IEEE Access
Multidisciplinary Rapid Review Open Access Journal

Received December 21, 2019, accepted January 11, 2020, date of publication January 20, 2020, date of current version January 30, 2020.

Digital Object Identifier 10.1109/ACCESS.2020.2967813

Linear Function Observers for Linear Time-Varying Systems With Time-Delay: A Parametric Approach

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This work was supported in part by the National Natural Science Foundation of China under Grant 61690210 and Grant 61690212.

ABSTRACT In this paper, a parametric approach to design a Luenberger functional observer for linear time-varying (LTV) systems with time-delay is investigated. Based on the solution to generalized Sylvester equation (GSE), the complete general parametric expressions for the functional observer gain matrices are established with the time-varying coefficient matrices, the time-varying closed-loop system and a group of arbitrary parameters. With the parametric method, the observation error system can be transformed into a linear system with the expected eigenstructure. Finally, a numerical simulation is provided to illustrate the effectiveness of the parametric approach.

INDEX TERMS Functional observers, LTV systems with time-delay, parametric method, Sylvester equation.

I. INTRODUCTION

Due to the complex work condition on-site, the state variables cannot be all measured such that it is difficult for the realization of control strategies. In this case, the concept of observer design is proposed by Luenberger [1], [2], in which the state of the system is reconstructed to achieve the corresponding control strategy [3]–[5].

The linear function observer has the advantage of greatly reducing the complexity of designing. Since the seminal theory of Luenberger, a significant quantity of results have been published to the problem of observing a linear function. For linear time-invariant (LTI) systems, Aldeen and Trinh solved the problem of designing a reduced-order function observer [6]. A method to construct a minimum-order functional observer is proposed and extended to a system with vector output by Korovin *et al.* [7]. Volkov and Demyanov proposed a novel approach to design functional observers via the solution to linear matrix inequalities (LMIs) [8]. Aimed at LTV systems, the conditions of existence for linear functional observers have been proposed [9]. Rotella offered the existence conditions for the minimum-order functional observer inspiring by linearly independent rows of time-varying matrices [10]. Different from the general approaches, a parametric approach is proposed to design a linear functional observer via the solution to GSE [11]. Moreover, there are also other results for functional observer (see [12]–[15]).

The phenomenon of time-delay exists widely in fields such as mechanical transmission [16], network control systems [17], communication [18], and chemical engineering [19]. The rate of state change for time-delay systems depends on the past state, which is an essential cause of the instability of systems. Time-delay systems have been extensively researched over the past few decades and have obtained a string of results. For stability analysis of the time-varying time-delay system, three classes of strict input-to-state stability Lyapunov-Krasovskii functions have been proposed by Zhou [20]. Some stability criteria have been proposed for time-varying time-delay systems, which based on Razumikhi and Krasovskii stability approaches in [21]. Moreover, there are also other researches in [22]–[25]. However, there still exits some problems to be solved, especially in observer design [26]–[30]. In recent, the LMIs approach has commonly used in observer design. The observer design problem can be transformed into the problem to be solved by LMIs, further, it can be expanded into an optimization problem to obtain the desired observer [31], [32]. Most of the results

The associate editor coordinating the review of this manuscript and approving it for publication was Yan-Jun Liu.

on time-delay observers are concentrated on LTI systems. However, for LTV systems with time-delay, the task of designing an observer is more challenging. Until now, there are only a few works focused on designing observers for LTV systems with time-delay [33]–[35]. As reported in [34] Briat *et al.* obtained the key characteristics of the observer error system through the nonlinear algebraic matrix. According to the conditions of these characteristics, the appropriate observer is designed under the LMIs method. An observer is proposed by J. G. Rueda-Escobedo *et al.*, which accelerated the convergence speed based on the structure similar to Gramian, making the method suitable for the delay in the time range of the system in [35].

Most researchers use LMIs to obtain observer parameters. Different from the above methods, we consider a parametric approach to design observer. Through some simple transformations, the parameters of the observer are transformed into the solution to GSE [36], [37]. Further, the completely parametric expression of the observer coefficient matrices has established. The stability of the observer error system is ensured by the selectable degrees of freedom in the design process.

The main contribution of the present work is to propose a parametric approach to design a linear function observer for LTV systems with time-delay. The proposed method simplifies the complexity of computation and provides a group of arbitrary parameters can be optimized to fulfill some additional system performance.

The remaining of this work is divided as follows. Some assumptions, background knowledge, and problem statement are reviewed in Section 2. Then, in Section 3 the corresponding existence conditions and the general parametric solutions of Luenberger observers for LTV systems with time-delay are proposed. In Section 4, a simple example is provided to prove the effectiveness of the parametric approach. Section 5 draws the conclusions of the proposed work.

Notation: We present some notations which will be used throughout this paper. $\mathbb{R}^{n\times r}$ denotes all real matrices of dimension $n \times r$, $\mathbb{R}^{n\times r}[s]$ represents all polynomial matrices of dimension $n \times r$ with real coefficients, $\mathbb{R}^{+}$, $\mathbb{C}$ denote the set of real number and complex number, $\mathrm{eig}(A)$ denotes the set of all eigenvalues of matrix A, $\deg(A(t, s))$ denotes the degree of polynomial $A(t, s)$ with respect to variable s, $\det(A)$ is the determinant of matrix A and $\mathrm{adj}(A)$ is the adjoint matrix of matrix A, σ_1 and σ_2 represent the highest degree of $d_{ij}(t, s)$ and $n_{ij}(t, s)$, σ denotes the maximum among σ_1 and σ_2.

II. PROBLEM STATEMENT

Consider the following LTV system with time-delay

$$\begin{cases} \dot{x}(t) = A(t)x(t) + A_d(t)x(t-\tau) + B(t)u(t), \\ y(t) = C(t)x(t), \\ x(t) = \phi(t),\ t \in [-\tau, 0], \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $x(t-\tau) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^m$ and $u(t) \in \mathbb{R}^r$ are the state vector, time-delay state vector, measured output vector and control vector, $A(t) \in \mathbb{R}^{n\times n}$, $A_d(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times r}$, $C(t) \in \mathbb{R}^{m\times n}$ are the system coefficient matrices where t is a time-varying parameter, $\phi(t)$ is the initial value of the system.

We consider a linear functional for the problem of observing

$$h(t) = \bar{L}(t)x(t), \tag{2}$$

where $h(t)$ is the state combination signal of system (1), $\bar{L}(t) \in \mathbb{R}^{l\times n}$ is the gain matrix.

Assumption 1: [10], [38], [39] $\{A(t), C(t)\}$ is observable.

For Assumption 1, let $\pounds(t)$ be the observability matrix for system (1) as

$$\pounds(t) = \left[\pounds_0^{\mathrm{T}}(t)\ \pounds_1^{\mathrm{T}}(t)\ \ldots\ \pounds_{j-1}^{\mathrm{T}}(t) \right]^{\mathrm{T}},$$

where

$$\pounds_0 = C(t),$$
$$\pounds_{(m-1)} = \pounds_{(m-1)}A(t) + \dot{\pounds}_{(m-1)}(t),\ m = 1, 2, \ldots, j-1.$$

System (1) is completely observable if

$$\mathrm{rank}\ \pounds(t) = m,\ \exists\ t \in \mathbb{R}^{+}.$$

It is uniformly observable if

$$\mathrm{rank}\ \pounds(t) = m,\ \forall\ t \in \mathbb{R}^{+}.$$

Assumption 2: $\mathrm{rank}\ C(t) = m,\ \mathrm{rank}\ \bar{L}(t) = l, \forall\ t \in \mathbb{R}^{+}$.

The Luenberger functional observer for system (1) design as follows

$$\begin{cases} \dot{z}(t) = F(t)z(t) + F_d(t)z(t-\tau) \\ \qquad\quad + L(t)y(t) + L_d(t)y(t-\tau) + H(t)u(t), \\ w(t) = M(t)z(t) + N(t)y(t), \end{cases} \tag{3}$$

where $z(t) \in \mathbb{R}^{\mu}$, $z(t-\tau) \in \mathbb{R}^{\mu}$ are the state vectors of observer, $F(t) \in \mathbb{R}^{\mu\times\mu}$, $F_d(t) \in \mathbb{R}^{\mu\times\mu}$, $L(t) \in \mathbb{R}^{\mu\times m}$, $L_d(t) \in \mathbb{R}^{\mu\times m}$, $H(t) \in \mathbb{R}^{\mu\times r}$, $M(t) \in \mathbb{R}^{l\times\mu}$, $N(t) \in \mathbb{R}^{l\times m}$ are the coefficient matrices of the observer which need to be designed. The observer output $w(t)$ and the signal $h(t)$ satisfying the following relation

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty} [w(t) - h(t)] = 0. \tag{4}$$

Consider GSE as follows

$$\sum_{i=0}^{\varphi} \mathcal{A}_i(t)\mathcal{V}(t)\mathcal{F}^i = \sum_{i=0}^{\varphi} \mathcal{B}_i(t)\mathcal{W}(t)\mathcal{F}^i, \tag{5}$$

1. where $\mathcal{A}_i(t) \in \mathbb{R}^{n\times q}$, $\mathcal{B}_i(t) \in \mathbb{R}^{n\times r}$, $i = 1, 2, \ldots, \varphi$, are matrix functions which are piecewisely continuous with respect to t, and $\mathcal{F} \in \mathbb{R}^{p\times p}$, are the parameter matrices;
2. where matrices $\mathcal{V}(t)$ and $\mathcal{W}(t)$ are need to be determined.

The polynomial matrices associated with the GSE (5) are

$$\begin{cases} \mathcal{A}(t, s) = \sum\limits_{i=0}^{\varphi} \mathcal{A}_i(t)s^i, \\ \mathcal{B}(t, s) = \sum\limits_{i=0}^{\varphi} \mathcal{B}_i(t)s^i, \end{cases} \tag{6}$$

Definition 1: [37] Let $\mathcal{A}(t,s) \in \mathbb{R}^{n\times q}[s]$, and $\mathcal{B}(t,s) \in \mathbb{R}^{n\times r}[s]$, $q+r>n$ be given as in (6), and $\mathcal{F} \in \mathbb{C}^{p\times p}$ be an arbitrary matrix. Then $\mathcal{A}(t,s)$ and $\mathcal{B}(t,s)$ are said to be $\mathcal{F}$-left coprime over $t \in \mathbb{R}^+$ if

$$\operatorname{rank}\left[\, \mathcal{A}(t,s)\ \mathcal{B}(t,s) \,\right] = n, \forall t \in \mathbb{R}^+, s \in \operatorname{eig}[F]. \quad (7)$$

Assumption 3: $\{F(t), M(t)\}$ is observable.

Problem 1: Given system (1) and the Luenberger functional observer (3) satisfying Assumptions 1–3. There exist the following problems

1. Propose the sufficient conditions for the existence of the functional observer (3).
2. According to the sufficient conditions, find the parametric forms of the coefficient matrices $F(t)$, $F_d(t)$, $L(t)$, $L_d(t)$, $H(t)$, $M(t)$, $N(t)$ to construct a functional observer as (3) for the system (1).

III. MAIN RESULTS

A. EXISTENCE CONDITIONS

According to the asymptotic stability problem for LTV systems with time-delay, there exists the following Lemma.

Lemma 1: $\dot{\varepsilon}(t) = F(t)\varepsilon(t) + F_d(t)\varepsilon(t-\tau)$ is asymptotically stable if there exist matrices $F(t)$, $F_d(t)$ and differentiable positive definite symmetric matrix $P(t)$, positive definite symmetric matrix $Q(t)$ satisfying the following inequality

$$\begin{bmatrix} \Phi(t) & P(t)F_d(t) \\ F_d^{\mathrm{T}}(t)P(t) & -Q(t) \end{bmatrix} < 0, \quad (8)$$

where

$$\Phi(t) = P(t)F(t) + F^{\mathrm{T}}(t)P(t) + Q(t) + \dot{P}(t).$$

Proof: Choose a Lyapunov function as

$$V(\varepsilon(t),t) = x^{\mathrm{T}}(t)P(t)x(t) + \int_{t-\tau(t)}^{t} x^{\mathrm{T}}(\theta)Q(t)x(\theta)\, d\theta,$$

and thus the derivative of $\dot{V}(\varepsilon(t),t)$ satisfies the relation

$$\begin{aligned}\dot{V}(\varepsilon(t),t) = {} & \varepsilon^{\mathrm{T}}(t)\Phi(t)\varepsilon(t) + \varepsilon^{\mathrm{T}}(t)P(t)F_d(t)\varepsilon(t-\tau) \\ & +\varepsilon^{\mathrm{T}}(t-\tau)F_d^{\mathrm{T}}(t)P(t)\varepsilon(t) \\ & -\varepsilon^{\mathrm{T}}(t-\tau)Q(t)\varepsilon^{\mathrm{T}}(t-\tau),\end{aligned}$$

then

$$\begin{aligned}\dot{V}(\varepsilon(t),t) = {} & \begin{bmatrix} \varepsilon(t) \\ \varepsilon(t-\tau) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Phi(t) & P(t)F_d(t) \\ F_d^{\mathrm{T}}(t)P(t) & -Q(t) \end{bmatrix} \\ & \times \begin{bmatrix} \varepsilon(t) \\ \varepsilon(t-\tau) \end{bmatrix} < 0, \end{aligned} \quad (9)$$

when $\begin{bmatrix} \varepsilon(t) \\ \varepsilon(t-\tau) \end{bmatrix} \neq 0$, if $\dot{V}(\varepsilon(t),t) < 0$, $\varepsilon \to 0$ as $t \to \infty$, therefore $\dot{\varepsilon}(t) = F(t)\varepsilon(t) + F_d(t)\varepsilon(t-\tau)$ is asymptotically stable. From Inequality (9) we obtain $\dot{V}(\varepsilon(t),t) < 0$, if the Inequality (8) is established. The proof of Lemma 1 is completed. □

There exist the following sufficient conditions for the existence of the functional observer for the system (1).

Theorem 1: The LTV observer of the form (3) for system (1) can be satisfied if there exists a continuously differentiable matrix $K(t)$ such that

$$K(t)A(t) - F(t)K(t) + \dot{K}(t) = L(t)C(t), \quad (10)$$
$$K(t)A_d(t) - F_d(t)K(t) = L_d(t)C(t), \quad (11)$$
$$H(t) = K(t)B(t), \quad (12)$$
$$M(t)K(t) + N(t)C(t) = \bar{L}(t), \quad (13)$$

where $\dot{\varepsilon}(t) = F(t)\varepsilon(t) + F_d(t)\varepsilon(t-\tau)$ is asymptotically stable and $F(t)$ is a Hurwitz matrix.

Proof: Denote

$$\begin{cases} \varepsilon(t) = z(t) - K(t)x(t), \\ e(t) = w(t) - \bar{L}(t)x(t). \end{cases} \quad (14)$$

From Equation (14), we have

$$\begin{aligned}\dot{\varepsilon}(t) &= \dot{z}(t) - \dot{K}(t)x(t) - K(t)\dot{x}(t) \\ &= F(t)z(t) + F_d(t)z(t-\tau) + L(t)y(t) \\ &\quad +L_d(t)y(t-\tau) - \dot{K}(t)x(t) - K(t)A(t)x(t) \\ &\quad -K(t)A_d(t)x(t-\tau) + H(t)u(t) - K(t)B(t)u(t) \\ &= F(t)(z(t) - K(t)x(t)) + F(t)K(t)x(t) \\ &\quad +F_d(t)(z(t-\tau) - K(t)x(t-\tau)) + L(t)C(t)x(t) \\ &\quad +F_d(t)K(t)x(t-\tau) + L_d(t)C(t)x(t-\tau) \\ &\quad -\dot{K}(t)x(t) - K(t)A(t)x(t) - K(t)A_d(t)x(t-\tau) \\ &\quad +H(t)u(t) - K(t)B(t)u(t) \\ &= F(t)\varepsilon(t) + F_d(t)\varepsilon(t-\tau) + (H(t) - K(t)B(t))u(t) \\ &\quad +(L(t)C(t) + F(t)K(t) - K(t)A(t) - \dot{K}(t))x(t) \\ &\quad +(L_d(t)C(t) + F_d(t)K(t) - K(t)A_d(t))x(t-\tau),\end{aligned}$$

and

$$\begin{aligned}e(t) &= w(t) - \bar{L}(t)x(t) \\ &= M(t)z(t) + N(t)y(t) - \bar{L}(t)x(t) \\ &= M(t)(z(t) - K(t)x(t)) + M(t)K(t)x(t) \\ &\quad +N(t)C(t)x(t) - \bar{L}(t)x(t) \\ &= M(t)\varepsilon(t) + (M(t)K(t) + N(t)C(t) - \bar{L}(t))x(t).\end{aligned}$$

Based on the above conditions, the error system can be rewritten as

$$\begin{cases} \dot{\varepsilon}(t) = F(t)\varepsilon(t) - F_d(t)\varepsilon(t-\tau), \\ e(t) = M(t)\varepsilon(t). \end{cases} \quad (15)$$

According to Lemma 1, $\dot{\varepsilon}(t) = F(t)\varepsilon(t) + F_d(t)\varepsilon(t-\tau)$ is asymptotically stable, $M(t)$ can be chosen arbitrarily, such that the error e can converge asymptotically to zero as $t \to \infty$. The proof of Theorem 1 is completed. □

B. DESIGN METHOD OF FUNCTIONAL OBSERVER

There exists the following time-varying right coprime factorization (RCF)

$$\mathcal{A}(t,s)N(t,s) - \mathcal{B}(t,s)D(t,s) = 0, \quad (16)$$

where $N(t,s) \in \mathbb{R}^{q\times\beta_0}[s]$ and $D(t,s) \in \mathbb{R}^{r\times\beta_0}[s]$ are a set of polynomial matrices. Denote $N(t,s) = \left[n_{ij}(t,s)\right]_{q\times\beta_0}$, $D(t,s) = \left[d_{ij}(t,s)\right]_{r\times\beta_0}$ and

$$\begin{cases}\sigma_1 = \max\left\{\deg(d_{ij}(t,s)),\right. \\ \qquad\qquad \left. i=1,2,\cdots,r,\ \ j=1,2,\cdots,\beta_0\right\}, \\ \sigma_2 = \max\left\{\deg(n_{ij}(t,s)),\right. \\ \qquad\qquad \left. i=1,2,\cdots,q,\ \ j=1,2,\cdots,\beta_0\right\}, \\ \sigma = \max\left\{\sigma_1,\sigma_2\right\},\end{cases}$$

then $N(t,s)$ and $D(t,s)$ are written as follows

$$\begin{cases}N(t,s) = \sum\limits_{i=0}^{\sigma} N_i(t)s^i,\ N_i(t) \in \mathbb{R}^{q\times\beta_0}, \\ D(t,s) = \sum\limits_{i=0}^{\sigma} D_i(t)s^i,\ D_i(t) \in \mathbb{R}^{r\times\beta_0}.\end{cases} \tag{17}$$

For the solution to function observers of LTV systems with time-delay, the following Theorem is proposed.

Theorem 2: The coefficient matrices of the function observer (3) for system (1) are parameterized as

$$\begin{cases}H(t) = K(t)B(t), \\ N(t) = (\bar{L}(t) - M(t)K(t))C^{\{1\}}(t), \\ L(t) = (W(t) + \dot{K}(t))C^{\{1\}}(t),\end{cases} \tag{18}$$

and

$$\begin{cases}K(t) = \sum\limits_{i=0}^{\sigma} F^{\sigma}(t)Z^{\mathrm{T}}(t)N_i^{\mathrm{T}}(t), \\ W(t) = \sum\limits_{i=0}^{\sigma} F^{\sigma}(t)Z^{\mathrm{T}}(t)D_i^{\mathrm{T}}(t),\end{cases} \tag{19}$$

where $Z(t) \in \mathbb{R}^{n\times\mu}$ and $M(t) \in \mathbb{R}^{l\times\mu}$ are arbitrary matrices satisfying

Constraint 1: $\operatorname{rank}\begin{bmatrix} C(t) \\ \bar{L}(t) - M(t)K(t) \\ W(t) + \dot{K}(t)\end{bmatrix} = \operatorname{rank} C(t)$,

and $C^{\{1\}}(t)$ is the generalized inverse of $C(t)$.

Proof: Denoting

$$W(t) = L(t)C(t) - \dot{K}(t). \tag{20}$$

By the transpose of Equation (10), we have

$$K(t)A(t) - F(t)K(t) = L(t)C(t) - \dot{K}(t), \tag{21}$$

thus Equation (21) can be rewritten as

$$K(t)A(t) - W(t) = F(t)K(t). \tag{22}$$

By transposition, Equation (22) can be rewritten as the GSE

$$A^{\mathrm{T}}(t)K^{\mathrm{T}}(t) - W^{\mathrm{T}}(t) = K^{\mathrm{T}}(t)F^{\mathrm{T}}(t). \tag{23}$$

The polynomial matrices associated with above GSE (23) are

$$\begin{cases}\mathcal{A}(t,s) = sI_n - A^{\mathrm{T}}(t), \\ \mathcal{B}(t,s) = -I_n.\end{cases} \tag{24}$$

Using (17), we have

$$\begin{aligned}(sI - A^{\mathrm{T}}(t))\sum_{i=0}^{\sigma} N_i(t)s^i &= \sum_{i=0}^{\sigma} N_i(t)s^{i+1} - \sum_{i=0}^{\sigma} A^{\mathrm{T}}(t)N_i(t)s^i \\ &= N_{\sigma}(t)s^{\sigma+1} + \sum_{i=1}^{\sigma}(N_{i-1}(t) \\ &\quad - A^{\mathrm{T}}(t)N_i(t))s^i - A^{\mathrm{T}}(t)N_0,\end{aligned}$$

and

$$-I_n\sum_{i=0}^{\sigma} D_i(t)s^i = -\sum_{i=1}^{\sigma} D_i(t)S^i - D_0(t),$$

substituting the above relations into Equation (16), we have

$$\begin{cases}A^{\mathrm{T}}(t)N_0 = D_0(t), \\ N_{i-1}(t) - A^{\mathrm{T}}(t)N(t) = -D_i(t), \\ N_w(t) = 0.\end{cases} \tag{25}$$

According to Equations (17) and (23), we have

$$\begin{aligned}&A^{\mathrm{T}}(t)K^{\mathrm{T}}(t) - W^{\mathrm{T}}(t) \\ &\quad = A^{\mathrm{T}}(t)\sum_{i=0}^{\sigma} N_i(t)Z(t)(F^{\sigma}(t))^{\mathrm{T}} \\ &\qquad - \sum_{i=0}^{\sigma} D_i(t)Z(t)(F^{\sigma}(t))^{\mathrm{T}} \\ &\quad = A^{\mathrm{T}}(t)N_0(t)Z(t) + \sum_{i=1}^{\sigma} A^{\mathrm{T}}(t)N_i(t)Z(t)(F^{\sigma}(t))^{\mathrm{T}} \\ &\qquad - \sum_{i=1}^{\sigma} D_i(t)Z(t)(F^{\sigma}(t))^{\mathrm{T}} - D_0(t)Z(t) \\ &\quad = \sum_{i=1}^{\sigma}(A^{\mathrm{T}}(t)N_i(t) - D_i(t))Z(t)(F^{\sigma}(t))^{\mathrm{T}} \\ &\qquad + (A^{\mathrm{T}}(t)N_0(t) - D_0(t))Z(t) \\ &\quad = \sum_{i=1}^{\sigma} N_{i-1}(t)Z(t)(F^{\sigma}(t))^{\mathrm{T}} = K^{\mathrm{T}}(t)F^{\mathrm{T}}(t).\end{aligned}$$

The matrices $K(t)$ and $W(t)$ are parameterized as Equation (19) satisfying the Equation (23).

According to the forms of $K(t)$ and $W(t)$ in Equation (19), the following equations can be obtained

$$\begin{cases}L(t)C(t) = W(t) - \dot{K}(t), \\ N(t)C(t) = \bar{L}(t) - M(t)K(t).\end{cases} \tag{26}$$

The approach relies on the solution of the Equation (26). The matrices $L(t)$ and $N(t)$ exist if

$$\operatorname{rank}\begin{bmatrix} C(t) \\ \bar{L}(t) - M(t)K(t) \\ W(t) + \dot{K}(t)\end{bmatrix} = \operatorname{rank} C(t). \tag{27}$$

The proof is completed. □

Based on Constraint 1, we can obtain the generally parameterized expressions of observer coefficient matrices $L(t)$

and $M(t)$. Parametric expression of coefficient matrix $L_d(t)$ is given as follows

$$L_d(t) = (K(t)A_d(t) - F_d(t)K(t))C^{\mathrm{T}}(t)(C(t)C^{\mathrm{T}}(t))^{-1}, \quad (28)$$

which satisfies the following Constraint 2.

Constraint 2:

$$\operatorname{rank}\begin{bmatrix} C(t) \\ K(t)A_d(t) - F_d(t)K(t) \end{bmatrix} = \operatorname{rank} C(t) = m.$$

Proof: Based on Constraint 2, the Equation (11) can be rewritten as follows

$$L_d(t)(C(t)C^{\mathrm{T}}(t)) = (K(t)A_d(t) - F_d(t)K(t))C^{\mathrm{T}}(t),$$

then

$$\begin{aligned} L_d(t)&(C(t)C^{\mathrm{T}}(t))(C(t)C^{\mathrm{T}}(t))^{-1} \\ &= (K(t)A_d(t) - F_d(t)K(t))C^{\mathrm{T}}(t)(C(t)C^{\mathrm{T}}(t))^{-1}, \end{aligned}$$

we can obtain

$$L_d(t) = (K(t)A_d(t) - F_d(t)K(t))C^{\mathrm{T}}(t)(C(t)C^{\mathrm{T}}(t))^{-1}.$$

The proof is completed. □

Remark 1: From the error system (15), we can deduce that the performance of the observer is determined by the matrices $F(t)$, $F_d(t)$, $M(t)$, from Theorem 1, a Hurwitz matrix $F(t)$ can be selected arbitrarily to determine the observer error system. The matrices $F_d(t)$ and $M(t)$ can be designed to determine the observer if the degrees of freedom still exist during the design of the observer.

Remark 2: The free parameter matrix $Z(t)$, which represents the degrees of freedom appearing linearly in the general solution. This property gives the convenience and advantages to solve the problem. The parameter matrix $Z(t)$ can be optimized to achieve some better performance in applications.

C. GENERAL PROCEDURE

Based on Theorems 1 and 2, a general procedure is proposed to solve the design problem of the functional observer for LTV systems with time-delay.

Step 1: Design the structure of matrix $F(t)$.

The structure of matrix $F(t)$ is usually chosen as a Hurwitz matrix, it is required that the eigenvalues of the matrix lie in the left-half s-plane.

Step 2: Select the matrices $F(t)$ and $F_d(t)$.

The matrices $F(t)$ and $F_d(t)$ can be selected to satisfy the Equation (15) and Constraint 2. Verify the matrices $F(t)$ and $F_d(t)$ meet the Inequality (8)? If Yes, go to **Step 3**, if No, go back to **Step 2**, select again.

Step 3: Obtain a pair of RCF $\{N(t,s), D(t,s)\}$.

From RCF (16), a pair of particular solutions can be given by

$$\begin{cases} N(t,s) = \operatorname{adj}(sI_n - A^{\mathrm{T}}(t)) * (-I_n), \\ D(t,s) = \det(sI_n - A^{\mathrm{T}}(t))I_n. \end{cases} \quad (29)$$

Step 4: Compute the matrices $K(t)$ and $W(t)$.

Compute the matrices $K(t)$ and $W(t)$ through the Equation (19), find that if there exists an arbitrary parameter $Z(t)$ satisfying Constraint 1, if Yes, go to **Step 5**, if No, go back to **Step 2**, select again.

Step 5: Calculate the coefficient matrices $M(t)$, $N(t)$, $L_d(t)$, $L(t)$, and $H(t)$.

Calculate the observer coefficient matrices $M(t)$, $N(t)$, $L_d(t)$, $L(t)$, and $H(t)$ through the formulas (18) and (28).

IV. EXAMPLE

A. GENERAL SOLUTION

Consider a LTV system with time-delay in [34].

$$A = \begin{bmatrix} 0 & -a_1(t) \\ -2 & -a_2(t) \end{bmatrix}, A_d = \begin{bmatrix} a_3(t) & 0.1 \\ a_4(t) & -0.3 \end{bmatrix},$$
$$B = \begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Let

$$\bar{L}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

we can easily deduce the RCF (16) obviously hold for the above system and

$$\begin{cases} N(t,s) = -I_2, \\ D(t,s) = \begin{bmatrix} s & 2 \\ a_1(t) & s + a_2(t) \end{bmatrix}. \end{cases} \quad (30)$$

Let

$$F = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}, F_d = \begin{bmatrix} -0.3 & 0 \\ 0 & -0.3 \end{bmatrix}. \quad (31)$$

and the Inequality (8) holds.

For convenience we denote

$$Z(t) = \begin{bmatrix} z_{11}(t) & z_{12}(t) \\ z_{21}(t) & z_{22}(t) \end{bmatrix}, \quad (32)$$

and

$$M(t) = \begin{bmatrix} m_{11}(t) & m_{12}(t) \\ m_{21}(t) & m_{22}(t) \end{bmatrix}. \quad (33)$$

With the above sets of parameter, we can obtain

$$\begin{cases} K(t) = \begin{bmatrix} -z_{11}(t) & -z_{21}(t) \\ -z_{12}(t) & -z_{22}(t) \end{bmatrix}, \\ W(t) = \begin{bmatrix} w_{11}(t) & w_{12}(t) \\ w_{21}(t) & w_{22}(t) \end{bmatrix}, \end{cases} \quad (34)$$

where

$$\begin{cases} w_{11}(t) = -3z_{11}(t) + 2z_{21}(t), \\ w_{12}(t) = -3z_{21}(t) + a_1(t)z_{11}(t) + a_2(t)z_{21}(t), \\ w_{21}(t) = -2z_{12}(t) + 2z_{22}(t), \\ w_{22}(t) = -2z_{22}(t) + a_1(t)z_{12}(t) + a_2(t)z_{22}(t), \end{cases} \quad (35)$$

and Constraint 1 can be expressed as

$$\begin{cases} m_{11}(t)z_{21}(t) + m_{12}(t)z_{22}(t) = 0, \\ 1 + m_{21}(t)z_{21}(t) + m_{22}(t)z_{22}(t) = 0, \\ a_1(t)z_{11}(t) - 3z_{21}(t) + a_2(t)z_{21}(t) - \dot{z_{21}}(t) = 0, \\ a_1(t)z_{12}(t) - 2z_{22}(t) + a_2(t)z_{22}(t) - \dot{z_{22}}(t) = 0, \end{cases} \quad (36)$$

where $z_{mn}(t), m = 1, 2, \ n = 1, 2$ are a group of arbitrary parameters.

According to Equations (18), (35) and (36), we have

$$\begin{cases} N(t) = \begin{bmatrix} 1 + m_{11}(t)z_{11}(t) + m_{12}(t)z_{12}(t) \\ m_{21}(t)z_{11}(t) + m_{22}(t)z_{12}(t) \end{bmatrix}, \\ L(t) = \begin{bmatrix} w_{11}(t) - \dot{z}_{11}(t) \\ w_{21}(t) - \dot{z}_{12}(t) \end{bmatrix}, \\ H(t) = \begin{bmatrix} -b_1(t)z_{11}(t) - b_2(t)z_{21}(t) \\ -b_1(t)z_{12}(t) - b_2(t)z_{22}(t) \end{bmatrix}. \end{cases}$$

Based on Equation (28), we have

$$L_d(t) = \begin{bmatrix} (0.3 - a_3)z_{11} - a_4(t)z_{21}(t) \\ -a_3(t)z_{12}(t) - a_4(t)z_{22}(t) \end{bmatrix}.$$

Further, specially choosing $z_{11}(t) = 0$, $z_{12}(t) = 0$, $z_{21}(t)$ and $z_{22}(t)$ can be obtained by Equation (36) as

$$z_{21}(t) = e^{-\frac{t^2}{20}}, z_{22}(t) = e^{t-\frac{t^2}{20}},$$

that is,

$$Z(t) = \begin{bmatrix} 0 & 0 \\ e^{-\frac{t^2}{20}} & e^{t-\frac{t^2}{20}} \end{bmatrix}, \quad (37)$$

thus, we can lead

$$\begin{cases} K(t) = \begin{bmatrix} 0 & -e^{-\frac{t^2}{20}} \\ 0 & -e^{t-\frac{t^2}{20}} \end{bmatrix}, L(t) = \begin{bmatrix} 2e^{-\frac{t^2}{20}} \\ 2e^{t-\frac{t^2}{20}} \end{bmatrix}, \\ N(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, W(t) = \begin{bmatrix} 2e^{-\frac{t^2}{20}} & (-0.1t)e^{-\frac{t^2}{20}} \\ 2e^{t-\frac{t^2}{20}} & (1-0.1t)e^{t-\frac{t^2}{20}} \end{bmatrix}, \\ H(t) = \begin{bmatrix} -(t+2)e^{-\frac{t^2}{20}} \\ -(t+2)e^{t-\frac{t^2}{20}} \end{bmatrix}. \end{cases}$$

and based on Equation (28), we have

$$L_d(t) = \begin{bmatrix} -(0.1t - 0.2)e^{-\frac{t^2}{20}} \\ -(0.1t - 0.2)e^{t-\frac{t^2}{20}} \end{bmatrix}.$$

B. NUMERICAL SIMULATION AND COMPARISON

The initial conditions are

$$\begin{cases} x(0) = \begin{bmatrix} 2 & -6 \end{bmatrix}^{\mathrm{T}}, \\ z(0) = \begin{bmatrix} 5 & 8 \end{bmatrix}^{\mathrm{T}}, \end{cases}$$

and $\tau(t) = 0.5, \forall t \in \mathbb{R}^+$, the control input $u(t)$ is given as

$$u(t) = \begin{cases} 0, \ t \leq 3, \\ \sin(10t), \ t > 3, \end{cases}$$

meanwhile other related parameters are given in Table 1. The simulation results are plotted in Figures 1–5.

TABLE 1. Parameters in the system.

Parameters	Values	Parameters	Values
$a_1(t)$	$-1-0.2t$	$m_{11}(t)$	$e^{t-\frac{t^2}{20}}$
$a_2(t)$	$3-0.1t$	$m_{12}(t)$	$-e^{-\frac{t^2}{20}}$
$a_3(t)$	$0.2t$	$m_{21}(t)$	$-e^{\frac{t^2}{20}}$
$a_4(t)$	$-0.2+0.1t$	$m_{22}(t)$	0
$b_1(t)$	$1+t$		
$b_2(t)$	$2+t$		

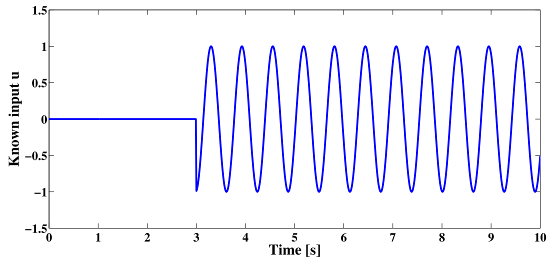

FIGURE 1. The variation diagram of control input *u*(*t*).

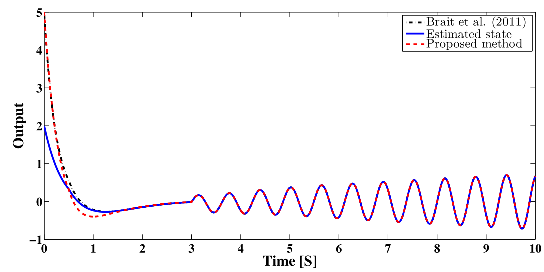

FIGURE 2. Comparison of the outputs among estimated state, the proposed method and Brait et al. (2011).

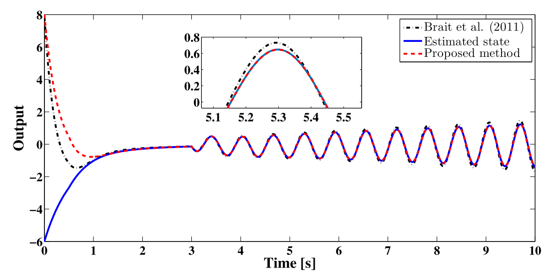

FIGURE 3. Comparison of the outputs among estimated state, the proposed method and Brait et al. (2011).

Figure 1 shows the variation diagram of control input u. The Luenberger functional observer output and the linear function state are depicted in Figures 2 and 3, we can see that, the Luenberger functional observer estimates the linear function state well. Figures 4 and 5 show the corresponding estimation error, we can see that the estimation error approaches to zero quickly. The simulation results show the effectiveness of the design approach.

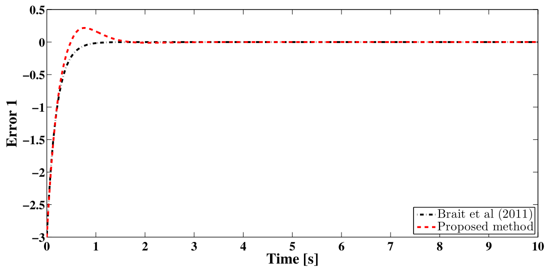

FIGURE 4. Estimation Error 1.

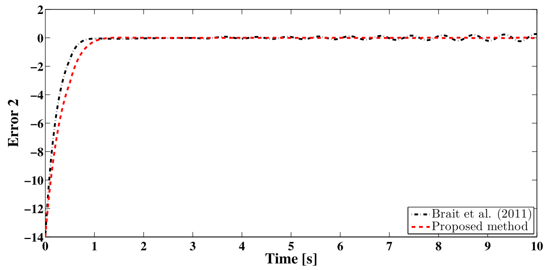

FIGURE 5. Estimation Error 2.

Compared with the outputs in [34], we can clearly see that the estimation of the parametric method is obviously more accurate from the above figures.

V. CONCLUSION

In this paper, we present a parametric approach to design Luenberger functional observer for LTV systems with time-delay. This approach provides the completely parameterized expression of Luenberger functional observer. With the proposed method, a group of arbitrary parameters can be obtained to provide the degrees of freedom. The Luenberger functional observer can accurately track the linear functional at an arbitrary desired convergence rate with selectable degrees of freedom. A numerical example has been offered to illustrate the effectiveness of the proposed approach.

The next major work is to extend the proposed approach for state delay and input delay systems based on observer-predictors.

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