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Structural Controllability for a Class of Complex Networks With Root Strongly Connected Components

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ABSTRACT Structural controllability of complex networks has been an attractive research area, and Minimum Inputs Theorem was proposed to identify the minimum driver nodes for complex networks with one-dimensional node dynamics. Then, the Minimum Inputs Theorem was extended to the complex network with multidimensional node dynamics by looking the multidimensional node as a subnetwork. However, when the structures of these subnetworks possess the root Strongly Connected Components that have perfect matching, the minimum driver nodes of these subnetworks are not sufficient to guarantee the full control, and some extra nodes are needed to be controlled. Therefore, in this paper, we study the structural controllability of complex networks with multidimensional node dynamics whose corresponding subnetworks possess such root Strongly Connected Components. First, we apply the Maximum Matching principle to the network topology to obtain which subnetworks we need to control. Then, an algorithm is proposed to identify a feasible minimum controlled node set of the subnetwork. Finally, by analyzing the structural features of the whole network and synthetically applying the proposed algorithm, Maximum Matching principle and Graphical Approach, a flowchart is given for identifying the minimum controlled node set of the whole network. By duality, the above results can also apply to the structural observability problem of such complex networks.

INDEX TERMS Complex networks, multidimensional node dynamics, perfect matching, root strongly connected components, structural controllability.

I. INTRODUCTION

A system is controllable if it is possible to drive it from any initial state to any desired final state in finite time [1]. And as a dual concept of controllability, observability describes the possibility to reconstruct the whole states of the system using the measured outputs. The controllability and observability analyses are fundamental issues in most complex systems and networks [2]–[4].

The concepts of state controllability and observability for linear time-invariant (LTI) systems were first introduced by Kalman [5]. To test whether an LTI system is controllable or observable, one could check the rank of the constant controllability or observability matrix of the system, referred to as the Kalman rank condition for controllability or observability [1], [6]. In practice, accurate system parameters are difficult to acquire, so that the Kalman rank condition is not applicable. To deal with this problem, and taking into account the system or network structures, the concept of structural controllability was proposed by Lin in [7], which provided a graph perspective in controllability analysis. In 2011, a pioneering work by Liu *et al.* [8] suggested that one could convert the structural controllability problem into a maximum matching problem on a bipartite graph, and a Minimum Inputs Theorem for determining the minimum number of inputs or equivalently the minimum number of driver nodes needed to fully control a directed network was presented. Later, Liu *et al.* [9] studied the problem of minimum number of sensors needed to observe a complex system, and introduced a complex system on an inference diagram.

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Then, the inference diagram is decomposed into a set of maximal strongly connected components (SCC). Finally, at least one node is chosen from each root SCC (the SCC that has no incoming edges) as the measured node, which is called the Graphical Approach (GA). The above works by Liu *et al.* established a connection between system controllability and observability and complex networks, which have recently attracted a renewal of interest in controllability and observability problems [10]–[20].

Besides the minimum number of driver nodes, the locations of these driver nodes, i.e., the minimum driver node set, were also determined by identifying the unmatched node set in [8]. However, for a class of networks, whose structure possesses some root SCCs (rSCCs) that have perfect matching, i.e., pm-rSCCs, the minimum driver node set may not be enough, and one needs to control some extra nodes to gain the full control on such networks. For example, although $\mathbf{x}_1$ is the sole driver node for both networks in Fig. 1, controlling $\mathbf{x}_1$ is enough to control the network in Fig. 1(a) while not enough to control the network in Fig. 1(d), because the latter possesses a pm-rSCC composed of $\mathbf{x}_3$. This standpoint was put forward by Liu *et al.* in [9]. So, identifying the minimum controlled node set for such networks is a significant research topic and has attracted a lot of interests [21]–[23]. In [21], based on the strategy that maximizing the number of unmatched nodes in different rSCCs, Pequito *et al.* tried to generate a minimum controlled node set for LTI systems. In [23], from the algebraic perspective, Yin *et al.* mapped the identification of a minimum controlled node set into the constrained combinatorial optimization problem which was solved by the branch and bound method. However, for these existing works on the structural controllability, most of them mainly focused on the complex networks with one-dimensional node dynamics, yet in most complex dynamical networks, the node often has multidimensional dynamics which will influence the network controllability and observability [24]–[30]. In [30], Wang *et al.* first studied the structural controllability problem of complex dynamical networks with multidimensional node dynamics and extended the Minimum Inputs Theorem to the complex dynamical network with multidimensional node dynamics by looking the multidimensional node as a subnetwork. However, when the structures of these subnetworks possess the pm-rSCCs, the minimum driver nodes of these subnetworks are not sufficient to guarantee the full control, and some extra nodes are needed to be controlled.

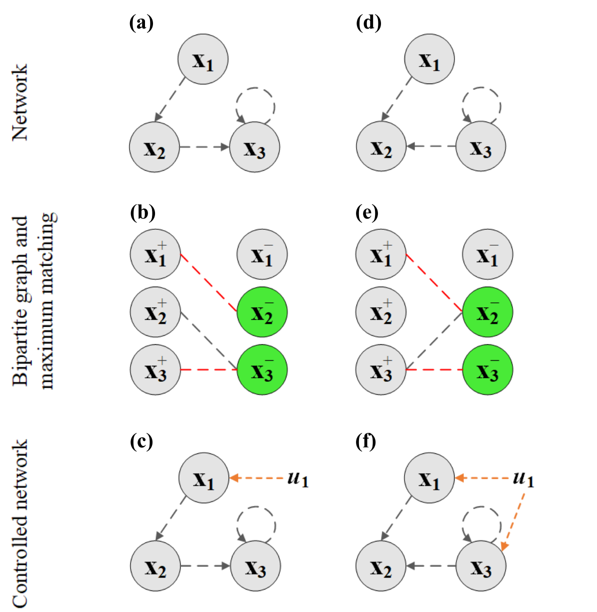

FIGURE 1. Illustrations of controlling networks. (a) The nodes are marked by grey circles and the interactive relations are marked by grey dashed arrows. This network has no pm-rSCCs. (b) Matching edges are marked in red, matched nodes are green, and unmatched nodes are grey. $\mathbf{x}_1$ is unmatched. (c) One node $\mathbf{x}_1$ is controlled by input u_1 that this network is controllable. (d) This network has a pm-rSCC composed of $\mathbf{x}_3$. (e) Similarly, $\mathbf{x}_1$ is the sole unmatched node. (f) Two nodes, $\mathbf{x}_1$ and $\mathbf{x}_3$, are controlled by u_1 that this network is controllable.

Motivated by the above discussions, in this paper, we take the multidimensional LTI node dynamics whose corresponding subnetworks possess the pm-rSCCs into account and address the structural controllability problem of such complex dynamical networks. First, we apply the Maximum Matching principle to the network topology to obtain which subnetworks we need to control. Then, we propose an algorithm to identify a set of minimum controlled nodes of the subnetwork. Finally, by analyzing the structural features of the whole network and synthetically applying the proposed algorithm, Maximum Matching principle and Graphical Approach, we design a flowchart to identify the minimum controlled node set of the whole network. By duality, the above results can also apply to the structural observability problem of such complex networks.

The rest of the paper is organized as follows. Section II describes the main problem to be studied in this paper. In Section III, an algorithm for identifying a minimum controlled node set of the subnetwork is proposed, and a flowchart for identifying the minimum controlled node set of the whole network is designed. In Section IV, two numerical examples are given to validate the theoretical results. Finally, Section V concludes the paper.

II. PROBLEM STATEMENT

We consider the following complex dynamical network consisting of N identical n-dimensional LTI nodes:

$$\dot{\mathbf{x}}_i(t) = \mathbf{A}\mathbf{x}_i(t) + \sum_{j=1}^{N} d_{ij}\mathbf{\Gamma}\mathbf{x}_j(t) + \mathbf{B}_i\mathbf{u}_i(t) \qquad (1)$$

where $\mathbf{x}_i(t) = (x_{i1}(t), \ldots, x_{in}(t))^T$ is the state vector of the i^{th} node at time t. $\mathbf{A} = (a_{ij})_{n\times n}$ is the coefficient matrix of the node system, describing the interactions between the states. $\mathbf{D} = (d_{ij})_{N\times N}$ is the outer coupling matrix: if there is a connection between node i and node j, then $d_{ij} \neq 0$ and $d_{ji} \neq 0$; otherwise, $d_{ij} = d_{ji} = 0$. $\mathbf{\Gamma} \in R^{n\times n}$ is the inner coupling matrix, which is assumed to be an identity matrix in this paper. $\mathbf{B}_i \in R^{n\times p}$ denotes the input matrix that

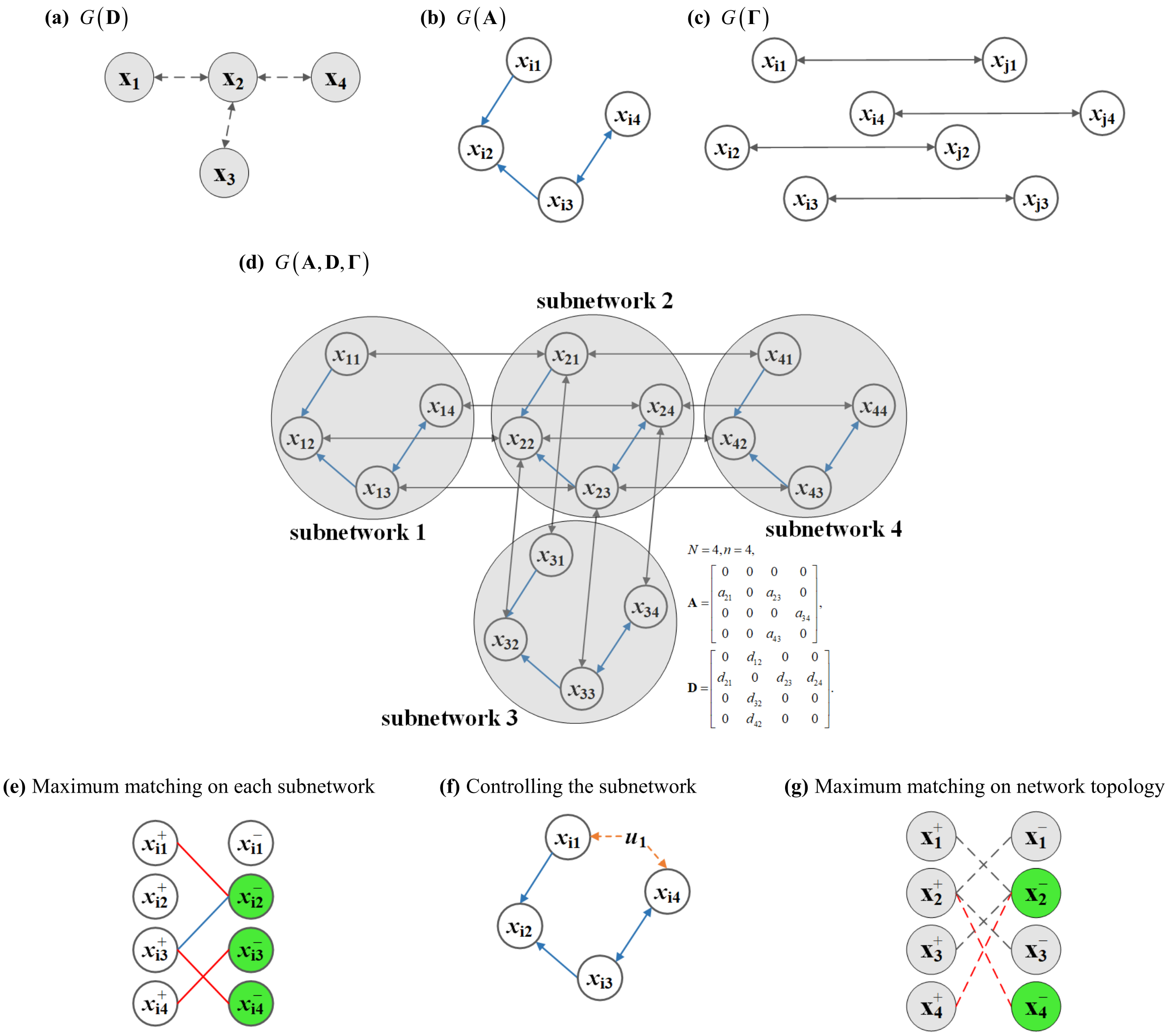

FIGURE 2. Illustration of a network with multidimensional node dynamic. (a) Network topology: each node is marked in grey and the paths that connect adjacent nodes are marked by grey dashed arrows. (b) Node dynamics (subnetwork): look the multidimensional node as a subnetwork, where the nodes of the subnetwork x_{i1}, x_{i2}, x_{i3}, and x_{i4}($i = 1, 2, 3, 4$) are marked by white circles and the interactive relations are marked by blue arrows. (c) Inner coupling configuration: the coupling relations between nodes of adjacent subnetworks are marked by grey arrows. (d) The whole network consists of 4 subnetworks. (e) Maximum matching on each subnetwork. x_{i1} is unmatched. (f) Two nodes, x_{i1} and x_{i4}, are controlled by u_1 that the subnetwork is controllable. (g) Maximum matching on network topology. Subnetwork 1 and 3 are needed to control.

identifies the controlled states of the i^{th} node where the control input is $\mathbf{u}_i(t) = (u_{i1}(t), \ldots, u_{ip}(t))^T$. $\mathbf{B}_i \neq \mathbf{O}$ and $\mathbf{u}_i \neq \mathbf{O}$ if the i^{th} node is controlled.

Without inputs injected, we introduce this network onto a digraph where the network topology is represented as $G(\mathbf{D})$ (see Fig. 2(a)) by drawing a directed edge $\mathbf{x}_i \to \mathbf{x}_j$ if $d_{ij} \neq 0$, then the node dynamics is represented as $G(\mathbf{A}) = (V_A, E_A)$ (see Fig. 2(b)) by looking the multidimensional node as a subnetwork, $V_A = \{x_1, \cdots, x_n\}$ is the node set and $E_A = \{(x_j, x_i) \,|a_{ij} \neq 0\}$ is the edge set, then the inner coupling matrix is represented as $G(\Gamma)$ (see Fig. 2(c)) by drawing n directed edges $x_{iq} \to x_{jq}$ $(q = 1, 2, \ldots, n)$ if $d_{ij} \neq 0$. Finally, this network is represented as $G(\mathbf{A}, \mathbf{D}, \mathbf{\Gamma})$ (see Fig. 2(d)).

When the structure of the subnetwork possesses the root Strongly Connected Components that have perfect matching, the minimum driver nodes of these subnetworks are not sufficient to guarantee the full control, and some extra nodes are needed to be controlled. Therefore, in this paper, we aim to study the structural controllability of complex dynamical networks with multidimensional node whose corresponding subnetworks possess such root Strongly Connected Components. First, we apply the Maximum Matching principle to the network topology to obtain which subnetworks we need to control, then we try to identify a minimum controlled node set of the subnetwork, finally, by analyzing the structural features of the whole network, we try to identify the minimum controlled node set of the whole network. By duality, the above results can also apply to the structural observability analysis of such complex dynamical networks.

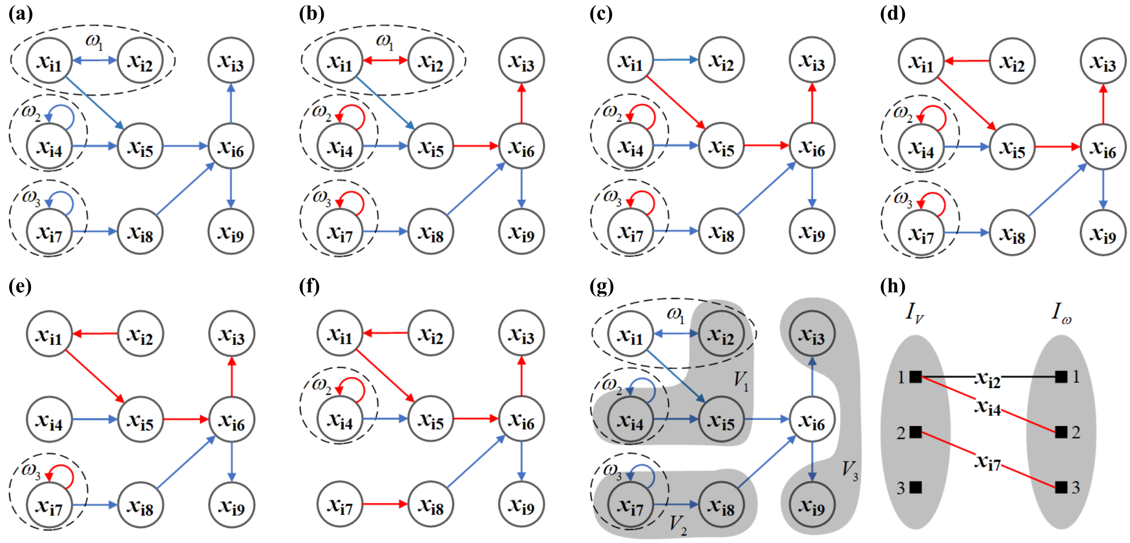

FIGURE 3. **Illustration of the Algorithm 1. (a) This subnetwork consists of 9 nodes and possesses 3 pm-rSCCs, denoted by ω_1, ω_2, ω_3, marked by black dashed circle. (b) The initial matching edges are marked in red, where $|M^*| = 6$. (The maximum matching shown here is not unique.) (c) Set the node x_{i1} as an independent controlled node by leaving out the edge (x_{i2}, x_{i1}), where $|M^{x_{i1}}| = 5 \neq |M^*|$, $V_{IC} = \{\}$. (d) Set the node x_{i2} as an independent controlled node by leaving out the edge (x_{i1}, x_{i2}), where $|M^{x_{i2}}| = 6 = = |M^*|$, $V_{IC} = \{x_{i2}, x_{i8}, x_{i9}\}$, the edge (x_{i1}, x_{i2}) is removed. (e) Set the node x_{i4} as an independent controlled node by leaving out the edge (x_{i4}, x_{i4}), where $|M^{x_{i4}}| = 5 \neq |M^*|$, V_{IC} is still $\{x_{i2}, x_{i8}, x_{i9}\}$. (f) Set the node x_{i7} as an independent controlled node by leaving out the edge (x_{i7}, x_{i7}), where $|M^{x_{i7}}| = 6 = = |M^*|$, $V_{IC} = \{x_{i2}, x_{i7}, x_{i9}\}$, the edge (x_{i7}, x_{i7}) is removed. (g) The sets of all alternatives to each element in $V_{IC} = \{x_{i2}, x_{i7}, x_{i9}\}$ are $V_1 = \{x_{i2}, x_{i4}, x_{i5}\}$, $V_2 = \{x_{i7}, x_{i8}\}$, and $V_3 = \{x_{i3}, x_{i9}\}$. (h) The maximum matching of the bipartite graph $BG(I_V, I_\omega, E_{I_V, I_\omega})$. Here, matching edges are $(1, 2, x_{i4})$ and $(2, 3, x_{i7})$. V_3 and ω_1 are not involved in the maximum matching.**

III. MAIN RESULTS

In this section, we present the main results in this work. The first result is an algorithm for identifying a minimum controlled node set of the subnetwork. Then by analyzing the structural features of the whole network, we find the set of the minimum controlled nodes of all controlled subnetworks may not be the minimum controlled node set of the whole network. Therefore, by synthetically applying the proposed algorithm, Maximum Matching principle, and Graphical Approach, the second result is a flowchart to identify the minimum controlled node set of the whole network.

A. MINIMUM CONTROLLED NODE SET OF THE SUBNETWORK

First, we recall some concepts from graph theory.

Definition 1 (Maximum Matching) [8]: For a digraph, a maximum matching M^* is a largest subset of edges that do not share common starting nodes or ending nodes.

A maximum matching M^* of $G(\mathbf{A})$ could be found efficiently from its corresponding bipartite graph, denoted as $BG(V_A^+, V_A^-, E_A')$, where $V_A^+ = \{x_1^+, \cdots, x_n^+\}$ and $V_A^- = \{x_1^-, \cdots, x_n^-\}$ are the sets of starting and ending nodes, respectively, and $E_A' = \{(x_j^+, x_i^-) \,|a_{ij} \neq 0\}$ (see Fig. 2(e)). A node in V_A^- is matched if it belongs to an edge in the matching; otherwise, it is unmatched. (According to the Minimum Inputs Theorem [8], the number of inputs equals the number of unmatched nodes which are called the driver nodes or independent controlled nodes. Here, to better distinguish from the controlled nodes, we use the appellation of independent controlled nodes.) A maximum matching is a perfect matching if every node is matched [31]. (In this case, only one input is needed.)

Definition 2 (Strongly Connected Component) [9]: For a digraph, a Strongly Connected Component (SCC) is a subgraph in which there is a directed path from each node to every other node.

A SCC is a root SCC (rSCC) if it has no incoming edges to its nodes from other nodes [9] (i.e., inaccessibility), e.g. x_{i3} and x_{i4} form a rSCC in Fig. 2(b).

From Fig. 2(e), it can be seen that x_{i1} is the sole unmatched node, and according to the Minimum Inputs Theorem [8], only one input is needed and x_{i1} is independently controlled by u_1. Here, besides x_{i1}, one also needs to extra control x_{i4} (or x_{i3}) (see Fig. 2(f)) because x_{i3} and x_{i4} compose a pm-rSCC, defined as follows.

Definition 3 (pm-rSCC): A pm-rSCC is a rSCC whose maximum matching is a perfect matching.

Remark 1: Given a digraph $G(\mathbf{A}) = (V_A, E_A)$, the SCC decomposition (which comes from the component graph that obtained by contracting all edges within each SCC of G) could be implemented by using two depth-first searches with $O(|V_A| + |E_A|)$ time [31]. So, the rSCCs are thereupon obtained and the pm-rSCCs could be obtained by testing the maximum matching of each rSCC.

For the pm-rSCC, the following result can be established.

Theorem 1: For a given digraph, the addition of a pm-rSCC pointing to this digraph will not change the number of unmatched nodes.

Proof: 1) *It will not increase the number of unmatched nodes.* Since the maximum matching of pm-rSCC is perfect, it has no unmatched nodes and so will not increase the number of unmatched nodes.

2) *It will not decrease the number of unmatched nodes.* From Definition 1, all the nodes in the pm-rSCC are ending nodes in the maximum matching, meanwhile, all the nodes are starting nodes. Now, consider the edge from the pm-rSCC to the existing digraph. If this edge is chosen as a matching edge, then one node in the pm-rSCC is used to match one node in the existing digraph, which probably reduces one unmatched node in the existing digraph. Yet, it also reduces the number of starting nodes in the maximum matching of the pm-rSCC by one, equivalently the number of ending nodes will be decreased by one, implying the appearance of one unmatched node in the pm-rSCC. So, it will not decrease the number of unmatched nodes. ■

Although the pm-rSCCs, denoted by $\omega_l, l \in I_\omega = \{1, \ldots, \theta\}$, will not change the number of unmatched nodes, i.e., the number of independent controlled nodes N_{IC}, due to its inaccessibility, it needs to control a random node of each pm-rSCC (the extra controlled nodes), i.e., it will increase the number of controlled nodes N_C. Consequently, in order to minimize the number of controlled nodes, we adopt the strategy that maximizing the number of pm-rSCCs which contain unmatched nodes in a maximum matching, and propose the Algorithm 1 for identifying a minimum controlled node set of the subnetwork.

Next, we will explain the Algorithm 1 with one illustrative example (see Fig. 3):

1) INITIAL INFORMATION

There are 3 pm-rSCCs in this subnetwork, denoted by $\omega_1, \omega_2, \omega_3$, and from the initial matching edges (the maximum matching here is not unique), we know $|M^*| = 6$ (see Fig. 3(a), (b)).

2) STEP 4 TO STEP 15

Firstly, select $\eta = x_{i1}$ from ω_1 and compute a maximum matching $M^{x_{i1}}$ while leaving out the edge (x_{i2}, x_{i1}) (i.e., set the node x_{i1} as an independent controlled node), which turns out that $|M^{x_{i1}}| = 5 \neq |M^*|$, so $V_{IC} = \{\}$ (see Fig. 3(c)). Secondly, select $\eta = x_{i2}$ from ω_1 and compute the maximum matching $M^{x_{i2}}$ while leaving out the edge (x_{i1}, x_{i2}), which turns out that $|M^{x_{i2}}| = 6 = = |M^*|$, so $V_{IC} = \{x_{i2}, x_{i8}, x_{i9}\}$ and the edge (x_{i1}, x_{i2}) is removed (see Fig. 3(d)). Thirdly, select $\eta = x_{i4}$ from ω_2 and compute a maximum matching $M^{x_{i4}}$ while leaving out the edge (x_{i4}, x_{i4}), which turns out that $|M^{x_{i4}}| = 5 \neq |M^*|$, so V_{IC} is still $\{x_{i2}, x_{i8}, x_{i9}\}$ (see Fig. 3(e)). Fourthly, select $\eta = x_{i7}$ from ω_3 and compute the maximum matching $M^{x_{i7}}$ while leaving out the edge (x_{i7}, x_{i7}), which turns out that $|M^{x_{i7}}| = 6 = = |M^*|$, so finally $V_{IC} = \{x_{i2}, x_{i7}, x_{i9}\}$ and the edge (x_{i7}, x_{i7}) is removed (see Fig. 3(f)). The V_{IC} obtained here is related to the permutation of the pm-rSCCs. To make the results more general, we do the following steps.

3) STEP 16 TO STEP 33

Drawing lessons from the conception of "natural constrained partitions" in [21], we compute the set of all alternatives to each element in V_{IC}, denoted by V_λ, $\lambda \in \{1, \ldots, |V_{IC}|\}$, by fixing the other elements as the unmatched nodes in the maximum matching (step 17 to step 22), where we could obtain $V_1 = \{x_{i2}, x_{i4}, x_{i5}\}$, $V_2 = \{x_{i7}, x_{i8}\}$, and $V_3 = \{x_{i3}, x_{i9}\}$ (see Fig. 3(g)). Next, in order to determine which nodes in V_λ could belong to which pm-rSCCs to satisfy the strategy we adopt, we consider the bipartite graph $BG(I_V, I_\omega, E_{I_V,I_\omega})$, where I_V is the set of indices of V_λ, $I_\omega = \{1, \ldots, \theta\}$ is the set of indices of the pm-rSCCs, E_{I_V,I_ω} is the set of edges (λ, l, η) which denotes the node η belongs to V_λ and ω_l simultaneously (step 23 to step 25), e.g. the edges $(1, 1, x_{i2})$, $(1, 2, x_{i4})$ and $(2, 3, x_{i7})$ in Fig. 3(h). Then, the nodes determined by the matching edges in the maximum matching of the $BG(I_V, I_\omega, E_{I_V,I_\omega})$ could serve as the elements of the independent controlled node set V_{IC} (step 26 to step 31), e.g. x_{i4} and x_{i7} in Fig. 3(h). After this, just select a node from each V_λ that is not involved in the maximum matching, to serve as the remaining elements, that the final independent controlled node set V_{IC} could be determined (step 32 to step 33), e.g. select x_{i3} from V_3 that $V_{IC} = \{x_{i3}, x_{i4}, x_{i7}\}$.

4) STEP 34 TO STEP 36

Besides the nodes of V_{IC}, to ensure the controllability, we still need to extra control the pm-rSCCs that are not involved in the maximum matching by selecting a node from each such pm-rSCC, e.g. select x_{i1} from ω_1 in Fig. 3(g), so the minimum controlled node set V_C could be determined by combining V_{IC} with the extra controlled nodes, e.g. $V_C = \{x_{i1}, x_{i3}, x_{i4}, x_{i7}\}$ in this illustrative example.

Remark 2: The time complexity of the Algorithm 1 is as follows: The complexity of computing the initial maximum matching M^* is $O\left(\sqrt{|V_A|}\,|E_A|\right)$. Getting the pm-rSCCs ω_l refers to Remark 1. Obtaining the initial V_{IC} has the complexity $\tilde{N} O\left(\sqrt{|V_A|}\,|E_A|\right)$ with $\tilde{N}$ being the number of nodes in the pm-rSCCs. The complexity of determining the V_λ and E_{I_V,I_ω}

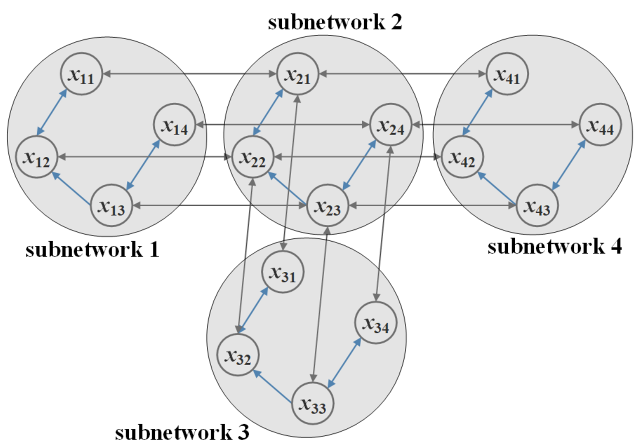

FIGURE 4. Illustration: the maximum matching of the subnetwork is perfect.

Algorithm 1 Find a Minimum Controlled Node Set of the Subnetwork

Input: $BG(V_A^+, V_A^-, E_A')$, M^* and ω_l.
Output: The independent controlled node set V_{IC} and the controlled node set V_C.
Initialize $V_{IC} = \{\}$, $V_C = \{\}$;
$V_r = \{\};$
$\bar{E}'_A = E'_A;$
for each $\eta \in \omega_1 \cup \cdots \cup \omega_\theta - V_r$
  Compute a maximum matching M^η (the independent controlled node set is V_η) associated with $BG(V_A^+, V_A^-, \bar{E}'_A - \{(x, \eta) : x \in V_A\})$;
  if $|M^\eta| = \ = |M^*|$
    $\bar{E}'_A = \bar{E}'_A - \{(x, \eta) : x \in V_A\};$
    $V_{IC} = V_\eta;$
    for each $l \in I_\omega$
      if $\eta \in \omega_l$
        $I_\omega = I_\omega - \{l\};$
        $V_r = V_r \cup \omega_l;$
        break;
$E_{I_V, I_\omega} = \{\};$
for each $\lambda \in \{1, \ldots, |V_{IC}|\}$
  $V_\lambda = \{\};$
  for each $\eta \in V_A$
    Compute a maximum matching M^η associated with $BG(V_A^+, V_A^-, E_A' - \{(x, \eta) : x \in V_A\} - \{(x, \mu) : x \in V_A, \mu \in (V_{IC} - \{v_\lambda\})\});$
    if $|M^\eta| = \ = |M^*|$
      $V_\lambda = V_\lambda \cup \{\eta\};$
      for each $l \in I_\omega$
        if $\eta \in \omega_l$
          $E_{I_V, I_\omega} = E_{I_V, I_\omega} \cup \{(\lambda, l, \eta)\};$
Compute a maximum matching M' associated with $BG(I_V, I_\omega, E_{I_V, I_\omega});$
$V_{IC} = \{\};$
for each $(\lambda, l, \eta) \in M'$
  $V_{IC} = V_{IC} \cup \{\eta\};$
  $I_V = I_V - \{\lambda\};$
  $I_\omega = I_\omega - \{l\};$
for each $\lambda \in I_V$
  Select a node η from V_λ and $V_{IC} = V_{IC} \cup \{\eta\};$
$V_C = V_{IC};$
for each $l \in I_\omega$
  Select a node η from ω_l and $V_C = V_C \cup \{\eta\};$

is $p\,|V_A|\,O\left(\sqrt{|V_A|}\,|E_A|\right)$. The complexity of computing the maximum matching of $BG(I_V, I_\omega, E_{I_V, I_\omega})$ is $O\left(\sqrt{p+\theta}p\tilde{N}\right)$. Finally, selecting nodes from the V_λ and ω_l that are not involved in the maximum matching is of constant complexity.

B. MINIMUM CONTROLLED NODE SET OF THE WHOLE NETWORK

For the network in Fig. 2, by applying the Maximum Matching principle to the network topology $G(\mathbf{D})$ (see Fig. 2(g)),

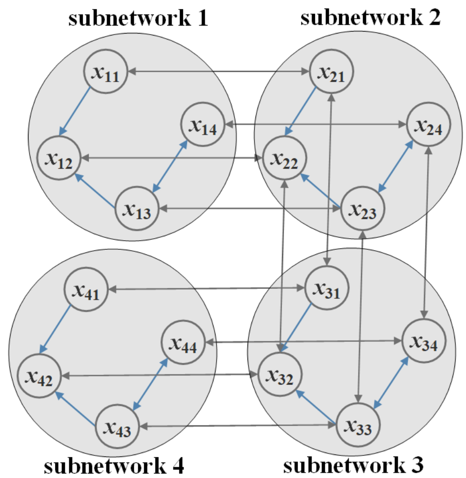

FIGURE 5. Illustration: the maximum matching of the network topology is perfect.

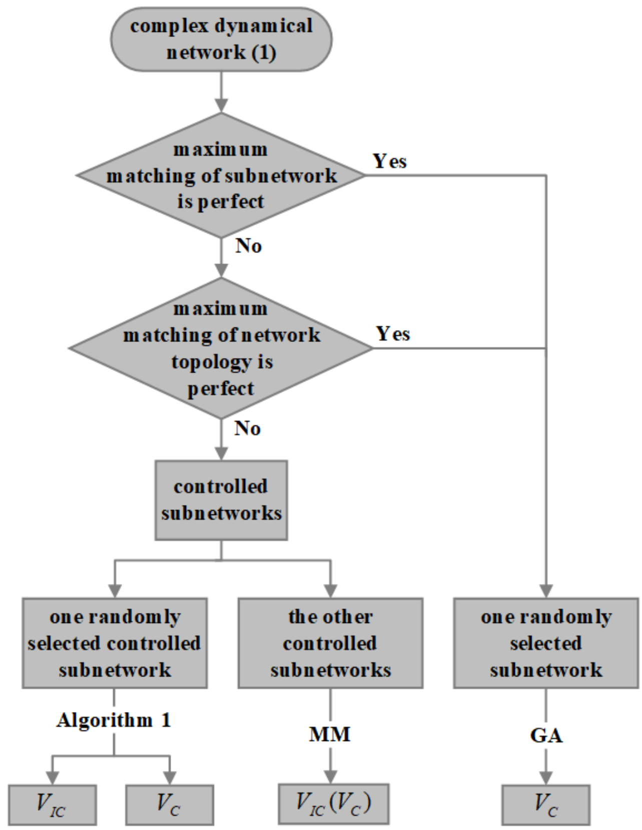

FIGURE 6. A flowchart for identifying the V_{IC} and V_C of complex dynamical network (1).

we can obtain that subnetwork 1 and 3 are needed to control. And by Algorithm 1, we can obtain $V_C = \{x_{i1}, x_{i4}\}$ (or $V_C = \{x_{i1}, x_{i3}\}$). Thus, the controlled node set of the whole network is $\{x_{11}, x_{14}, x_{31}, x_{34}\}$. However, this controlled node set is not the minimum controlled node set of the whole network.

The reason is that the corresponding pm-rSCCs of each subnetwork, along with the edges among themselves,

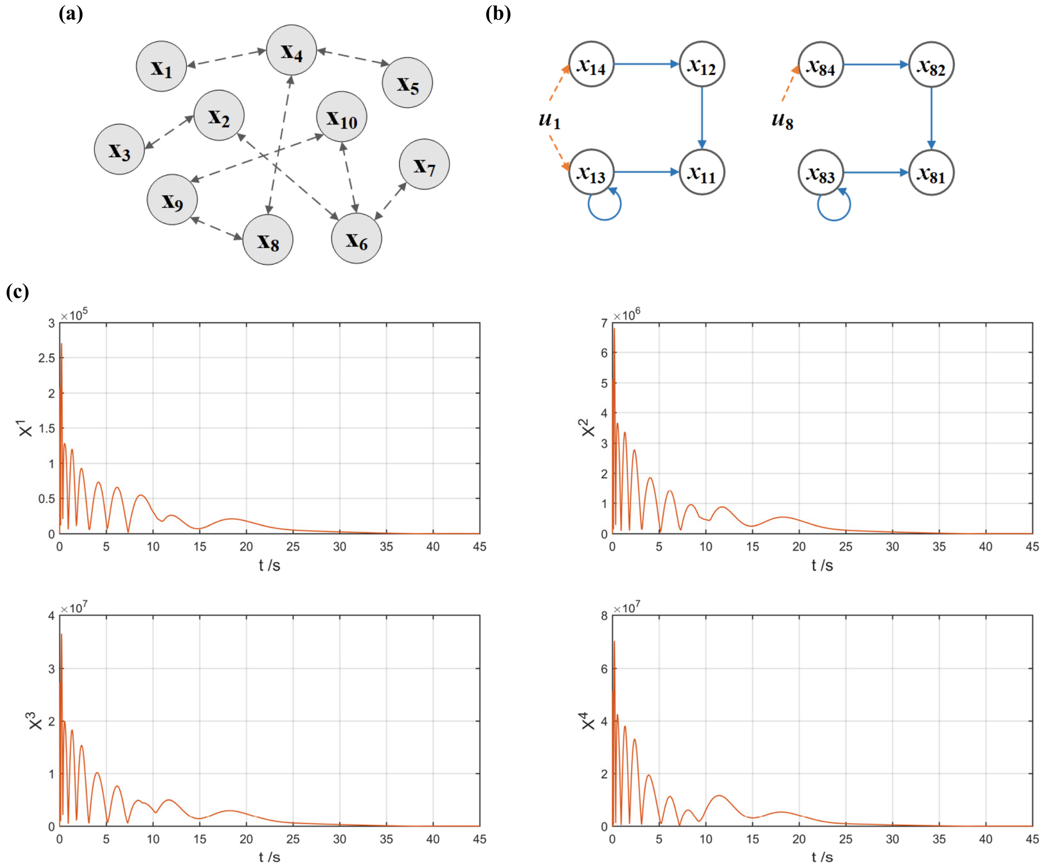

FIGURE 7. Example 1. (a) The network topology. (b) Two subnetworks, subnetwork 1 and 8, are controlled, where x_{13} and x_{14} are controlled by u_1, x_{84} is controlled by u_8. (c) All node states are controlled to zero.

constitute the pm-rSCCs of the whole network, so it only needs to control a random node from each pm-rSCC of the whole network, e.g. x_{i3} and $x_{i4} (i = 1, 2, 3, 4)$ form a pm-rSCC of the subnetwork (see Fig. 2(b)), simultaneously, x_{i3} and x_{i4} along with the edges among themselves constitute a pm-rSCC of the whole network (see Fig. 2(d)), so we just need to control one node of x_{14} and x_{34}. Therefore, when the number of the controlled subnetworks is more than one, the set of the minimum controlled nodes of all controlled subnetworks is not the minimum controlled node set of the whole network.

Based on this finding, for complex dynamical network (1), we could apply the Algorithm 1 to one randomly selected controlled subnetwork and apply the Maximum Matching principle (MM) to the other controlled subnetworks, then the independent controlled node set V_{IC} and the minimum controlled node set V_C could be determined, e.g. apply Algorithm 1 to subnetwork 1, and apply the MM to subnetwork 3 in Fig. 2(d), getting $V_{IC} = \{x_{11}, x_{31}\}$ and $V_C = \{x_{11}, x_{14}, x_{31}\}$.

Note that there could be a special case that the maximum matching of the subnetwork (see Fig. 4) or the network topology $G(\mathbf{D})$ (see Fig. 5) is perfect. In this case, every node of the subnetworks is matched, so it only needs to apply the Graphical Approach (GA) that choosing one node from each rSCC of one randomly selected controlled subnetwork as the controlled nodes (e.g. x_{14} in Fig. 4, x_{11} and x_{14} in Fig. 5), to fully control the whole network.

The above analysis could be summarized as the flowchart in Fig. 6, showing the procedure of identifying the independent controlled node set V_{IC} and the minimum controlled node set V_C of complex dynamical network (1).

Remark 3: The multidimensional node dynamic complicates the structural controllability problem of complex dynamical networks. Although one could regard such a network as a large-scale system (Nn dimensions) for analysis, it may create the dispersal of the controlled nodes, leading to a situation that one input signal is needed to control the nodes of different subnetworks. However, this situation may be unreasonable in practice due to the long physical distance between the different subnetworks. So, the input configurations in this work (see Fig. 6) are localized onto individual subnetworks, which could avoid the above undesirable situation.

IV. NUMERICAL EXAMPLES

Here, two numerical examples are given to validate the theoretical results in this paper.

Example 1: To control a network with the identified minimum controlled node set.

Consider the following complex dynamical network that the maximum matchings of the subnetwork and the network topology are not perfect (see Fig. 7(a), (b)):

$$\dot{\mathbf{x}}_i(t) = \mathbf{A}\mathbf{x}_i(t) + \sum_{j=1}^{10} d_{ij}\Gamma\mathbf{x}_j(t) + \mathbf{B}_i\mathbf{u}_i(t) \tag{2}$$

where $n = 4$, $\mathbf{\Gamma} = I_n$, $\mathbf{A} = \begin{bmatrix} 0 & 3.7 & -2.7 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -0.002 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and $\mathbf{B}_i$ will be determined later.

In this complex dynamical network, the subnetworks we need to control are subnetwork 1 and 8 (the maximum matching of network topology is not unique), and the pm-rSCC of each subnetwork is $\{x_{i3}\}$. According to the flowchart, we apply the Algorithm 1 to subnetwork 1 to obtain $V_{IC} = \{x_{14}\}$, $V_C = \{x_{13}, x_{14}\}$, and then apply the Maximum Matching principle to subnetwork 8 to obtain $V_{IC} = V_C = \{x_{84}\}$. So, $\mathbf{B}_i$ are set as follows (see Fig. 7(b)):

$$\mathbf{B}_i = \begin{cases} \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}^T & i = 1 \\ \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T & i = 8 \\ \mathbf{O} & else \end{cases} \tag{3}$$

where $\mathbf{O}$ is a zero matrix of suitable dimension.

Here, we try to control each node state to zero, then the network (2) can be rewritten as

$$\begin{aligned} \dot{\mathbf{X}} &= \mathbf{A}^*\mathbf{X} + \mathbf{B}^*\mathbf{U}^* \\ &= \mathbf{A}^*\mathbf{X} - \mathbf{B}^*\mathbf{K}\mathbf{X} \\ &= (\mathbf{A}^* - \mathbf{B}^*\mathbf{K})\mathbf{X} \end{aligned} \tag{4}$$

where $\mathbf{X} = \left(\mathbf{x}_1(t)^T, \ldots, \mathbf{x}_{10}(t)^T\right)^T$, $\mathbf{A}^* = I_N \otimes \mathbf{A} + \mathbf{D} \otimes \mathbf{\Gamma}$, $\mathbf{B}^* = \begin{bmatrix} \mathbf{B}_1^T & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{B}_8^T & \mathbf{O} & \mathbf{O} \end{bmatrix}^T$, and $\mathbf{K}$ is the feedback gain to be determined.

Define a Lyapunov function as follows:

$$V = \mathbf{X}^T\mathbf{P}\mathbf{X} \tag{5}$$

Differentiating V gives

$$\begin{aligned} \dot{V} &= \dot{\mathbf{X}}^T\mathbf{P}\mathbf{X} + \mathbf{X}^T\mathbf{P}\dot{\mathbf{X}} \\ &= \mathbf{X}^T(\mathbf{A}^* - \mathbf{B}^*\mathbf{K})^T\mathbf{P}\mathbf{X} + \mathbf{X}^T\mathbf{P}(\mathbf{A}^* - \mathbf{B}^*\mathbf{K})\mathbf{X} \\ &= \mathbf{X}^T\left[(\mathbf{A}^* - \mathbf{B}^*\mathbf{K})^T\mathbf{P} + \mathbf{P}(\mathbf{A}^* - \mathbf{B}^*\mathbf{K})\right]\mathbf{X} \end{aligned} \tag{6}$$

According to the Lyapunov stability theory, (4) is asymptotically stable if and only if $\dot{V} < 0$, i.e.

$$(\mathbf{A}^* - \mathbf{B}^*\mathbf{K})^T\mathbf{P} + \mathbf{P}(\mathbf{A}^* - \mathbf{B}^*\mathbf{K}) < 0 \tag{7}$$

Thus, the feedback gain $\mathbf{K}$ could be obtained by solving (7). Here, $\mathbf{K} \in R^{2\times 40}$, and due to the page space, it is omitted.

In the simulation, the node state information is specified as $\mathbf{X}^l(t) = \sum_{i=1}^{10} |x_{il}(t)|$, $l = 1, 2, 3, 4$, which is shown in Fig. 7(c), and it is obvious that the node states all converge to zero, implying that the network (2) is successfully controlled.

Example 2: To observe a network with the identified minimum measured node set.

Consider the following complex dynamical network:

$$\begin{aligned} \dot{\mathbf{x}}_i(t) &= \mathbf{A}\mathbf{x}_i(t) + \sum_{j=1}^{10} d_{ij}\mathbf{\Gamma}\mathbf{x}_j(t) \\ \mathbf{y}_i(t) &= \mathbf{C}_i\mathbf{x}_i(t) \end{aligned} \tag{8}$$

where $n = 3$, $\mathbf{\Gamma} = I_n$, $\mathbf{A} = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & -1.7 \\ 0 & 2.4 & 0 \end{bmatrix}$, and $\mathbf{C}_i$ are the output matrices which will be determined later.

In this complex dynamical network, the network topology is a nearest-neighbor coupled structure (see Fig. 8(a)). Apparently, its maximum matching is perfect. According to the flowchart, we apply the Graphical Approach to one randomly selected subnetwork (here is the subnetwork 1) to

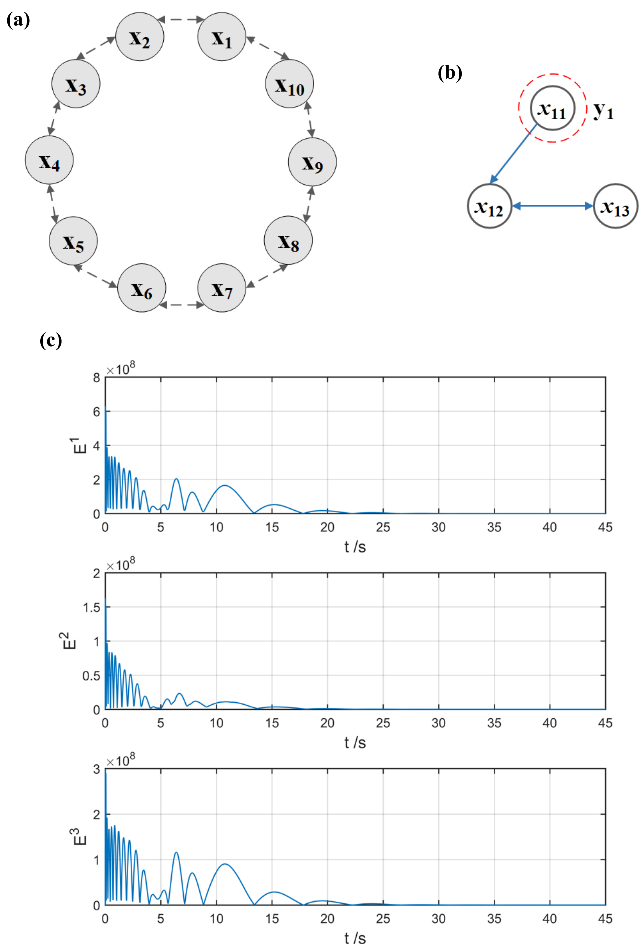

FIGURE 8. Example 2. (a) The network topology. (b) Subnetwork 1 is measured, where $x_{1\,1}$ is measured by y_1, marked by red dashed circle. (c) The observation errors all converge to zero.

obtain $V_M = \{x_{11}\}$ (V_M denotes the minimum measured node set). So, $\mathbf{C}_i$ are set as follows (see Fig. 8(b)):

$$\mathbf{C}_i = \begin{cases} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} & i = 1 \\ \mathbf{O} & else \end{cases} \tag{9}$$

Similarly, the network (8) could be rewritten as

$$\begin{aligned} \dot{\mathbf{X}} &= \mathbf{A}^* \mathbf{X} \\ \mathbf{Y} &= \mathbf{C}^* \mathbf{X} \end{aligned} \tag{10}$$

where $\mathbf{X} = \left(\mathbf{x}_1(t)^T, \ldots, \mathbf{x}_{10}(t)^T\right)^T$, $\mathbf{A}^* = I_N \otimes \mathbf{A} + \mathbf{D} \otimes \mathbf{\Gamma}$, and $\mathbf{C}^* = \begin{bmatrix} \mathbf{C}_1 & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \end{bmatrix}$.

We could build an observer for this network, and similar to Example 1, the observer gain $\mathbf{L} \in R^{30\times 1}$ can be obtained by applying the Lyapunov stability theory to the error system, here due to the page space, it is omitted.

In the simulation, the observation errors are specified as $E^l(t) = \sum_{i=1}^{10} \left|x_{il}(t) - \hat{x}_{il}(t)\right|$, $l = 1, 2, 3, 4$, which are shown in Fig. 8(c). It is obvious that all observation errors converge to zero, implying that the network (8) is successfully observed.

From these two numerical examples, the proposed flowchart for identifying the minimum controlled/measured node set is demonstrated to be feasible and effective for controlling/observing the complex dynamical network with multidimensional node dynamics.

V. CONCLUSION

In this paper, by looking the multidimensional node as a subnetwork, we consider the complex dynamical network with multidimensional node dynamics whose corresponding subnetworks possess the root Strongly Connected Components that have perfect matching, and address the structural controllability problem of such complex dynamical networks. By analyzing the structural features of the whole network, a flowchart is designed for identifying the minimum controlled node set of the whole network.

Here, the inner coupling matrix is an identity matrix, as commonly assumed in the studies of complex dynamical network. Yet, this matrix is known to play an important role in structural controllability of complex dynamical networks [24], [25], [28], so the case with the general inner coupling matrix should be further investigated in the future.

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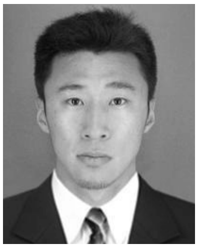

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