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Sharp Threshold for the Dynamics of a SIRS Epidemic Model With General Awareness-Induced Incidence and Four Independent Brownian Motions

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ABSTRACT In this paper, a stochastic SIRS epidemic model with general awareness-induced and four independent Brownian Motions is established. We verify the global existence of a unique positive solution and find out the noise modified reproduction number R_0^S which is a sharp threshold for the dynamics: If $R_0^S < 1$, the disease will die out; if $R_0^S > 1$, the disease persists and there exists a global asymptotically stable stationary distribution under parameter restrictive conditions. Numerical simulations are presented to illustrate the theoretical results.

INDEX TERMS Stochastic modeling, stability analysis, necessary and sufficient condition, stochastic stabilization, stationary distribution.

I. INTRODUCTION

Compartmental epidemic models for infectious disease are established to illustrate the transmission behaviour in a host population. Classical SIR model partitions the host population into the susceptible compartment S, the infectious compartment I and the recovered compartment R. At a time t, the size of population in compartments S, I, and R are denoted by $S(t)$, $I(t)$, and $R(t)$, respectively. Ordinary differential equations that describe the changes of sizes in each compartment can be written as

$$\begin{aligned}\frac{\mathrm{d}S}{\mathrm{d}t} &= \Lambda - d_1 S - (\beta_1 - \beta_2 f(I))SI + \delta R\\ \frac{\mathrm{d}I}{\mathrm{d}t} &= (\beta_1 - \beta_2 f(I))SI - (d_2 + \gamma + \alpha)I\\ \frac{\mathrm{d}R}{\mathrm{d}t} &= \gamma I - (d_3 + \delta)R. \end{aligned} \tag{1}$$

Parameter Λ denotes the influx of susceptible hosts, γ and δ are the rate of transfers of hosts from compartment I to R and from R to S, respectively. Accordingly, $1/\gamma$ is the mean infectious period and $1/\delta$ is the mean immune period.

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Parameter d_1, d_2, d_3 are background mortality rates for compartment S, I and R [1]. Considering the effects of media coverage though the transmission process, the incidence of the disease in this model is given by a general awareness-induced incidence bilinear expression $(\beta_1 - \beta_2 f(I))SI$ [2], where β_1 is the direct contact rate, β_2 is the maximum reduced contact rate due to the effects of media coverage to protected individuals. Based on biological considerations, $\beta_1 \geq \beta_2 > 0$, the function $f(I)$ satisfies the following basic assumptions:

(H1) $f(0) = 0, f'(t) \geq 0$;
(H2) $\lim_{I\to\infty} f(I) = 1$.

The total population $N(t) = S(t) + I(t) + R(t)$ satisfies the equation

$$N'(t) = A - d_1 S(t) - (d_2 + \alpha)I(t) - d_3 R(t),$$

so $N(t)$ can vary with time. The basic reproduction number of the deterministic model (1) is

$$R_0 = \beta_1/(\gamma + d_2 + \alpha),$$

which measures the average number of secondary infections from a single infective host in an entirely susceptible population during the host's infectious period [3]. More specifically,

if $R_0 \leq 1$, the disease-free equilibrium $P_0 = (A/d_1, 0, 0)$ is globally asymptotically stable which means the disease will die out; if $R_0 > 1$, P_0 is unstable and there exist an endemic equilibrium $P^* = (S^*, I^*, R^*)$ which is globally asymptotically stable, the endemic equilibrium $P^* = (S^*, I^*, R^*)$ satisfies the following equations:

$$\begin{aligned}
&\Lambda - d_1 S^* - (\beta_1 - \beta_2 f(I^*))S^*I^* + \delta R^* = 0;\\
&(\beta_1 - \beta_2 f(I^*))S^*I^* - (d_2 + \gamma + \alpha)I^* = 0;\\
&\gamma I^* - (d_3 + \delta)R^* = 0.
\end{aligned}$$

To account for variability of the environment and stochasticity in the disease transmission process, as well as uncertainty in measurement of model parameters, various noise terms have been introduced into model (1), an ODE model becomes a system of stochastic differential equations (SDE). There are generally two approaches to derive a SDE model. To describe demographic stochasticity, Allen [4] discussed the approach of deriving stochastic differential equations from the forward Kolmogorov equation of continuous time Markov chain models. Schramm and Dimitrov [5] describe threshold as a random process, develop an extension to differential equation models of dynamical systems. To incorporate stochasticity in measurement and estimation of model parameters, noise terms have been introduced into deterministic models as perturbations to model parameters [6]–[13].

Some stochastic epidemic models exist disease-free equilibrium, the stochastic threshold of the model has been obtained [6]–[10]. Some stochastic epidemic models have no disease-free equilibrium anymore, the asymptotic behavior around the deterministic model's disease-free equilibrium and endemic equilibrium has been obtained [11]–[13].

It remains an open question find an appropriate form of a sharp stochastic threshold(reproduction number) for the dynamics of SDE epidemic models that plays the role of R_0 for ODE models.

In the present paper, we add four different white noises into the deterministic system (1) by perturbing model parameters $\beta_1, d_1, d_2 + \alpha$ and d_3 to $\beta_1 + \sigma_4\dot{B}_4(t), d_1 + \sigma_1\dot{B}_1(t), d_2 + \alpha + \sigma_2\dot{B}_2(t)$ and $d_3 + \sigma_3\dot{B}_3(t)$, where $\sigma_1, \sigma_2, \sigma_3$ are perturb intensities, then the deterministic model becomes to

$$\begin{aligned}
\mathrm{d}S &= [\Lambda - d_1 S - (\beta_1 - \beta_2 f(I))SI + \delta R]\mathrm{d}t\\
&\quad + \sigma_1 S\mathrm{d}B_1(t) - \sigma_4 SI\mathrm{d}B_4(t),\\
\mathrm{d}I &= [(\beta_1 - \beta_2 f(I))SI - (d_2 + \gamma + \alpha)I]\mathrm{d}t\\
&\quad + \sigma_4 SI\mathrm{d}B_4(t) + \sigma_2 I\mathrm{d}B_2(t),\\
\mathrm{d}R &= [\gamma I - (d_3 + \delta)R]\mathrm{d}t\\
&\quad + \sigma_3 R\mathrm{d}B_3(t),
\end{aligned} \tag{2}$$

We found out an improved stochastic noise reproduction number R_0^S which determine the extinction and persistence of the disease.

$$\begin{aligned}
R_0^S &= \frac{\frac{\beta_1\Lambda}{d_1} - \frac{\sigma_2^2}{2} - \frac{\sigma_4^2\Lambda^2}{2d_1^2 - d_1\sigma_1^2}}{\gamma + d_2 + \alpha}\\
&= R_0 - \frac{\sigma_2^2}{2}\frac{1}{\gamma + d_1 + \alpha}\\
&\quad - \frac{\sigma_4^2}{2}\frac{\Lambda^2}{(d_1^2 - \frac{d_1\sigma_1^2}{2})(\gamma + d_1 + \alpha)}.
\end{aligned} \tag{3}$$

More specifically, if $R_0^S < 1$, the disease dies out almost surely; if $R_0^S > 1$, the disease will weakly persistence and there exist a stationary distribution. The main contributions of this paper are summarized as follows:

- Compared with the previous works on stochastic modeling of epidemic models, the general SIRS model(2) is stochastic perturbed by four independent noises which enable the non-existence of disease-free equilibrium. It expends the adding method of parameter perturbation, contain previous works as special cases.
- The stability necessary and sufficient condition of the general SIRS model is obtained which is a sharp threshold for the disease dynamics. Our definition of R_0^S contains the deterministic model's R_0^D,some stochastic SIS models' stochastic reproduction number and SIRS models's stochastic reproduction number as special cases.
- Stochastic perturbations paly a positive role in disease extinction as they lower the reproduction number R_0^S. This agrees with control strategies on stochastic stabilization of a given unstable system [14]–[19], and provide a new thinking of disease Control.

In section 2, we prove the global existence of positive solutions. Stability analysis of the disease-free equilibrium is carried out in Section 3. In section 4, we give the persistence results in two versions: weak stochastic persistence, the existence of a globally stable stationary distribution under the condition $R_0^S > 1$. In section 5, An example and numerical simulations are provided to substantiate our theoretical results.

II. EXISTENCE OF THE GLOBAL POSITIVE SOLUTION

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ be a complete probability space with a filtration$\{\mathcal{F}_t\}_{t\geq 0}$. We use $(B_1(t), B_2(t), B_3(t)$ and $(B_4(t)$ to denote independent Brownian motions defined on the probability space. To establish that model (2) is well-posed, we show that the model has a unique global positive solution. Uniqueness of the solution follows from the Lipschitz properties of the coefficients of drift term and diffusion term. Global existence of solutions typically requires linear growth condition [14], which does not hold in model (2) because of the bilinear incidence. In this section, we first show that there exist a unique maximal positive solution, and then prove that this solution is global.

Theorem 2.1: For any initial data $(S(0), I(0), R(0)) \in R_+^3$, the stochastic SIRS epidemic model (2) has a unique positive

solution $(S(t), I(t), R(t)) \in R_+^3$ that exists for all $t \geq 0$ with probability one.

Proof: For any given initial data $(S(0), I(0), R(0)) \in R_+^3$, Lipschitz property of the right-hand-side functions implies that a unique local positive solution $(S(t), I(t), R(t)) \in R_+^3$ exists in a maximal interval $[0, \tau_e)$, where $\tau_e \leq +\infty$ is the escape time from $\mathbb{R}_+^3$ [14]. For $t \in [0, \tau_e)$, a solution can be expressed as

$$\begin{aligned} S(t) &= \varphi_1(t)\Big[S_0 + \int_0^t (\Lambda + \delta R(s))\varphi_1^{-1}(s)\mathrm{d}s\Big], \\ I(t) &= I_0\varphi_2(t), \\ R(t) &= \varphi_3(t)[R_0 + \gamma \int_0^t I(s)\varphi_3^{-1}(s)\mathrm{d}s], \end{aligned}$$

where

$$\begin{aligned} \varphi_1(t) &= \exp\Big\{ \int_0^t \Big[-(\beta_1 - \beta_2 f(I(u))I(u) - d_1 - \frac{\sigma_1^2}{2} \\ &\quad - \frac{\sigma_4^2 I^2(u)}{2}\Big]\mathrm{d}u + \sigma_1 B_1(t) - \int_0^t \sigma_4 I(u)\mathrm{d}B_4(u)\Big\}, \\ \varphi_2(t) &= \exp\Big\{ \int_0^t \Big[(\beta_1 - \beta_2 f(I(u))S(u) - d_2 - \alpha - \gamma \\ &\quad - \frac{\sigma_4^2 S^2(u) + \sigma_2^2}{2}\Big]\mathrm{d}u + \sigma_2 B_2(t) \\ &\quad + \int_0^t \sigma_4 S(u)\mathrm{d}B_4(u)\Big\}, \\ \varphi_3(t) &= \exp\Big[(-d_3 - \delta - \frac{\sigma_3^2}{2})t \\ &\quad + \sigma_3 B_3(t)\Big]. \end{aligned}$$

Therefore, with probability 1, a sample path of model (2) starting in R_+^3 will remain in R_+^3 for as long as the solution is defined. The escape time τ_e is then the blow-up time, and we need to show that $\tau_e = \infty$ almost surely. Let k_0 be sufficiently large such that $S(0), I(0), R(0)$ all lie with the interval$(0, k_0)$, For each integer $k > k_0$, define the stopping time

$$\tau_k = \inf\{t \in [0, \tau_e)\ \ S(t) + I(t) + R(t) \geq k\},$$

where $\inf \emptyset = \infty$. Set $\tau_\infty = \lim_{k\to\infty} \tau_k$, then $\tau_\infty \leq \tau_e$. It suffices to prove that $\tau_\infty = \infty$ a.s. for all $t \geq 0$. Suppose on the contrary that $\tau_\infty < \infty$. Then there exists pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that $\mathrm{P}\{\tau_\infty \leq T\} > \varepsilon$, and thus there exists integer $k_1 \geq k_0$ such that $\mathrm{P}\{\tau_k \leq T\} \geq \varepsilon$ for all $k \geq k_1$.

Consider function $V(S, I, R) = S + I + R$. Using the Itô formula, for any $t \in [0, T]$ and $k \geq k_1$

$$\begin{aligned} &\mathrm{E}V(S(t \wedge \tau_k), I(t \wedge \tau_k), R(t \wedge \tau_k)) \\ &\quad = V(S(0), I(0), R(0)) + \mathrm{E}\int_0^{t\wedge\tau_k} \mathrm{L}V(S(s), I(s), R(s))\mathrm{d}s, \end{aligned}$$

where $\mathrm{L}V$ satisfies

$$\mathrm{L}V(S, I, R) = A - d_1 S - (d_2 + \alpha)I - d_3 R \leq A.$$

Therefore,

$$\mathrm{E}V(S(t), I(t), R(t)) \leq V(S(0), I(0), R(0)) + AT,$$

here $t \in [0, T]$. Set $\Omega_k = \{\tau_k \leq T\} \subset \Omega$, then $\omega \in \Omega_k$ and $k \geq k_1$ imply that $S(\tau_k) + I(\tau_k) + R(\tau_k) \geq k$, hence

$$\begin{aligned} &V(S(0), I(0), R(0)) + AT \\ &\quad \geq \mathrm{E}[\mathrm{I}_{\Omega_k}(\omega)V(S(\tau_k), I(\tau_k), R(\tau_k))] \\ &\quad \geq \varepsilon k. \end{aligned}$$

Letting $k \to \infty$, we have $V(S(0), I(0), R(0)) + AT > \infty$, a contradiction. □

III. THE BASIC REPRODUCTION NUMBER AND THRESHOLD THEOREM

Let R_0^S be given in (3). We establish in this section that R_0^S is a sharp threshold for the stability of the disease-free solution $\bar{P}_0 = (\bar{S}, 0, 0)$, where $\bar{S}$ is the solution of

$$\mathrm{d}S(t) = (\Lambda - d_1 S)\mathrm{d}t + \sigma_1 S\mathrm{d}B_1(t).$$

Before proving the main theorem we put forward a lemma.

Lemma 3.1: Considering the following stochastic differential equation

$$\mathrm{d}x(t) = (\Lambda - d_1 x(t))\mathrm{d}t + \sigma_1 x(t)\mathrm{d}B_1(t). \tag{4}$$

where A, d_1, σ_1 are constants. The solution is stable in distribution, ergodic and satisfies

$$\begin{aligned} \lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)\mathrm{d}t &= \frac{\Lambda}{d_1}, \\ \lim_{T\to\infty} \frac{1}{T}\int_0^T x^2(t)\mathrm{d}t &= \frac{2\Lambda^2}{2d_1^2 - d_1\sigma_1^2}. \end{aligned} \tag{5}$$

Proof: The integral form of (4) can be written as

$$x(T) = x(0) + \Lambda T - d_1 \int_0^T x(t)\mathrm{d} + \sigma_1 \int_0^T x(t)\mathrm{d}B(t).$$

Dividing both sides by T and sending $T \to \infty$, applying the ergodic property of the stationary distribution [20] and also the large number theorem of martingales, we have the result that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)\mathrm{d}t = \frac{\Lambda}{d_1}. \tag{6}$$

Then we need to prove that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T x^2(t)\mathrm{d}t = \frac{2\Lambda^2}{2d_1^2 - d_1\sigma_1^2}. \tag{7}$$

To find the second moment of $x(t)$,we use Itô's formula

$$\mathrm{d}(x^2(t)) = (\sigma_1^2 - 2d_1)x^2(t)\mathrm{d}t + 2\Lambda x(t)\mathrm{d}t + 2\sigma_1 x^2(t)\mathrm{d}B_1(t).$$

Using the same method and combining the mean moment of the stationary distribution, we obtain

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T x^2(t)\mathrm{d}t = \frac{2\Lambda^2}{2d_1^2 - d_1\sigma_1^2}.$$

We now prove the main theorem.

Theorem 3.2: The disease-free solution $\bar{P}_0 = (\bar{S}, 0, 0)$ is almost surely asymptotically stable if $R_0^S < 1$ and is almost surely unstable if $R_0^S > 1$.

Proof: We use the method of linearization for the stability analysis. Consider the linearized system at $\bar{P}_0$:

$$\begin{aligned} \mathrm{d}x(t) &= [-d_1\, x(t) - \beta_1 \bar{S} y(t) + \delta z(t)]\mathrm{d}t \\ &\quad + \sigma_1 x(t)\mathrm{d}B_1(t) - \sigma_4 \bar{S} y(t)\mathrm{d}B_4(t), \\ \mathrm{d}y(t) &= [\beta_1 \bar{S} - \gamma - d_2 - \alpha] y(t)\mathrm{d}t \\ &\quad + \sigma_2 y(t)\mathrm{d}B_2(t) + \sigma_4 \bar{S} y(t)\mathrm{d}B_4(t), \\ \mathrm{d}z(t) &= [\gamma y(t) - (d_3 + \delta) z(t)]\mathrm{d}t \\ &\quad + \sigma_3 z(t)\mathrm{d}B_3(t). \end{aligned} \tag{8}$$

The analytic solution of the second equation in (8) is

$$\begin{aligned} y(t) = y(0)\exp\Big\{ \int_0^t \Big[\beta_1 \bar{S}(u) - \gamma - d_2 \\ - \alpha - \frac{\sigma_2^2}{2} - \frac{\sigma_4^2}{2}\bar{S}^2(u)\Big]\mathrm{d}u \\ + \sigma_1 B_1(t) + \sigma_4 \int_0^t \bar{S}(u)\mathrm{d}B_4(u)\Big\}. \end{aligned} \tag{9}$$

From lemma 3.1, $\bar{S}(t)$ satisfies

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \bar{S}(u)\mathrm{d}u = \frac{\Lambda}{d_1}; \tag{10}$$

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \bar{S}^2(u)\mathrm{d}u = \frac{2\Lambda^2}{2d_1^2 - d_1\sigma_1^2}; \tag{11}$$

Combining with the fact that $B_i(t)$ satisfy $B_i(t)/t \to 0$ a.s. as $t \to \infty$, we arrive at

$$\begin{aligned} \limsup_{t\to\infty} \frac{1}{t}\log \mid y(t) \mid = \frac{\beta_1\Lambda}{d_1} - \gamma - d_2 - \alpha \\ - \frac{\sigma_2^2}{2} - \frac{\sigma_4^2\Lambda^2}{2d_1^2 - d_1\sigma_1^2}. \end{aligned}$$

Let

$$\begin{aligned} \lambda_1 &= \frac{\beta\Lambda}{d_1} - \gamma - d_2 - \alpha - \frac{\sigma_2^2}{2} - \frac{\sigma_4^2\Lambda^2}{2d_1^2 - d_1\sigma_1^2}, \\ \lambda_2 &= -d_3 - \delta - \frac{\sigma_3^2}{2}. \end{aligned} \tag{12}$$

Then $\lambda_2 < 0$. If $R_0^S > 1$, then $\lambda_1 > 0$, and $y(t) \geq \exp(\lambda_1 t)$ a.e. for sufficiently large t. Therefore $y(t) \to \infty$ exponentially, and $\bar{P}_0$ is a.e. unstable.

Suppose that $R_0^S < 1$. Then $\lambda_1 < 0$. This implies that $y(t) \to 0$ exponentially a.s. as $t \to \infty$. Hence for any $0 < \epsilon_1 < -\lambda_1$, there a $\xi > 0$ such that

$$|y(t)| \leq \xi \exp\Big[(\lambda_1 + \epsilon_1)t\Big]. \tag{13}$$

From the third equation of system (8) and by the Itȯ's formula, we can derive the following relation

$$z(t) = e^{\lambda_2\, t + \sigma_3 B_3(t)}\Big[z(0) + \int_0^t \gamma y(s) e^{-\lambda_2\, s - \sigma_3 B_3(s)}\mathrm{d}s\Big]. \tag{14}$$

Since $B_i(t)/t \to \infty, i = 1, 2, 3$, there exist $T > 0$ and $\epsilon_2 > 0$ such that

$$|B_i(t)| \leq \epsilon_2\, t, \quad t \geq T, \ a.s.$$

with

$$\lambda_1 + \epsilon_1 + 2|\sigma_3|\epsilon_2 < 0 \tag{15}$$

$$\lambda_2 + |\sigma_3|\epsilon_2 < 0 \tag{16}$$

Substituting (13) into (14) we obtain

$$\begin{aligned} |z(t)| &\leq \Big[|z(0)| + \int_0^T \gamma |y(s)| \\ &\quad \times e^{-\lambda_2\, s - \sigma_3 B_3(s)}\mathrm{d}s\Big] e^{\lambda_2 t - \sigma_3 B_3(t)} \\ &\quad + e^{\lambda_2 t - \sigma_3 B_3(t)} \int_T^t \gamma\xi e^{(\lambda_1 + \epsilon_1 - \lambda_2)s + |\sigma_3|\epsilon_2\, s} ds \\ &\leq \Big[|z(0)| + \int_0^T \gamma |y(s)| \\ &\quad \times e^{-\lambda_2\, s - \sigma_3 B_3(s)}\mathrm{d}s\Big] e^{\lambda_2 t - \sigma_3 B_3(t)} \\ &\quad + \gamma\, \xi e^{(\lambda_2 + |\sigma_3|\epsilon_2)t} \int_T^t e^{(\lambda_1 + \epsilon_1 - \lambda_2 + \epsilon_2|\sigma_3|)s} ds \\ &\leq (C_1 + C_2) e^{(\lambda_2 + |\sigma_3|\epsilon_2)t} + C_3 e^{(\lambda_1 + \epsilon_1 + 2|\sigma_3|\epsilon_2)t}, \end{aligned}$$

where

$$\begin{aligned} C_1 &= |z(0)| + \int_0^T \gamma |y(s)| e^{-\lambda_2\, s - \sigma_3 B_3(s)}\mathrm{d}s, \\ C_2 &= \frac{\gamma|\xi|}{|\lambda_1 - \lambda_2\lambda + \epsilon_1 + \epsilon_2\sigma_3|} e^{(\lambda_1 - \lambda_2\lambda + \epsilon_1 + \epsilon_2\sigma_3)T}, \\ C_3 &= \frac{\gamma|\xi|}{|\lambda_1 - \lambda_2\lambda + \epsilon_1 + \epsilon_2\sigma_3|}. \end{aligned}$$

Therefore, $|z(t)| \to 0$ exponentially a.s. as $t \to \infty$. A similar argument can be used to show that $|x(t)| \to 0$ exponentially a.s. as $t \to \infty$. Summarize all the three results, the largest Lyapunov exponents of the linearized system (8) are negative. By the Oseledec Multiplicative Ergodic Theorem [21], we conclude the disease-free solution $(\bar{S}, 0, 0)$ of system (2) is almost sure exponentially stable if $R_0^S < 1$. □

Remark 1: In theorem 2, we gain necessary and sufficient conditions related to the basic reproduction number R_0^S (3) for the stochastic SIRS model's stability behaviour. Our method is different from Lyapunov function method which typically provides only sufficient conditions of the disease-free equilibrium.

Remark 2: If $d_1 = d_2 = d_3 = \mu$ and $\alpha = \sigma_4 = 0$, our stochastic SIRS model(2) becomes to the model in reference [11]. They point out that R_0^D does not exceed a critical level, which has been proved theoretically in this paper. The stochastic reproduction number is

$$R_0^S = \frac{\beta_1\Lambda}{\mu(\mu+\gamma)} - \frac{\sigma_2^2}{2(\mu+\gamma)} = R_0^D - \frac{\sigma_2^2}{2(\mu+\gamma)}.$$

Remark 3: If $f(I) = \frac{\alpha\beta I}{\beta_2(1+\alpha I)}$, $\beta_1 = \beta, \alpha = \varepsilon, d_1 = d_2 = d_3 = \mu, \sigma_4 = \sigma$ and $\sigma_1 = \sigma_2 = \sigma_3 = 0$, our stochastic SIRS model(2) becomes to the model in Jiang et al. [7]. They have obtained a threshold of the stochastic system

$$\tilde{R}_0 = \frac{\beta_1\Lambda}{\mu(\mu+\gamma+\varepsilon)} - \frac{\sigma^2\Lambda^2}{2\mu^2(\mu+\gamma+\varepsilon)}$$

If $\tilde{R}_0 < 1$,$I(t)$ tends to zero exponentially. This threshold $\tilde{R}_0$ is equivalent to our R_0^S.

Remark 4: If $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0$, our stochastic SIRS model(2) becomes to the deterministic model(1). There is a disease-free equilibrium $(\frac{\Lambda}{d_1}, 0, 0)$, the stochastic reproduction number R_0^S equals to R_0^D. By theorem 3.1, the disease-free equilibrium is asymptotically stable.

Regarding the global asymptotic stability of $\bar{P}_0$, we have the following result.

Theorem 3.2: Assume that the stochastic reproduction number $R_0^S < 1$ and the direct contact transmission coefficient satisfies $\beta_1 < \sigma_4\sqrt{2(\gamma + d_2 + \alpha + \frac{\sigma_2^2}{2})}$, Then for any given initial data $(S(0), I(0), R(0)) \in \mathbb{R}^3_+$, the solution $(S(t), I(t), R(t))$ of model (2) has the property

$$\limsup_{t\to\infty} \frac{1}{t}\log(I(t)) < 0, \quad a.s.$$

which means the disease will die out globally.

Proof: By the Itô formula, we have

$$\begin{aligned}\log(I(t)) = \log(I(0)) + \int_0^t [(\beta_1 - \beta_2 f(I(u)))S(u) \\ - \gamma - d_2 - \alpha - \frac{\sigma_4^2 S(u)^2}{2} - \frac{\sigma_2^2}{2}]\mathrm{d}u \\ - \sigma_2 B_2(t) + \bar{M}(t), \end{aligned} \tag{17}$$

where $\bar{M}(t) = \int_0^t \sigma_4 S(u)\mathrm{d}B_4(u)$ is a continuous local martingale. By the exponential martingale inequality (see Mao [14], Theorem 7.4 on P44) and Borel-Cantelli's Lemma, we get that for almost all $\omega \in \Omega$, there exits a random integer $k_0(\omega)$ such that if $k > k_0$,

$$\bar{M}(t) \le \frac{1}{2}v\int_0^t \sigma_4^2 S^2(u)\mathrm{d}u + \frac{2}{v}\ln k, \tag{18}$$

for all $t \in [0, k]$. Combining (17) and (18) we obtain

$$\begin{aligned}\log(I(t)) = \log(I(0)) + \int_0^t [(\beta_1 - \beta_2 f(I(u)))S(u) \\ - \gamma - d_2 - \alpha - \frac{(1-v)\sigma_4^2 S(u)^2}{2} \\ - \frac{\sigma_2^2}{2}]\mathrm{d}u - \sigma_2 B_2(t) + \frac{2}{v}\ln k. \end{aligned}$$

We define $LV(x)$ as

$$\begin{aligned}LV(x) = (\beta_1 - \beta_2 f(x))S(u) - \gamma - d_2 - \alpha \\ - \frac{(1-v)\sigma_4^2 S(u)^2}{2} - \frac{\sigma_2^2}{2}, \end{aligned}$$

Noting that

$$\begin{aligned}LV(I(u)) &\le \beta_1 S(u) - \gamma - d_2 - \alpha \\ &\quad - \frac{(1-v)\sigma_4^2 S(u)^2}{2} - \frac{\sigma_2^2}{2} \\ &\le \frac{\beta_1^2}{2(1-v)\sigma_4^2} - \gamma - d_2 - \alpha - \frac{\sigma_2^2}{2}, \end{aligned}$$

it follows that

$$\begin{aligned}\log(I(t)) = \log(I(0)) + [\frac{\beta_1^2}{2(1-v)\sigma_4^2} - \gamma - d_2 - \alpha - \frac{\sigma_2^2}{2}]t \\ + \frac{2}{v}\ln k - \sigma_2 B_2(t). \end{aligned}$$

Thus for $k - 1 \le t \le k$, we have

$$\begin{aligned}\limsup_{t\to\infty} \frac{1}{t}\log(I(t)) \le \frac{\beta_1^2}{2(1-v)\sigma_4^2} - \gamma - d_2 - \alpha - \frac{\sigma_2^2}{2} \\ - \limsup_{t\to\infty} \frac{\sigma_2 B_2(t)}{t}. \end{aligned}$$

By the low of large numbers to the Brownian motion and sending $v \to 0$

$$\begin{aligned}\limsup_{t\to\infty} \frac{1}{t}\log(I(t)) &\le \frac{\beta_1^2}{2(1-v)\sigma_4^2} - \gamma - d_2 - \alpha - \frac{\sigma_2^2}{2} \\ &\le 0, \end{aligned}$$

which means $I(t)$ is almost sure exponentially stable in the large. Whence the proof is complete. □

IV. STOCHASTIC PERSISTENCE

In this section we establish that the disease persists in the population if $R_0^S > 1$. There are several concepts of stochastic persistence [6], [10]. Firstly, We demonstrate weakly persistence of our stochastic SIRS model.

Theorem 4.1: If $R_0^S > 1$,then the disease will be weakly persistent.

Proof: If it's not true, there is a $\varepsilon \in (0, 1)$,such that $p(\Omega_1) > \varepsilon$.

$$\Omega_1 = \{\omega | \limsup_{t\to\infty} I(t) = 0\}$$

so for every $\omega \in \Omega_1$, there is a $T_1 = T_1(\omega) > 0$,such that

$$I(t) = 0, \ t > T_1,$$

as we know

$$\begin{aligned}R(t,\omega) &= e^{-(d_3+\delta+\frac{\sigma_3^2}{2})t+\sigma_3 B_3(t)}[R(0,\omega) \\ &\quad + \int_0^t \gamma I(s,\omega)e^{(d_3+\delta+\frac{\sigma_3^2}{2})s-\sigma_3 B_3(s)}\mathrm{d}s] \\ &= e^{-(d_3+\delta+\frac{\sigma_3^2}{2})t+\sigma_3 B_3(t)}[R(0,\omega) \\ &\quad + \int_0^{T_1} \gamma I(s,\omega)e^{(d_3+\delta+\frac{\sigma_3^2}{2})s-\sigma_3 B_3(s)}\mathrm{d}s], \end{aligned}$$

so for any $\varepsilon > 0$, there exist $T_2 = T_2(\omega) > 0$, such that

$$R(t) \le \varepsilon, \ t > T_2.$$

For $t > T = max\{T_1, T_2\}$, using Lemma 3.1 we know

$$\begin{aligned}\lim_{t\to\infty} \frac{1}{t}\int_0^t S(u)\mathrm{d}u = \frac{\Lambda + \delta\varepsilon}{d_1}; \\ \lim_{t\to\infty} \frac{1}{t}\int_0^t S^2(u)\mathrm{d}u = \frac{2(\Lambda+\delta\varepsilon)^2}{2d_1^2 - d_1\sigma_1^2}; \end{aligned} \tag{19}$$

Moreover by the large number of martingales, there is also a set Ω_2 such that for every $\omega \in \Omega_2$,

$$\lim_{t\to\infty}[\frac{1}{t}\int_0^t \sigma_4 S(s)\mathrm{d}B_4(s) + \frac{\sigma_2 B_2(t)}{t}] = 0.$$

Since $p(\Omega_2) = 1$, we can find some $\omega \in \Omega_1 \cap \Omega_2$. Now fixed any $\omega_1 \in \Omega_1 \cap \Omega_2$ for $t > T(\omega_1)$,

$$\begin{aligned} f(I(t,\omega)) &= 0 \\ \log(I(t,\omega)) &= \log(I(0)) + \int_0^t [\beta_1(S(s,\omega) - \gamma - d_2 \\ &\quad -\alpha - \frac{\sigma_2^2}{2} - \frac{\sigma_4^2 S^2(s,\omega)}{2}]\mathrm{d}s \\ &\quad + \int_0^t \sigma_4 S(s)\mathrm{d}B_4(s) + \sigma_2 B_2(t) \end{aligned}$$

Using (19) and sending $\varepsilon \to 0$, the above equation yields

$$\begin{aligned} \liminf_{t\to\infty} \frac{1}{t}\log(I(t,\omega)) &= \frac{\beta_1\Lambda}{d_1} - \gamma - d_2 - \alpha \\ &\quad -\frac{\sigma_1^2}{2} - \frac{\sigma_4^2 A^2}{2d_1^2 - d_1\sigma_1^2} \\ &> 0 \end{aligned}$$

which means $\lim_{t\to\infty} I(t) = \infty$, however this contradicts $I(t) = 0$. The proof is complete. □

Next, we concentrate on the existence of stationary distribution. To prove the main results, we need Khasminskii's stationary distribution theorem [21]. let $X(t)$ be a regular time-homogeneous Markov process described by the stochastic differential equation

$$\mathrm{d}X(t) = b(X)\mathrm{d}t + \sum_{r=1}^{k} \sigma_r(X)\mathrm{d}B_r(t).$$

The diffusion matrix is defined as follows:

$$A(x) = (a_{ij}(x)),\ a_{ij}(x) = \sum_{r=1}^{k} \sigma_r^i(x)\sigma_r^j(x)$$

Lemma 4.2: The Markov process $X(t)$ has a unique stationary distribution μ if there exists a open bounded domain $U \subset \mathbb{R}^l$,and the condition are satisfied.

(A) In the domain U and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix $A(x)$ is bounded away from zero.

(B) If $x \in \mathbb{R}^l$ U, the mean time τ at which a path issuing from x reaches the set U is finite, and $\sup_{x\in K} E^x\tau < \infty$ for every compact subset $K \subset \mathbb{R}^l$. Let $f(\cdot)$ be a function integrable with respect to the measure μ. Then

$$P(lim_{T\to\infty}\frac{1}{T}\int_0^T f(X^x(t))\mathrm{d}t = \int_{\mathbb{R}^l} f(x)\mu((d)x)) = 1,$$

for all $x \in \mathbb{R}^l$.

Before the main proving, Let us introduce five parameters which will be used in Theorem 4.3

$$\begin{aligned} \omega_1 &= \frac{2d_2(\delta + d_1 + d_3)}{\delta} - [\frac{2(d_1 + d_2 + \alpha)}{\beta}] \\ &\quad + \frac{2(d_1 + d_3)(d_1 + d_2 + \alpha + \gamma)}{\delta} \\ &\quad + \frac{2(d_1 + d_3 + \delta)}{\delta I^*}]\frac{I^*\sigma_1^2}{2}, \\ \omega_2 &= \frac{2\delta(d_2 + \alpha) + 2(d_1 + d_3)(\alpha + d_2 + \gamma)}{\delta} \\ &\quad - \frac{\delta + d_1 + d_3}{\delta}\sigma_2^2, \\ \omega_3 &= \frac{2[d_3\gamma + (d_2 + \alpha - d_1)(d_3 + \delta)]}{\gamma} \\ &\quad - \frac{\gamma + d_2 + \alpha - d_1}{\gamma}\sigma_3^2, \\ \varpi &= \frac{a_2 I^*\sigma_2^2}{2} + \frac{(d_1 + d_3)^2(S^*)^2}{\omega_1\delta^2} + \frac{a_2^2(S^*)^2}{\omega_2} \\ &\quad + \frac{(d_2 + \alpha - d_1)^2(R^*)^2}{\gamma^2\omega_3} - \frac{(d_1 + d_3)(S^*)^2}{\delta} \\ &\quad - a_2(I^*)^2 - \frac{(d_2 + \alpha - d_1)(R^*)^2}{\gamma} \\ a_2 &= \frac{2(d_1 + d_2 + \alpha)}{\beta} \\ &\quad + \frac{2(d_1 + d_3)(d_1 + d_2 + \alpha + \gamma)}{\delta}. \end{aligned}$$

Theorem 4.3: If $R_0^S > 1$ and $\varpi < \frac{(d_1+d_3)^2(S^*)^2}{\delta^2\omega_1} \wedge \frac{a_2^2(I^*)^2}{\omega_2} \wedge \frac{(d_2+\alpha-d_1)^2(R^*)^2}{\gamma^2\omega_3}$, Then there exists a stationary distribution for the stochastic system (2).

Proof: Since $R_0 \geq R_0^S > 1$,then there exists a positive equilibrium (S^*, I^*, R^*) of the deterministic SIRS model, it satisfies

$$\begin{aligned} \Lambda &= d_1 S^* + (\beta_1 - \beta_2 f(I^*))S^* I^* - \delta R^*, \\ (\beta_1 - \beta_2 f(I^*))S^* &= \gamma + d_2 + \alpha, \\ \gamma I^* &= (d_3 + \delta)R^*. \end{aligned} \tag{20}$$

Define a positive function as follows

$$\begin{aligned} V(S,I,R) &= (S + I + R - S^* - I^* - R^*)^2 \\ &\quad + a_1(S + I - S^* - I^*)^2 \\ &\quad + a_2(I - I^* - I^* ln\frac{I}{I^*}) \\ &\quad + a_3(R - R^*)^2 \\ &= V_1 + a_1 V_2 + a_2 V_3 + a_3 V_4. \end{aligned}$$

Here a_1, a_2, a_3 are constants to be specified later. By Itô's formula, we obtain

$$\begin{aligned} \mathrm{L}V_1 &= 2(S - S^* + I - I^* + R - R^*)[A \\ &\quad - d_1 S - (d_2 + \alpha)I - d_3 R] + \sigma_1^2 S^2 \\ &\quad + \sigma_2^2 I^2 + \sigma_3^2 R^2. \end{aligned}$$

Adding the three equations in (20), we get $A = d_1S^* + (d_2 + \alpha)I^* + d_3R^*$, using this to replace A gives

$$\begin{aligned}LV_1 &= 2(S - S^* + I - I^* + R - R^*)[-d_1(S - S^*) \\ &\quad -(d_2 + \alpha)(I - I^*) - d_3(R - R^*)] \\ &\quad +\sigma_1^2S^2 + \sigma_2^2I^2 + \sigma_3^2R^2 \\ &= -2d_1(S - S^*)^2 - 2(d_2 + \alpha)(I - I^*)^2 \\ &\quad -2d_3(R - R^*)^2 + \sigma_1^2S^2 + \sigma_2^2I^2 + \sigma_3^2R^2 \\ &\quad -2(d_1 + d_2 + \alpha)(S - S^*)(I - I^*) \\ &\quad -2(d_1 + d_3)(S - S^*)(R - R^*) \\ &\quad -2(d_2 + d_3 + \alpha)(I - I^*)(R - R^*).\end{aligned}$$

To calculate V_2

$$\begin{aligned}LV_2 &= 2(S - S^* + I - I^*)[A - d_1S - (\gamma + d_2 + \alpha)I \\ &\quad + \delta R] + \sigma_1^2S^2 + \sigma_2^2I^2. \qquad (21)\end{aligned}$$

Substituting (20) into (21) yield

$$\begin{aligned}LV_2 &= 2(S - S^* + I - I^*)[A - d_1S - (\gamma + d_2 + \alpha)I \\ &\quad +\delta R] + \sigma_1^2S^2 + \sigma_2^2I^2 \\ &= 2(S - S^* + I - I^*)[-d_1(S - S^*) \\ &\quad -(\alpha + d_2 + \alpha)(I - I^*) \\ &\quad +\delta(R - R^*)] + \sigma_1^2S^2 + \sigma_2^2I^2 \\ &= -2d_1(S - S^*)^2 - 2(\alpha + d_2 + \gamma)(I - I^*)^2 \\ &\quad +\sigma_1^2S^2 + \sigma_2^2I^2 \\ &\quad -2(d_1 + d_2 + \alpha + \gamma)(S - S^*)(I - I^*) \\ &\quad +2\delta(S - S^*)(R - R^*) + 2\delta(I - I^*)(R - R^*)\end{aligned}$$

we calculate V_3 and use (20) as above we get

$$\begin{aligned}LV_3 &= (1 - \frac{I^*}{I})(\beta S - \gamma - d_2 - \alpha)I + \frac{I^*}{2}(\sigma_4^2S^2 + \sigma_2^2) \\ &= \beta(S - S^*)(I - I^*) + \frac{I^*}{2}(\sigma_4^2S^2 + \sigma_2^2),\end{aligned}$$

we calculateV_4

$$\begin{aligned}LV_4 &= 2(R - R^*)[\gamma I - (d_3 + \delta)R] + \sigma_3^2R^2 \\ &= 2\gamma(R - R^*)(I - I^*) - 2(d_3 + \delta)(R - R^*)^2 \\ &\quad +\sigma_3^2R^2.\end{aligned}$$

Selecting coefficients $a_1 = \frac{d_1+d_3}{\delta}, a_2 = \frac{2(d_1+d_2+\alpha)}{\beta} + \frac{2(d_1+d_3)(d_1+d_2+\alpha+\gamma)}{\delta}, a_3 = \frac{d_2+\alpha-d_1}{\gamma}$,we obtain

$$\begin{aligned}LV &= -\frac{2d_1(\delta + d_1 + d_3)}{\delta}(S - S^*)^2 \\ &\quad -\frac{2\delta(d_2 + \alpha) + 2(d_1 + d_3)(\alpha + d_2 + \gamma)}{\delta}(I - I^*)^2 \\ &\quad -\frac{2[d_3\gamma + (d_2 + \alpha - d_1)(d_3 + \delta)]}{\gamma}(R - R^*)^2 \\ &\quad +[\frac{2(d_1 + d_2 + \alpha)}{\beta} \\ &\qquad + \frac{2(d_1 + d_3)(d_1 + d_2 + \alpha + \gamma)}{\delta}]\frac{I^*\sigma_2^2}{2} \\ &\quad +[\frac{2(d_1 + d_2 + \alpha)}{\beta} + \frac{2(d_1 + d_3)(d_1 + d_2 + \alpha + \gamma)}{\delta} \\ &\qquad + \frac{2(d_1 + d_3 + \delta)}{\delta I^*}]\frac{I^*\sigma_1^2}{2}S^2 \\ &\quad +\frac{\delta + d_1 + d_3}{\delta}\sigma_2^2I^2 + \frac{\gamma + d_2 + \alpha - d_1}{\gamma}\sigma_3^2R^2.\end{aligned}$$

Using the coefficients determined above, we obtain

$$LV = -\omega_1(S - \bar{S})^2 - \omega_2(I - \bar{I})^2 - \omega_3(R - \bar{R})^2 + \varpi.$$

If ϖ satisfies the following condition

$$\varpi < \omega_1\bar{S}^2 \wedge \omega_2\bar{I}^2 \wedge \omega_3\bar{R}^2,$$

then we find out the ellipsoid U

$$\omega_1(S - \bar{S})^2 + \omega_2(I - \bar{I})^2 + \omega_3(R - \bar{R})^2 = \varpi$$

stay in R_+^3. Taking U to be a neighborhood of the ellipsoid such that $\bar{U} \in R_+^3$, then the inequality $(S, I, R) \in R_3^+ \setminus U, LV < 0$ holds, which implies the condition (B2) in reference [21] is satisfied. On the other hand, it is to see that the diffusion matrix is uniformly elliptic in U. Thus the stochastic system (2) has a stationary distribution $\mu(\cdot)$ and it is ergodic. □

V. NUMERICAL SIMULATIONS ANALYSIS

In this section, we provide three numerical simulation results for the stochastic model(2) to substantiate theoretical findings: threshold theorem in section 3 and persistence theorem in section 4. Using Milstein's higher order method [22], [23], the numerical equations are

$$\begin{aligned}S_{k+1} &= S_k + (\Lambda - d_1S_k - (\beta_1 - \beta_2f(I_k))S_kI_k + \delta R_k)\Delta t \\ &\quad +\sigma_1S_k\sqrt{\Delta t}\xi_{1,k} + \frac{1}{2}\sigma_1^2S_k^2(\xi_{1,k}^2 - 1)\Delta t \\ &\quad -\sigma_4S_kI_k\sqrt{\Delta t}\xi_{4,k} + \frac{1}{2}\sigma_4^2S_k^2I_k^2(\xi_{4,k}^2 - 1)\Delta t, \\ I_{k+1} &= I_k + [(\beta_1 - \beta_2f(I_k))S_kI_k - (d_2 + \gamma + \alpha)I_k]\Delta t \\ &\quad +\sigma_2I_k\sqrt{\Delta t}\xi_{2,k} + \frac{1}{2}\sigma_2^2I_k^2(\xi_{2,k}^2 - 1)\Delta t \\ &\quad +\sigma_4S_kI_k\sqrt{\Delta t}\xi_{4,k} + \frac{1}{2}\sigma_4^2S_k^2I_k^2(\xi_{4,k}^2 - 1)\Delta t, \\ R_{k+1} &= R_k + [\gamma I_k - (d_3 + \delta)R_k]\Delta t \\ &\quad +\sigma_3R_k\sqrt{\Delta t}\xi_{3,k} + \frac{1}{2}\sigma_3^2R_k^2(\xi_{3,k}^2 - 1)\Delta t, \qquad (22)\end{aligned}$$

where $\xi_{1,k}, \xi_{1,k}, \xi_{1,k}, \xi_{1,k}, k = 1, 2, \ldots, n$ are independent stochastic variables N(0,1).

Note that

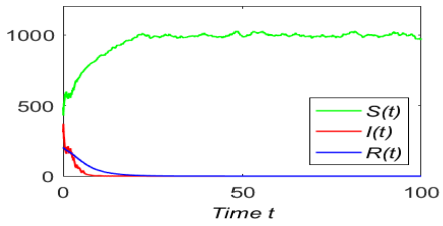
$$\begin{aligned}R_0^S &= \frac{\frac{\beta_1 A}{d_1} - \frac{\sigma_2^2}{2} - \frac{\sigma_4^2A^2}{2d_1^2 - d_1\sigma_1^2}}{\gamma + d_2 + \alpha} \\ &= R_0 - \frac{\sigma_2^2}{2}\frac{1}{\gamma + d_1 + \alpha} \\ &\quad -\frac{\sigma_4^2}{2}\frac{A^2}{(d_1^2 - \frac{d_1\sigma_1^2}{2})(\gamma + d_1 + \alpha)}.\end{aligned}$$

TABLE 1. The fixed parameters of model (2).

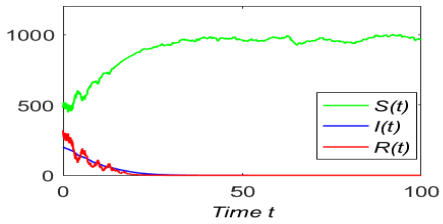
Parameters	Λ	d_1	d_2	d_3	δ	α	γ
Value	100	0.10	0.11	0.09	0.10	0.20	0.10
Unit	person	day^{-1}	day^{-1}	day^{-1}	day^{-1}	day^{-1}	day^{-1}

TABLE 2. The variable parameters of model (2).

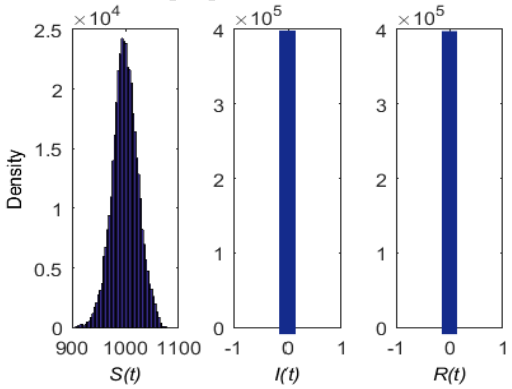
R_0	R_0^S	β_1	β_2	σ_1	σ_2	σ_3	σ_4
0.5818	0.8869	0.0004	0.0003	0.0100	0.0100	0.0100	0.0005
0.9466	1.3304	0.0006	0.0003	0.0100	0.0300	0.0100	0.00056
1.2815	1.3304	0.0006	0.0003	0.0100	0.0100	0.0100	0.0002



(a)One sample path of the stochastic SIRS model



(b)Another sample path of the stochastic SIRS model

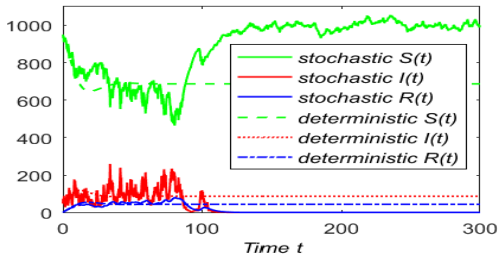


(c)Density of the stochastic SIRS model's solution

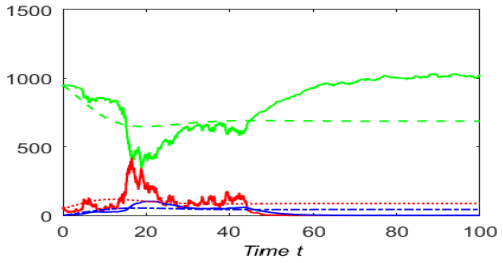
FIGURE 1. Extinction behaviour under the conditions: $R_0^S < 1$ and $R_0 < 1$.

According to the structure of the reproduction number, we assume some fixed parameters in table 1 and some variable parameters in table 2. Modifications of the parameters in table 2 will cause the fluctuation of R_0 and R_0^S around 1.

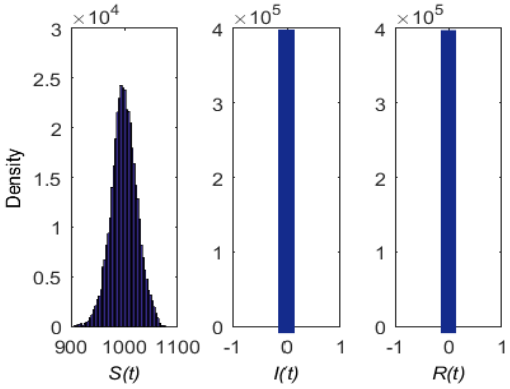
Firstly, we give two examples to support the threshold theorem 3.1: under the condition of $R_0^S < 1$, disease will die out no matter the value of R_0 lower or bigger than 1. we assume that $f(I) = \frac{I}{I+1}$ and the initial values are (480,320,200). In figure 1, we choose the first row's parameters in table 2 which guarantee $R_0^S \leq R_0 < 1$. Fig.1(a) and Fig.1(b) present two different sample paths of $S(t), I(t), R(t)$. Both of the infective class go to 0. To illustrate the extinction situation in probability, we also statistics 40000 times simulation



(a)One sample path of the stochastic SIRS and the deterministic SIRS model



(b) Another sample path of the stochastic SIRS model



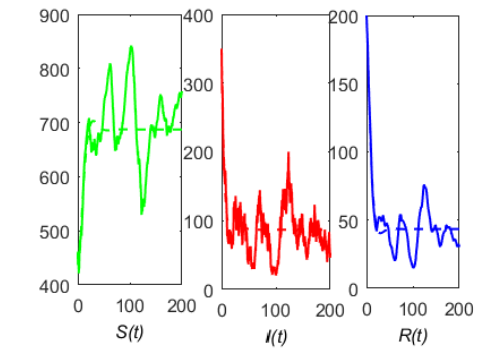
(c)Density of the stochastic model's solution

FIGURE 2. Extinction behaviour under the conditions: $R_0^S < 1$ but $R_0 > 1$.

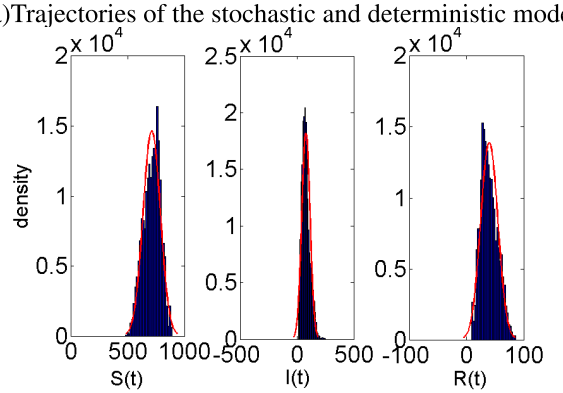
results on $S(10000)$,$I(10000)$ and $R(10000)$, the frequency histogram show the extinction of disease.

In figure 2, we choose the second row's parameters in table 2 which guarantee $R_0^S < 1$. Since $R_0 > 1$, the deterministic model(1) has an endemic equilibrium E^* which is globally asymptotical stable. By theorem 3.1, I(t) of system(2) tends to 0. we can observe that the sample paths of the solution converge to $(\bar{S},0,0)$ in Fig.2(a) and Fig.2(b), where $\bar{S}$ is the solution of $\mathrm{d}S = (\Lambda - d_1 S)\mathrm{d}t + \sigma_1 S\mathrm{d}B(t)$. 40000 times stochastic simulations on $S(10000)$,$I(10000)$ and $R(10000)$ also be presented in Figure 2 (c), it clearly show the extinction of disease.

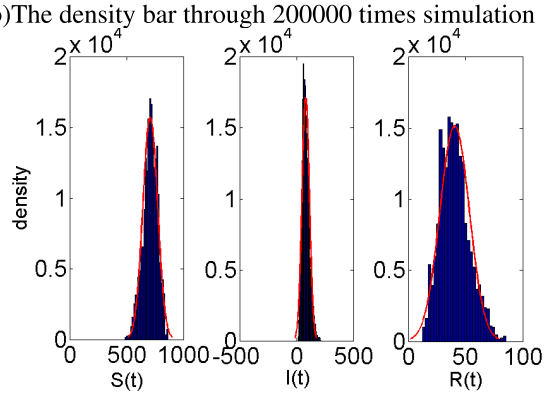
To investigate the persistence behaviour, we use the third row's parameter in table 2 which satisfy the condition in Theorem 4.1 and 4.2. We can compute the deterministic model's endemic equilibrium is (687.2436, 86.8768, 43.4384). In figure 3(a), solution fluctuate around the endemic equilibrium which means the disease persistence. In figure 3(b), we present the frequency histograms based on 200000 stochastic simulations for $I(t)$ at time t = 10000.



(a)Trajectories of the stochastic and deterministic model



(b)The density bar through 200000 times simulation



(c)The density bar though continuous-time section

FIGURE 3. **Trajectories and stationary distribution with the conditions: $R_0^S > 1$ and $R_0 > 1$.**

In figure 3(c), we collect $S(t), I(t), R(t)$ from $t = 300001$ to $t = 500000$ step on $\Delta t = 1$. Comparing the curves of figure 3(b) and figure 3(c), we can conclude that there exists a stationary distribution for the stochastic system(2).

VI. CONCLUSION

In this paper, we investigated the dynamics of a stochastic SIRS model with general awareness-induced incidence and four independent Brownian motions. Firstly, we have verified the existence and uniqueness of the global positive solution. Then we have derived the noise modified basic reproduction number R_0^S for the stochastic model and show that it is a sharp threshold. More specifically, if $R_0^S < 1$, the disease-free solution P_0 is asymptotically stable; if $R_0^S > 1$, the disease is weakly persistent and there is a stationary distribution under a parameter restrictive condition. The sharp threshold is a new result which gives a sufficient and necessary condition about the stability of disease-free solution since there is no disease-free equilibrium for the general stochastic SIRS model(2). We have shown wide range numerical investigation results of the stochastic model to substantiate the sharp threshold R_0^S and the existence of the stationary distribution. Our results provide a new thinking to disease preventive strategy: stochastic perturbation, efforts should be made to prevent the disease to spread widely in the population.

Some interesting topics deserve further investigations, it is also interesting to consider the global stability behaviour of the disease-free solution since simulation show the stability property under different initial values. We leave these for further investigations and look forward to solving them in the near future.

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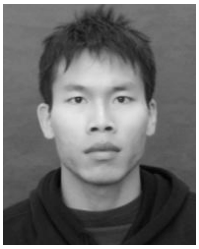

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