IEEE Access
Multidisciplinary : Rapid Review : Open Access Journal

Received January 2, 2020, accepted January 20, 2020, date of publication January 22, 2020, date of current version January 30, 2020.

Digital Object Identifier 10.1109/ACCESS.2020.2968760

Novel Stability Results for Halanay Inequality and Applications to Delay Neural Networks

DEHAO RUAN [1,2], WEIGUO LIU [3], MINRAN YANG [4], ZHEHAO HUANG [1], AND XIAOXIA GUO [5]

[1] Guangzhou International Institute of Finance and Guangzhou University, Guangzhou 510405, China
[2] School of Mathematics and Statistics, Qiannan Normal University for Nationalities, Duyun 558000, China
[3] School of Mathematics and Statistics, Guangdong University of Finance and Economics, Guangzhou 510006, China
[4] Library, Guangzhou University, Guangzhou 510006, China
[5] School of Mathematics and Information Sciences, Guangzhou University, Guangzhou 510006, China

Corresponding author: Xiaoxia Guo (xxguo91@163.com)

This work was supported in part by the China Postdoctoral Science Foundation under Grant 2019M660196, and in part by the National Natural Science Foundation of China under Grant 11701115 and Grant 11802063.

ABSTRACT This work investigates the stability of Halanay inequality. Some novel results are obtained by means of constructing an auxiliary differential equation. Some previous works are improved and extended. After that, the obtained results are applied to investigate the stability of neural networks with time-varying and distributed delays. At last, some examples along with numerical simulations are presented to illustrate the validity of the theoretical results.

INDEX TERMS Stability, Halanay inequality, delay neural networks.

I. INTRODUCTION

In recent years, dynamical systems have come to play a more and more important role in natural scientific, social scientific and engineering. The main reason for this is that it allows us to model some kinds of natural scientific, social scientific and engineering problems appropriately. Due to their great applications, dynamical systems have been developed very fast, see for example [1]–[30].

In many real systems including telecommunication systems, manufacturing systems and network control systems, time delays often occur due to the limitation of transmission or switching speed. It is well known that the delay may cause divergence, oscillation, instability and chaos in systems. Therefore, it is necessary to consider the influence of the time delays in the investigation of these systems, and lots of related literatures have been published [8]–[30].

In the study of the stability of time-delay systems, one excellent technique is Lyapunov's method (see for example [9], [11], [12], [17]–[19], [22], [23], [29]). The key to Lyapunov's method is to construct a suitable Lyapunov function or functional. However, finding a suitable Lyapunov function or functional is not an easy case. On the other hand, many kinds of differential inequalities such as Halanay inequality and its generalisations can also be applied to study the stability of time-delay systems (see for instance [8], [10], [13]–[16], [24]–[27], [30]). Especially, [10], [13]–[16], [24]–[26] the authors proposed many kinds of generalised Halanay inequalities to consider the stability of neural networks (which can be seen as a multidimensional differential equation) with delays.

We mention here that the existing works [8], [10], [14], [16], [24]–[27], [30] all required some similar conditions, i.e.,

$$D^{+}u(t) \leq -a(t)u(t) + b(t) \sup_{t-\tau(t)\leq s\leq t} u(s) \tag{1}$$

and

$$a(t) > 0, \quad b(t) \geq 0, \ -a(t) + b(t) < 0 \text{ for } t \geq 0.$$

In [13], the authors studied the above inequality and obtained some novel results, i.e., $a(t)$ and $b(t)$ can be negative in some interval. However, the authors required the boundedness of $a(t)$, $b(t)$ and $\tau(t)$. It should be pointed out that when $-a(t)+b(t)$ has no upper bound or lower bound, for example:

$$a(t) = 0.5t - t\cos t \quad \text{and } b(t) = e^{-4t},$$

then the existing techniques can't be used to deal with this case. The main difficulty is that $a(t)$ sometimes is positive and sometimes is negative, the usual method characteristic equation is no longer applicable.

The associate editor coordinating the review of this manuscript and approving it for publication was Wangli He.

Inspired by the above discussion, we also investigate the stability of Halanay inequality (1). The major contributions of this work are as follows. (i) The uniform positiveness of coefficient function, and the boundedness of coefficient and delay functions are no longer required. (ii) The decay rate of inequality (1) is considered. (iii) The results in [13] are improved and extended.

The contents of this paper are organised as follows. In section 2, the main theoretical results and their proof are provided. In section 3, the theoretical results are applied to investigate the stability of neural networks with time-varying and distributed delays. In section 4, some examples along with their numerical simulations are presented to illustrate the validity of the theoretical results.

Natation: Throughout this paper, $\mathbb{R} = (-\infty, +\infty)$, $J_n = \{1, 2, \ldots, n\}$ and D^+ stands for Dini derivative. For a function $b(t)$ define $b^+(t) := \max\{0, b(t)\}$. $BC((-\infty, 0], \mathbb{R})$ is the space of all bounded continuous functions. $\mathbb{R}^n$ stands for the n-dimensional Euclidean space. A^T denotes the transpose of a vector or a matrix A. $|\cdot|$ denotes the Euclidean vector norm in $\mathbb{R}^n$. Let $\|y\| := \max_{i\in J_n} |y_i|$, where $y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n$. $\mathcal{C} := \mathcal{C}([-\tau, 0]; \mathbb{R}^n)$ stands for the space of all continuous functions ψ from $[-\tau, 0]$ into $\mathbb{R}^n$ equipped with the norm $\|\psi\|_{\mathcal{C}} := \sup_{\theta\in[-\tau,0]} \|\psi(\theta)\|$. For a function $F(t)$ define on $[0, +\infty)$, we complementally define

$$F^*(t) = \begin{cases} F(t), & t \geq 0, \\ 0, & t < 0. \end{cases}$$

II. MAIN RESULTS

The main contribution of this section is to investigate the stability of the following Halanay inequality:

$$\begin{cases} D^+u(t) \leq -a(t)u(t) + b(t) \sup\limits_{t-\tau(t)\leq s\leq t} u(s), \\ \qquad t \in [0, +\infty), \\ u(t) = \varphi(t) \in BC((-\infty, 0], \mathbb{R}), \quad t \in (-\infty, 0] \end{cases} \tag{2}$$

where $u(t) \geq 0$ for $t \in \mathbb{R}$, $a(t)$ and $b(t)$ are two scalar functions and $\tau(t) \geq 0$ is the delay function.

Remark 1 Without loss of generality, if delay function is bounded by a constant τ, we define $u(t) = \varphi(-\tau)$ for $t < -\tau$, then system (2) is well defined.

Theorem 1: If $b(t) \geq 0$ and

$$\lim_{t\to+\infty} \int_0^t [-a(v) + b(v)h(v)]dv = -\infty, \tag{3}$$

where

$$h(t) = e^{\sup\limits_{t-\tau(t)\leq s\leq t} \int_s^t a^*(v)dv}.$$

Then,

$$u(t) \to 0 \quad as\ t \to +\infty.$$

In addition, if there exists a nonnegative function $\gamma(t)$ such that

$$\lim_{t\to+\infty} \int_0^t [-a(v) + b(v)h(v) + \gamma(v)]dv = -\infty, \tag{4}$$

then, there is a constant $K \in [1, +\infty)$ such that

$$u(t) \leq K \sup_{\theta\leq 0} |\varphi(\theta)| e^{-\int_0^t \gamma(s)ds}, \quad t \in [0, +\infty).$$

Proof: Firstly, we show that

$$u(t) \leq \sup_{\theta\leq 0} |\varphi(\theta)| e^{\int_0^t [-a^*(v)+b^*(v)h(v)]dv}, \quad t \in \mathbb{R}. \tag{5}$$

In order to prove the above inequality, we need to construct the following differential equation

$$\begin{cases} \dfrac{dy^\varepsilon(t)}{dt} = [-a(t) + b(t)h(t)]y(t)dt, & t \in [0, +\infty), \\ y^\varepsilon(t) = (1+\varepsilon) \sup\limits_{\theta\leq 0} |\varphi(\theta)|, & t \in (-\infty, 0], \end{cases} \tag{6}$$

where ε is an arbitrary positive constant. Obviously,

$$y^\varepsilon(t) = (1+\varepsilon) \sup_{\theta\leq 0} |\varphi(\theta)| e^{\int_0^t [-a^*(v)+b^*(v)h(v)]dv}, \quad t \in \mathbb{R},$$

and

$$u(t) < y^\varepsilon(t), \quad t \in (-\infty, 0].$$

Suppose there exists a $t^* > 0$ such that

$$u(t) < y^\varepsilon(t), \quad t \in (-\infty, t^*)$$

and

$$u(t^*) = y^\varepsilon(t^*).$$

By the nonnegativity of $b^*(t)$ and $h(t)$, we have

$$\begin{aligned} y^\varepsilon(t - \tau(t)) &= (1+\varepsilon) \sup_{\theta\leq 0} |\varphi(\theta)| e^{\int_0^{t-\tau(t)} [-a^*(v)+b^*(v)h(v)]dv} \\ &= (1+\varepsilon) \sup_{\theta\leq 0} |\varphi(\theta)| e^{\int_0^t [-a^*(v)+b^*(v)h(v)]dv} \\ &\quad \times e^{\int_{t-\tau(t)}^t [a^*(v)-b^*(v)h(v)]dv} \\ &\leq y^\varepsilon(t) e^{\int_{t-\tau(t)}^t a^*(v)dv}. \end{aligned}$$

Then we have

$$\sup_{t-\tau(t)\leq s\leq t} y^\varepsilon(s) \leq y^\varepsilon(t)h(t), \quad t \in [0, +\infty).$$

By the above inequality, we get

$$\begin{aligned} &\left(D^+u(t) - \frac{dy^\varepsilon(t)}{dt} \right)\Big|_{t=t^*} \\ &\leq -a(t^*)[u(t^*) - y^\varepsilon(t^*)] \\ &\quad + b(t^*)[\sup_{t^*-\tau(t^*)\leq s\leq t^*} u(s) - y^\varepsilon(t^*)h(t^*)] \\ &\leq -a(t^*)[u(t^*) - y^\varepsilon(t^*)] \\ &\quad + b(t^*) \sup_{t^*-\tau(t^*)\leq s\leq t^*} [u(s) - y^\varepsilon(s)] = 0, \end{aligned}$$

which is a contradiction. This means that

$$u(t) < y^\varepsilon(t), \quad t \in \mathbb{R}. \tag{7}$$

By letting $\varepsilon \to 0^+$ on both sides of (7), we derive

$$u(t) \leq \sup_{\theta\leq 0} |\varphi(\theta)| e^{\int_0^t [-a^*(v)+b^*(v)h(v)]dv}, \quad t \in \mathbb{R}. \tag{8}$$

From (3) and (8), we have

$$u(t) \to 0 \quad \text{as } t \to +\infty.$$

In addition, from (4), we can find a $T_* \in [0, +\infty)$ such that

$$\int_0^t [-a(v) + b(v)h(v)]dv \le -\int_0^t \gamma(v)dv, \quad t \in [T_*, +\infty),$$

then

$$u(t) \le \sup_{\theta \le 0} |\varphi(\theta)| e^{-\int_0^t \gamma(v)dv}, \quad t \in [T_*, +\infty).$$

For $t \in [0, T_*]$, we have

$$u(t) \le \sup_{\theta \le 0} |\varphi(\theta)| e^{\int_0^t [-a(v)+b(v)h(v)+\gamma(v)]dv} e^{-\int_0^t \gamma(v)dv} \le Ke^{-\int_0^t \gamma(v)dv},$$

where,

$$K := e^{\sup\limits_{s\in[0,T_*]} \int_0^s [-a(v)+b(v)h(v)+\gamma(v)]dv},$$

which implies

$$u(t) \le K \sup_{\theta \le 0} |\varphi(\theta)| e^{-\int_0^t \gamma(v)dv}, \quad t \in [0, +\infty).$$

The proof is completed.

Remark 2: For $\lambda > 0$, if we replace $\gamma(s)$ by λ, $\frac{\lambda}{t+1}$ and $\frac{\lambda}{(e+s-1)\ln(e+s-1)}$, respectively. Then system (2) is exponential, polynomial and logarithmic stability, respectively.

Theorem 2: If we don't require the nonnegativity of $b(t)$, Theorem 1 remains true for replacing $b(t)$ by $b^+(t)$.

Proof: Since $u(t) \ge 0$ and $b^+(t) \ge b(t)$, then we get the following delay differential inequality:

$$D^+u(t) \le -a(t)u(t) + b^+(t) \sup_{t-\tau(t)\le s\le t} u(s).$$

The proof is similar to the proof of Theorem 1. The proof is easy, so we omit it.

Remark 3: It should be pointed out that we do not require the boundedness of $a(t)$ or $\tau(t)$, which is imposed in [13], i.e., $|a(t)| \le M_a$ and $|\tau(t)| \le \tau$. Even in this special case, our condition is

$$\lim_{t\to+\infty} \int_0^t [-a(v) + b^+(v)h(v)]dv = -\infty.$$

However, the corresponding condition in [13] is

$$\lim_{t\to+\infty} \int_0^t \left[-a(v) + b^+(v)e^{M_a\tau}\right] dv = -\infty.$$

Obviously, $h(v) \le e^{M_a\tau}$ for $v \in [0, +\infty)$. In this sense, this paper improves the results in [13]. In addition, we don't have harsh restrictions on delay function $\tau(t)$. Obviously, $|a(t)| \le M_a$, $0 \le b^+(t) \le M_b$, $|\tau(t)| \le \tau$, $\delta > 0$, $\tau \in \left(0, \frac{1}{M_a}\ln(1 + \frac{\sigma}{M_b T})\right)$, and there exist $t_0 \ge 0$ and $T > 0$ such that

$$\int_{t_0+kT}^{t_0+(k+1)T} [-a(v) + b^+(v)]dv \le -\delta$$

for $k \in \mathbb{N}$ imply

$$\lim_{t\to+\infty} \int_0^t [-a(v) + b^+(v)h(v) + \lambda]dv = -\infty,$$

where

$$\lambda = \frac{[\delta - M_b T(e^{M_a\tau} - 1)]}{T}.$$

Moreover, we also consider the decay rate of (2), which is not considered in [13]. Based on the above discussion, this paper improves and extends the results in [13].

III. STABILITY OF DELAY NEURAL NETWORKS

In this section, we apply the obtained results in section 2 to consider the following non-autonomous neural networks with time-varying and distributed delays:

$$\begin{aligned} dx_i(t) = \Bigg[&- a_i(t)x_i(t) + \sum_{j=1}^n b_{ij}(t)f_j(x_j(t)) \\ &+ \sum_{j=1}^n c_{ij}(t)g_j(x_j(t - \tau_{ij}(t))) \\ &+ \sum_{j=1}^n d_{ij}(t)\int_{t-r_{ij}(t)}^t h_j(x_j(v))dv + I_i(t)\Bigg]dt, \\ & t \in [0, +\infty), \quad i \in J_n, \end{aligned} \tag{9}$$

In order to set the stability problem, we assume the following assumptions hold:

- (**A.1**) The functions $a_i(t)$, $b_{ij}(t)$, $c_{ij}(t)$, $d_{ij}(t)$ and $I_i(t)$ are all integrable functions for $t \in [0, +\infty)$, where $i, j \in J_n$.
- (**A.2**) The delay functions $\tau_{ij}(t)$ and $r_{ij}(t)$ satisfy
  $$\tau_{ij} : [0, +\infty) \to [0, \tau], \quad r_{ij} : [0, +\infty) \to [0, \tau],$$
  where $i, j \in J_n$ and $\tau > 0$.
- (**A.3**) The functions f_j, g_j and h_j satisfy Lipschitz condition with Lipschitz's constants L_j, M_j and N_j, respectively, where $j \in J_n$.

Definition 1 [9]: The system (9) is said to be globally exponentially stable, if there exists a pair of positive constants β and λ such that

$$\|x^{(1)}(t) - x^{(2)}(t)\| \le \beta\|\phi^{(1)} - \phi^{(2)}\|_{\mathcal{C}} e^{-\lambda t}, \quad t \in [0, +\infty),$$

where $x^{(1)}(t) = (x_1^{(1)}(t), x_2^{(1)}(t), \ldots, x_n^{(1)}(t))^T$ and $x^{(2)}(t) = (x_1^{(2)}(t), x_2^{(2)}(t), \ldots, x_n^{(2)}(t))^T$ are two different solutions of system (9) starting form different initial values $\phi^{(1)}(t) = (\phi_1^{(1)}(t), \phi_2^{(1)}(t), \ldots, \phi_n^{(1)}(t))^T$ and $\phi^{(2)}(t) = (\phi_1^{(2)}(t), \phi_2^{(2)}(t), \ldots, \phi_n^{(2)}(t))^T$.

*Theorem 3: Suppose assumptions (**A.1**)-(**A.3**) hold, then system (9) is globally exponentially stable if there is a constant $\lambda \in (0, +\infty)$ such that*

$$\begin{aligned} \lim_{t\to+\infty} \int_0^t \Bigg\{ &- a_i(v) + \sum_{j=1}^n |b_{ij}(v)|L_j \\ &+ \sum_{j=1}^n \Big[|c_{ij}(v)|M_j + |d_{ij}(v)|\tau N_j\Big]h_i(v) + \lambda\Bigg\}dv \\ = -\infty&, \end{aligned} \tag{10}$$

where

$$h_i(t) = e^{\sup\limits_{\theta\in[-\tau,0]} \int_{t+\theta}^{t}[a_i(v)-\sum\limits_{j\in J_n}|b_{ij}(v)|L_j]^* dv},$$

$i \in J_n$.

Proof: For system (9), we get

$$\begin{aligned}\frac{d[x_i^{(1)}(t)-x_i^{(2)}(t)]}{dt} &= -a_i(t)\Big[x_i^{(1)}(t)-x_i^{(2)}(t)\Big] \\ &\quad + \sum_{j=1}^{n} b_{ij}(t)\Big[f_j(x_j^{(1)}(t))-f_j(x_j^{(2)}(t))\Big] \\ &\quad + \sum_{j=1}^{n} c_{ij}(t)\Big[g_j(x_j^{(1)}(t-\tau_{ij}(t)))-g_j(x_j^{(2)}(t-\tau_{ij}(t)))\Big] \\ &\quad + \sum_{j=1}^{n} d_{ij}(t)\int_{t-r_{ij}(t)}^{t}\Big[h_j(x_j^{(1)}(v))-h_j(x_j^{(2)}(v))\Big]dv, \\ &\quad t\in[0,+\infty), \quad i\in J_n.\end{aligned}$$

Define

$$z_i(t) := |x_i^{(1)}(t)-x_i^{(2)}(t)|, \quad t\in[-\tau,+\infty), \quad i\in J_n.$$

From (**A.1**)-(**A.3**), we can obtain the following inequalities:

$$\begin{aligned} D^+ z_i(t) &\le -a_i(t)z_i(t) + \sum_{j=1}^{n}|b_{ij}(t)|L_j z_j(t) \\ &\quad + \sum_{j=1}^{n}|c_{ij}(t)|M_j \sup_{\theta\in[-\tau,0]} z_j(t+\theta) \\ &\quad + \sum_{j=1}^{n}|d_{ij}(t)|\tau N_j \sup_{\theta\in[-\tau,0]} z_j(t+\theta), \\ &\quad t\in[0,+\infty), \quad i\in J_n.\end{aligned}$$

Define

$$U(t) := \max_{i\in J_n}\{z_i(t)\}, \quad t\in[-\tau,+\infty).$$

For any $t \in [0,+\infty)$, let $\bar{i}_t$ stand for the index such that $U(t) = |z_{\bar{i}_t}(t)|$. For $t \in [0,+\infty)$, we get

$$\begin{aligned} D^+U(t) &\le -a_{\bar{i}_t}(t)U(t) + \sum_{j=1}^{n}|b_{\bar{i}_t j}(t)|L_j z_j(t) \\ &\quad + \sum_{j=1}^{n}|c_{\bar{i}_t j}(t)|M_j \sup_{\theta\in[-\tau,0]} z_j(t+\theta) \\ &\quad + \sum_{j=1}^{n}|d_{\bar{i}_t j}(t)|\tau N_j \sup_{\theta\in[-\tau,0]} z_j(t+\theta) \\ &\le -a_{\bar{i}_t}(t)U(t) + \sum_{j=1}^{n}|b_{\bar{i}_t j}(t)|L_j U(t) \\ &\quad + \sum_{j=1}^{n}\Big[|c_{\bar{i}_t j}(t)|M_j + |d_{\bar{i}_t j}(t)|\tau N_j\Big] \sup_{\theta\in[-\tau,0]} U(t+\theta) \\ &\le \Big[-a_{\bar{i}_t}(t) + \sum_{j=1}^{n}|b_{\bar{i}_t j}(t)|L_j\Big]U(t) \\ &\quad + \sum_{j=1}^{n}\Big[|c_{\bar{i}_t j}(t)|M_j + |d_{\bar{i}_t j}(t)|\tau N_j\Big] \sup_{\theta\in[-\tau,0]} U(t+\theta).\end{aligned}$$

Let

$$a(t) = a_{\bar{i}_t}(t) - \sum_{j=1}^{n}|b_{\bar{i}_t j}(t)|L_j$$

and

$$b(t) = \sum_{j=1}^{n}\Big[|c_{\bar{i}_t j}(t)|M_j + |d_{\bar{i}_t j}(t)|\tau N_j\Big],$$

then we get the following delay differential inequality:

$$\begin{cases} D^+U(t) \le -a(t)U(t) + b(t)\sup\limits_{\theta\in[-\tau,0]} U(t+\theta), \\ \qquad t\in[0,+\infty), \\ U(t) = \sup\limits_{s\in[-\tau,0]} U(s), \quad t\in[-\tau,0]. \end{cases}$$

From (10), we can obtain

$$\lim_{t\to+\infty}\int_0^t[-a(v)+b(v)h(v)+\lambda]dv = -\infty,$$

where

$$h(t) = e^{\sup\limits_{\theta\in[-\tau,0]}\int_{t+\theta}^{t} a^*(s)ds}.$$

From Theorem 1, we can find a constant $\beta \in (0,+\infty)$ such that

$$U(t) \le \beta \sup_{s\in[-\tau,0]} U(s)e^{-\lambda t}, \quad t\in[0,+\infty).$$

The proof is completed.

IV. EXAMPLES AND NUMERICAL SIMULATIONS

The main contribution of this section is to provide some examples with numerical simulations to illustrate the effectiveness of the obtained results in sections 2 and 3.

Example 1: Consider a delay differential equation:

$$\begin{cases} \dfrac{du(t)}{dt} = \Big(-\dfrac{1}{\sqrt{1+t}}+\cos t\Big)u(t) + \dfrac{u(t-\pi|\cos t|)}{2e^5\sqrt{1+t}}, \\ \qquad t\in[0,+\infty), \\ u(t) = 1, \quad t\in[-\pi,0]. \end{cases} \tag{11}$$

Obviously,

$$a(t) = \frac{1}{\sqrt{1+t}} - \cos t, \quad b(t) = \frac{1}{2e^5\sqrt{1+t}},$$

and

$$h(t) \le e^5, \quad t \ge 0.$$

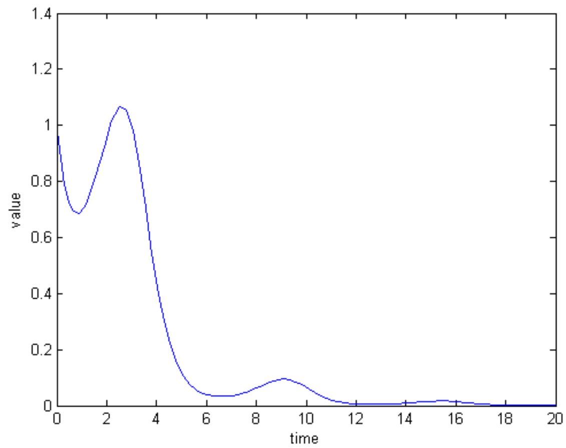


FIGURE 1. **The solution u(t) of (11).**

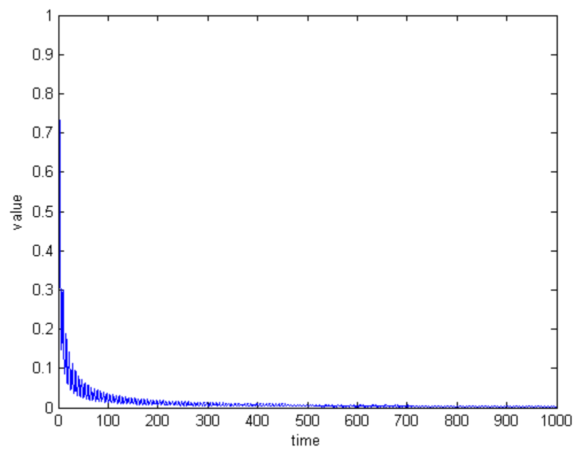


FIGURE 2. **The solution u(t) of (12).**

Then,

$$\lim_{t\to+\infty}\int_0^t [-a(v)+b(v)h(v)]dv = -\infty.$$

Consequently, all the assumptions of Theorem 1 are satisfied, it is shown in FIGURE 1 *that the solution of (11) is asymptotically stable.*

Remark 4: We mention here that the delay function $\tau(t) = |\pi\cos(t)|$ *are neither monotonous nor differentiable.*

Remark 5: Obviously, there is no constant $\lambda^* > 0$ *satisfies that*

$$\lim_{t\to+\infty}\int_0^t [-a(v)+b(v)h(v)+\lambda^*]dv = -\infty.$$

This shows that system (11) is asymptotically stable but not exponentially stable.

Example 2: Consider a delay differential equation:

$$\begin{cases} \dfrac{du(t)}{dt} = (-\dfrac{1}{t+1}+\sin t)u(t) + \dfrac{u(0.5t)}{4(t+1)e^2}, \\ \qquad t \in [0,+\infty), \\ u(0) = 1. \end{cases} \tag{12}$$

Obviously,

$$a(t) = \frac{1}{t+1} - \sin t, \quad b(t) = \frac{1}{4(t+1)e^2},$$

and

$$h(t) \le 2e^2, \quad t \ge 0.$$

Then,

$$\lim_{t\to+\infty}\int_0^t [-a(v)+b(v)h(v)]dv = -\infty.$$

Consequently, all the assumptions of Theorem 1 are satisfied, it is shown in FIGURE 2 that the solution of (12) is asymptotically stable.

Remark 6: We mention that the delay function of (12) is a proportional delay function which is an unbounded delay function.

Remark 7: It should be pointed out that the existing works [24]–[26] also considered the Halanay inequalities with proportional delay. However, they all required uniformly positive of $a(t)$*. In this example,* $a(t)$ *sometimes is positive and sometimes is negative, which means that the existing works [24]–[26] are invalid for this example.*

Example 3: Consider a delay differential equation:

$$\begin{cases} \dfrac{du(t)}{dt} = (-0.5+\sin t)u(t) \\ \qquad +0.4e^{-0.5\pi-2}u(t-\pi|\cos t|), \\ \qquad t \in [0,+\infty), \\ u(t) = 1, \quad t \in [-\pi, 0]. \end{cases} \tag{13}$$

Obviously,

$$a(t) = 0.5 - \sin t, \quad b(t) = 0.4e^{-0.5\pi-2},$$

and

$$h(t) \le e^{0.5\pi+2}, \quad t \ge 0.$$

Then,

$$\lim_{t\to+\infty}\int_0^t [-a(v)+b(v)h(v)+0.05]dv = -\infty.$$

Consequently, all the assumptions of Theorem 1 are satisfied, it is shown in FIGURE 3 that the solution of (13) converges exponentially.

Remark 8: Obviously, the function $a(t) < 0$ *for* $t \in (2k\pi + \frac{\pi}{6}, 2k\pi + \frac{5\pi}{6})$*,* $k \in \mathbb{N}$*, which means that* $-a(t)+b(t)$ *can be positive in some intervals. Therefore the results in [8], [10], [14], [16], [27], [30]) can not be applicable to this example.*

Remark 9: Obviously,

$$|a(t)| \le 1.5, \quad t \in [0,+\infty).$$

Though $a(t)$ *is a bounded function,*

$$\lim_{t\to+\infty}\int_0^t \left(-0.5+\sin v + 0.4e^{-0.5\pi-2}\cdot e^{1.5\pi}\right)dv = +\infty.$$

Therefore the results in [13] can not be applicable to this example.

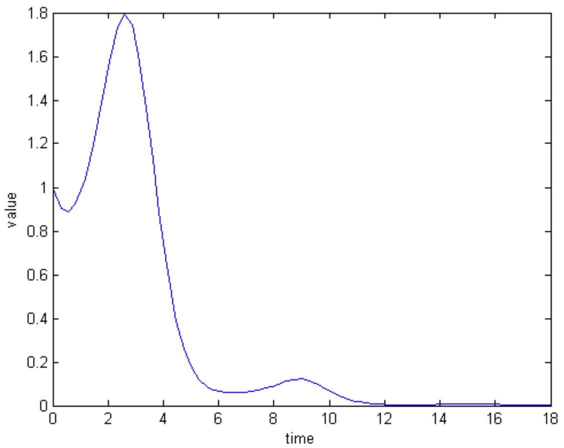


FIGURE 3. The solution u(t) of (13).

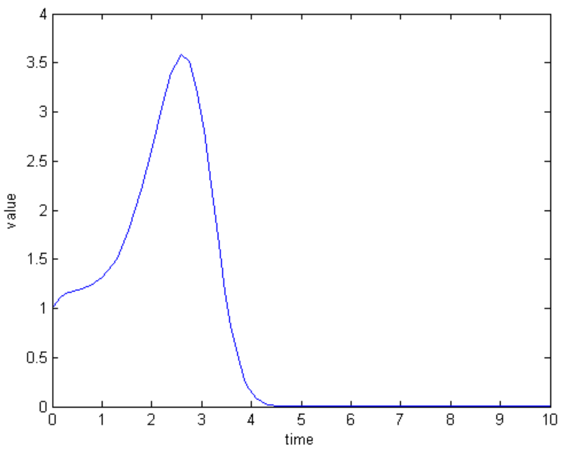


FIGURE 4. The solution u(t) of (14).

Example 4: Consider a delay differential equation:

$$\begin{cases} \frac{du(t)}{dt} = (-0.5t + t\sin t)u(t) + e^{-4t}u(t-\pi|\cos t|), \\ \qquad t \in [0, +\infty), \\ u(t) = 1, \quad t \in [-\pi, 0]. \end{cases} \tag{14}$$

Obviously,

$$a(t) = 0.5t - t\sin t, \quad b(t) = e^{-4t}$$

and

$$h(t) \leq e^{4t}, \quad t \geq 0,$$

then

$$\lim_{t\to+\infty}\int_0^t [a(v) + b(v)h(v) + 0.4t]dv = -\infty.$$

Consequently, all the assumptions of Theorem 1 are satisfied, it is shown in FIGURE 4 that the solution of (14) converges very fast for $t \geq 3$.

Remark 10: It should be pointed that $a(t) = 0.5t - t\cos t$ has no upper bound or lower bound, which means that the existing works [8], [10], [13]–[16], [27], [30] can not be applied to this situation.

Example 5: Consider 2-dimensional non-autonomous neural networks with time-varying and distributed delays:

$$\begin{aligned} dx_i(t) = \Big[&- a_i(t)x_i(t) + \sum_{j=1}^{2} b_{ij}(t)f_j(x_j(t)) \\ &+ \sum_{j=1}^{2} c_{ij}(t)g_j(x_j(t-\tau_{ij}(t))) \\ &+ \sum_{j=1}^{2} d_{ij}(t)\int_{t-r_{ij}(t)}^{t} h_j(x_j(v))dv + I_i(t)\Big]dt, \\ & t \in [0, +\infty), \quad i = 1, 2, \end{aligned} \tag{15}$$

where

$$\begin{aligned} &a_1(t) = -\sin t + 0.5, \quad a_2(t) = -2\sin t + 1, \\ &b_{11}(t) = 0.1, \quad b_{12}(t) = 0.1, \; b_{21}(t) = 0.2, \; b_{22}(t) = 0.2, \\ &c_{11}(t) = c_{12}(t) = d_{11}(t) = d_{12}(t) = 0.03e^{-1.3}, \\ &c_{21}(t) = c_{22}(t) = d_{21}(t) = d_{22}(t) = 0.06e^{-2.6}, \\ &I_1(t) = \cos t, \quad I_2(t) = \sin t, \\ &\tau_{11}(t) = \tau_{12}(t) = \tau_{21}(t) = \tau_{22}(t) \\ &\quad = r_{11}(t) = r_{12}(t) = r_{21}(t) = r_{22}(t) = |cos(t)|. \end{aligned}$$

For each $v \in \mathbb{R}$, $f_1(v) = f_2(v) = \arctan v$, $g_1(v) = g_2(v) = \frac{v}{1+v^2}$, $h_1(v) = h_2(v) = v$.

It is easy to see that $L_1 = L_2 = M_1 = M_2 = N_1 = N_2 = 1$

$$-a_1(t) + \sum_{j=1}^{2}|b_{1j}(t)|L_j = -0.3 + \sin t,$$

$$-a_2(t) + \sum_{j=1}^{2}|b_{2j}(t)|L_j = -0.6 + 2\sin t,$$

$$\sum_{j=1}^{2}[|c_{1j}(t)|M_j + |d_{1j}(t)|\tau N_j] = 0.12e^{-1.3},$$

$$\sum_{j=1}^{2}[|c_{2j}(t)|M_j + |d_{2j}(t)|\tau N_j] = 0.24e^{-2.6}.$$

and

$$h_1(t) \leq e^{1.3}, \quad h_2(t) \leq e^{2.6}, \quad t \geq 0.$$

Then,

$$\begin{aligned} \lim_{t\to+\infty}\int_0^t \Big\{ &- a_i(v) + \sum_{j=1}^{2}|b_{ij}(v)|L_j \\ &+ \sum_{j=1}^{2}\Big[|c_{ij}(v)|M_j + |d_{ij}(v)|\tau N_j\Big]h_i(v) + 0.1\Big\}dv \\ = -\infty, &\quad i = 1, 2. \end{aligned}$$

Then, all the assumption of Theorem 3 are satisfied, we consider the dynamical behaviour of two solutions $x^{(1)}(t) =$

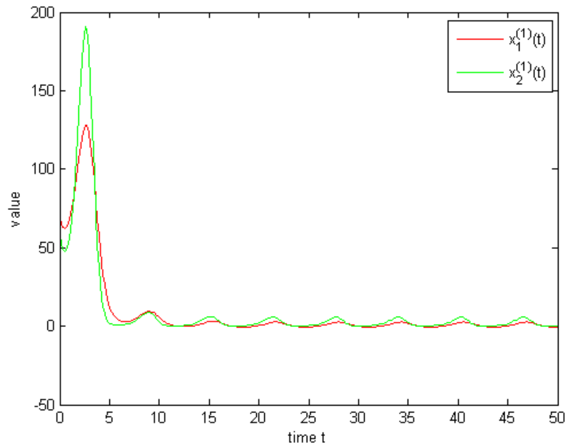


FIGURE 5. The states response $x_1^{(1)}(t)$ and $x_2^{(1)}(t)$ of (15).

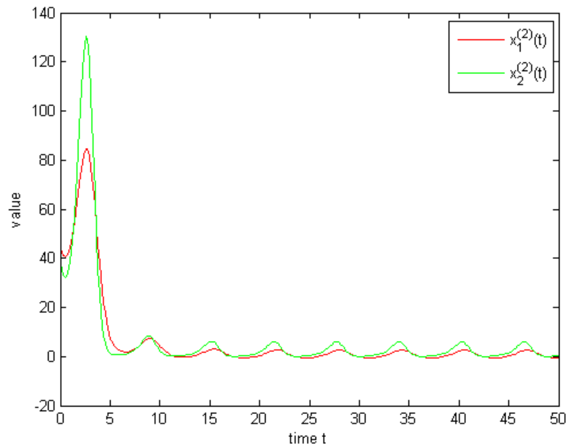


FIGURE 6. The states response $x_1^{(2)}(t)$ and $x_2^{(2)}(t)$ of (15).

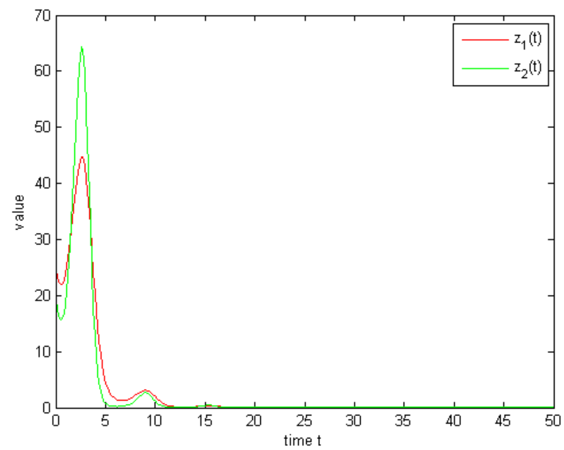


FIGURE 7. The states response $z_1(t)$ and $z_2(t)$ of (15).

$(x_1^{(1)}, x_2^{(1)})^T$, $x^{(2)}(t) = (x_1^{(2)}, x_2^{(2)})^T$ of (15) with different initial values for $t \in [-1, 0]$, respectively, which are as follows

$$\begin{cases} \psi_1^{(1)}(t) = 20 + 50e^{2t}, & \psi_1^{(2)}(t) = 40 + 20e^{2t}, \\ \psi_2^{(1)}(t) = 30 + 15e^{0.4t}, & \psi_2^{(2)}(t) = 20 + 20e^{-2t}. \end{cases}$$

Define

$$\begin{aligned} z(t) :&= (z_1(t), z_2(t))^T \\ &= (|x_1^{(1)}(t) - x_1^{(2)}(t)|, |x_2^{(1)}(t) - x_2^{(2)}(t)|)^T. \end{aligned}$$

It is shown in Figure 7 that $z(t)$ converges exponentially as pointed out by Theorem 3.

Remark 11: Obviously, the function $a_1(t) < 0$ and $a_2(t) < 0$ for $t \in (2k\pi + \frac{\pi}{6}, 2k\pi + \frac{5\pi}{6})$, $k \in \mathbb{N}$, which means that $a_1(t)$ and $a_2(t)$ can be negative in some intervals. Therefore the existing works [10], [15], [16] can not be applied to this situation.

V. CONCLUSION

In this work, some novel stability results for Halanay inequality and delay neural networks have been derived by means of constructing an auxiliary differential equation. The obtained results have shown that the coefficient functions can be positive or negative in some intervals, and has no upper bound or lower bound. It is noteworthy that our results have improved and extended the results in [13]. At last, some examples with numerical simulations have been presented to illustrate the effectiveness of our main results. In the future, we will investigate the neural networks whose coefficient functions and delay functions are all unbounded.

REFERENCES

[1] X. Hu, G. Feng, S. Duan, and L. Liu, "Multilayer RTD-memristor-based cellular neural networks for color image processing," *Neurocomputing*, vol. 162, pp. 150–162, Aug. 2015.

[2] D. Liu and A. N. Michel, "Cellular neural networks for associative memories," *IEEE Trans. Circuits Syst. II, Analog Digit. Signal Process.*, vol. 40, no. 2, pp. 119–121, Feb. 1993.

[3] T. Matsumoto, L. Chua, and H. Suzuki, "CNN cloning template: Connected component detector," *IEEE Trans. Circuits Syst.*, vol. 37, no. 5, pp. 633–635, May 1990.

[4] T. Roska and L. O. Chua, "Cellular neural networks with nonlinear and delay-type template elements," *Int. J. Circuit Theory Appl.*, vol. 20, no. 5, pp. 469–481, Jan. 1992.

[5] V. Dičpinigaitienė and L. Novickytė, "Application of systemic risk measurement methods: A systematic review and meta-analysis using a network approach," *Quant. Finance Econ.*, vol. 2, no. 4, pp. 798–820, 2018.

[6] V. Donskoy, "BOMD: Building optimization models from data (neural networks based approach)," *Quant. Finance Econ.*, vol. 3, no. 3, pp. 608–623, 2019.

[7] G. Rigatos, P. Siano, T. Ghosh, and D. Sarno, "A nonlinear optimal control approach to stabilization of a macroeconomic development model," *Quant. Finance Econ.*, vol. 2, no. 2, pp. 373–387, 2018.

[8] L. V. Hien, V. N. Phat, and H. Trinh, "New generalized Halanay inequalities with applications to stability of nonlinear non-autonomous time-delay systems," *Nonlinear Dyn.*, vol. 82, nos. 1–2, pp. 563–575, Oct. 2015.

[9] H. Jiang and Z. Teng, "Global exponential stability of cellular neural networks with time-varying coefficients and delays," *Neural Netw.*, vol. 17, no. 10, pp. 1415–1425, Dec. 2004.

[10] M. Jiang, J. Mu, and D. Huang, "Globally exponential stability and dissipativity for nonautonomous neural networks with mixed time-varying delays," *Neurocomputing*, vol. 205, pp. 421–429, Sep. 2016.

[11] M. Jiang and Y. Shen, "Stability of non-autonomous bidirectional associative memory neural networks with delay," *Neurocomputing*, vol. 71, nos. 4–6, pp. 863–874, Jan. 2008.

[12] J. Li, F. Zhang, and J. Yan, "Global exponential stability of nonautonomous neural networks with time-varying delays and reaction-diffusion terms," *J. Comput. Appl. Math.*, vol. 233, no. 2, pp. 241–247, Nov. 2009.

[13] B. Liu, W. Lu, and T. Chen, "Stability analysis of some delay differential inequalities with small time delays and its applications," *Neural Netw.*, vol. 33, pp. 1–6, Sep. 2012.

[14] B. Liu, W. Lu, and T. Chen, "New criterion of asymptotic stability for delay systems with time-varying structures and delays," *Neural Netw.*, vol. 54, pp. 103–111, Jun. 2014.

[15] S. Long, H. Li, and Y. Zhang, "Dynamic behavior of nonautonomous cellular neural networks with time-varying delays," *Neurocomputing*, vol. 168, pp. 846–852, Nov. 2015.

[16] B. Lu, H. Jiang, A. Abdurahman, and C. Hu, "Global generalized exponential stability for a class of nonautonomous cellular neural networks via generalized Halanay inequalities," *Neurocomputing*, vol. 214, pp. 1046–1052, Nov. 2016.

[17] X. Wang, M. Jiang, and S. Fang, "Stability analysis in Lagrange sense for a non-autonomous Cohen–Grossberg neural network with mixed delays," *Nonlinear Anal., Theory, Methods Appl.*, vol. 70, no. 12, pp. 4294–4306, Jun. 2009.

[18] X. Wei, D. Zhou, and Q. Zhang, "On asymptotic stability of discrete-time non-autonomous delayed hopfield neural networks," *Comput. Math. Appl.*, vol. 57, nos. 11–12, pp. 1938–1942, Jun. 2009.

[19] Y. Huang, D. Xu, and Z. Yang, "Dissipativity and periodic attractor for non-autonomous neural networks with time-varying delays," *Neurocomputing*, vol. 70, nos. 16–18, pp. 2953–2958, Oct. 2007.

[20] X. Nie, J. Liang, and J. Cao, "Multistability analysis of competitive neural networks with Gaussian-wavelet-type activation functions and unbounded time-varying delays," *Appl. Math. Comput.*, vol. 356, pp. 449–468, Sep. 2019.

[21] X. Yang, Q. Song, J. Cao, and J. Lu, "Synchronization of coupled Markovian reaction–diffusion neural networks with proportional delays via quantized control," *IEEE Trans. Neural Netw. Learn.Syst.*, vol. 30, no. 3, pp. 951–958, Mar. 2019.

[22] C. Xu, X. Yang, J. Lu, J. Feng, F. E. Alsaadi, and T. Hayat, "Finite-time synchronization of networks via quantized intermittent pinning control," *IEEE Trans. Cybern.*, vol. 48, no. 10, pp. 3021–3027, Oct. 2018.

[23] L. Shi, X. Yang, Y. Li, and Z. Feng, "Finite-time synchronization of nonidentical chaotic systems with multiple time-varying delays and bounded perturbations," *Nonlinear Dyn.*, vol. 83, nos. 1–2, pp. 75–87, Jan. 2016.

[24] H. Li, C. Li, W. Zhang, and J. Xu, "Global dissipativity of inertial neural networks with proportional delay via new generalized Halanay inequalities," *Neural Process. Lett.*, vol. 48, no. 3, pp. 1543–1561, Dec. 2018.

[25] C. Zheng, N. Li, and J. Cao, "Matrix measure based stability criteria for high-order neural networks with proportional delay," *Neurocomputing*, vol. 149, pp. 1149–1154, Feb. 2015.

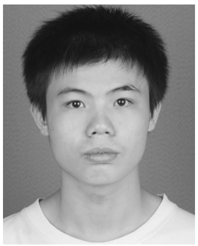
[26] L. Van Hien, D. T. Son, and H. Trinh, "On global dissipativity of nonautonomous neural networks with multiple proportional delays," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 29, no. 1, pp. 225–231, Jan. 2018.

[27] C. Baker and A. Tang, "Generalized Halanay inequalities for volterra functional differential equations and discretized versions," in *Proc. Volterra Centennial Meeting*, Jun. 1996, pp. 39–55.

[28] J.-J. Xiong and G. Zhang, "Improved stability criterion for recurrent neural networks with time-varying delays," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 29, no. 11, pp. 5756–5760, Nov. 2018.

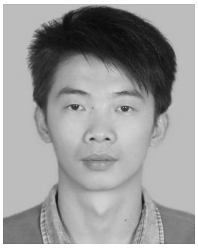
[29] J.-J. Xiong, G.-B. Zhang, J.-X. Wang, and T.-H. Yan, "Improved sliding mode control for finite-time synchronization of nonidentical delayed recurrent neural networks," *IEEE Trans. Neural Netw. Learn. Syst.*, to be published, doi: 10.1109/tnnls.2019.2927249.

[30] A. Halanay, *Differential Euqations*. New York, NY, USA: Academic, 1996.

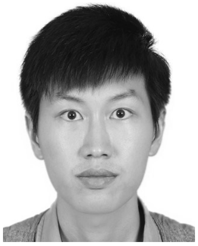


DEHAO RUAN was born in Guangdong, China, in 1988. He received the B.S. degree in mathematics and applied mathematics from Guangzhou University, Guangzhou, China, and the M.S. and Ph.D. degrees in probability and statistics from Guangzhou University, in 2014 and 2018, respectively. From 2018 to 2019, he held a postdoctoral position at the Guangzhou International Institute of Finance and Guangzhou University. His current research interests include stochastic differential equations, delay differential equations, neural networks, control theory, and mathematical biology.

WEIGUO LIU was born in Hengyang, Hunan, China, in 1979. He received the B.S. degree in physical education from the Hunan University of Science and Technology, Xiangtan, in 2002, and the M.S. and Ph.D. degrees in probability and statistics from Guangzhou University, Guangzhou, in 2015. From 2002 to 2009, he was a Middle School Teacher with the Linzhang Middle School, Handan. Since 2016, he has been a University Teacher with the Guangdong University of Finance and Economics, Guangzhou. His research interests include stochastic differential equations, neural networks, stochastic analysis and mathematical finance, and stochastic process.



MINRAN YANG was born in Guangdong, China, in 1991. He received the B.E. degree in chemical engineering and technology from the Zhongkai University of Agriculture and Engineering, Guangzhou, China, and the M.S. degree in chemical technology from Guangzhou University, China, where he is currently pursuing the Ph.D. degree. His current research interests include dynamical systems, statistical analysis, and library information analysis.



ZHEHAO HUANG was born in Guangdong, China, in 1988. He received the B.S., M.S., and Ph.D. degrees in applied mathematics from the South China University of Technology, Guangzhou, China. He held a postdoctoral position at the Guangzhou International Institute of Finance, from 2016 to 2018. His current research interest includes stochastic differential equations and applications in economics and finance and continuous time finance.

XIAOXIA GUO was born in Shanxi, China, in 1991. She received the B.S. degree in mathematics and applied mathematics from Taiyuan Normal University, China, and the M.S. degree from Shanxi Normal University. She is currently pursuing the Ph.D. degree with Guangzhou University. Her current research interests include neural networks, mathematical biology, and stochastic differential equations.