(\kappa) = \kappa$, we attain inequality (I.14) of Theorem 13.

(e) by taking $h(\kappa) = \kappa^s$, $p = 1$ and $\alpha = 0$, we attain inequality (2.1) of Theorem 2.1 in [3].

(f) by taking $h(\kappa) = \kappa^s$ and $p = 1$, we attain inequality (I.12) of Theorem 11. (g) by taking $h(\kappa) = \kappa$, $\alpha = 0$ and $p = 1$, we attain inequality (I.1).

Theorem 19: Let $\zeta : \mathcal{Y} \to \mathbb{R}$ be a differentiable function on $\mathcal{Y}^\circ$ and $y_1, y_2 \in \mathcal{Y}$ with $y_1 < y_2$ and $\zeta' \in L_1([y_1, y_2])$. If $|\zeta'|^q$ is exponentially (p, h)-convex on $[y_1, y_2]$ for $q \geq 1$ and $\alpha \in \mathbb{R}$, then

$$\left|\frac{\zeta(y_1) + \zeta(y_2)}{2} - \frac{p}{y_2^p - y_1^p}\int_{y_1}^{y_2}\frac{\zeta(w)}{w^{1-p}}dw\right| \leq \frac{y_2^p - y_1^p}{2p}T_1^{1-\frac{1}{q}}\left[T_2\left|\frac{\zeta'(y_1)}{e^{\alpha y_1}}\right|^q + T_3\left|\frac{\zeta'(y_2)}{e^{\alpha y_2}}\right|^q\right]^{\frac{1}{q}}, \quad \text{(II.8)}$$

where

$$T_1 = \frac{1}{4}\left(\frac{y_1^p + y_2^p}{2}\right)^{\frac{1}{p}-1}\left[{}_2F_1\left(1 - \frac{1}{p}, 2; 3; \frac{y_1^p - y_2^p}{y_1^p + y_2^p}\right) + {}_2F_1\left(1 - \frac{1}{p}, 2; 3; \frac{y_2^p - y_1^p}{y_1^p + y_2^p}\right)\right],$$

$$T_2 = \int_0^1 \frac{|1 - 2\kappa|h(\kappa)}{\left[\kappa y_1^p + (1-\kappa)y_2^p\right]^{1-\frac{1}{p}}}d\kappa,$$

and

$$T_3 = \int_0^1 \frac{|1 - 2\kappa|h(1-\kappa)}{\left[\kappa y_1^p + (1-\kappa)y_2^p\right]^{1-\frac{1}{p}}}d\kappa.$$

Proof 20: Using inequality (I.7) of Lemma 6, we find

$$\begin{aligned}&\left|\frac{\zeta(y_1)+\zeta(y_2)}{2}-\frac{p}{y_2^p-y_1^p}\int_{y_1}^{y_2}\frac{\zeta(w)}{w^{1-p}}dw\right|\\&\le\frac{y_2^p-y_1^p}{2p}\int_0^1\left|\frac{1-2\kappa}{[\kappa y_1^p+(1-\kappa)y_2^p]^{1-\frac{1}{p}}}\right|\\&\quad\times\left|\zeta'\left([\kappa y_1^p+(1-\kappa)y_2^p]^{\frac{1}{p}}\right)\right|d\kappa\\&\le\frac{y_2^p-y_1^p}{2p}\left(\int_0^1\frac{|1-2\kappa|}{[\kappa y_1^p+(1-\kappa)y_2^p]^{1-\frac{1}{p}}}d\kappa\right)^{1-\frac{1}{q}}\\&\quad\times\left(\int_0^1\frac{|1-2\kappa|}{[\kappa y_1^p+(1-\kappa)y_2^p]^{1-\frac{1}{p}}}\right.\\&\quad\left.\times\left|\zeta'\left([\kappa y_1^p+(1-\kappa)y_2^p]^{\frac{1}{p}}\right)\right|^q d\kappa\right)^{\frac{1}{q}}.\end{aligned}\tag{II.9}$$

Since $|\zeta'|^q$ is exponentially (p, h)-convex on $[y_1, y_2]$, we achieve

$$\begin{aligned}&\left|\frac{\zeta(y_1)+\zeta(y_2)}{2}-\frac{p}{y_2^p-y_1^p}\int_{y_1}^{y_2}\frac{\zeta(w)}{w^{1-p}}dw\right|\\&\le\frac{y_2^p-y_1^p}{2p}\left(\int_0^1\frac{|1-2\kappa|}{[\kappa y_1^p+(1-\kappa)y_2^p]^{1-\frac{1}{p}}}d\kappa\right)^{1-\frac{1}{q}}\\&\times\left(\int_0^1\frac{||1-2\kappa||\left[h(\kappa)\left|\frac{\zeta'(y_1)}{e^{\alpha y_1}}\right|^q+h(1-\kappa)\left|\frac{\zeta'(y_2)}{e^{\alpha y_2}}\right|^q\right]}{[\kappa y_1^p+(1-\kappa)y_2^p]^{1-\frac{1}{p}}}d\kappa\right)^{\frac{1}{q}}\\&\le\frac{y_2^p-y_1^p}{2p}T_1^{1-\frac{1}{q}}\left[T_2\left|\frac{\zeta'(y_1)}{e^{\alpha y_1}}\right|^q+T_3\left|\frac{\zeta'(y_2)}{e^{\alpha y_2}}\right|^q\right]^{\frac{1}{q}}.\end{aligned}\tag{II.10}$$

Observe that,

$$\begin{aligned}&\int_0^1\frac{|1-2\kappa|}{[\kappa y_1^p+(1-\kappa)y_2^p]^{1-\frac{1}{p}}}d\kappa\\&=\frac{1}{4}\left(\frac{y_1^p+y_2^p}{2}\right)^{\frac{1}{p}-1}\left[{}_2F_1\left(1-\frac{1}{p},2;3;\frac{y_1^p-y_2^p}{y_1^p+y_2^p}\right)\right.\\&\quad\left.+{}_2F_1\left(1-\frac{1}{p},2;3;\frac{y_2^p-y_1^p}{y_1^p+y_2^p}\right)\right].\end{aligned}$$

Hence proved.

Note that once we let $h(\kappa) = \kappa$, we find

$$\begin{aligned}T_2=&\frac{1}{24}\left(\frac{y_1^p+y_2^p}{2}\right)^{\frac{1}{p}-1}\left[{}_2F_1\left(1-\frac{1}{p},2;4;\frac{y_1^p-y_2^p}{y_1^p+y_2^p}\right)\right.\\&+6\,{}_2F_1\left(1-\frac{1}{p},2;3;\frac{y_2^p-y_1^p}{y_1^p+y_2^p}\right)\\&\left.+{}_2F_1\left(1-\frac{1}{p},2;4;\frac{y_2^p-y_1^p}{y_1^p+y_2^p}\right)\right],\end{aligned}$$

and

$$T_3 = T_1 - T_2.$$

Thus we have following remark.

Remark 21: In Theorem 19,

(a) by letting $h(\kappa) = \kappa$, one gets inequality (2.8) of Theorem 2.3 in [12].

(b) by taking $\alpha = 0$ and $h(\kappa) = \kappa$, one gets the Theorem 7 in [9].

(c) by letting $h(\kappa) = \kappa^s$ and $p = 1$, one gets inequality (3.20) of Theorem 3.6 in [12].

(d) by letting $h(\kappa) = \kappa$ and $p = 1$, one gets the Theorem 5 in [1].

Corollary 22: Let $\zeta : \mathcal{Y} \to \mathbb{R}$ be a differentiable function on $\mathcal{Y}^\circ$ and $y_1, y_2 \in \mathcal{Y}$, $y_1 < y_2$, and $\zeta' \in L_1([y_1, y_2])$. If $|\zeta'|$ is exponentially (p, h)-convex on $[y_1, y_2]$, then

$$\begin{aligned}&\left|\frac{\zeta(y_1)+\zeta(y_2)}{2}-\frac{p}{y_2^p-y_1^p}\int_{y_1}^{y_2}\frac{\zeta(w)}{w^{1-p}}dw\right|\\&\quad\le\frac{y_2^p-y_1^p}{2p}\left[T_2\left|\frac{\zeta'(y_1)}{e^{\alpha y_1}}\right|+T_3\left|\frac{\zeta'(y_2)}{e^{\alpha y_2}}\right|\right],\end{aligned}\tag{II.11}$$

here T_2 and T_3 are as in Theorem 19.

Remark 23: In Corollary 22,

(a) by choosing $h(\kappa) = \kappa$, we find the Corollary 2.4 in [12].

(b) by choosing $h(\kappa) = \kappa$ and $\alpha = 0$, we find the Corollary 1 in [9].

(c) by choosing $h(\kappa) = \kappa^s$ and $p = 1$, we find the Theorem 3.5 in [12].

(d) by choosing $h(\kappa) = \kappa^s$, $p = 1$ and $\alpha = 0$, we find the Theorem 1 in [10].

(e) by taking $h(\kappa) = \kappa$ and $p = 1$, we attain the Theorem 3 in [1].

Theorem 24: Let $\zeta : \mathcal{Y} \to \mathbb{R}$ be a differentiable function on $\mathcal{Y}^\circ$. Let $y_1, y_2 \in \mathcal{Y}$, $y_1 < y_2$, and $\zeta' \in L_1([y_1, y_2])$. If $|\zeta'|^q$ is exponentially (p, h)-convex on $[y_1, y_2]$, and $q, l > 1$, $1/q + 1/l = 1$, and $\alpha \in \mathbb{R}$, then

$$\begin{aligned}&\left|\frac{\zeta(y_1)+\zeta(y_2)}{2}-\frac{p}{y_2^p-y_1^p}\int_{y_1}^{y_2}\frac{\zeta(w)}{w^{1-p}}dw\right|\\&\le\frac{y_2^p-y_1^p}{2p}\left(\frac{1}{l+1}\right)^{\frac{1}{l}}\left[T_4\left|\frac{\zeta'(y_1)}{e^{\alpha y_1}}\right|^q+T_5\left|\frac{\zeta'(y_2)}{e^{\alpha y_2}}\right|^q\right]^{\frac{1}{q}},\end{aligned}\tag{II.12}$$

where

$$T_4=\int_0^1\frac{h(\kappa)}{[\kappa y_1^p+(1-\kappa)y_2^p]^{q-\frac{q}{p}}}d\kappa,$$

and

$$T_5=\int_0^1\frac{h(1-\kappa)}{[\kappa y_1^p+(1-\kappa)y_2^p]^{q-\frac{q}{p}}}d\kappa.$$

Proof 25: Applying Hölder's inequality on (I.7) of Lemma 6 and using the exponential (p, h)-convexity of $|\zeta'|^q$

on $[y_1, y_2]$, we find

$$\left|\frac{\zeta(y_1)+\zeta(y_2)}{2} - \frac{p}{y_2^p - y_1^p}\int_{y_1}^{y_2}\frac{\zeta(w)}{w^{1-p}}dw\right|$$
$$\leq \frac{y_2^p - y_1^p}{2p}\left(\int_0^1 |1-2\kappa|^l d\kappa\right)^{\frac{1}{l}}$$
$$\times\left(\int_0^1 \frac{1}{[\kappa y_1^p + (1-\kappa)y_2^p]^{q(1-\frac{1}{p})}}\right.$$
$$\left.\times\left|\zeta'\left(\left[\kappa y_1^p + (1-\kappa)y_2^p\right]^{\frac{1}{p}}\right)\right|^q d\kappa\right)^{\frac{1}{q}}$$
$$\leq \frac{y_2^p - y_1^p}{2p}\left(\frac{1}{l+1}\right)^{\frac{1}{l}}$$
$$\times\left(\int_0^1 \frac{h(\kappa)\left|\frac{\zeta'(y_1)}{e^{\alpha y_1}}\right|^q + h(1-\kappa)\left|\frac{\zeta'(y_2)}{e^{\alpha y_2}}\right|^q}{[\kappa y_1^p + (1-\kappa)y_2^p]^{q-\frac{q}{p}}}d\kappa\right)^{\frac{1}{q}}$$
$$\leq \frac{y_2^p - y_1^p}{2p}\left(\frac{1}{l+1}\right)^{\frac{1}{l}}\left[T_4\left|\frac{\zeta'(y_1)}{e^{\alpha y_1}}\right|^q + T_5\left|\frac{\zeta'(y_2)}{e^{\alpha y_2}}\right|^q\right]^{\frac{1}{q}}. \quad \text{(II.13)}$$

Hence proved.

In above theorem once we let $h(\kappa) = \kappa$, we observe that

$$T_4 = \int_0^1 \frac{\kappa}{[\kappa y_1^p + (1-\kappa)y_2^p]^{q-\frac{q}{p}}}d\kappa$$
$$= \begin{cases} \frac{1}{2y_1^{qp-q}}\, {}_2F_1\left(q-\frac{q}{p}, 1; 3; 1-(\frac{y_2}{y_1})^p\right), & p<0 \\ \frac{1}{2y_2^{qp-q}}\, {}_2F_1\left(q-\frac{q}{p}, 2; 3; 1-(\frac{y_1}{y_2})^p\right), & p>0, \end{cases}$$
$$T_5 = \int_0^1 \frac{1-\kappa}{[\kappa y_1^p + (1-\kappa)y_2^p]^{q-\frac{q}{p}}}d\kappa$$
$$= \begin{cases} \frac{1}{2y_1^{qp-q}}\, {}_2F_1\left(q-\frac{q}{p}, 2; 3; 1-(\frac{y_2}{y_1})^p\right), & p<0 \\ \frac{1}{2y_2^{qp-q}}\, {}_2F_1\left(q-\frac{q}{p}, 1; 3; 1-(\frac{y_1}{y_2})^p\right), & p>0. \end{cases}$$

Thus we have following remark.

Remark 26: In Theorem 24,

(a) by letting $h(\kappa) = \kappa$, we find the Theorem 2.5 in [12].

(b) by letting $h(\kappa) = \kappa$ and $\alpha = 0$, we find the Theorem 8 in [9].

(c) by letting $h(\kappa) = \kappa$ and $p = 1$, we find the Theorem 4 in [1].

(d) by letting $h(\kappa) = \kappa^s$ and $p = 1$, we find the Theorem 3.7 in [12].

(e) by letting $h(\kappa) = \kappa^s$, $p = 1$ and $\alpha = 0$, we get Remark 3.4(a) in [12].

Theorem 27: Let $\zeta : \mathcal{Y} \to \mathbb{R}$ be a differentiable function on $\mathcal{Y}^\circ$ and $y_1, y_2 \in \mathcal{Y}$, $y_1 < y_2$, and $\zeta' \in L_1([y_1, y_2])$. If $|\zeta'|^q$ is exponentially (p, h)-convex on $[y_1, y_2]$, and $q, l > 1$, $1/q + 1/l = 1$, and $\alpha \in \mathbb{R}$, then

$$\left|\frac{\zeta(y_1)+\zeta(y_2)}{2} - \frac{p}{y_2^p - y_1^p}\int_{y_1}^{y_2}\frac{\zeta(w)}{w^{1-p}}dw\right|$$
$$\leq \frac{y_2^p - y_1^p}{2p}T_6^{\frac{1}{l}}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left(\frac{T_7\left|\frac{\zeta'(y_1)}{e^{\alpha y_1}}\right|^q + T_8\left|\frac{\zeta'(y_2)}{e^{\alpha y_2}}\right|^q}{2}\right)^{\frac{1}{q}}, \quad \text{(II.14)}$$

where

$$T_6 = \begin{cases} \frac{1}{2y_1^{pl-l}}\, {}_2F_1\left(l-\frac{l}{p}, 1; 2; 1-(\frac{y_2}{y_1})^p\right), & p<0 \\ \frac{1}{2y_2^{pl-l}}\, {}_2F_1\left(l-\frac{l}{p}, 1; 2; 1-(\frac{y_1}{y_2})^p\right), & p>0, \end{cases}$$
$$T_7 = \int_0^1 h(\kappa)|1-2\kappa|^q d\kappa,$$

and

$$T_8 = \int_0^1 h(1-\kappa)|1-2\kappa|^q d\kappa.$$

Proof 28: Using Hölder's inequality on (I.7) of Lemma 6 and then applying the exponential (p, h)-convexity of $|\zeta'|^q$ on $[y_1, y_2]$, we get

$$\left|\frac{\zeta(y_1)+\zeta(y_2)}{2} - \frac{p}{y_2^p - y_1^p}\int_{y_1}^{y_2}\frac{\zeta(w)}{w^{1-p}}dw\right|$$
$$\leq \frac{y_2^p - y_1^p}{2p}\left(\int_0^1 \frac{1}{[\kappa y_1^p + (1-\kappa)y_2^p]^{l-\frac{l}{p}}}d\kappa\right)^{\frac{1}{l}}$$
$$\times\left(\int_0^1 |1-2\kappa|^q\left|\zeta'\left(\left[\kappa y_1^p + (1-\kappa)y_2^p\right]^{\frac{1}{p}}\right)\right|^q d\kappa\right)^{\frac{1}{q}}$$
$$\leq \frac{y_2^p - y_1^p}{2p}B_6^{\frac{1}{l}}\left(\int_0^1 |1-2\kappa|^q\right.$$
$$\left.\times\left[h(\kappa)\left|\frac{\zeta'(y_1)}{e^{\alpha y_1}}\right|^q + h(1-\kappa)\left|\frac{\zeta'(y_2)}{e^{\alpha y_2}}\right|^q\right]d\kappa\right)^{\frac{1}{q}}$$
$$= \frac{y_2^p - y_1^p}{2p}T_6^{\frac{1}{l}}\left(T_7\left|\frac{\zeta'(y_1)}{e^{\alpha y_1}}\right|^q + T_8\left|\frac{\zeta'(y_2)}{e^{\alpha y_2}}\right|^q\right)^{\frac{1}{q}}, \quad \text{(II.15)}$$

note that

$$T_6 = \int_0^1 \frac{1}{[\kappa y_1^p + (1-\kappa)y_1^p]^{l-\frac{l}{p}}}d\kappa$$
$$= \begin{cases} \frac{1}{2y_1^{pl-l}}\, {}_2F_1\left(l-\frac{l}{p}, 1; 2; 1-(\frac{y_2}{y_1})^p\right), & p<0 \\ \frac{1}{2y_2^{pl-l}}\, {}_2F_1\left(l-\frac{l}{p}, 1; 2; 1-(\frac{y_1}{y_2})^p\right), & p>0. \end{cases} \quad \text{(II.16)}$$

By substituting (II.16) in (II.15), we get (II.14).

In above theorem once we let $h(\kappa) = \kappa$, we get

$$\int_0^1 \kappa|1-2\kappa|^q d\kappa = \int_0^1 (1-\kappa)|1-2\kappa|^q d\kappa = \frac{1}{2(q+1)}.$$

Thus we have following remark.

Remark 29: In Theorem 27,
(a) by letting $h(\kappa)=\kappa$, we obtain the Theorem 2.6 in [12].
(b) by letting $h(\kappa)=\kappa$ and $\alpha=0$, we obtain the Theorem 9 in [9].
Now for the next two results we take $h_1, h_2 : \mathcal{J} \subseteq \mathbb{R} \to \mathbb{R}$ be a positive function.

Theorem 30: Let $\zeta_1, \zeta_2 : \mathcal{Y} \to \mathbb{R}$ be integrable exponentially (p, h_1)- and (p, h_2)-convex functions, respectively. Let $y_1, y_2 \in \mathcal{Y}$ with $y_1 < y_2$. Let $\zeta_1\zeta_2 \in L_1([y_1, y_2])$ and $h_1h_2 \in L_1([0,1])$. Then for $\alpha \in \mathbb{R}$, we have

$$\frac{p}{y_2^p - y_1^p}\int_{y_1}^{y_2}\frac{\zeta_1(w)\zeta_2(w)}{w^{1-p}e^{\alpha w}}dw \le M(y_1,y_2)\int_0^1 h_1(\kappa)h_2(\kappa)d\kappa + N(y_1,y_2)\int_0^1 h_1(\kappa)h_2(1-\kappa)d\kappa. \tag{II.17}$$

where

$$M(y_1,y_2)=\frac{\zeta_1(y_1)}{e^{\alpha y_1}}\frac{\zeta_2(y_1)}{e^{\alpha y_1}}+\frac{\zeta_1(y_2)}{e^{\alpha y_2}}\frac{\zeta_2(y_2)}{e^{\alpha y_2}},$$

and

$$N(y_1,y_2)=\frac{\zeta_1(y_1)}{e^{\alpha y_1}}\frac{\zeta_2(y_2)}{e^{\alpha y_2}}+\frac{\zeta_1(y_2)}{e^{\alpha y_2}}\frac{\zeta_2(y_1)}{e^{\alpha y_1}}.$$

Proof 31: Since ζ_1 and ζ_2 are exponentially (p, h_1)- and (p, h_2)-convex functions, respectively, we have

$$\begin{aligned}\frac{\zeta_1\left(\left[\kappa y_1^p+(1-\kappa)y_2^p\right]^{\frac{1}{p}}\right)}{e^{\alpha(\kappa y_1^p+(1-\kappa)y_2^p)^{\frac{1}{p}}}} &\le \zeta_1\left(\left[\kappa y_1^p+(1-\kappa)y_2^p\right]^{\frac{1}{p}}\right)\\ &\le h_1(\kappa)\frac{\zeta_1(y_1)}{e^{\alpha y_1}}+h_1(1-\kappa)\frac{\zeta_1(y_2)}{e^{\alpha y_2}},\end{aligned} \tag{II.18}$$

and

$$\begin{aligned}\frac{\zeta_2\left(\left[\kappa y_1^p+(1-\kappa)y_2^p\right]^{\frac{1}{p}}\right)}{e^{\alpha(\kappa y_1^p+(1-\kappa)y_2^p)^{\frac{1}{p}}}} &\le \zeta_2\left(\left[\kappa y_1^p+(1-\kappa)y_2^p\right]^{\frac{1}{p}}\right)\\ &\le h_2(\kappa)\frac{\zeta_2(y_1)}{e^{\alpha y_1}}+h_2(1-\kappa)\frac{\zeta_2(y_2)}{e^{\alpha y_2}}.\end{aligned} \tag{II.19}$$

From (II.18) and (II.19), we get

$$\begin{aligned}&\frac{\zeta_1\left(\left[\kappa y_1^p+(1-\kappa)y_2^p\right]^{\frac{1}{p}}\right)\zeta_2\left(\left[\kappa y_1^p+(1-\kappa)y_2^p\right]^{\frac{1}{p}}\right)}{e^{\alpha(\kappa y_1^p+(1-\kappa)y_2^p)^{\frac{1}{p}}}}\\ &\le \left[h_1(\kappa)\frac{\zeta_1(y_1)}{e^{\alpha y_1}}+h_1(1-\kappa)\frac{\zeta_1(y_2)}{e^{\alpha y_2}}\right]\\ &\quad\times\left[h_2(\kappa)\frac{\zeta_2(y_1)}{e^{\alpha y_1}}+h_2(1-\kappa)\frac{\zeta_2(y_2)}{e^{\alpha y_2}}\right]\\ &= h_1(\kappa)h_2(\kappa)\frac{\zeta_1(y_1)}{e^{\alpha y_1}}\frac{\zeta_2(y_1)}{e^{\alpha y_1}}+h_1(\kappa)h_2(1-\kappa)\frac{\zeta_1(y_1)}{e^{\alpha y_1}}\frac{\zeta_2(y_2)}{e^{\alpha y_2}}\\ &\quad+h_1(1-\kappa)h_2(\kappa)\frac{\zeta_1(y_2)}{e^{\alpha y_2}}\frac{\zeta_2(y_1)}{e^{\alpha y_1}}\\ &\quad+h_1(1-\kappa)h_2(1-\kappa)\frac{\zeta_1(y_2)}{e^{\alpha y_2}}\frac{\zeta_2(y_2)}{e^{\alpha y_2}}.\end{aligned} \tag{II.20}$$

By integrating inequality (II.20), on both sides, over $\kappa \in [0,1]$, we find

$$\begin{aligned}&\int_0^1\frac{\zeta_1\left(\left[\kappa y_1^p+(1-\kappa)y_2^p\right]^{\frac{1}{p}}\right)\zeta_2\left(\left[\kappa y_1^p+(1-\kappa)y_2^p\right]^{\frac{1}{p}}\right)}{e^{\alpha(ry_1^p+(1-r)y_2^p)^{\frac{1}{p}}}}d\kappa\\ &\le \int_0^1\left[h_1(\kappa)\frac{\zeta_1(y_1)}{e^{\alpha y_1}}+h_1(1-\kappa)\frac{\zeta_1(y_2)}{e^{\alpha y_2}}\right]\\ &\quad\times\left[h_2(\kappa)\frac{\zeta_2(y_1)}{e^{\alpha y_1}}+h_2(1-\kappa)\frac{\zeta_2(y_2)}{e^{\alpha y_2}}\right]d\kappa\\ &=\frac{\zeta_1(y_1)}{e^{\alpha y_1}}\frac{\zeta_2(y_1)}{e^{\alpha y_1}}\int_0^1 h_1(\kappa)h_2(\kappa)d\kappa\\ &\quad+\frac{\zeta_1(y_1)}{e^{\alpha y_1}}\frac{\zeta_2(y_2)}{e^{\alpha y_2}}\int_0^1 h_1(\kappa)h_2(1-\kappa)d\kappa\\ &\quad+\frac{\zeta_1(y_2)}{e^{\alpha y_2}}\frac{\zeta_2(y_1)}{e^{\alpha y_1}}\int_0^1 h_1(1-\kappa)h_2(\kappa)d\kappa\\ &\quad+\frac{\zeta_1(y_2)}{e^{\alpha y_2}}\frac{\zeta_2(y_2)}{e^{\alpha y_2}}\int_0^1 h_1(1-\kappa)h_2(1-\kappa)d\kappa.\end{aligned}$$

Then

$$\begin{aligned}&\frac{p}{y_2^p-y_1^p}\int_{y_1}^{y_2}\frac{\zeta_1(w)\zeta_2(w)}{w^{1-p}e^{\alpha w}}dw\\ &\le M(y_1,y_2)\int_0^1 h_1(\kappa)h_2(\kappa)d\kappa\\ &\quad+N(y_1,y_2)\int_0^1 h_1(\kappa)h_2(1-\kappa)d\kappa.\end{aligned} \tag{II.21}$$

where

$$M(y_1,y_2)=\frac{\zeta_1(y_1)}{e^{\alpha y_1}}\frac{\zeta_2(y_1)}{e^{\alpha y_1}}+\frac{\zeta_1(y_2)}{e^{\alpha y_2}}\frac{\zeta_2(y_2)}{e^{\alpha y_2}},$$

and

$$N(y_1,y_2)=\frac{\zeta_1(y_1)}{e^{\alpha y_1}}\frac{\zeta_2(y_2)}{e^{\alpha y_2}}+\frac{\zeta_1(y_2)}{e^{\alpha y_2}}\frac{\zeta_2(y_1)}{e^{\alpha y_1}}.$$

Corollary 32: Under the similar assumptions of Theorem 30,
(i) for $h_1(\kappa)=h_2(\kappa)=\kappa$, we get

$$\frac{p}{y_2^p-y_1^p}\int_{y_1}^{y_2}\frac{\zeta_1(w)\zeta_2(w)}{w^{1-p}e^{\alpha w}}dw \le \frac{1}{3}M(y_1,y_2)+\frac{1}{6}N(y_1,y_2); \tag{II.22}$$

(ii) for $h_1(\kappa)=h_2(\kappa)=\kappa$ and $p=1$, we get

$$\frac{1}{y_2-y_1}\int_{y_1}^{y_2}\frac{\zeta_1(w)\zeta_2(w)}{e^{\alpha w}}dw \le \frac{1}{3}M(y_1,y_2)+\frac{1}{6}N(y_1,y_2); \tag{II.23}$$

(iii) for $h_1(\kappa)=h_2(\kappa)=\kappa^s$, we get

$$\begin{aligned}&\frac{p}{y_2^p-y_1^p}\int_{y_1}^{y_2}\frac{\zeta_1(w)\zeta_2(w)}{w^{1-p}e^{\alpha w}}dw\\ &\le \frac{1}{2s+1}M(y_1,y_2)+\frac{s}{(s+1)(2s+1)}N(y_1,y_2);\end{aligned} \tag{II.24}$$

(iv) for $h_1(\kappa) = h_2(\kappa) = \kappa^s$ and $p = 1$, we get

$$\frac{1}{y_2 - y_1}\int_{y_1}^{y_2}\frac{\zeta_1(w)\zeta_2(w)}{e^{\alpha w}}dw \leq \frac{1}{2s+1}M(y_1, y_2) + \frac{s}{(s+1)(2s+1)}N(y_1, y_2) \quad \text{(II.25)}$$

(v) for $h_1(\kappa) = \kappa$ and $h_2(\kappa) = \kappa^s$, we get

$$\frac{p}{y_2^p - y_1^p}\int_{y_1}^{y_2}\frac{\zeta_1(w)\zeta_2(w)}{w^{1-p}e^{\alpha w}}dw \leq \frac{1}{s+1}M(y_1, y_2) + \frac{1}{(s+1)(s+2)}N(y_1, y_2); \quad \text{(II.26)}$$

(vi) for $h_1(\kappa) = \kappa$, $h_2(\kappa) = \kappa^s$ and $p = 1$, we get

$$\frac{1}{y_2 - y_1}\int_{y_1}^{y_2}\frac{\zeta_1(w)\zeta_2(w)}{e^{\alpha w}}dw \leq \frac{1}{s+2}M(y_1, y_2) + \frac{1}{(s+1)(s+2)}N(y_1, y_2). \quad \text{(II.27)}$$

where $M(y_1, y_2)$ and $N(y_1, y_2)$ defined in Theorem 30.

Remark 33: (a) In Corollary 32 (ii), if we let $\alpha = 0$, we get inequality (1) of Theorem 1 in [15].

(b) In Corollary 32 (vi), if we let $\alpha = 0$, we get the inequality of Theorem 5 in [10].

III. CONCLUSION

This research investigation includes some Hermite-Hadamard type inequalities for exponentially (p, h)-convex function. Some special cases are discussed, which implies new and previous results.

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