IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received December 22, 2019, accepted February 9, 2020, date of publication February 11, 2020, date of current version February 26, 2020.

Digital Object Identifier 10.1109/ACCESS.2020.2973289

On the Some New Preconditioned Generalized AOR Methods for Solving Weighted Linear Least Squares Problems

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ABSTRACT Recently, in the paper [Z.G. Huang, L.G. Wang, Z. Xu, J.J. Cui, Some new preconditioned generalized AOR methods for solving weighted linear least squares problems, Computational and Applied Mathematics, 37(2018) 415–438.], Huang et al, by using the generalized accelerated over-relaxation (GAOR) methods, proposed some new preconditioners for solving weighted linear least squares problems and discuss their comparison results. In this paper, we present a new model of GAOR methods to solve the weighted linear least squares problems. We prove that the new model is superior to the existing mentioned methods. Numerical examples are also reported to confirm our theoretical analysis.

INDEX TERMS Preconditioned GAOR method, weighted linear least squares problems, linear system, convergence, comparison theorem.

I. INTRODUCTION

Let us consider the weighted linear least squares problem (WLLSP):

$$\min_{x\in R^n}(Ax-b)^T W^{-1}(Ax-b) \tag{1}$$

where, W is the variance–covariance matrix; (see [1]–[4] and references therein).

Least squares (LS) method for resolving linear equation systems and WLLSP as a generalization of the LS arises in many practical applications including linear programming, convex quadratic programming, linear regression, geodetic symmetrical transformations, electrical networks, boundary value problems, analysis of large-scale structures and the inequality constrained least squares problems [4]–[8]. For solving this problem, we can solve a linear system as follows:

$$Hy = f, \tag{2}$$

where,

$$H = \begin{bmatrix} (I-B)_{p\times p} & (U)_{p\times(n-p)} \\ (L)_{(n-p)\times p} & (I-C)_{(n-p)(n-p)} \end{bmatrix},$$

is a nonsingular matrix with $B = (b_{ij})_{p\times p}$, $C = (c_{ij})_{q\times q}$, $L = (l_{ij})_{q\times p}$, $U = (u_{ij})_{p\times q}$, $q = n - p$, and I is an identity matrix.

The basic stationary iterative method to solve the system of linear equations $Ax = d$, with any splitting, A=M-N that $\det(M) \neq 0$ is as follows:

$$x^{(t+1)} = M^{-1}Nx^{(t)} + M^{-1}b. \quad t = 0, 1, \ldots \tag{3}$$

where, $M^{-1}N$ is called the iteration matrix. This iterative scheme converges to the unique solution if and only if $\rho(M^{-1}N) < 1$, where $\rho(A)$ is the spectral radius of A.

Based on the model of Eq. (3), there are some popular iterative methods; see [9]–[18] and references therein. However, if we solve Eq. (2) by the model of Eq. (3), we need the inverses of $I - B$ and $I - C$, which is the main drawback of these methods.

To solve Eq. (2), Yuan and Jin [3] proposed the generalized AOR (GAOR) method for solving the WLLSP. Consider the following splitting for the matrix H:

$$H = D - C_L - C_u = D\,(I - L - U)\,,$$

where, $D = diag(H)$, and C_L and C_u are strictly lower and upper triangular matrices of H, respectively. Now, to solve the Eq. (2) using the GAOR method, the matrix H is split as;

$$H = (I)_{n\times n} - \begin{bmatrix} (0)_{p\times p} & (0)_{p\times(n-p)} \\ (-C)_{(n-p)\times p} & (0)_{(n-p)(n-p)} \end{bmatrix} - \begin{bmatrix} (B)_{p\times p} & (-U)_{p\times(n-p)} \\ (0)_{(n-p)\times p} & (K)_{(n-p)(n-p)} \end{bmatrix}. \tag{4}$$

Then, for $w \neq 0$, the GAOR method is as follows:

$$y^{(t+1)} = \mu_{w,r}y^{(t)} + wg. \quad t = 0, 1, \ldots \tag{5}$$

The associate editor coordinating the review of this manuscript and approving it for publication was Heng Zhang.

where,

$$\mu_{w,r} = M^{-1}N = \underbrace{\begin{bmatrix} I_1 & 0 \\ rC & I_2 \end{bmatrix}^{-1}}_{M^{-1}} \times \underbrace{\left((I-w)I+(w-r)\begin{bmatrix} 0 & 0 \\ -C & 0 \end{bmatrix}+w\begin{bmatrix} B & -U \\ 0 & K \end{bmatrix}\right)}_{N}, \tag{6}$$

I_1 and I_2 are identity matrices of orders (p) and (n-p) respectively, and:

$$g = \begin{bmatrix} I_1 & (0)_{p\times(n-p)} \\ -rC & I_2 \end{bmatrix} f.$$

Recently, several scholars established numerous models to solve WLLSP [19]–[26]. Very recently, Huang *et al.* [26], to decrease the spectral radius of $\mu_{w,r}$, presented some models of the preconditioned GAOR method. These authors, using the splitting of Eq. (4), presented two preconditioning models as follow:

Consider the preconditioner form of Eq. (2) as:

$$\bar{H}y = \bar{f},$$

where $\bar{H} = (I + \bar{S})H$ and $\bar{f} = (I + \bar{S})f$ with;

$$(I)\ \bar{S} = \begin{bmatrix} W_i & (0)_{p\times(n-p)} \\ K_i & (0)_{(n-p)(n-p)} \end{bmatrix}, \quad i = 1, 2, \tag{7}$$

where K_i is a $q \times p$ matrix and the form of K_1 is as follows:

$$K_1 = \begin{pmatrix} -\mu_1 l_{11} & 0 & \dots & 0 \\ -\mu_2 l_{21} & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ -\mu_q l_{q1} & 0 & \dots & 0 \end{pmatrix}, \tag{8}$$

Also, for K2:

If q< p, then:

$$K_2 = \begin{pmatrix} -\nu_1 l_{11} & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & -\nu_2 l_{22} & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -\nu_q l_{qq} & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{9}$$

If q = p, then:

$$K_2 = \begin{pmatrix} -\nu_1 l_{11} & 0 & \dots & 0 \\ 0 & -\nu_2 l_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -\nu_q l_{qq} \end{pmatrix}. \tag{10}$$

If q > p, then:

$$K_2 = \begin{pmatrix} -\nu_1 l_{11} & 0 & \dots & 0 \\ 0 & -\nu_2 l_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -\nu_q l_{qq} \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}. \tag{11}$$

And,

$$W_1 = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \gamma_2 b_{21} & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ \gamma_p b_{p1} & 0 & \dots & 0 \end{pmatrix},$$
$$W_2 = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ \delta_2 b_{21} & 0 & \dots & 0 & 0 \\ 0 & \delta_3 b_{32} & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & \delta_p b_{p,p-1} & 0 \end{pmatrix} \tag{12}$$

for $\mu_i, \nu_i, \gamma_i, \delta_i > 0, i = 2, \dots, p$.

Then, the preconditioned matrix with the preconditioner (7) can be decomposed by the following splitting

$$\bar{H}_i = I - \begin{bmatrix} (0)_{p\times p} & (0)_{p\times(n-p)} \\ -\tilde{L}_i & (0)_{(n-p)(n-p)} \end{bmatrix} - \begin{bmatrix} \tilde{B}_i & -\tilde{U}_i \\ (0)_{(n-p)\times p} & \tilde{C}_i \end{bmatrix}, \tag{13}$$

where, $\tilde{B}_i = B-W_i(I-B)$, $\tilde{U}_i = (I+W_i)U$, $\tilde{L}_i = L+K_i(I-B)$ and $\tilde{C}_i = C - K_iU$.

Furthermore, its iteration matrix defined as:

$$\begin{aligned} &\tilde{\mu}_{w,r} \\ &= \tilde{M}^{-1}\tilde{N} = \underbrace{\begin{bmatrix} I_1 & 0 \\ r\tilde{L}_i & I_2 \end{bmatrix}^{-1}}_{\tilde{M}^{-1}} \\ &\quad\times\left((I-w)I+(w-r)\begin{bmatrix} 0 & 0 \\ -\tilde{L}_i & 0 \end{bmatrix}+w\begin{bmatrix} \tilde{B}_i & -\tilde{U}_i \\ 0 & \tilde{C}_i \end{bmatrix}\right) \\ &= \begin{bmatrix} (1-w)I_1 + w\tilde{B}_i & -w\tilde{U}_i \\ w(r-1)\tilde{L}_i - wr\tilde{L}_i\tilde{B}_i & (1-w)I_2 + w\tilde{C}_i + wr\tilde{L}_i\tilde{U}_i \end{bmatrix}. \end{aligned} \tag{14}$$

Also, the second class of preconditioner is;

$$(II)\ \bar{S} = \begin{bmatrix} S_i & (0)_{p\times(n-p)} \\ (0)_{(n-p)\times p} & V_i \end{bmatrix}, \quad i = 1, 2. \tag{15}$$

where for $\alpha_i, \beta_i, \tau_i, \sigma_i > 0$;

$$S_1 = \begin{pmatrix} 0 & \alpha_2 b_{12} & \cdots & 0 & 0 \\ \beta_2 b_{21} & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & \alpha_p b_{p-1,p} \\ 0 & 0 & \dots & \beta_p b_{p,p-1} & 0 \end{pmatrix} \tag{16}$$

$$S_2 = \begin{pmatrix} 0 & \alpha_2 b_{12} & \cdots & \alpha_p b_{1p} \\ \beta_2 b_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \beta_p b_{p1} & 0 & \cdots & 0 \end{pmatrix} \tag{17}$$

$$V_1 = \begin{pmatrix} 0 & \tau_2 c_{12} & \cdots & 0 & 0 \\ \sigma_2 c_{21} & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & \tau_q c_{q-1,q} \\ 0 & 0 & \cdots & \sigma_q c_{q,q-1} & 0 \end{pmatrix} \tag{18}$$

$$V_2 = \begin{pmatrix} 0 & \sigma_2 c_{12} & \cdots & \sigma_q c_{1q} \\ \sigma_2 c_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_q c_{q1} & 0 & \cdots & 0 \end{pmatrix} \tag{19}$$

Then, the preconditioned matrix with preconditioner (15) can be decomposed by the following splitting:

$$\bar{H}_i = I - \begin{bmatrix} (0)_{p\times p} & (0)_{p\times(n-p)} \\ -\hat{L}_i & (0)_{(n-p)(n-p)} \end{bmatrix} - \begin{bmatrix} \hat{B}_i & -\hat{U}_i \\ (0)_{(n-p)\times p} & \hat{C}_i \end{bmatrix}, \tag{20}$$

where, $\hat{B}_i = B - S_i(I - B)$, $\hat{U}_i = (I + S_i)U$, $\hat{L}_i = (I + V_i)L$ and $\hat{C}_i = C - V_i(I - C)$.

The iteration matrix of Eq. (20) also defined as:

$$\begin{aligned} &\bar{\mu}_{w,r} \\ &= \bar{M}^{-1}\bar{N} = \underbrace{\begin{bmatrix} I_1 & 0 \\ r\hat{L}_i & I_2 \end{bmatrix}^{-1}}_{\bar{M}^{-1}} \\ &\quad\times\left((I-w)I + (w-r)\begin{bmatrix} 0 & 0 \\ -\hat{L}_i & 0 \end{bmatrix} + w\begin{bmatrix} \hat{B}_i & -\hat{U}_i \\ 0 & \hat{C}_i \end{bmatrix}\right) \\ &= \begin{bmatrix} (1-w)I_1 + w\hat{B}_i & -w\hat{U}_i \\ w(r-1)\hat{L}_i - wr\hat{L}_i\hat{B}_i & (1-w)I_2 + w\hat{C}_i + wr\hat{L}_i\hat{U}_i \end{bmatrix}. \end{aligned} \tag{21}$$

In this paper, we propose some new splittings of H and $\bar{H}$. Moreover, we prove that the new splittings compare with the splittings of (4), (13) and (20) work better.

II. PREREQUISITE

We start with some basic notation and preliminary results which we allude to later.

Definition 1 ([9], [10]):

(a) A matrix H is *Z-matrix* if for any $i \neq j$, $h_{ij} \leq 0$.

(b) A *Z-matrix* is *M-matrix*, if H is nonsingular, and if $H^{-1} \geq 0$.

(c) A square matrix $H = h_{ij}$ is *M-matrix* if

$$H = \alpha I - B; B \geq 0 \quad and, \ \alpha > \rho(B).$$

Definition 2 ([9], [10]): The splitting $H = M - N$ is called

(a) Convergent if $\rho\ (M^{-1}N) < 1$;

(b) Regular if $M^{-1} \geq 0$, $N \geq 0$;

(c) Nonnegative if $M^{-1}N \geq 0$;

(d) M-splitting if M is a nonsingular *M-matrix* and $N \geq 0$;

Lemma 3 ([27], [28]): Let $H = M - N$ be an *M-splitting* of H. Then $\rho\ (M^{-1}N) < 1$ if and only if H is *M- matrix*.

Lemma 4 ([27], [28]): Let A, B are *Z-matrix* and A is an *M-matrix*, if $A \leq B$ then B is an *M-matrix* too.

Lemma 5 [9]: Let $H = M_1 - N_1 = M_2 - N_2$ be two regular splittings of H, where $H^{-1} \geq 0$. If $M_1^{-1} \geq M_2^{-1}$, then:

$$\rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2) < 1.$$

Lemma 6 [10]: Let H be a *Z-matrix*. Then H is an *M-matrix* if and only if there is a positive vector x such that $Hx > 0$.

III. MAIN RESULT

Consider the following splittings;

$$B = D_1 - L_1 - U_1, C = D_2 - L_2 - U_2,$$

where D_i, $-L_i$, $-U_i$ $(i = 1, 2)$ are diagonal, strictly lower and strictly upper triangular parts of B and C, respectively. Furthermore,

$$D_1 = D_{11} + D_{12};\ where\ D_{11} < I_1,$$

and,

$$D_2 = D_{21} + D_{22};\ where\ D_{22} < I_2.$$

Then, to solve the linear system (1), we consider the following splitting

$$\begin{aligned} H = &\begin{bmatrix} I_1 - D_{11} & (0)_{p\times(n-p)} \\ (0)_{(n-p)\times p} & I_2 - D_{22} \end{bmatrix} - \begin{bmatrix} -L_1 & (0)_{p\times(n-p)} \\ -L & -L_2 \end{bmatrix} \\ &- \begin{bmatrix} D_{12} - U_1 & -U \\ (0)_{(n-p)\times p} & D_{21} - U_2 \end{bmatrix}. \end{aligned} \tag{22}$$

So, the iteration matrix of GAOR method with the splitting of Eq. (22) is:

$$\begin{aligned} &\mu_{w,r} \\ &= \underbrace{\begin{bmatrix} (I_1 - D_{11}) + rL_1 & 0 \\ rL & (I_2 - D_{22}) + rL_2 \end{bmatrix}^{-1}}_{M^{-1}} \\ &\times \underbrace{\left((I-w)\begin{bmatrix} I_1 - D_{11} & 0 \\ 0 & I_2 - D_{22} \end{bmatrix} + (w-r)\begin{bmatrix} -L_1 & 0 \\ -L & -L_2 \end{bmatrix} + w\begin{bmatrix} D_{12} - U_1 & -U \\ 0 & D_{21} - U_2 \end{bmatrix}\right)}_{N} \end{aligned} \tag{23}$$

In the following we will compare our splitting with the splitting of Eq.(4).

Theorem 7: Consider the matrix H in Eq. (2) and let $\mu_{w,r}^{(1)}$ and $\mu_{w,r}^{(2)}$ be the iteration matrices of the GAOR method by splittings of (4) and (22), respectively. If $C \leq 0$, $U \leq 0$, $B_i \geq 0$; $(i = 1, 2)$ and $\rho(\mu_{w,r}^{(1)}) < 1$, then for $0 < w \leq 1$ and $0 \leq r < 1$ we have; $\rho(\mu_{w,r}^{(2)}) \leq \rho(\mu_{w,r}^{(1)})$.

Proof: From Eq. (6) and Definition 2, we can see that $\mu_{w,r}^{(1)}$ is *M-splitting* of H. Then, from the Lemma 3 H is *M-matrix*. Also, from (6), (23) and Definition 2, these splittings are regular.

Now, consider $\mu_{w,r}^{(1)} = M_1^{-1}N_1$, and $\mu_{w,r}^{(2)} = M_2^{-1}N_2$. Then;

$$M_2 - M_1 = \begin{bmatrix} (I_1 - D_{11}) + rL_1 & 0 \\ rL & (I_2 - D_{22}) + rL_2 \end{bmatrix} - \begin{bmatrix} I_1 & 0 \\ rL & I_2 \end{bmatrix} \le \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore;

$$M_2 \le M_1.$$

Since H is *M-matrix*, from Lemma 4 M_1, M_2 are *M-matrices*. Therefore, $M_2^{-1} \ge M_1^{-1}$.

And by Lemma 5 the proof is completed. ■

Similarly, we consider the following model for the preconditioned matrix $\bar{H}$. We only consider the first kind of preconditioners; another class can be analogously verified. By Eq. (13):

$$\bar{H}_i = \begin{bmatrix} (I_1 + W_i)(I_1 - B) & (I_1 + W_i)U \\ K_i(I_1 - B) + L & K_iU + (I_2 - C) \end{bmatrix}, \quad (24)$$

$$(I_1 + W_i)(I_1 - B) = (I_1 + W_i) - (I_1 - W_i)B, \quad (25)$$

Now, consider the following splittings;

$$(I_1 - W_i)B = D_1 - L_1 - U_1,$$
$$K_iU - C = D_2 - L_2 - U_2,$$

where D_i, $-L_i$, $-U_i$ $(i = 1, 2)$ are diagonal, strictly lower and strictly upper triangular parts of $(I_1 - W_i)B$ and $K_iU - C$, respectively. Furthermore,

$$D_1 = D_{11} + D_{12};\ where\ D_{11} < I_1.$$

And,

$$D_2 = D_{21} + D_{22};\ where\ D_{22} < I_2.$$

Then, we have the following splitting of $\bar{H}$:

$$\bar{H} = \begin{bmatrix} I_1 - D_{11} & 0 \\ 0 & I_2 - D_{22} \end{bmatrix} - \begin{bmatrix} -L_1 - W_i & 0 \\ -(K_i(I_1 - B) + L) & -L_2 \end{bmatrix} - \begin{bmatrix} D_{12} - U_1 & -(I_1 + W_i)U \\ 0 & D_{21} - U_2 \end{bmatrix}. \quad (26)$$

Then iteration matrix of preconditioned GAOR method with the splitting of (26) is

$$\bar{\bar{\mu}}_{w,r} = \bar{\bar{M}}^{-1}\bar{\bar{N}} = \underbrace{\begin{bmatrix} (I_1 - D_{11}) + rL_1 + rW_i & 0 \\ r(K_i(I_1 - B) + L) & (I_2 - D_{22}) + rL_2 \end{bmatrix}^{-1}}_{\bar{\bar{M}}^{-1}}$$
$$\times \underbrace{\left((I - w)\begin{bmatrix} I_1 - D_{11} & 0 \\ 0 & I_2 - D_{22} \end{bmatrix} + (w - r)\begin{bmatrix} -L_1 - W_i & 0 \\ -(K_i(I_1 - B) + L) & -L_2 \end{bmatrix} + w\begin{bmatrix} D_{12} - U_1 & -(I_1 + W_i)U \\ 0 & D_{21} - U_2 \end{bmatrix} \right)}_{\bar{\bar{N}}} \quad (27)$$

Theorem 8: Let $\bar{\mu}_{w,r}$ and $\bar{\bar{\mu}}_{w,r}$ be the iteration matrices of the preconditioned GAOR method by splittings of (13) and (26), respectively. If the conditions of Theorem 7 are satisfied, then we have,

$$\rho(\bar{\bar{\mu}}_{w,r}) \le \rho(\bar{\mu}_{w,r}) < 1.$$

Proof: We know that H is *M-matrix*. Then by Lemma 6 it is easy to see that $\bar{H}$ is *M-matrix* too. Therefore, similar to the proving process of Theorem 7, we conclude that these splittings are regular. Moreover, let

$$\bar{\mu}_{w,r} = \bar{M}^{-1}\bar{N}, \quad \bar{\bar{\mu}}_{w,r} = \bar{\bar{M}}^{-1}\bar{\bar{N}}.$$

Then;

$$\bar{\bar{M}} - \bar{M} = \begin{bmatrix} (I_1 - D_{11}) + rL_1 + rW_i & 0 \\ r(K_i(I_1 - B) + L) & (I_2 - D_{22}) + rL_2 \end{bmatrix} - \begin{bmatrix} I_1 & 0 \\ r(L + K_i(I - B)) & I_2 \end{bmatrix} \le \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

And by Lemma 6 the proof is completed. ■

IV. NUMERICAL EXAMPLE

In this section we test the effectiveness of two mentioned splitting methods. The experimental results were obtained using an Intel Celeron with a 2.8GHz 32-bit processor and 1GB RAM memory running Windows 7.

The computational platform used was the MATLAB environment. $y^{(0)}$ is zero vector and the f was selected such that the exact solution of the system (2) is $y^T = (1, 2, 3, \ldots, n)^T \in R^n$. The stopping criterion is $\varepsilon < 10^{-6}$.

The number of iterations (denoted by IT) and spectral radius and CPU time are reported in the following tables in order to show the efficiency of our splitting methods.

Example 9: Consider the matrix H of Eq. (2) with the following conditions:

$$b_{ii} = 1/\xi(i+1); \quad i = 1, \cdots, p,$$
$$b_{ij} = (1/\lambda) - (1/\lambda j + i);$$
$$i < j,\ i = 1, \cdots, p-1, j = 2, \ldots, p,$$
$$b_{ij} = (1/\lambda) - (1/[\lambda(i - j + 1) + i];$$
$$i > j,\ i = 2, \cdots, p, j = 1, \ldots, p-1,$$
$$c_{ii} = 1/\xi(p + i + 1); \quad i = 1, \cdots, n-p,$$
$$c_{ij} = (1/\lambda) - (1/[\lambda(p + j) + p + i]);$$
$$i < j,\ i = 1, \cdots, n-p+1, j = 2, \ldots, n-p,$$
$$c_{ij} = (1/\lambda) - (1/[\lambda(i - j + 1) + p + i]);$$

TABLE 1. The spectral radius and number of iterations of the GAOR with two different splitting.

n	w	r	p	ξ	λ	Splitting(4)		Splitting(22)	
						$\rho(\mu_{w,r}^{(1)})$	Iter	$\rho(\mu_{w,r}^{(2)})$	Iter
10	0.9	0.85	5	5	9	0.8357	81	0.7866	63
20	0.75	0.65	10	8	20	0.8668	100	0.8420	85
30	0.65	0.55	16	10	30	0.9145	154	0.9012	134
40	0.9	0.9	20	15	37	0.9749	491	0.9659	372
50	0.95	0.9	35	25	47	0.9727	452	0.9612	330

TABLE 2. The results of iterations of the preconditioned GAOR with two different splittings.

n	w	r	p	ξ	λ	Splitting(13)		Splitting(26)	
						$\rho(\mu_{w,r}^{(1)})$	Iter	$\rho(\mu_{w,r}^{(2)})$	Iter
10	0.9	0.85	5	5	9	0.8123	72	0.7660	57
20	0.75	0.65	10	8	20	0.8583	94	0.8350	81
30	0.65	0.55	16	10	30	0.9108	148	0.8982	131
40	0.9	0.9	20	15	37	0.9741	478	0.9651	364
50	0.95	0.9	35	25	47	0.9720	441	0.9605	324

TABLE 3. The results of the preconditioned GAOR with two different splittings.

n	w	r	p	ξ	λ	Splitting(18)		Splitting(26)	
						$\rho(\mu_{w,r}^{(1)})$	Iter	$\rho(\mu_{w,r}^{(2)})$	Iter
10	0.9	0.85	5	5	9	0.7946	65	0.7528	54
20	0.75	0.65	10	8	20	0.8527	91	0.8302	79
30	0.65	0.55	16	10	30	0.9086	144	0.8962	128
40	0.9	0.9	20	15	37	0.9734	465	0.9643	357
50	0.95	0.9	35	25	47	0.9715	434	0.9598	319

$$\begin{aligned} &\quad i > j,\ i = 2, \cdots, n-p,\ j = 1, \ldots, n-p-1, \\ l_{ij} &= (1/[\lambda(p+i-j+1)+p+i]) - (1/\lambda); \\ &\quad i = 1, \cdots, n-p, j = 1, \ldots, p, \\ u_{ij} &= (1/[\lambda(p+j)+i]) - (1/\lambda); \quad i = 1, \cdots, p, \\ &\quad j = 1, \ldots, n-p,\ \xi, \lambda \in \Re. \end{aligned}$$

In Table 1, we reported the number of iterations and the spectral radius of the corresponding iterative methods with different splittings.

In Table 2, we show the results of the corresponding iterative schemes with the following preconditioner ($\mu_i = \gamma_i = 1$) and the splittings of (13) and (26):

$$\bar{P} = I + \bar{S} = \begin{bmatrix} I_1 + W_1 & 0 \\ K_1 & I_2 \end{bmatrix} \tag{28}$$

In Table 3, we show the results of the corresponding iterative methods with the following preconditioner ($\alpha_i = \beta_i = \tau_i = \sigma_i = 1$) and the splittings of (20) and (26):

$$\bar{P} = I + \bar{S} = \begin{bmatrix} I_1 + S_1 & 0 \\ 0 & I_2 + V_i \end{bmatrix} \tag{29}$$

The results also show that the scalability of our algorithms is suitable. For example, from Table 3, we can see that when *n=10*, the iteration steps of Splitting (26) $\approx$ 0.83 $\times$ Splitting(18), but, when n is increased to 50, the iteration steps of Splitting (26) $\approx$ 0.73 $\times$ Splitting(18).

From tables 1-3 we can see that, the iterative methods with our splitting, without extra cost per iteration step, perform much better than existing methods for solving weighted linear least squares problems. Furthermore, an iterative method with the splitting (22) performs much better than preconditioned methods with the splittings (13) and (20).

V. CONCLUSION

In this paper, we modified the solution procedures to solve the weighted linear least squares problem and improved the convergence rates of iterative methods. From the numerical results and theoretical analysis, we can conclude that the performance of the modified iterative method is much better in comparison with the existing methods.

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