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Bipartite Consensus Problems on Second-Order Signed Networks With Heterogeneous Topologies

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ABSTRACT This paper is devoted to the convergence problem for second-order signed networks that are associated with two signed graphs in the presence of heterogeneous topologies. An eigenvalue analysis approach is presented to develop convergence results for second-order signed networks, which employs a sign-consistency property for signed graph pairs. When the sign-consistency of two heterogeneous signed graphs and the connectivity of their union are given, bipartite consensus (respectively, state stability) can be derived for second-order signed networks if and only if the union signed graph is structurally balanced (respectively, unbalanced). Two examples are provided to illustrate the effectiveness of the obtained results.

INDEX TERMS Bipartite consensus, eigenvalue analysis, heterogeneous topology, signed network, structural balance.

I. INTRODUCTION

Networks involving multiple nodes (vertices or agents) have received considerable attention from various application fields recently, such as unmanned aerial vehicles, formation satellites and mobile robots. Traditionally, networks refer to cooperative networks that involve nodes to implement collaborative tasks, where consensus (or agreement) of nodes plays a fundamental role (see, e.g., [1], [2] for more explanations). For communications among nodes, the traditional networks resort to unsigned graphs whose positive adjacency edge weights can interpret the collaborative relations among nodes. However, there are many situations, especially in the area of social networks, for which the antagonistic relations among nodes should be also noticed since the unavoidable relations like, e.g., approve/disapprove, like/dislike, or trust/distrust may be encountered. This leads to a new class of signed networks that may have both cooperative and antagonistic relations. Compared to traditional networks, signed networks use signed graphs to model communications among nodes, with positive/negative adjacency edge weights to represent cooperative/antagonistic relations, respectively.

Signed networks barely reach consensus, and instead the so-called bipartite consensus usually emerges, which means that all nodes can reach agreement with two camps and the agreement values of different camps are opposite [3]. Recently, bipartite consensus problems for signed networks have been extensively discussed (see, e.g., [4]–[8]). Many additional problems have been also investigated for signed networks, such as finite-time bipartite consensus [9]–[11], modulus consensus [12]–[14], bipartite containment [15], [16], interval bipartite consensus [17]–[19] and dynamic behaviors of signed networks with switching topologies [20]. At the same time, many promising approaches to the behavior analysis of signed networks have been proposed, see, e.g., [21] for a lifting approach, [22], [23] for a feedback approach, [24] for an M-matrix approach, and [25] for a frequency-domain approach. The aforementioned results most contribute to signed networks under homogeneous signed graphs. However, in practical networked systems, the communication topologies of signed networks may be heterogeneous due to the disturbance of external environments and the restriction of transmission abilities. Networked systems with heterogeneous topologies may be more suitable than these systems under homogeneous topologies in actual applications (see [26]–[28] for more details). Though in [29], an attempt has been made to accommodate bipartite consensus problems for signed networks with heterogeneous topologies, it is achieved only by extending the network-to-network control results of [30]. In fact, it is even unclear what the eigenvalues are distributed for second-order

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signed networks. How to explore the relationship between the eigenvalue distribution and the behaviour analysis of second-order signed networks is also to be solved especially in the presence of heterogeneous topologies.

In this paper, bipartite consensus problems on second-order signed networks with heterogeneous topologies are considered. We introduce a class of sign-consistency properties for pairs of signed graphs, based on which an eigenvalue analysis approach to exploring the convergence results of the second-order signed networks is developed. Moreover, the relation among structural balance, connectivity and eigenvalue distribution is established for second-order signed networks with sign-consistent heterogeneous topologies. When the two signed graphs representing heterogeneous topologies are sign-consistent and their union is connected, second-order signed networks can achieve bipartite consensus (respectively, state stability) if and only if the union of the two signed graphs is structurally balanced (respectively, unbalanced). This is also demonstrated with simulation tests.

The organization of the reminder of our paper is as follows. In Section II, signed graphs and problems on signed networks are introduced. We present necessary and sufficient conditions for convergence behaviours of second-order signed networks in Section III. Simulations and conclusions are given in Sections IV and V, respectively.

Notations: Denote $\mathscr{I}_n = \{1, 2, \cdots, n\}$, $1_n = [1, 1, \cdots, 1]^T \in \mathbb{R}^n$, $0_n = [0, 0, \cdots, 0]^T \in \mathbb{R}^n$ and $\mathscr{D}_n = \{D = \operatorname{diag}\{d_1, d_2, \cdots, d_n\} : d_i \in \{1, -1\}, i = 1, 2, \cdots, n\}$. For $A = [a_{ij}] \in \mathbb{R}^{p\times q}$, $|A| = [|a_{ij}|]$ is nonnegative, namely, $|A| \geq 0$. If $p = q$, then $\pi_+(A)$, $\pi_-(A)$, and $\pi_0(A)$ denote the number of eigenvalues (counted with the algebraic multiplicity) of A that have the positive, negative and zero real parts, respectively. For a complex number $b \in \mathbb{C}$, let $\operatorname{Re}(b)$ be the real part of b. We denote I_n and $0_{n\times n}$ as the identity and null matrices with m dimensions, respectively.

II. PRELIMINARIES AND PROBLEM OF SIGNED NETWORKS

A. SIGNED GRAPHS

A signed graph is denoted by a triple $\mathscr{G} = (\mathscr{V}, \mathscr{E}, \mathscr{A})$ [31], including a node set $\mathscr{V} = \{v_i : \forall i \in \mathscr{I}_n\}$, an edge set $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V} = \left\{(v_i, v_j) : \forall i, j \in \mathscr{I}_n\right\}$, and an adjacency weighted matrix $\mathscr{A} = \left[a_{ij}\right] \in \mathbb{R}^{n\times n}$, where $a_{ij} \neq 0 \Leftrightarrow \left(v_j, v_i\right) \in \mathscr{E}$ and $a_{ij} = 0$, otherwise. Let $\mathscr{G}$ have no self-loops, namely, $a_{ii} = 0, \forall i \in \mathscr{I}_n$. Each edge $\left(v_j, v_i\right) \in \mathscr{E}, \forall j \neq i$ indicates v_i with a neighbour v_j, and let $\mathscr{N}_i = \left\{j : (v_j, v_i) \in \mathscr{E}\right\}$ be the index set of all neighbours of v_i. Assume $\mathscr{A} = \mathscr{A}^T$, i.e., $\mathscr{G}$ is an undirected signed graph. If $\mathscr{G}$ has sequential edges $(v_i, v_{l_1}), (v_{l_1}, v_{l_2}), \cdots, (v_{l_{m-1}}, v_j)$ for distinct nodes $v_i, v_{l_1}, \cdots, v_{l_{m-1}}, v_j$, then $\mathscr{G}$ has a path between v_i and v_j. Thus, $\mathscr{G}$ is connected if there exists a path between each pair of distinct nodes. In addition, we specifically denote the signed graph $\mathscr{G}$ associated with $\mathscr{A}$ as $\mathscr{G}(\mathscr{A})$, and $\mathscr{L}_{\mathscr{A}} = \left[l_{ij}^{\mathscr{A}}\right] \in \mathbb{R}^{n\times n}$ defines its Laplacian matrix, where

$$l_{ij}^{\mathscr{A}} = \begin{cases} \sum_{k\in\mathscr{N}_i} |a_{ik}|, & j = i \\ -a_{ij}, & j \neq i. \end{cases}$$

By following [3], $\mathscr{G}(\mathscr{A})$ is structurally balanced if $\mathscr{V}$ admits two disjoint sets $\mathscr{V}^{(1)}$ and $\mathscr{V}^{(2)}$ such that $a_{ij} \geq 0$, $\forall v_i, v_j \in \mathscr{V}^{(1)}$ or $\forall v_i, v_j \in \mathscr{V}^{(2)}$ and $a_{ij} \leq 0$, $\forall v_i \in \mathscr{V}^{(1)}$, $\forall v_j \in \mathscr{V}^{(2)}$ or $\forall v_i \in \mathscr{V}^{(2)}$, $\forall v_j \in \mathscr{V}^{(1)}$; and it is structurally unbalanced, otherwise. The structural balance of $\mathscr{G}(\mathscr{A})$ equivalently implies $D\mathscr{A}D = |\mathscr{A}|$ for some $D \in \mathscr{D}_n$.

For $\mathscr{G}(\mathscr{A})$, together with two other signed graphs $\mathscr{G}(\mathscr{A}^a) = (\mathscr{V}, \mathscr{E}^a, \mathscr{A}^a)$ and $\mathscr{G}(\mathscr{A}^b) = (\mathscr{V}, \mathscr{E}^b, \mathscr{A}^b)$, if $\mathscr{E} = \mathscr{E}^a \cup \mathscr{E}^b$, then $\mathscr{G}(\mathscr{A})$ is called the union of $\mathscr{G}(\mathscr{A}^a)$ and $\mathscr{G}\left(\mathscr{A}^b\right)$, for which we denote $\mathscr{G}(\mathscr{A}) = \mathscr{G}(\mathscr{A}^a) \cup \mathscr{G}\left(\mathscr{A}^b\right)$. For $\mathscr{A}^a = \left[a_{ij}^a\right] \in \mathbb{R}^{n\times n}$ and $\mathscr{A}^b = \left[a_{ij}^b\right] \in \mathbb{R}^{n\times n}$, if $a_{ij}^a a_{ij}^b \geq 0$, $\forall i, j \in \mathscr{I}_n$, then we say that $\mathscr{G}(\mathscr{A}^a)$ and $\mathscr{G}\left(\mathscr{A}^b\right)$ are sign-consistent; and we say that they are sign-inconsistent, otherwise.

B. NETWORK DYNAMICS

We consider second-order signed networks under two signed graphs $\mathscr{G}(\mathscr{A}^c)$ and $\mathscr{G}\left(\mathscr{A}^d\right)$. Let $\mathscr{A}^c = \left[a_{ij}^c\right] \in \mathbb{R}^{n\times n}$ and $\mathscr{A}^d = \left[a_{ij}^d\right] \in \mathbb{R}^{n\times n}$, and the neighbour index sets of v_i in $\mathscr{G}(\mathscr{A}^c)$ and $\mathscr{G}\left(\mathscr{A}^d\right)$ be $\mathscr{N}_i^c$ and $\mathscr{N}_i^d$, respectively. Then the dynamics of node v_i, $\forall i \in \mathscr{I}_n$ is given by

$$\dot{x}_i(t) = \sum_{j\in\mathscr{N}_i^c} a_{ij}^c \left[x_j(t) - \operatorname{sgn}\left(a_{ij}^c\right) x_i(t)\right] + u_i(t) \quad (1)$$

where $x_i(t) \in \mathbb{R}$ is the state, and $u_i(t) \in \mathbb{R}$ is the driving input that satisfies

$$\dot{u}_i(t) = -ku_i(t) + \sum_{j\in\mathscr{N}_i^d} a_{ij}^d \left[x_j(t) - \operatorname{sgn}\left(a_{ij}^d\right) x_i(t)\right] \quad (2)$$

for some $k > 0$. Clearly, (1) and (2) form a second-order signed network with nonidentical topologies that are described by two signed graphs $\mathscr{G}(\mathscr{A}^c)$ and $\mathscr{G}\left(\mathscr{A}^d\right)$.

The problem addressed in this paper is to find conditions of $\mathscr{G}(\mathscr{A}^c)$ and $\mathscr{G}\left(\mathscr{A}^d\right)$ such that the states of the signed network (1) and (2) satisfies

$$\lim_{t\to\infty} x_i(t) \in \{\pm\xi\} \text{ and } \lim_{t\to\infty} u_i(t) = 0, \quad \forall i \in \mathscr{I}_n$$

where $\xi \geq 0$ is closely related to $x_i(0)$ and $u_i(0)$ for all $i \in \mathscr{I}_n$. If, for all $x_i(0)$ and $u_i(0)$, $i \in \mathscr{I}_n$, $\xi = 0$ holds, then we say that the signed network (1) and (2) achieves the (state) stability; and otherwise, we say that it achieves the bipartite consensus.

III. MAIN RESULTS

Next, we aim at exploring convergence analysis for second-order signed networks given by (1) and (2). We can rewrite (1) and (2) in a compact vector form of

$$\begin{bmatrix}\dot{x}(t)\\ \dot{u}(t)\end{bmatrix}=\underbrace{\begin{bmatrix}-\mathscr{L}_{\mathscr{A}^c} & I\\ -\mathscr{L}_{\mathscr{A}^d} & -kI\end{bmatrix}}_{\triangleq A}\underbrace{\begin{bmatrix}x(t)\\ u(t)\end{bmatrix}}_{\triangleq X(t)} \tag{3}$$

where $x(t) = [x_1(t), x_2(t), \cdots, x_n(t)]^T$, $u(t) = [u_1(t), u_2(t), \cdots, u_n(t)]^T$, and $\mathscr{L}_{\mathscr{A}^c}$ and $\mathscr{L}_{\mathscr{A}^d}$ denote the Laplacian matrices of $\mathscr{G}(\mathscr{A}^c)$ and $\mathscr{G}(\mathscr{A}^d)$, respectively. Let us denote $X_0 = X(0)$ as the initial state of the system (3). Then the solution to (3) is $X(t) = e^{At}X_0$, by which it is of vital importance to develop the eigenvalue distribution of A in order to determine the convergence of $X(t)$ as $t \to \infty$. For this purpose, we show a similarity transformation of A as

$$\begin{aligned}\Omega &= \begin{bmatrix}kI & I\\ I & 0_{n\times n}\end{bmatrix}A\begin{bmatrix}kI & I\\ I & 0\end{bmatrix}^{-1}\\ &= \begin{bmatrix}0 & -\left(k\mathscr{L}_{\mathscr{A}^c}+\mathscr{L}_{\mathscr{A}^d}\right)\\ I & -\left(kI+\mathscr{L}_{\mathscr{A}^c}\right)\end{bmatrix}.\end{aligned} \tag{4}$$

With the above discussions, the following theorem presents convergence conditions for second-order signed networks with sign-consistent heterogeneous topologies.

Theorem 1: Let two signed graphs $\mathscr{G}(\mathscr{A}^c)$ and $\mathscr{G}(\mathscr{A}^d)$ be sign-consistent, and the union $\mathscr{G}(\mathscr{A}^c)\cup\mathscr{G}(\mathscr{A}^d)$ be connected. Then the signed network (1) and (2) converges, and further two convergence results hold as follows.

1) Bipartite consensus can be accomplished if and only if $\mathscr{G}(k\mathscr{A}^c+\mathscr{A}^d)$ is structurally balanced, and moreover,

$$\lim_{t\to\infty} x(t) = \left[\frac{1}{n}1_n^T D\left(x(0)+k^{-1}u(0)\right)\right]D1_n,$$
$$\lim_{t\to\infty} u(t) = 0$$

where $D \in \mathscr{D}_n$ satisfies $D\left(k\mathscr{A}^c+\mathscr{A}^d\right)D = \left|k\mathscr{A}^c+\mathscr{A}^d\right|$.

2) Stability is achieved if and only if $\mathscr{G}(k\mathscr{A}^c+\mathscr{A}^d)$ is structurally unbalanced.

Remark 1: Based on Theorem 1, the convergence problems are solved for second-order signed networks in the presence of heterogeneous topologies given by two distinct signed graphs. Moreover, bipartite consensus and stability correspond to the mutually exclusive structural balance and structural unbalance properties of the union of the two signed graphs, respectively. Theorem 1 extends the existing convergence results of signed networks whose topology is associated with one single graph (see, e.g., [3]). Since traditional unsigned (or cooperative) networks are a special case of signed networks, Theorem 1 also extends existing second-order consensus results that only admit cooperative interactions among nodes.

To prove Theorem 1, we introduce some helpful lemmas, especially by noting (3) and (4).

Lemma 1: For any $k > 0$, if $\mathscr{G}(\mathscr{A}^c)$ and $\mathscr{G}(\mathscr{A}^d)$ are sign-consistent signed graphs, then

$$\mathscr{G}(\mathscr{A}^c)\cup\mathscr{G}\left(\mathscr{A}^d\right)=\mathscr{G}\left(k\mathscr{A}^c+\mathscr{A}^d\right). \tag{5}$$

Further, if $\mathscr{G}(\mathscr{A}^c)\cup\mathscr{G}(\mathscr{A}^d)$ is connected, then

1) $k\mathscr{L}_{\mathscr{A}^c}+\mathscr{L}_{\mathscr{A}^d}$ is positive semi-definite and has exactly one zero eigenvalue if and only if $\mathscr{G}(k\mathscr{A}^c+\mathscr{A}^d)$ is structurally balanced;
2) $k\mathscr{L}_{\mathscr{A}^c}+\mathscr{L}_{\mathscr{A}^d}$ is positive definite if and only if $\mathscr{G}(k\mathscr{A}^c+\mathscr{A}^d)$ is structurally unbalanced.

Proof: From the definition for sign-consistency of any two signed graphs, (5) can be developed straightforwardly. Further, we can verify that $k\mathscr{L}_{\mathscr{A}^c}+\mathscr{L}_{\mathscr{A}^d} = \mathscr{L}_{k\mathscr{A}^c+\mathscr{A}^d}$ is the Laplacian matrix associated with $k\mathscr{A}^c+\mathscr{A}^d$. Thus, we can benefit from [3, Lemma 1 and Corollary 2] to establish the results (1) and (2) as a consequence.

Remark 2: From (5), $\mathscr{G}(k\mathscr{A}^c+\mathscr{A}^d) = \mathscr{G}(k\mathscr{A}^c)\cup\mathscr{G}(\mathscr{A}^d)$ also holds. In fact, if we denote $\mathscr{G}(\mathscr{A}^a) = \mathscr{G}(k\mathscr{A}^c)\cup\mathscr{G}(\mathscr{A}^d)$ and $\mathscr{G}(\mathscr{A}^b) = \mathscr{G}(\mathscr{A}^c)\cup\mathscr{G}(\mathscr{A}^d)$, then the entries of $\mathscr{A}^a$ and $\mathscr{A}^b$ fulfil $sign(a_{ij}^a) = sign(ka_{ij}^b)$, $\forall i,j \in \mathscr{I}_n$, $\forall k > 0$. It implies that if $\mathscr{G}(\mathscr{A}^c)$ and $\mathscr{G}(\mathscr{A}^d)$ are sign-consistent, then for any $k > 0$, $\mathscr{G}(\mathscr{A}^c)\cup\mathscr{G}(\mathscr{A}^d)$, $\mathscr{G}(k\mathscr{A}^c)\cup\mathscr{G}(\mathscr{A}^d)$, $\mathscr{G}(k\mathscr{A}^c+\mathscr{A}^d)$ and $\mathscr{G}(\mathscr{A}^c+k^{-1}\mathscr{A}^d)$ have the same properties of connectivity and structural balance.

Lemma 2: Consider $k > 0$ and two sign-consistent signed graphs $\mathscr{G}(\mathscr{A}^c)$ and $\mathscr{G}(\mathscr{A}^d)$. If $\mathscr{G}(k\mathscr{A}^c+\mathscr{A}^d)$ is structurally balanced, then each of $\mathscr{G}(\mathscr{A}^c)$ and $\mathscr{G}(\mathscr{A}^d)$ is structurally balanced. Equivalently, if $D\left(k\mathscr{A}^c+\mathscr{A}^d\right)D = \left|k\mathscr{A}^c+\mathscr{A}^d\right|$ holds for some $D \in \mathscr{D}_n$, then $D\mathscr{A}^cD = |\mathscr{A}^c|$ and $D\mathscr{A}^dD = |\mathscr{A}^d|$ hold simultaneously.

Proof: Based on [3, Lemma 1], the structural balance of $\mathscr{G}(k\mathscr{A}^c+\mathscr{A}^d)$ equivalently implies that there exists some $D \in \mathscr{D}_n$ satisfying

$$D\left(k\mathscr{A}^c+\mathscr{A}^d\right)D=\left|k\mathscr{A}^c+\mathscr{A}^d\right|. \tag{6}$$

Since $\mathscr{G}(\mathscr{A}^c)$ and $\mathscr{G}(\mathscr{A}^d)$ are sign-consistent, we can validate

$$\left|k\mathscr{A}^c+\mathscr{A}^d\right| = k\left|\mathscr{A}^c\right|+\left|\mathscr{A}^d\right|$$

which, together with (6), $D\mathscr{A}^cD \le |\mathscr{A}^c|$ and $D\mathscr{A}^dD \le |\mathscr{A}^d|$, leads to $D\mathscr{A}^cD = |\mathscr{A}^c|$ and $D\mathscr{A}^dD = |\mathscr{A}^d|$ immediately.

Remark 3: It is worth highlighting that Lemmas 1 and 2 take advantage of the sign-consistency property between two signed graphs. The results 1) and 2) of Lemma 1 also benefit from the diagonal dominance of the Laplacian matrices of signed graphs (see also [3]). In addition, a consequence of this diagonal dominance property of the Laplacian matrices is that $kI+\mathscr{L}_{\mathscr{A}^c}$ is positive definite for any $k > 0$.

Lemma 3: Let matrices $U \in \mathbb{R}^{n\times n}$, $V \in \mathbb{R}^{n\times n}$ and $W \in \mathbb{R}^{n\times n}$ be symmetric. If U is nonsingular and V is positive definite, then for a quadratic matrix polynomial as

$$F(\lambda) = U\lambda^2+V\lambda+W$$

the eigenvalues satisfy $\pi_+(F) = \pi_-(U)+\pi_-(W)$, $\pi_-(F) = \pi_+(U)+\pi_+(W)$ and $\pi_0(F) = \pi_0(W)$, where $\pi_+(F)+\pi_-(F)+\pi_0(F) = 2n$, and $\pi_+(F)$, $\pi_-(F)$, and

$\pi_0(F)$ are the numbers of eigenvalues of $F(\lambda)$ (i.e., the zeros of $\det(F(\lambda))$) which have positive, negative and zero real parts, respectively.

Proof: This is a direct consequence of [32, Lemma 4.4] and [33, Theorem 3].

By Lemmas 1-3, we can provide the proof of Theorem 1 as follows.

Proof of Theorem 1: Sufficiency: We begin with transforming A into the Jordan canonical form as follows:

$$J = P^{-1}AP \tag{7}$$

with

$$P = \left[w_1, w_2, \cdots, w_{2n}\right] \text{ and } P^{-1} = \begin{bmatrix} y_1^T \\ y_2^T \\ \vdots \\ y_{2n}^T \end{bmatrix}$$

where w_k denotes one of the right eigenvectors or generalized eigenvectors of A associated with $\lambda_k(A)$, and y_l and w_k satisfy $y_l^T w_k = 1$ for $l = k$ and $y_l^T w_k = 0$ for $l \neq k$. Let us denote the eigenvalues of A as $\lambda_1, \lambda_2, \cdots, \lambda_{2n}$. Without any loss of generality, we apply (7) to reformulate e^{At} in the form of

$$e^{At} = P \begin{bmatrix} e^{\lambda_1 t} & & & & & \\ & \ddots & & & & \\ & & e^{\lambda_m t} & & & \\ & & & J_1 & & \\ & & & & \ddots & \\ & & & & & J_r \end{bmatrix} P^{-1} \tag{8}$$

with

$$J_i = \begin{bmatrix} e^{\lambda_{m+i}t} & te^{\lambda_{m+i}t} & \frac{1}{2!}t^2e^{\lambda_{m+i}t} & \cdots & \frac{1}{(s_i-1)!}t^{s_i-1}e^{\lambda_{m+i}t} \\ & e^{\lambda_{m+i}t} & te^{\lambda_{m+i}t} & \cdots & \vdots \\ & & e^{\lambda_{m+i}t} & \cdots & \vdots \\ & & & \ddots & \vdots \\ & & & & e^{\lambda_{m+i}t} \end{bmatrix}$$

where $J_i \in \mathbb{R}^{s_i \times s_i}$ for $i \in \mathscr{I}_r$ and $m + \sum_{i=1}^{r} s_i = 2n$. For convenience, denote $Q(\lambda) = [I\lambda^2 + (kI + \mathscr{L}_{\mathscr{A}^c})\lambda + k\mathscr{L}_{\mathscr{A}^c} + \mathscr{L}_{\mathscr{A}^d}]$. Two cases are divided to obtain this sufficiency.

Case 1): $\mathscr{G}(\mathscr{A}^c) \cup \mathscr{G}(\mathscr{A}^d)$ is structurally balanced.

By considering $Q(\lambda) = I\lambda^2 + (kI + \mathscr{L}_{\mathscr{A}^c})\lambda + k\mathscr{L}_{\mathscr{A}^c} + \mathscr{L}_{\mathscr{A}^d}$, it follows from Lemmas 1 and 2 as well as Remark 3 that for any $k > 0$, $kI + \mathscr{L}_{\mathscr{A}^c}$ and $k\mathscr{L}_{\mathscr{A}^c} + \mathscr{L}_{\mathscr{A}^d}$ are positive definite and positive semi-definite, respectively. It ensures that Lemma 3 applies to $Q(\lambda)$. Thus, we again consider the property 1) of Lemma 1 for the eigenvalues distribution of $k\mathscr{L}_{\mathscr{A}^c} + \mathscr{L}_{\mathscr{A}^d}$ and can deduce

$$\begin{aligned} \pi_+(Q) &= \pi_-(I) + \pi_-\left(k\mathscr{L}_{\mathscr{A}^c} + \mathscr{L}_{\mathscr{A}^d}\right) = 0 \\ \pi_-(Q) &= \pi_+(I) + \pi_+\left(k\mathscr{L}_{\mathscr{A}^c} + \mathscr{L}_{\mathscr{A}^d}\right) = 2n-1 \\ \pi_0(Q) &= \pi_0\left(k\mathscr{L}_{\mathscr{A}^c} + \mathscr{L}_{\mathscr{A}^d}\right) = 1. \end{aligned} \tag{9}$$

In addition, with the identical structural balance of $\mathscr{G}(\mathscr{A}^c) \cup \mathscr{G}(\mathscr{A}^d)$ and $\mathscr{G}(k\mathscr{A}^c + \mathscr{A}^d)$ from Lemma 1 and Remark 2, it can be obtained from Lemma 2 that there exists some $D \in \mathscr{D}_n$ satisfying $D\left(k\mathscr{A}^c + \mathscr{A}^d\right)D = \left|k\mathscr{A}^c + \mathscr{A}^d\right|$ such that

$$\begin{aligned} D\mathscr{L}_{\mathscr{A}^c}D1_n &= \left|\mathscr{A}^c\right|1_n - D\mathscr{A}^cD1_n = 0 \\ D\mathscr{L}_{\mathscr{A}^d}D1_n &= \left|\mathscr{A}^d\right|1_n - D\mathscr{A}^dD1_n = 0. \end{aligned} \tag{10}$$

It is worth noting (see, e.g., [33] and [34]) that the eigenvalues of $Q(\lambda)$ coincide with those of the companion matrix Φ defined by

$$\Phi = \Omega^T = \begin{bmatrix} 0 & I \\ -\left(k\mathscr{L}_{\mathscr{A}^c} + \mathscr{L}_{\mathscr{A}^d}\right) & -\left(kI + \mathscr{L}_{\mathscr{A}^c}\right) \end{bmatrix}.$$

Thus, A, Ω, Φ and $Q(\lambda)$ share identical eigenvalues, including exactly one zero eigenvalue and others with negative real parts. We denote them as $\lambda_1 = 0$ and $\mathrm{Re}(\lambda_i) < 0$ for $i \in \{2, 3, \cdots, 2n\}$. Based on (10), we can validate $A\left[1_n^TD, 0\right]^T = 0$, which inspires us to take $w_1 = \left[1_n^TD, 0\right]^T$.

We proceed to determine y_1. It is not difficult to deduce

$$\begin{aligned} \begin{bmatrix} D1_n \\ \frac{1}{k}D1_n \end{bmatrix}^T (\lambda_i I - A) &= \begin{bmatrix} D1_n \\ \frac{1}{k}D1_n \end{bmatrix}^T \begin{bmatrix} \lambda_i I + \mathscr{L}_{\mathscr{A}^c} & -I \\ \mathscr{L}_{\mathscr{A}^d} & (\lambda_i + k)I \end{bmatrix} \\ &= \lambda_i \begin{bmatrix} D1_n \\ \frac{1}{k}D1_n \end{bmatrix}^T, \quad \forall i \in \mathscr{I}_{2n}\backslash\{1\}. \end{aligned} \tag{11}$$

By noticing $\lambda_i \neq 0$ and $(\lambda_i I - A)w_i = 0$ for any $i \in \mathscr{I}_{2n}\backslash\{1\}$, as well as (11), it follows that $\left[1_n^TD\ \frac{1}{k}1_n^TD\right]w_i = 0$ holds for any $i \in \mathscr{I}_{2n}\backslash\{1\}$. With this observation, we conclude from (7) that $y_1^T = \frac{1}{n}\left[1_n^TD\ \frac{1}{k}1_n^TD\right]$. Based on $\mathrm{Re}(\lambda_i) < 0$, $\forall i \in \mathscr{I}_{2n}\backslash\{1\}$, we can validate that $\lim_{t\to\infty} t^q e^{\lambda_i t} = 0$ holds for any nonnegative integer $q \geq 0$. Thus, we employ (8) to get

$$\begin{aligned} \lim_{t\to\infty} e^{At} &= P\begin{bmatrix} 1 & 0_{1\times(2n-1)} \\ 0_{(2n-1)\times 1} & 0_{(2n-1)\times(2n-1)} \end{bmatrix}P^{-1} \\ &= w_1 y_1^T. \end{aligned} \tag{12}$$

The substitution of $w_1 = \left[1_n^TD, 0\right]^T$ and $y_1^T = \frac{1}{n}\left[1_n^TD\ \frac{1}{k}1_n^TD\right]$ into (12) results in

$$\lim_{t\to\infty} x(t) = \frac{1}{n}\left[1_n^TD\left(x(0) + k^{-1}u(0)\right)\right]D1_n$$

and $\lim_{t\to\infty} u(t) = 0$ immediately. Namely, the second-order signed network (1) and (2) achieves bipartite consensus.

Case 2): $\mathscr{G}(\mathscr{A}^c) \cup \mathscr{G}(\mathscr{A}^d)$ is structurally unbalanced.

For $Q(\lambda)$ in this case, we can easily conclude from Lemma 1 and Remark 3 that for any $k > 0$, both $kI + \mathscr{L}_{\mathscr{A}^c}$ and $k\mathscr{L}_{\mathscr{A}^c} + \mathscr{L}_{\mathscr{A}^d}$ are positive definite. We thus apply Lemma 3 to $Q(\lambda)$ and can deduce

$$\begin{aligned} \pi_+(Q) &= \pi_-(I) + \pi_-\left(k\mathscr{L}_{\mathscr{A}^c} + \mathscr{L}_{\mathscr{A}^d}\right) = 0 \\ \pi_-(Q) &= \pi_+(I) + \pi_+\left(k\mathscr{L}_{\mathscr{A}^c} + \mathscr{L}_{\mathscr{A}^d}\right) = 2n \\ \pi_0(Q) &= \pi_0\left(k\mathscr{L}_{\mathscr{A}^c} + \mathscr{L}_{\mathscr{A}^d}\right) = 0. \end{aligned} \tag{13}$$

Clearly, (13) implies that the eigenvalues of $Q(\lambda)$, and consequently of A, Ω and Φ, are all in the open left half plane of the complex plane. We hence have $\mathrm{Re}(\lambda_i) < 0$ for all $i \in \mathcal{I}_{2n}$. Since $\lim_{t\to\infty} t^q e^{\lambda_i t} = 0$ holds for any nonnegative integer $q \geq 0$ and any $i \in \mathcal{I}_{2n}$, we reconsider (8) for e^{At}, and $\lim_{t\to\infty} e^{At} = 0$ directly follows, resulting in $\lim_{t\to\infty} X(t) = 0$. This means that the stability is achieved for the second-order signed network (1) and (2).

Necessity: We use the proof-by-contradiction to obtain this necessity. If $\mathcal{G}(\mathcal{A}^c) \cup \mathcal{G}(\mathcal{A}^d)$ is structurally unbalanced, then $\mathcal{G}(k\mathcal{A}^c + \mathcal{A}^d)$ is also structurally unbalanced from Remark 2, which produces $\lim_{t\to\infty} X(t) = 0$ and leads to a contradiction with bipartite consensus of the signed network (1) and (2). Conversely, we can conclude that bipartite consensus of the signed network (1) and (2) necessarily needs the structural balance of $\mathcal{G}(\mathcal{A}^c) \cup \mathcal{G}(\mathcal{A}^d)$. Likewise, we can deduce that the stability of (1) and (2) needs the structural unbalance of $\mathcal{G}(\mathcal{A}^c) \cup \mathcal{G}(\mathcal{A}^d)$ as a necessary condition.

Remark 4: For the proof of Theorem 1, we take advantage of the properties of quadratic matrix polynomials to investigate the eigenvalue distribution of the second-order signed network given by (3) with heterogeneous topologies. This exploits the eigenvalue properties of Laplacian matrices of signed graphs. Moreover, it provides an efficient eigenvalue analysis approach in the time domain to overcome the effect from heterogeneous topologies on convergence of second-order signed networks.

There exists an interesting special situation where $\mathcal{A}^c = 0$. Then the system (3) collapses into a second-order (or double-integrator) system with the form of

$$\dot{X}(t) = \begin{bmatrix} 0 & I \\ -\mathcal{L}_{\mathcal{A}^d} & -kI \end{bmatrix} X(t). \tag{14}$$

With (14), we can develop the second-order bipartite consensus results for signed networks in the following corollary.

Corollary 1: Consider any $k > 0$ and any connected signed graph $\mathcal{G}\left(\mathcal{A}^d\right)$. Then the following results can be obtained for any $x(0) \in \mathbb{R}^n$ and $u(0) \in \mathbb{R}^n$.

1) The second-order system (14) admits bipartite consensus if and only if $\mathcal{G}\left(\mathcal{A}^d\right)$ is structurally balanced. Moreover, the solution of (14) satisfies

$$\lim_{t\to\infty} x(t) = \left[\frac{1}{n} 1_n^T D\left(x(0) + k^{-1}u(0)\right)\right] D1_n,$$
$$\lim_{t\to\infty} u(t) = 0$$

for some $D \in \mathcal{D}_n$ such that $D\mathcal{A}^d D = \left|\mathcal{A}^d\right|$.

2) The second-order system (14) achieves stability if and only if $\mathcal{G}\left(\mathcal{A}^d\right)$ is structurally unbalanced.

Proof: Due to $\mathcal{A}^c = 0$, the sign-consistency still keeps for $\mathcal{G}\left(\mathcal{A}^c\right)$ and $\mathcal{G}\left(\mathcal{A}^d\right)$, and connectivity of $\mathcal{G}\left(\mathcal{A}^d\right)$ ensures that $\mathcal{G}\left(\mathcal{A}^c\right) \cup \mathcal{G}\left(\mathcal{A}^d\right)$ is connected. Hence, (14) is obviously a special case of (3), and this corollary can be derived directly with the help of Theorem 1.

Remark 5: The system (14) is similar to [25, Example 1], which discusses a special case of dynamic network systems. What such two systems in common is that only one graph $\mathcal{G}\left(\mathcal{A}^d\right)$ matters for nodes. As Corollary 1 formulates, the sign-consistent property always keeps for any signed graph $\mathcal{G}\left(\mathcal{A}^d\right)$. Thanks to Theorem 1, if we only consider the static interactions in [25, Example 1], then it can be viewed as a special case of system (3) and holds the same results as Corollary 1. Nevertheless, its bipartite consensus and stability results can be developed from a WSPR (short for weakly strictly positive real) property, instead of sign-consistency. Actually, for any $k > 0$, the WSPR property of $a_{ij}^c + (s+k)^{-1} a_{ij}^d$ is equivalent to the sign-consistency of $\mathcal{G}\left(\mathcal{A}^c\right)$ and $\mathcal{G}\left(\mathcal{A}^d\right)$. Thus, our results may provide new insights into the dynamic distributed control of signed networks considered in, e.g., [25].

In a trivial case that $\mathcal{G}\left(\mathcal{A}^c\right)$ and $\mathcal{G}\left(\mathcal{A}^d\right)$ have no antagonistic interactions, the consensus problems are analogous to the network-to-network ones in [30]. In this case, sign-consistency of $\mathcal{G}\left(\mathcal{A}^c\right)$ and $\mathcal{G}\left(\mathcal{A}^d\right)$ always keeps apparently. By definition of structural balance, this case can be viewed as a special one of structural balance. In other words, $\mathcal{V}^{(l)} = \emptyset$ and $\mathcal{V}^{(q)} = \mathcal{V}$ for $l \neq q$ and $l, q \in \{1, 2\}$, and therefore $\mathcal{A}^c \geq 0$ and $\mathcal{A}^d \geq 0$ are both nonnegative. Under such circumstance, we present a corollary as follows to make this trivial case clear.

Corollary 2: Consider any $k > 0$ and any traditional graphs $\mathcal{G}\left(\mathcal{A}^c\right)$ and $\mathcal{G}\left(\mathcal{A}^d\right)$ with $\mathcal{A}^c \geq 0$ and $\mathcal{A}^d \geq 0$. If the union $\mathcal{G}\left(\mathcal{A}^c\right) \cup \mathcal{G}\left(\mathcal{A}^d\right)$ is connected, then for any $x(0) \in \mathbb{R}^n$ and $u(0) \in \mathbb{R}^n$, the system (3) achieves consensus such that

$$\lim_{t\to\infty} x(t) = \left[\frac{1}{n}\sum_{i=1}^{n}\left(x_i(0) + k^{-1}u_i(0)\right)\right] 1_n,$$
$$\lim_{t\to\infty} u(t) = 0. \tag{15}$$

Proof: Because this situation belongs to structurally balanced cases, and the gauge transformation matrix $D = I$ holds in Theorem 1, we derive from the result 1) of Theorem 1 that (15) follows immediately. Namely, consensus is achieved.

Remark 6: The result of Corollary 2 is consistent with the traditional second-order consensus results. It also suggests that the second-order consensus results can be extended to traditional networks in the presence of heterogeneous topologies. For heterogeneous traditional networks, they are invariably sign-consistent. Compared with the consensus results of network-to-network systems in [30], the second-order consensus is greatly generalized to signed networks with a precondition of sign-consistent property.

IV. SIMULATION RESULTS

In this section, we give two examples for the signed network (1) and (2) with twelve nodes to illustrate the proposed results. Without loss of generality, we adopt $k = 2 > 0$ and choose the initial conditions of (1) and (2) as

$$x(0) = [5, -3, -9, 2, -8, -5, 3, -7, -1, 6, 1, -6]^T,$$
$$u(0) = [4, -3, 6, -2, 1, -5, 2, -1, -2, 3, -4, -1]^T.$$

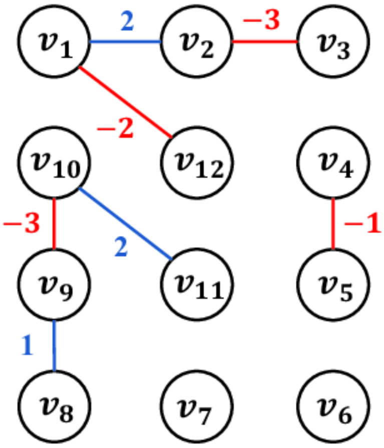

FIGURE 1. (Example 1). The signed digraph $\mathcal{G}(\mathcal{A}^c)$.

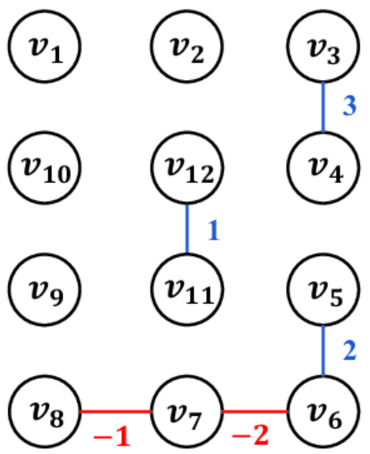

FIGURE 2. (Example 1). The signed digraph $\mathcal{G}\left(\mathcal{A}^d\right)$.

Example 1: Consider (1) and (2) under two signed graphs in Figs. 1 and 2. We can verify for Figs. 1 and 2 that

- $\mathcal{G}(\mathcal{A}^c)$ and $\mathcal{G}\left(\mathcal{A}^d\right)$ are sign-consistent;
- $\mathcal{G}(\mathcal{A}^c) \cup \mathcal{G}\left(\mathcal{A}^d\right)$ is structurally balanced and strongly connected.

A gauge transformation can be determined for the union signed graph as $D = \text{diag}\{1, 1, -1, -1, 1, 1, -1, 1, 1, -1, -1, -1\}$. By the result 1) of Theorem 1, bipartite consensus can be achieved for all nodes, which leads to

$$\lim_{t\to\infty} x_i(t) = -1.75, i \in \{1, 2, 5, 6, 8, 9\}$$
$$\lim_{t\to\infty} x_j(t) = 1.75, j \in \{3, 4, 7, 10, 11, 12\}.$$

For this case, the state evolution of $x_i(t)$, $\forall i \in \mathcal{I}_{12}$ is plotted in Fig. 3. It can be easily seen from Fig. 3 that bipartite consensus on the quantity with modulus equal to 1.75 is achieved for all nodes. Consequently, the illustration of Fig. 3 coincides with the bipartite consensus result of Theorem 1.

Example 2: We consider two signed graphs in Figs. 4 and 5 for the signed network (1) and (2). Obviously, $\mathcal{G}(\mathcal{A}^c)$ and $\mathcal{G}\left(\mathcal{A}^d\right)$ in Figs. 4 and 5 are sign-consistent and give a connected union $\mathcal{G}(\mathcal{A}^c) \cup \mathcal{G}\left(\mathcal{A}^d\right)$. But, different from

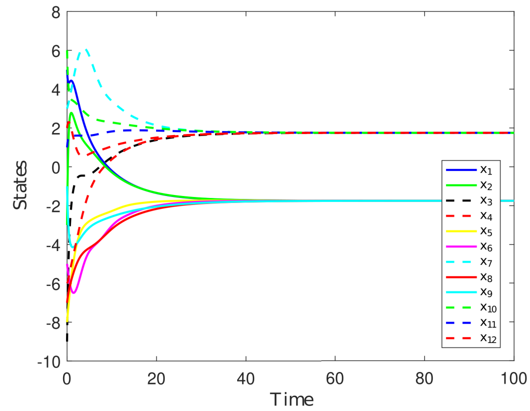

FIGURE 3. (Example 1). Bipartite consensus of (1) with heterogeneous topologies given by two signed graphs in Figs. 1 and 2.

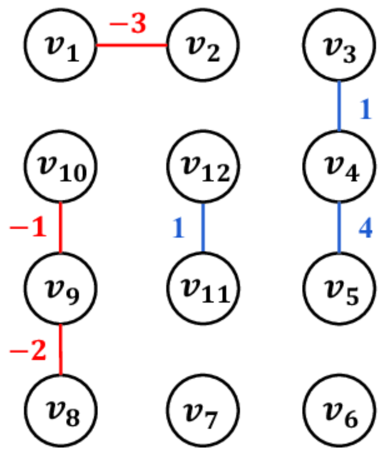

FIGURE 4. (Example 2). The signed digraph $\mathcal{G}(\mathcal{A}^c)$.

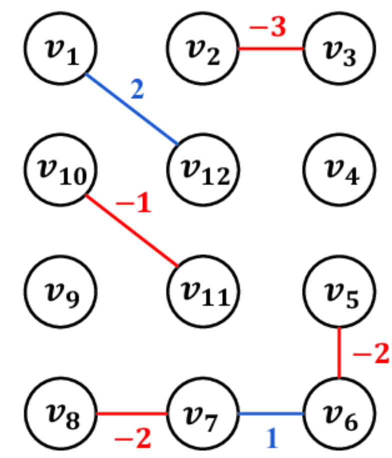

FIGURE 5. (Example 2). The signed digraph $\mathcal{G}\left(\mathcal{A}^d\right)$.

Example 1, $\mathcal{G}(\mathcal{A}^c) \cup \mathcal{G}\left(\mathcal{A}^d\right)$ is a structurally unbalanced signed graph. From the result 2) of Theorem 1, we know that all nodes described by (1) and (2) achieve the stability. In Fig. 6, we depict the state evolution of $x_i(t)$, $\forall i \in \mathcal{I}_{12}$, by which the stability is clearly realized. This illustrates the stability result 2) of Theorem 1.

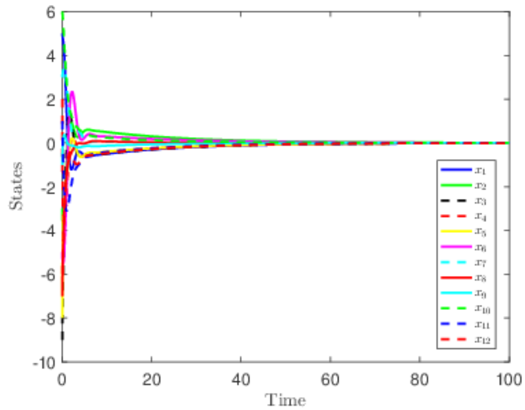

FIGURE 6. (Example 2). Stability of (1) with heterogeneous topologies described by two signed graphs in Figs. 4 and 5.

Through Examples 1 and 2, we illustrate our convergence results developed for second-order signed networks subject to heterogeneous topologies. These illustrations demonstrate that given the sign-consistency of heterogeneous topologies, the behaviours of second-order signed networks relate closely to the structural balance of signed graphs.

V. CONCLUSION

In this paper, the bipartite consensus problems have been discussed upon second-order signed networks subject to the heterogeneous topologies. We have introduced a class of sign-consistency properties for pairs of signed graphs. Consequently, an eigenvalue-based approach has been presented to implement the convergence analysis, which may be of independent interest for the higher-order signed networks. These help us to provide necessary and sufficient conditions for second-order signed networks to reach bipartite consensus or stability of all their nodes. In particular, two special cases of second-order signed networks have been involved and discussed. Simulation tests have verified the validity of our established results.

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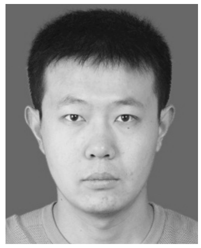

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