Received December 4, 2019, accepted January 9, 2020, date of publication January 14, 2020, date of current version January 27, 2020.

*Digital Object Identifier 10.1109/ACCESS.2020.2966525*

# Robust Synchronization for Discrete-Time Coupled Markovian Jumping Neural Networks With Mixed Time-Delays

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This work was supported in part by the National Natural Science Foundation of China under Grant 61702356, Grant 61672124, Grant 61370145, and Grant 61503375, and in part by the Password Theory Project of the 13th Five-Year Plan National Cryptography Development Fund under Grant MMJJ20170203.

**ABSTRACT** This paper concerns the robust synchronization problems for discrete-time coupled neural networks with discrete time delay and distributed time delays. Inner parameters in individual neural network are subject to be uncertain and both coupled matrixes and weight matrixes are supposed to switch from one mode to another because of the markovian jumping chain. Mixed time delays contain discrete and distributed time delays and the mixed time delays not only exist in the individual neural cell, but also exist in the coupled cells. By using the novel Lyapunov-Krasovskii functional method and Kronecker product as tools, mean square stability conditions are provided in terms of linear matrix inequalities. In numerical simulations, two examples (with and without unknown parameters) are given and simulation results show the robustness and effectiveness of our methods.

**INDEX TERMS** Discrete-time coupled neural networks, Markovian jumping chain, mixed time delays, linear matrix inequality, Lyapunov-Krasovskii functional method.

## I. INTRODUCTION

In the past decades, dynamical neural networks have been widely applied in a variety of areas, such as signal processing, image processing, pattern recognition, combinatorial optimization problems and so on (see, for instance [1]–[5]). In the study of such kind of dynamical neural networks, complicated dynamics (e.g. chaos, which has been deeply studied in low dimensional system and single system) attract researchers in recent years. Especially, since Pecora and Carroll achieved synchronization between two chaotic oscilators by PC method and proposed the concept of chaotic synchronization for the first time [6], synchronization, as an effective way, has attracted people's attention in the research of chaotic systems, coupled spatiotemporal chaotic systems, dynamical neural networks and complex dynamical networks (see [7]–[12] and references therein).

It is worth pointing out that information latching problems commonly exist in neural networks and can be handled by extracting finite-state patterns [13]. Markovian jumping chain is a finite state set and can govern the switching between different modes. So for a class of neural networks with finite states, markovian jumping chain is an effective tool to deal with the mode switching problems. Recently, dynamical properties with markovian jumping chain have been applied into the research of dynamical recurrent neural networks, complex dynamical networks and other complicated dynamical networks [14]–[18].

On the other hand, because of the finite speed of information transmission and traffic jam in networks, time delays commonly exists in the dynamical networks. Thus, the study of dynamic properties with time delay is of great significance and importance. Time delays in the neural dynamical networks can be generally divided into discrete time delay and distributed time delay. Compared with the study of distributed time delay, behaviors with discrete time delay in the neural networks are widely studied in the past few years and a lot of sufficient conditions to make systems convergence are achieved. However, due to the parallel pathways of a number of lengths and axon sizes in networks [19], distributed

The associate editor coordinating the review of this manuscript and approving it for publication was Jin-Liang Wang.

time delays attract more and more researchers' attentions. Recently, dynamic behaviors with mixed time delays (discrete time delay and distributed time delay) attract people's initial interests [20]–[22]. Compared with continuous network, discrete-time neural network has more applications in digital field and attract people's attentions [23]–[28]. However, the aforementioned discussions about the discrete neural network are not universal because mixed time delays exist not only in the inner neural cells but also in the outer coupled connections. In addition, parameters in real network are always unknown or uncertain and the neural network topology is not constant. This means practical discrete neural network should be established with mixed delays, unknown parameters and unfixed topology. The neural networks model in our study comes from the novel continuous coupled neural networks proposed by Zhang *et al.* [29] and it is convinced that this novel coupled neural network is an ideal model.

Although some sufficient conditions for stability problems of discrete neural networks have been derived by some researchers, as far as we know, there has been no literatures investigate the synchronization problem for discrete-time coupled Markovian jumping neural networks with unknown parameters and mixed time-delays both in the inner neural cell and in the outer coupled neural cells. Motivated by above discussions, this paper considers the stability analysis and robust synchronization problems for a class of discrete-time coupled Markovian jumping neural networks with mixed time-delays. Contributions can be listed as follows: 1. Mixed time delays not only exist in the individual neural cell, but also exist in the coupled links between different cells. 2. Parameters in the individual neural cell are subject to unknown. 3. System parameters of the discrete coupled neural networks are switching according to the Markov jumping chain.

The rest paper is organized as follows. Section 2 introduces the basic models, preliminaries and lemmas. In section 3, stability analysis and sufficient conditions with LMI (Linear matrix inequality) are presented. Section 4 gives some numerical simulations and examples to show the robustness and effectiveness of our methods. Finally, some concluding remarks are given in section 5.

*Notations*: Throughout the paper, $R^n$ represents the $n$-dimensional Euclidean space. $R^{n\times m}$ is the set of $n\times m$ real matrices. $T$ means the transpose of the corresponding matrix and the symmetric matrix $X\geq 0$ (respectively, $X>0$) means that $X$ is positive semidefinite (respectively, positive definite). $I$ denotes the identity matrix. $A\otimes B$ stands for the Kronecker product of matrices $A$and $B$; $diag\{\cdots\}$ represents a block-diagonal matrix and $*$ is used to represent a term induced by symmetry. $E[x]$ represents the expectation of $x$ and $E[y|x]$ means the expectation of $y$ on condition $x$. If not explicitly stated, matrices dimensions are assumed to be compatible for algebraic operations.

## II. THE SYSTEM MODEL AND PRELIMINARIES

In this paper, based on the structure of coupled neural networks with mixed time-delays presented in [29], we consider the following neural networks consisting of $N$coupled nodes with mixed time delays:

$$\begin{aligned} x_i(k+1) = &-\tilde{J}_{r(k)}x_i(k)+\tilde{A}_{r(k)}f(x_i(k)) \\ &+\tilde{B}_{r(k)}h(x_i(k-\tau_{1,r(k)})) \\ &+\tilde{C}_{r(k)}\sum_{v=1}^{\tau_{2,r(k)}}o(x_j(k-v)) \\ &+\sum_{j=1}^{N}G_{ij}^{(1)}D_{r(k)}^{(1)}x_j(k) \\ &+\sum_{j=1}^{N}G_{ij}^{(2)}D_{r(k)}^{(2)}x_j(k-\tau_{1,r(k)}) \\ &+\sum_{j=1}^{N}G_{ij}^{(3)}D_{r(k)}^{(3)}\sum_{v=1}^{\tau_{2,r(k)}}x_j(k-v), \end{aligned} \tag{1}$$

where nonlinear functions are

$$\begin{aligned} f(x_i(k)) &= (f_1(x_{i1}(k)),f_2(x_{i2}(k)),\cdots,f_n(x_{in}(k)))^T, \\ h(x_i(k)) &= (h_1(x_{i1}(k)),h_2(x_{i2}(k)),\cdots,h_n(x_{in}(k)))^T, \\ o(x_i(k)) &= (o_1(x_{i1}(k)),o_2(x_{i2}(k)),\cdots,o_n(x_{in}(k)))^T, \end{aligned}$$

$x_i(k)=(x_{i1}(k),x_{i2}(k),\cdots,x_{in}(k))^T\in R^n, i=1,2,\cdots,W$ is the state vector of $ith$ neural cell at $kth$ iteration. $W$is the number of neural cells and $\tilde{J}_{r(k)}$ is the unknown state feedback diagonal matrix. $\tilde{A}_{r(k)},\tilde{B}_{r(k)},\tilde{C}_{r(k)}\in R^{n\times n}$ are unknown connection weight matrices in mode $r(k)$. $r(k)(k\geq 0)$ is a discrete Markovian process and take values in the finite state set $S=\{1,2,\cdots,N\}$ with probability transition matrix $\Pi=(\pi_{ab})_{N\times N}$ given by

$$\Pr\{r(k+1)=b|r(k)=a\}=\begin{cases}\pi_{ab}, & a\neq b\\ 1+\pi_{ab}, & a=b,\end{cases}$$

where $\pi_{ab}\geq 0(a,b\in S)$ is the transition probability from mode $a$to mode $b$ and

$$\pi_{aa}=-\sum_{b=1,b\neq a}^{N}\pi_{ab},\quad \widehat{\pi}=min\{\pi_{aa}|a\in S\}.$$

For convenience, we set $r(k)=m$. $G^{(\iota)}=(G_{ij}^{(\iota)})_{W\times W},\iota=1,2,3$ represents outer coupling matrices between neural cells and satisfies zero row sum and symmetrical, that is to say

$$G_{ii}^{(\iota)}=-\sum_{j=1,j\neq i}^{N}G_{ij}^{(\iota)},\quad i,j=1,2,\cdots,M$$

and $G_{ij}^{(\iota)}=G_{ji}^{(\iota)}\geq 0, i\neq j$. $D_m^{(\iota)}\in R^{n\times n}$ is the inner coupling matrix in mode $r(k)$. $\tau_{1,m}$ represents the discrete time delay and $\tau_{2,m}$ is the distributed time delay in mode $m$. The mixed time delays satisfy

$$\widehat{\tau}_1\leq\tau_{1,m}\leq\breve{\tau}_1,\quad \widehat{\tau}_2\leq\tau_{2,m}\leq\breve{\tau}_2,$$

where $\widehat{\tau}_1$, $\breve{\tau}_1$, $\widehat{\tau}_2$ and $\breve{\tau}_2$ are known positive integers. In mode $m$, the individual unknown parameters are represented as $\tilde{J}_m=J_m+\Delta J_m,\tilde{A}_m=A_m+\Delta A_m,\tilde{B}_m=B_m+\Delta B_m$

and $\tilde{C}_m = C_m + \Delta C_m$, where $J_m$, $A_m$, $B_m$ and $C_m$ are certain matrices. Parameter uncertainties can be expressed as

$$[\Delta J_m \ \Delta A_m \ \Delta B_m \ \Delta C_m \ ] = \ M_m \Upsilon_m [E_m^J \, E_m^A \, E_m^B \, E_m^C \ ],$$

where $M_m$, $E_m^J$, $E_m^A$, $E_m^B$ and $E_m^C$ are known constant matrices in mode $m$ and $\Upsilon_m$ is unknown diagonal matrix which satisfies $\Upsilon_m^T \Upsilon_m \leq I$.

For convenience, we set

$$\begin{aligned} F(x(k)) &= (f^T(x_1(k)), f^T(x_2(k)), \cdots, f^T(x_W(k)))^T, \\ H(x(k)) &= (h^T(x_1(k)), h^T(x_2(k)), \cdots, h^T(x_W(k)))^T, \\ O(x(k)) &= (o^T(x_1(k)), o^T(x_2(k)), \cdots, o^T(x_W(k)))^T, \end{aligned}$$

and use the Kronecker product to rewrite the system (1) in mode $a$ as

$$\begin{aligned} & x_i(k+1) \\ &= -(I_W \otimes \tilde{J}_a)x_i(k) + (I_W \otimes \tilde{A}_a) f(x_i(k)) \\ &\quad + (I_W \otimes \tilde{B}_a) h(x_i(k-\tau_{1,a})) + (I_W \otimes \tilde{C}_a) \sum_{v=1}^{\tau_{2,a}} o(x_i(k-v)) \\ &\quad + (G^{(1)} \otimes D_a^{(1)}) x_j(k) + (G^{(2)} \otimes D_a^{(2)}) x_j(k-\tau_{1,a}) \\ &\quad + (G^{(3)} \otimes D_a^{(3)}) \sum_{v=1}^{\tau_{2,a}} x_j(k-v) \end{aligned} \tag{2}$$

where $I_W$ is a $W \times W$ identity matrix.

In this paper, we set the following assumptions, lemmas and definitions.

*Assumption:* For above neural networks (2), $F(\cdot)$, $H(\cdot)$ and $O(\cdot)$ are bounded activation function and satisfies $F(0) = H(0) = O(0) = 0$, there exists constants $\widehat{\varsigma}_i$,$\breve{\varsigma}_i$, $\widehat{\varphi}_i$,$\breve{\varphi}_i$$\widehat{\phi}_i$,$\breve{\phi}_i$ such that

$$\begin{aligned} \widehat{\varsigma}_i &\leq \frac{f_i(\alpha) - f_i(\beta)}{\alpha - \beta} \leq \breve{\varsigma}_i, \\ \widehat{\varphi}_i &\leq \frac{h_i(\alpha) - h_i(\beta)}{\alpha - \beta} \leq \breve{\varphi}_i, \\ \widehat{\phi}_i &\leq \frac{\sum\limits_j o_i(\alpha_j) - \sum\limits_j o_i(\beta_j)}{\sum\limits_j (\alpha_j - \beta_j)} \leq \breve{\phi}_i. \end{aligned}$$

For convenience, we set

$$\begin{aligned} \Re_1 &= diag(\widehat{\varsigma}_1 \breve{\varsigma}_1, \widehat{\varsigma}_2 \breve{\varsigma}_2, \cdots, \widehat{\varsigma}_n \breve{\varsigma}_n), \\ \Re_2 &= diag((\widehat{\varsigma}_1 + \breve{\varsigma}_1)\big/2, (\widehat{\varsigma}_2 + \breve{\varsigma}_2)\big/2, \cdots, (\widehat{\varsigma}_n + \breve{\varsigma}_n)\big/2), \\ \aleph_1 &= diag(\widehat{\varphi}_1 \breve{\varphi}_1, \widehat{\varphi}_2 \breve{\varphi}_2, \cdots, \widehat{\varphi}_n \breve{\varphi}_n), \\ \aleph_2 &= diag((\widehat{\varphi}_1 + \breve{\varphi}_1)\big/2 (\widehat{\varphi}_2 + \breve{\varphi}_2)\big/2, \cdots, (\widehat{\varphi}_n + \breve{\varphi}_n)\big/2), \\ \Im_1 &= diag(\widehat{\phi}_1 \breve{\phi}_1, \widehat{\phi}_2 \breve{\phi}_2, \cdots, \widehat{\phi}_n \breve{\phi}_n), \\ \Im_2 &= diag((\widehat{\phi}_1 + \breve{\phi}_1)\big/2 (\widehat{\phi}_2 + \breve{\phi}_2)\big/2, \cdots, (\widehat{\phi}_n + \breve{\phi}_n)\big/2). \end{aligned}$$

*Lemma 1:* Vectors $X$ and $Y$ are in $R^n$, and positive semidefinite matrix $P \in R^{n \times n} (P^T = P, P \geq 0)$. Then, the following matrix inequality holds:

$$2X^T PY \leq X^T PX + Y^T PY.$$

*Lemma 2 (Schur Complement):* Given the following matrix

$$\Sigma = \begin{bmatrix} \Sigma_1 & \Sigma_3^T \\ \Sigma_3 & -\Sigma_2 \end{bmatrix},$$

where $\Sigma_1$is a non-singular matrix and $\Sigma_1 = \Sigma_1^T$, $\Sigma_2 > 0$and $\Sigma_3$ is a constant matrix, then we say $\Sigma_1 + \Sigma_3^T \Sigma_2^{-1} \Sigma_3$ is the schur complement of $\Sigma$ about $\Sigma_1$ and we have the following conclusion:

$$\Sigma_1 + \Sigma_3^T \Sigma_2^{-1} \Sigma_3 < 0,$$

holds if and only if the following schur complement holds

$$\Sigma < 0.$$

*Lemma 3:* Let $\otimes$ be the Kronecker product, then we have the following conclusions:

(1) $(\alpha A) \otimes B = A \otimes (\alpha B)$,
(2) $A \otimes (B + C) = A \otimes B + A \otimes C$,
(3) $(A \otimes B)^T = A^T \otimes B^T$,
(4) $(A + B) \otimes (C + D) = (AC) \otimes (BD)$.

*Lemma 4:* For above coupled neural networks (2), $\wp = diag(\eta_1, \eta_2, \cdots, \eta_n)$ is a positive semidefinite diagonal matrix, the *ith* neural cell is $x_i = (x_{i1}, x_{i2}, \cdots, x_{in})^T \in R^n$, $1 \leq i \leq W$, and $v(x_i) = (v_1(x_{i1}), v_2(x_{i2}), \cdots, v_n(x_{in}))^T \in R^n$ are continuous functions satisfying aforementioned *assumption* ($\widehat{l}_u \leq \frac{v_u(\alpha) - v_u(\beta)}{\alpha - \beta} \leq \breve{l}_u$),$1 \leq u \leq n$, one has

$$\begin{bmatrix} x_i - x_j \\ v(x_i) - v(x_j) \end{bmatrix}^T \times \begin{bmatrix} \wp L_1 & -\wp L_2 \\ -\wp L_2 & \wp \end{bmatrix} \times \begin{bmatrix} x_i - x_j \\ v(x_i) - v(x_j) \end{bmatrix} \leq 0,$$

$1 \leq i \leq j \leq W$, where $L_1 = diag(\widehat{l}_1 \breve{l}_1, \widehat{l}_2 \breve{l}_2, \cdots, \widehat{l}_n \breve{l}_n)$, $L_2 = diag((\widehat{l}_1 + \breve{l}_1)\big/2 (\widehat{l}_2 + \breve{l}_2)\big/2, \cdots, (\widehat{l}_n + \breve{l}_n)\big/2)$.

*Proof:* From *Assumption*, we can get

$$\left[ v_u(x_{iu}) - v_u(x_{ju}) - \breve{l}_u (x_{iu} - x_{ju}) \right] \times \left[ v_u(x_{iu}) - v_u(x_{ju}) - \widehat{l}_u (x_{iu} - x_{ju}) \right] \leq 0,$$

that is to say

$$\begin{bmatrix} x_i - x_j \\ v(x_i) - v(x_j) \end{bmatrix}^T \times \begin{bmatrix} \breve{l}_u \widehat{l}_u e_u e_u^T & -\frac{\breve{l}_u + \widehat{l}_u}{2} e_u e_u^T \\ -\frac{\breve{l}_u + \widehat{l}_u}{2} e_u e_u^T & e_u e_u^T \end{bmatrix} \times \begin{bmatrix} x_i - x_j \\ v(x_i) - v(x_j) \end{bmatrix} \leq 0$$

where $e_u$ denotes the unit column vector where the items in *uth* row are all 1 and other items are all zeros. Because $\wp$ is positive semidefinite, we can get the following inequality by multiplying both sides by $\sum\limits_{u=1}^{n} \eta_u$,

$$\begin{bmatrix} x_i - x_j \\ v(x_i) - v(x_j) \end{bmatrix}^T \times \begin{bmatrix} \wp L_1 & -\wp L_2 \\ -\wp L_2 & \wp \end{bmatrix} \times \begin{bmatrix} x_i - x_j \\ v(x_i) - v(x_j) \end{bmatrix} \leq 0.$$

This completes the proof of **Lemma 4**. ■

*Lemma 5 [29]:* Consider a matrix defined as

$$U = \begin{bmatrix} W-1 & -1 & \cdots & -1 \\ -1 & W-1 & \cdots & -1 \\ \cdots & \cdots & \cdots & \cdots \\ -1 & \cdots & -1 & W-1 \end{bmatrix}_{W\times W},$$

$P \in R^{n\times n}$, $x = (x_1^T, x_2^T, \cdots, x_W^T)$ and $y = (y_1^T, y_2^T, \cdots, y_W^T)$. Then we have

$$x^T(U\otimes P)y = \sum_{1\le u\le v\le W} (x_i - x_j)^T P(y_i - y_j), \quad 1 \le i \le j \le W.$$

*Lemma 6 [24]:* Consider a symmetric positive-semidefinite matrix $\Psi \in R^{n\times n}$ (that is to say, $\Psi^T = \Psi > 0$ ), scalar $a_i \ge 0 (i = 1, 2, \cdots)$ and vector $x_i \in R^n$. We can get the following inequality

$$(\sum_{i=1}^{+\infty} a_i x_i)^T \Psi (\sum_{i=1}^{+\infty} a_i x_i) \le (\sum_{i=1}^{+\infty} a_i) \sum_{i=1}^{+\infty} a_i x_i^T \Psi x_i.$$

## III. STABILITY ANALYSIS AND MAIN RESULTS

In this section, we will deal with the synchronization problem of the aforementioned coupled neural networks (2). First, we will give the main result in this paper as follows.

*Theorem* Under the aforementioned *Assumption*, in mode $a$, the dynamical neural networks (2) will be robustly synchronized in the mean square if there exist positive matrices $P_a > 0$, $Q > 0$ and $R > 0$, three diagonal matrices $\Psi > 0$, $\Xi > 0$ and $\Omega > 0$, and scalar $\lambda > 0$ such that the following LMI holds for all $1 \le i \le j \le W$.

$$\Phi_{ij} = \begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & \Pi_{14} & \Pi_{15} & \Pi_{16} & -J_a^T\bar{P}_a & 0 \\ * & \Pi_{22} & \Pi_{23} & 0 & \Pi_{25} & 0 & 0 & 0 \\ * & * & \Pi_{33} & 0 & 0 & \Pi_{36} & 0 & 0 \\ * & * & * & \Pi_{44} & \Pi_{45} & \Pi_{46} & A_a^T\bar{P}_a & 0 \\ * & * & * & * & \Pi_{55} & \Pi_{56} & B_a^T\bar{P}_a & 0 \\ * & * & * & * & * & \Pi_{66} & C_a^T\bar{P}_a & 0 \\ * & * & * & * & * & * & \Pi_{77} & \bar{P}_a M_a \\ * & * & * & * & * & * & 0 & -\lambda I \end{bmatrix} \quad (3)$$

where

$$r(k) = a,$$

$$\Pi_{11} = W(G_{ij}^{(1)} - G_{ij}^{(1)}G_{ij}^{(1)})D_a^{(1)^T}\bar{P}_a D_a^{(1)} - P_a + \varpi Q + \sigma_a R - \Psi\Re_1 + \lambda(E_a^J)^T E_a^J$$

$$\Pi_{22} = W(G_{ij}^{(2)} - G_{ij}^{(2)}G_{ij}^{(2)})D_a^{(2)^T}\bar{P}_a D_a^{(2)} - Q - \Xi\aleph_1,$$

$$\Pi_{33} = W(G_{ij}^{(3)} - G_{ij}^{(3)}G_{ij}^{(3)})D_a^{(3)^T}\bar{P}_a D_a^{(3)} - R/\tau_{2,a} - \Omega\Im_1,$$

$$\Pi_{44} = -\Psi + \lambda(E_a^A)^T E_a^A, \quad \Pi_{55} = -\Xi + \lambda(E_a^B)^T E_a^B,$$

$$\Pi_{66} = -\Omega + \lambda(E_a^C)^T E_a^C,$$

$$\Pi_{77} = -(1 + WG_{ij}^{(1)} + WG_{ij}^{(2)} + WG_{ij}^{(3)})^{-1}\bar{P}_a,$$

$$\Pi_{12} = -WG_{ij}^{(1)}G_{ij}^{(2)}D_a^{(1)^T}\bar{P}_a D_a^{(2)},$$

$$\Pi_{13} = -WG_{ij}^{(1)}G_{ij}^{(3)}D_a^{(1)^T}\bar{P}_a D_a^{(3)}, \Pi_{14} = \Psi\Re_2 - \lambda(E_a^J)^T E_a^A,$$

$$\Pi_{15} = -\lambda(E_a^J)^T E_a^B, \quad \Pi_{16} = -\lambda(E_a^J)^T E_a^C,$$

$$\Pi_{23} = -WG_{ij}^{(2)}G_{ij}^{(3)}D_a^{(2)^T}\bar{P}_a D_a^{(3)}, \Pi_{25} = \Xi\aleph_2, \Pi_{36} = \Omega\Im_2,$$

$$\Pi_{45} = \lambda(E_a^A)^T E_a^B, \Pi_{46} = \lambda(E_a^A)^T E_a^C, \Pi_{56} = \lambda(E_a^B)^T E_a^C.$$

$$\varpi = (1 - \widehat{\pi})(\breve{\tau} - \widehat{\tau}) + 1, \quad \bar{P}_a = \sum_{b=1}^{N} \pi_{ab} P_b$$

$$\sigma_a = \tau_{2,a} + (1 - \pi_{ii})(\breve{\tau}_2 - \widehat{\tau}_2) + \frac{1}{2}(1 - \widehat{\pi})(\breve{\tau}_2 - \widehat{\tau}_2)(\breve{\tau}_2 + \widehat{\tau}_2 - 1),$$

*Proof:* In order to deal with the synchronization problem of the neural networks (2), we introduce the following Lyapunov-Krasovskii functional:

$$V(k, a) = V_1(k, a) + V_2(k, a) + V_3(k, a) + V_4(k, a) + V_5(k, a), \quad (4)$$

where

$$V_1(k, a) = x^T(k)(U \otimes P_a)x(k), \quad (5)$$

$$V_2(k, a) = \sum_{v=k-\tau_{1,a}}^{k-1} x^T(v)(U \otimes Q)x(v), \quad (6)$$

$$V_3(k, a) = (1 - \widehat{\pi}) \sum_{\rho=\widehat{\tau}_{1,a}}^{\breve{\tau}_{1,a}-1} \sum_{v=k-\rho}^{k-1} x^T(v)(U \otimes Q)x(v), \quad (7)$$

$$V_4(k, a) = \sum_{\rho=1}^{\tau_{2,a}} \sum_{v=k-\rho}^{k-1} x^T(v)(U \otimes R)x(v), \quad (8)$$

$$V_5(k, a) = (1 - \widehat{\pi}) \sum_{\gamma=\widehat{\tau}_{2,a}+1}^{\breve{\tau}_{2,a}} \sum_{\rho=1}^{\gamma-1} \sum_{v=k-\rho}^{k-1} x^T(v)(U \otimes R)x(v), \quad (9)$$

and the matrix $U$ is defined in the **Lemma 5**. By taking the mathematical expectation, we can get the difference of $V(k, a)$ along the solutions of system (2)

$$\begin{aligned} & E[V(k+1, r(k+1) \\ & = b|r(k) = a) - V(k, a)] \\ & = E[V_1(k+1, b|a) - V_1(k, a)] + E[V_2(k+1, b|a) - V_2(k, a)] \\ & \quad + E[V_3(k+1, b|a) - V_3(k, a)] \\ & \quad + E[V_4(k+1, b|a) - V_4(k, a)] \\ & \quad + E[V_5(k+1, b|a) - V_5(k, a)], \end{aligned} \quad (10)$$

where

$$\begin{aligned} & E[V_1(k+1, b|a) - V_1(k, r(k))] \\ & = \sum_{b=1}^{N} \pi_{ab} x^T(k+1)(U \otimes P_b)x(k+1) - x^T(k)(U \otimes P_a)x(k) \\ & = \big[ -(I_W \otimes \tilde{J}_a)x_i(k) + (I_W \otimes \tilde{A}_a)f(x_i(k)) + (I_W \otimes \tilde{B}_a) \\ & \quad \times h(x_i(k - \tau_{1,a})) + (I_W \otimes \tilde{C}_a) \sum_{v=1}^{\tau_{2,a}} o(x_i(k-v)) \end{aligned}$$

$$
\begin{aligned}
&+(G^{(1)}\otimes D_a^{(1)})x_j(k)+(G^{(2)}\otimes D_a^{(2)})x_j(k-\tau_{1,r(k)})\\
&+(G^{(3)}\otimes D_a^{(3)})\sum_{v=1}^{\tau_{2,a}}x_j(k-v)]^T\\
&\times(U\otimes\bar{P}_a)\big[-(I_W\otimes\tilde{J}_a)x_i(k)+(I_W\otimes\tilde{A}_a)f(x_i(k))\\
&+(I_W\otimes\tilde{B}_a)h(x_i(k-\tau_{1,a}))\\
&+(I_W\otimes\tilde{C}_a)\sum_{v=1}^{\tau_{2,a}}o(x_i(k-v))+(G^{(1)}\otimes D_a^{(1)})x_j(k)\\
&+(G^{(2)}\otimes D_a^{(2)})\times x_j(k-\tau_{1,r(k)})\\
&+(G^{(3)}\otimes D_a^{(3)})\sum_{v=1}^{\tau_{2,a}}x_j(k-v)]\\
&-x^T(k)(U\otimes P_a)x(k)\\
=&\ \zeta^T(k,a)\Theta^T(a)\bar{P}_a\Theta(a)\zeta(k,a)-x^T(k)(U\otimes P_a)x(k),
\end{aligned}
\tag{11}
$$

and vector in the mathematical expectation (11) can be represented as

$$
\begin{aligned}
\zeta(k,a)=&\Bigg[x^T(k),x^T(k-\tau_{1,a}),\sum_{v=1}^{\tau_{2,a}}x^T(k-v),f^T(x(k)),\\
&h^T(x(k-\tau_{1,a})),\sum_{v=1}^{\tau_{2,a}}o^T(x_i(k-v))\Bigg].\\
\Theta(a)=&\begin{bmatrix}-\tilde{J}_a & 0 & 0 & \tilde{A}_a & \tilde{B}_a & \tilde{C}_a\end{bmatrix}.
\end{aligned}
$$

And other expectations are listed as:

$$
\begin{aligned}
&E[V_2(k+1,r(k+1)=b|r(k)=a)-V_2(k,r(k))]\\
=&\sum_{b=1}^{N}\pi_{ab}\sum_{v=k+1-\tau_{1,b}}^{k}x^T(v)(U\otimes Q)x(v)\\
&-\sum_{v=k-\tau_{1,a}}^{k-1}x^T(v)(U\otimes Q)x(v)\\
=&\ x^T(k)(U\otimes Q)x(k)+\sum_{b=1}^{N}\pi_{ab}\sum_{v=k+1-\tau_{1,b}}^{k-1}x^T(v)(U\otimes Q)x(v)\\
&-x^T(k-\tau_{1,a})(U\otimes Q)x(k-\tau_{1,a})\\
&-\sum_{v=k+1-\tau_{1,a}}^{k-1}x^T(v)(U\otimes Q)x(v)\\
=&\ x^T(k)(U\otimes Q)x(k)-x^T(k-\tau_{1,a})(U\otimes Q)x(k-\tau_{1,a})\\
&+\sum_{b\neq a}\pi_{ab}[\sum_{v=k+1-\tau_{1,b}}^{k-1}x^T(v)(U\otimes Q)x(v)\\
&-\sum_{v=k+1-\tau_{1,a}}^{k-1}x^T(v)(U\otimes Q)x(v)]\\
\leq&\ x^T(k)(U\otimes Q)x(k)-x^T(k-\tau_{1,a})(U\otimes Q)x(k-\tau_{1,a})\\
&+\sum_{b\neq a}\pi_{ab}\sum_{v=k+1-\breve{\tau}_1}^{k-\widehat{\tau}_1}x^T(v)(U\otimes Q)x(v)\\
\leq&\ x^T(k)(U\otimes Q)x(k)-x^T(k-\tau_{1,a})(U\otimes Q)x(k-\tau_{1,a})\\
&+(1-\widehat{\pi})\sum_{v=k+1-\breve{\tau}_1}^{k-\widehat{\tau}_1}x^T(v)(U\otimes Q)x(v),
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
&E[V_3(k+1,r(k+1)=b|r(k)=a)-V_3(k,r(k))]\\
=&\ (1-\widehat{\pi})[\sum_{\rho=\widehat{\tau}_1}^{\breve{\tau}_1-1}\sum_{v=k+1-\rho}^{k}x^T(v)(U\otimes Q)x(v)\\
&-\sum_{\rho=\widehat{\tau}_1}^{\breve{\tau}_1-1}\sum_{v=k-\rho}^{k-1}x^T(v)(U\otimes Q)x(v)]\\
=&\ (1-\widehat{\pi})\sum_{\rho=\widehat{\tau}_1}^{\breve{\tau}_1-1}[\sum_{v=k+1-\rho}^{k}x^T(v)(U\otimes Q)x(v)\\
&-\sum_{v=k-\rho}^{k-1}x^T(v)(U\otimes Q)x(v)]\\
=&\ (1-\widehat{\pi})\sum_{\rho=\widehat{\tau}_1}^{\breve{\tau}_1-1}[x^T(k)(U\otimes Q)x(k)\\
&-x^T(k-\rho)(U\otimes Q)x(k-\rho)]\\
=&\ (1-\widehat{\pi})(\breve{\tau}_1-\widehat{\tau}_1)x^T(k)(U\otimes Q)x(k)\\
&-(1-\widehat{\pi})\sum_{v=k+1-\breve{\tau}_1}^{k-\widehat{\tau}_1}x^T(v)(U\otimes Q)x(v),
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
&E[V_4(k+1,r(k+1)=b|r(k)=a)-V_4(k,r(k))]\\
=&\sum_{b=1}^{N}\pi_{ab}\sum_{\rho=1}^{\tau_{2,b}}\sum_{v=k+1-\rho}^{k}x^T(v)(U\otimes R)x(v)\\
&-\sum_{\rho=1}^{\tau_{2,a}}\sum_{v=k-\rho}^{k-1}x^T(v)(U\otimes R)x(v)\\
=&\ \pi_{aa}(\sum_{\rho=1}^{\tau_{2,a}}\sum_{v=k+1-\rho}^{k}x^T(v)(U\otimes R)x(v)\\
&-\sum_{\rho=1}^{\tau_{2,a}}\sum_{v=k-\rho}^{k-1}x^T(v)(U\otimes R)x(v))\\
&+\sum_{b\neq a}\pi_{ab}(\sum_{\rho=1}^{\tau_{2,b}}\sum_{v=k+1-\rho}^{k}x^T(v)(U\otimes R)x(v)\\
&-\sum_{\rho=1}^{\tau_{2,a}}\sum_{v=k-\rho}^{k-1}x^T(v)(U\otimes R)x(v))\\
=&\ \pi_{aa}(\sum_{\rho=1}^{\tau_{2,a}}\sum_{v=k+1-\rho}^{k}x^T(v)(U\otimes R)x(v)\\
&-\sum_{\rho=1}^{\tau_{2,a}}\sum_{v=k-\rho}^{k-1}x^T(v)(U\otimes R)x(v))
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{b\neq a}\pi_{ab}(\sum_{\rho=1}^{\tau_{2,a}}\sum_{v=k+1-\rho}^{k}x^T(v)(U\otimes R)x(v)\\
&-\sum_{\rho=1}^{\tau_{2,a}}\sum_{v=k-\rho}^{k-1}x^T(v)(U\otimes R)x(v))\\
&+\sum_{b\neq a}\pi_{ab}(\sum_{\rho=1}^{\tau_{2,b}}\sum_{v=k+1-\rho}^{k}x^T(v)(U\otimes R)x(v)\\
&-\sum_{\rho=1}^{\tau_{2,a}}\sum_{v=k+1-\rho}^{k}x^T(v)(U\otimes R)x(v))\\
\leq &\sum_{\rho=1}^{\tau_{2,a}}(x^T(k)(U\otimes R)x(k)-x^T(k-\rho)(U\otimes R)x(k-\rho))\\
&+\sum_{b\neq a}\pi_{ab}(\sum_{\rho=\widehat{\tau}_2+1}^{\breve{\tau}_2}\sum_{v=k+1-\rho}^{k}x^T(v)(U\otimes R)x(v))\\
=&\ \tau_{2,a}x^T(k)(U\otimes R)x(k)-\sum_{\rho=1}^{\tau_{2,a}}x^T(k-\rho)(U\otimes R)x(k-\rho)\\
&+(1-\pi_{aa})\sum_{\rho=\widehat{\tau}_2+1}^{\breve{\tau}_2}\sum_{v=k+1-\rho}^{k}x^T(v)(U\otimes R)x(v)\\
\leq &\ (\tau_{2,a}+(1-\pi_{aa})(\breve{\tau}_2-\widehat{\tau}_2))x^T(k)(U\otimes R)x(k)\\
&-\sum_{\rho=1}^{\tau_{2,a}}x^T(k-\rho)(U\otimes R)x(k-\rho)\\
&+(1-\widehat{\pi})\sum_{\rho=\widehat{\tau}_2+1}^{\breve{\tau}_2}\sum_{v=k+1-\rho}^{k-1}x^T(v)(U\otimes R)x(v),
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
&E[V_5(k+1,r(k+1)=b|r(k)=a)-V_5(k,r(k))]\\
&=(1-\widehat{\pi})[\sum_{\gamma=\widehat{\tau}_2+1}^{\breve{\tau}_2}\sum_{\rho=1}^{\gamma-1}\sum_{v=k+1-\rho}^{k}x^T(v)(U\otimes R)x(v)\\
&\quad-\sum_{\gamma=\widehat{\tau}_2+1}^{\breve{\tau}_2}\sum_{\rho=1}^{\gamma-1}\sum_{v=k-\rho}^{k-1}x^T(v)(U\otimes R)x(v)]\\
&=(1-\widehat{\pi})\sum_{\gamma=\widehat{\tau}_2+1}^{\breve{\tau}_2}\sum_{\rho=1}^{\gamma-1}(x^T(k)(U\otimes R)x(k)\\
&\quad-x^T(k-\rho)(U\otimes R)x(k-\rho))\\
&=(1-\widehat{\pi})[\frac{1}{2}(\breve{\tau}_2-\widehat{\tau}_2)(\breve{\tau}_2+\widehat{\tau}_2-1)x^T(k)(U\otimes R)x(k)\\
&\quad-\sum_{\rho=\widehat{\tau}_2+1}^{\breve{\tau}_2}\sum_{v=k+1-\rho}^{k-1}x^T(v)(U\otimes R)x(v)].
\end{aligned}
\tag{15}
$$

For convinence, we set $x(k)$ as index 1, $x(k-\tau_{1,a})$ as index 2, $\sum_{v=1}^{\tau_{2,a}}x(k-v)$ as index 3, $f(x(k))$ as index 4, $h(x(k-\tau_{1,a}))$ as index 5 and $\sum_{v=1}^{\tau_{2,a}}o(x_i(k-v))$ as index 6. Substitute expectations (11)-(15) into system (10), one can get the quadratic terms and cross terms listed as follows:

$$
\begin{aligned}
E(V_1)_{11}&=x^T(k)[U\otimes(\tilde{J}_a^T\bar{P}_a\tilde{J}_a)-U\otimes P_a-(WG^{(1)})\\
&\quad\otimes(D_a^{(1)^T}\bar{P}_a\tilde{J}_a)-(WG^{(1)})\otimes(\tilde{J}_a^T\bar{P}_aD_a^{(1)})\\
&\quad+(WG^{(1)}G^{(1)})\otimes(D_a^{(1)^T}\bar{P}_a\tilde{D}_a^{(1)})]x(k),\\
E(V_1)_{22}&=x^T(k-\tau_{1,a})[(WG^{(2)}G^{(2)})\\
&\quad\otimes(D_a^{(2)^T}\bar{P}_aD_a^{(2)})]x(k-\tau_{1,a}),\\
E(V_1)_{33}&=\sum_{v=1}^{\tau_{2,a}}x^T(k-v)[(WG^{(3)}G^{(3)}\\
&\quad\otimes(D_a^{(3)^T}\bar{P}_aD_a^{(3)})]\sum_{v=1}^{\tau_{2,a}}x(k-v),\\
E(V_1)_{44}&=F^T(x(k))[U\otimes(\tilde{A}_a^T\bar{P}_a\tilde{A}_a)]F(x(k))\\
E(V_1)_{55}&=H^T(x(k-\tau_{1,a}))[U\otimes(\tilde{B}_a^T\bar{P}_a\tilde{B}_a)]H(x(k-\tau_{1,a})),\\
E(V_1)_{66}&=\sum_{v=1}^{\tau_{2,a}}O^T(x(k-v))[U\otimes(\tilde{C}_a^T\bar{P}_a\tilde{C}_a)]\\
&\quad\times\sum_{v=1}^{\tau_{2,a}}O(x(k-v)),\\
E(V_1)_{12}&=2x^T(k)[(WG^{(1)}G^{(2)})\otimes(D_a^{(1)^T}\bar{P}_aD_a^{(2)})\\
&\quad-(WG^{(2)})\otimes(\tilde{J}_a^T\bar{P}_aD_a^{(2)})]x(k-\tau_{1,a}),\\
E(V_1)_{13}&=2x^T(k)[(WG^{(1)}G^{(3)})\otimes(D_a^{(1)^T}\bar{P}_aD_a^{(3)})\\
&\quad-(WG^{(3)})\otimes(\tilde{J}_a^T\bar{P}_aD_a^{(3)})]\sum_{v=1}^{\tau_{2,a}}x(k-v),\\
E(V_1)_{14}&=2x^T(k)[(WG^{(1)})\otimes(D_a^{(1)^T}\bar{P}_a\tilde{A}_a)\\
&\quad-U\otimes(\tilde{J}_a^T\bar{P}_a\tilde{A}_a)]F(x_i(k)),\\
E(V_1)_{15}&=2x^T(k)[(WG^{(1)})\otimes(D_a^{(1)^T}\bar{P}_a\tilde{B}_a)\\
&\quad-U\otimes(\tilde{J}_a^T\bar{P}_a\tilde{B}_a)]H(x_i(k-\tau_{1,a})),\\
E(V_1)_{16}&=2x^T(k)[(WG^{(1)})\otimes(D_a^{(1)^T}\bar{P}_a\tilde{C}_a)\\
&\quad-U\otimes(\tilde{J}_a^T\bar{P}_a\tilde{C}_a)]\sum_{v=1}^{\tau_{2,a}}O(x(k-v)),\\
E(V_1)_{23}&=2x(k-\tau_{1,a})^T[(WG^{(2)}G^{(3)})\otimes(D_a^{(2)^T}\bar{P}_aD_a^{(3)})]\\
&\quad\times\sum_{v=1}^{\tau_{2,a}}x(k-v),\\
E(V_1)_{24}&=2x(k-\tau_{1,a})^T[(WG^{(2)})\otimes(D_a^{(2)^T}\bar{P}_a\tilde{A})]F(x_i(k))\\
E(V_1)_{25}&=2x(k-\tau_{1,a})^T[(WG^{(2)})\\
&\quad\otimes(D_a^{(2)^T}\bar{P}_a\tilde{B})]H(x_i(k-\tau_{1,a})),\\
E(V_1)_{26}&=2x(k-\tau_{1,a})^T[(WG^{(2)})\\
&\quad\otimes(D_a^{(2)^T}\bar{P}_a\tilde{C})]\sum_{v=1}^{\tau_{2,a}}O(x(k-v)),\\
E(V_1)_{34}&=2\sum_{v=1}^{\tau_{2,a}}x(k-v)^T[(WG^{(3)})\otimes(D_a^{(3)^T}\bar{P}_a\tilde{A})]F(x_i(k)),
\end{aligned}
$$

$$E(V_1)_{35} = 2\sum_{v=1}^{\tau_{2,a}} x(k-v)^T[(WG^{(3)}) \otimes (D_a^{(3)^T}\bar{P}_a\tilde{B})]H(x_i(k-\tau_{1,a})),$$

$$E(V_1)_{36} = 2\sum_{v=1}^{\tau_{2,a}} x(k-v)^T[(WG^{(3)}) \otimes (D_a^{(3)^T}\bar{P}_a\tilde{C})]\sum_{v=1}^{\tau_{2,a}} O(x(k-v)),$$

$$E(V_1)_{45} = 2F^T(x(k))[U\otimes(\tilde{A}_a^T\bar{P}_a\tilde{B}_a)]\times H(x_i(k-\tau_{1,a})),$$

$$E(V_1)_{46} = 2F^T(x(k))[U\otimes(\tilde{A}_a^T\bar{P}_a\tilde{C}_a)]\times\sum_{v=1}^{\tau_{2,a}} O(x(k-v)),$$

$$E(V_1)_{56} = 2H^T(x_i(k-\tau_{1,a}))[U \otimes(\tilde{B}_a^T\bar{P}_a\tilde{C}_a)]\sum_{v=1}^{\tau_{2,a}} O(x(k-v)),$$

$$E(V_2,V_3)_{11} = \varpi x^T(k)(U\otimes Q)x(k),$$

$$E(V_2,V_3)_{22} = -x^T(k-\tau_{1,a})(U\otimes Q)x(k-\tau_{1,a}),$$

$$E(V_4,V_5)_{11} = \sigma O^T(x(k))(U\otimes R)O(x(k),$$

$$E(V_4,V_5)_{33} = -\sum_{v=1}^{\tau_{2,a}} x^T(k-v)(U\otimes R)Ox(k-v).$$

where

$$E(V_2,V_3) = E(V_2)+E(V_3)$$
$$E(V_4,V_5) = E(V_4)+E(V_5).$$

Other terms in $E(V_1)$, $E(V_2,V_3)$ and $E(V_4,V_5)$ are zeros.

From Lemma 6 and $E(V_4,V_5)_{33}$, it is easy to get the following inequality

$$E(V_4,V_5)_{33} \le -\frac{1}{\tau_{2,a}}\sum_{v=1}^{\tau_{2,a}} x^T(k-v)(U\otimes R)\sum_{v=1}^{\tau_{2,a}} x(k-v). \tag{16}$$

And based on Lemma5 and system (16), the above nonzero terms are translated into

$$E(V_1)_{11} = \sum_{1\le i\le j\le W}\{(x_i(k)-x_j(k))^T[\tilde{J}_a^T\bar{P}_a\tilde{J}_a + WG_{ij}^{(1)}(D_a^{(1)^T}\bar{P}_a\tilde{J}_a+\tilde{J}_a^T\bar{P}_aD_a^{(1)}) - P_a - WG_{ij}^{(1)}G_{ij}^{(1)}D_a^{(1)^T}\bar{P}_aD_a^{(1)}](x_i(k)-x_j(k))\},$$

$$E(V_1)_{22} = \sum_{1\le i\le j\le W}\{(x_i(k-\tau_{1,a})-x_j(k-\tau_{1,a}))^T \times[-WG_{ij}^{(2)}G_{ij}^{(2)}D_a^{(2)^T}\bar{P}_aD_a^{(2)}](x_i(k-\tau_{1,a}) - x_j(k-\tau_{1,a}))\},$$

$$E(V_1)_{33} = \sum_{1\le i\le j\le W}\{(\sum_{v=1}^{\tau_{2,a}}(x_i(k-v)-x_j(k-v)))^T \times[-WG_{ij}^{(3)}G_{ij}^{(3)}D_a^{(3)^T}\bar{P}_aD_a^{(3)}](\sum_{v=1}^{\tau_{2,a}}(x_i(k-v) - x_j(k-v)))\},$$

$$E(V_1)_{44} = \sum_{1\le i\le j\le W}\{(f(x_i(k))-f(x_j(k)))^T \times[\tilde{A}_a^T\bar{P}_a\tilde{A}_a](f(x_i(k))-f(x_j(k)))\},$$

$$E(V_1)_{55} = \sum_{1\le i\le j\le W}\Big\{(h(x_i(k-\tau_{1,a}))-h(x_j(k-\tau_{1,a})))^T \times[\tilde{B}_a^T\bar{P}_a\tilde{B}_a](h(x_i(k-\tau_{1,a}))-h(x_j(k-\tau_{1,a})))\Big\},$$

$$E(V_1)_{66} = \sum_{1\le i\le j\le W}\{(\sum_{v=1}^{\tau_{2,a}}(o(x_i(k-v))-o(x_j(k-v))))^T \times[\tilde{C}_a^T\bar{P}_a\tilde{C}_a](\sum_{v=1}^{\tau_{2,a}}(o(x_i(k-v))-o(x_j(k-v))))\},$$

$$E(V_1)_{12} = 2\sum_{1\le i\le j\le W}\{(x_i(k)-x_j(k))^T[-WG_{ij}^{(1)}G_{ij}^{(2)} \times D_a^{(1)^T}\bar{P}_aD_a^{(2)} + WG_{ij}^{(2)}\tilde{J}_a^T\bar{P}_aD_a^{(2)}] \times(x_i(k-\tau_{1,a})-x_j(k-\tau_{1,a}))\},$$

$$E(V_1)_{13} = 2\sum_{1\le i\le j\le W}\{(x_i(k)-x_j(k))^T[-WG_{ij}^{(1)}G_{ij}^{(3)}D_a^{(1)^T} \times\bar{P}_aD_a^{(3)} + WG_{ij}^{(3)}\tilde{J}_a^T\bar{P}_aD_a^{(3)}](\sum_{v=1}^{\tau_{2,a}}(x_i(k-v) - x_j(k-v)))\},$$

$$E(V_1)_{14} = 2\sum_{1\le i\le j\le W}\{(x_i(k)-x_j(k))^T[-WG_{ij}^{(1)}D_a^{(1)^T}- \times P_a\tilde{A}_a - \tilde{J}_a^T\bar{P}_a\tilde{A}_a](f(x_i(k))-f(x_j(k)))\},$$

$$E(V_1)_{15} = 2\sum_{1\le i\le j\le W}\{(x_i(k)-x_j(k))^T[-WG_{ij}^{(1)}D_a^{(1)^T} \times\bar{P}_a\tilde{B}_a - \tilde{J}_a^T\bar{P}_a\tilde{B}_a](h(x_i(k-\tau_{1,a})) - h(x_j(k-\tau_{1,a})))\},$$

$$E(V_1)_{16} = 2\sum_{1\le i\le j\le W}\{(x_i(k)-x_j(k))^T[-WG_{ij}^{(1)}D_a^{(1)^T} \times\bar{P}_a\tilde{C}_a - \tilde{J}_a^T\bar{P}_a\tilde{C}_a](\sum_{v=1}^{\tau_{2,a}}(o(x_i(k-v)) - o(x_j(k-v))))\},$$

$$E(V_1)_{23} = 2\sum_{1\le i\le j\le W}\{(x_i(k-\tau_{1,a})-x_j(k-\tau_{1,a}))^T \times[-WG_{ij}^{(2)}G_{ij}^{(3)}D_a^{(2)^T}\bar{P}_aD_a^{(3)}](\sum_{v=1}^{\tau_{2,a}}(x_i(k-v) - x_j(k-v)))\},$$

$$E(V_1)_{24} = 2\sum_{1\le i\le j\le W}\{(x_i(k-\tau_{1,a})-x_j(k-\tau_{1,a}))^T \times[-WG_{ij}^{(2)}\tilde{A}_a^T\bar{P}_aD_a^{(2)}](f(x_i(k))-f(x_j(k)))\},$$

$$E(V_1)_{25} = 2\sum_{1\le i\le j\le W}\{(x_i(k-\tau_{1,a})-x_j(k-\tau_{1,a}))^T \times[-WG_{ij}^{(2)}\tilde{B}_a^T\bar{P}_aD_a^{(2)}](h(x_i(k-\tau_{1,a}))$$

$$
\begin{aligned}
& -h(x_j(k-\tau_{1,a})))\}, \\
E(V_1)_{26} =& 2\sum_{1\le i\le j\le W}\{(x_i(k-\tau_{1,a})-x_j(k-\tau_{1,a}))^T \\
& \times[-WG_{ij}^{(2)}\tilde{C}_a^T\bar{P}_aD_a^{(2)}](\sum_{v=1}^{\tau_{2,a}}(o(x_i(k-v)) \\
& -o(x_j(k-v))))\}, \\
E(V_1)_{34} =& 2\sum_{1\le i\le j\le W}\{(\sum_{v=1}^{\tau_{2,a}}(x_i(k-v)-x_j(k-v)))^T \\
& \times[-WG_{ij}^{(3)}\tilde{A}_a^T\bar{P}_aD_a^{(3)}](f(x_i(k))-f(x_j(k)))\}, \\
E(V_1)_{35} =& 2\sum_{1\le i\le j\le W}\{(\sum_{v=1}^{\tau_{2,a}}(x_i(k-v)-x_j(k-v)))^T \\
& \times[-WG_{ij}^{(3)}\tilde{B}_a^T\bar{P}_aD_a^{(3)}](h(x_i(k-\tau_{1,a})) \\
& -h(x_j(k-\tau_{1,a})))\}, \\
E(V_1)_{36} =& 2\sum_{1\le i\le j\le W}\{(\sum_{v=1}^{\tau_{2,a}}(x_i(k-v)-x_j(k-v)))^T \\
& \times[-WG_{ij}^{(3)}\tilde{C}_a^T\bar{P}_aD_a^{(3)}](\sum_{v=1}^{\tau_{2,a}}(o(x_i(k-v)) \\
& -o(x_j(k-v))))\}, \\
E(V_1)_{45} =& 2\sum_{1\le i\le j\le W}\{(f(x_i(k))-f(x_j(k)))^T[\tilde{A}_a^T\bar{P}_a\tilde{B}_a] \\
& \times(h(x_i(k-\tau_{1,a}))-h(x_j(k-\tau_{1,a})))\}, \\
E(V_1)_{46} =& 2\sum_{1\le i\le j\le W}\{(f(x_i(k))-f(x_j(k)))^T[\tilde{A}_a^T\bar{P}_a\tilde{C}_a] \\
& \times(\sum_{v=1}^{\tau_{2,a}}(o(x_i(k-v))-o(x_j(k-v))))\},. \\
E(V_2,V_3)_{11} =& ((1-\pi)(\bar{\tau}-\tau)+1)\sum_{1\le i\le j\le W}\{(x_i(k) \\
& -x_j(k))^TQ(x_i(k)-x_j(k))\}, \\
E(V_2,V_3)_{22} =& \sum_{1\le i\le j\le W}\{(x_i(k-\tau_{1,a})-x_j(k-\tau_{1,a}))^T \\
& \times[-Q](x_i(k-\tau_{1,a})-x_j(k-\tau_{1,a}))\}, \\
E(V_4,V_5)_{11} =& [\frac{1}{2}(1-\pi)(\bar{\tau}-\tau)(\bar{\tau}+\tau-1)+\tau_{2,a} \\
& +(1-\pi_{aa})(\bar{\tau}-\tau)]\sum_{1\le i\le j\le W}\{(x_i(k) \\
& -x_j(k))^TR(x_i(k)-x_j(k))\}, \\
E(V_4,V_5)_{33} =& -\sum_{v=1}^{\tau_{2,a}}x^T(k-v)(U\otimes R)x(k-v) \\
=& -\sum_{1\le i\le j\le W}\sum_{v=1}^{\tau_{2,a}}\{(x_i(k-v)-x_j(k-v))^T \\
& \times R(x_i(k-v)-x_j(k-v))\} \\
\le& -\frac{1}{\tau_{2,a}}\sum_{1\le i\le j\le W}\{(\sum_{v=1}^{\tau_{2,a}}(x_i(k-v)-x_j(k-v)))^T \\
& \times R(\sum_{v=1}^{\tau_{2,a}}(x_i(k-v)-x_j(k-v)))\} \\
=& \sum_{1\le i\le j\le W}\{(\sum_{v=1}^{\tau_{2,a}}(x_i(k-v)-x_j(k-v)))^T \\
& \times[-\frac{1}{\tau_{2,a}}R](\sum_{v=1}^{\tau_{2,a}}(x_i(k-v)-x_j(k-v)))\}
\end{aligned}
$$

So the expection of the whole system is translated into

$$
\begin{aligned}
& E[V(k+1,r(k+1)=b|r(k)=a)-V(k,a)] \\
& =\sum_{i=1}^{6}\sum_{j=1}^{6}E(V_1)_{ij}+\sum_{i=1}^{6}\sum_{j=1}^{6}E(V_2,V_3)_{ij} \\
& \quad +\sum_{i=1}^{6}\sum_{j=1}^{6}E(V_4,V_5)_{ij},
\end{aligned} \tag{17}
$$

From ***Assumption***, **lemma 4**, we can get the following inequalities (18)-(20), shown at the bottom of this page.

For convenience, substitute Eqs. (18)-(20) into system (17) and the expectation can be represented as

$$
\begin{aligned}
& E[V(k+1,r(k+1)=b|r(k)=a)-V(k,a)] \\
& =\sum_{1\le i\le j\le W}\{\zeta_{ij}^T(k,a)[\Phi_{ij}^{(1)}+\Theta^T(a)\bar{P}\Theta(a) \\
& \quad -WG_{ij}^{(1)}(D_a^{(1)^T}\bar{P}_a\Theta(a)+\Theta^T(a)\bar{P}D_a^{(1)}) \\
& \quad -WG_{ij}^{(2)}(D_a^{(2)^T}\bar{P}_a\Theta(a)+\Theta^T(a)\bar{P}D_a^{(2)}) \\
& \quad -WG_{ij}^{(3)}(D_a^{(3)^T}\bar{P}_a\Theta(a)+\Theta^T(a)\bar{P}D_a^{(3)})]\zeta_{ij}(k,a)\}
\end{aligned} \tag{21}
$$

$$
\begin{bmatrix} x_i(k)-x_j(k) \\ f(x_i(k))-f(x_j(k)) \end{bmatrix}^T \begin{bmatrix} \Psi\Re_1 & -\Psi\Re_2 \\ -\Psi\Re_2 & \Psi \end{bmatrix} \begin{bmatrix} x_i(k)-x_j(k) \\ f(x_i(k))-f(x_j(k)) \end{bmatrix} \le 0, \tag{18}
$$

$$
\begin{bmatrix} x_i(k-\tau_{1,a})-x_j(k-\tau_{1,a}) \\ h(x_i(k-\tau_{1,a}))-h(x_j(k-\tau_{1,a})) \end{bmatrix}^T \begin{bmatrix} \Xi\aleph_1 & -\Xi\aleph_2 \\ -\Xi\aleph_2 & \Xi \end{bmatrix} \begin{bmatrix} x_i(k-\tau_{1,a})-x_j(k-\tau_{1,a}) \\ h(x_i(k-\tau_{1,a}))-h(x_j(k-\tau_{1,a})) \end{bmatrix} \le 0, \tag{19}
$$

$$
\begin{bmatrix} \sum_{v=1}^{\tau_{2,a}}(x_i(k-v)-x_j(k-v)) \\ \sum_{v=1}^{\tau_{2,a}}(o(x_i(k-v))-o(x_j(k-v))) \end{bmatrix}^T \begin{bmatrix} \Omega\Im_1 & -\Omega\Im_2 \\ -\Omega\Im_2 & \Omega \end{bmatrix} \begin{bmatrix} \sum_{v=1}^{\tau_{2,a}}(x_i(k-v)-x_j(k-v)) \\ \sum_{v=1}^{\tau_{2,a}}(o(x_i(k-v))-o(x_j(k-v))) \end{bmatrix} \le 0, \tag{20}
$$

where

$$\zeta_{ij}(k,a) = \big[(x_i(k)-x_j(k))^T, (x_i(k-\tau_{1,a})-x_j(k-\tau_{1,a}))^T, (\sum_{v=1}^{\tau_{2,a}}(x_i(k-v)-x_j(k-v)))^T, (f(x_i(k))-f(x_j(k)))^T, (h(x_i(k-\tau_{1,a}))-h(x_j(k-\tau_{1,a})))^T, (\sum_{v=1}^{\tau_{2,a}}(o(x_i(k-v))-o(x_j(k-v))))^T\big],$$

and the matrix inequality is

$$\Phi_{ij}^{(1)} = \begin{bmatrix} \Pi_{11}^{(1)} & \Pi_{12}^{(1)} & \Pi_{13}^{(1)} & \Pi_{14}^{(1)} & 0 & 0 \\ * & \Pi_{22}^{(1)} & \Pi_{23}^{(1)} & 0 & \Pi_{25}^{(1)} & 0 \\ * & * & \Pi_{33}^{(1)} & 0 & 0 & \Pi_{36}^{(1)} \\ * & * & * & \Pi_{44}^{(1)} & 0 & 0 \\ * & * & * & * & \Pi_{55}^{(1)} & 0 \\ * & * & * & * & * & \Pi_{66}^{(1)} \end{bmatrix}, \quad (22)$$

where

$$\begin{aligned}
\Pi_{11}^{(1)} &= -WG_{ij}^{(1)}G_{ij}^{(1)}D_a^{(1)^T}\bar{P}_aD_a^{(1)} - P_a + \varpi Q + \sigma_a R - \Psi\Re_1,\\
\Pi_{22}^{(1)} &= -WG_{ij}^{(2)}G_{ij}^{(2)}D_a^{(2)^T}\bar{P}_aD_a^{(2)} - Q - \Xi\aleph_1,\\
\Pi_{33}^{(1)} &= -WG_{ij}^{(3)}G_{ij}^{(3)}D_a^{(3)^T}\bar{P}_aD_a^{(3)} - R/\tau_{2,a} - \Omega\Im_1,\\
\Pi_{44}^{(1)} &= -\Psi,\\
\Pi_{55}^{(1)} &= -\Xi,\\
\Pi_{66}^{(1)} &= -\Omega,\\
\Pi_{12}^{(1)} &= -WG_{ij}^{(1)}G_{ij}^{(2)}D_a^{(1)^T}\bar{P}_aD_a^{(2)},\\
\Pi_{13}^{(1)} &= -WG_{ij}^{(1)}G_{ij}^{(3)}D_a^{(1)^T}\bar{P}_aD_a^{(3)},\\
\Pi_{14}^{(1)} &= \Psi\Re_2,\\
\Pi_{23}^{(1)} &= -WG_{ij}^{(2)}G_{ij}^{(3)}D_a^{(2)^T}\bar{P}_aD_a^{(3)},\\
\Pi_{25}^{(1)} &= \Xi\aleph_2,\\
\Pi_{36}^{(1)} &= \Omega\Im_2.
\end{aligned}$$

By using the **lemma 1** and the definition of $G_{ij}^{(\iota)} = G_{ji}^{(\iota)} \geq 0, i \neq j$, following inequalities can be represented as

$$\begin{aligned}
&\sum_{1\leq i\leq j\leq W}\{-WG_{ij}^{(\iota)}\zeta_{ij}^T(k,a)(D_a^{(\iota)^T}\bar{P}_a\Theta(a) + \Theta^T(a)\bar{P}D_a^{(\iota)})\zeta_{ij}(k,a)\}\\
&\leq \sum_{1\leq i\leq j\leq W}\{WG_{ij}^{(\iota)}\zeta_{ij}^T(k,a)(D_a^{(\iota)^T}\bar{P}_aD_a^{(\iota)} + \Theta^T(a)\bar{P}\Theta(a))\zeta_{ij}(k,a)\}, \quad \iota = 1,2,3.
\end{aligned}$$

So system (21) is translated into

$$\begin{aligned}
&E[V(k+1, r(k+1) = b|r(k)=a) - V(k,a)]\\
&= \sum_{1\leq i\leq j\leq W}\{\zeta_{ij}^T(k,a)[\Phi_{ij}^{(2)} + (1+WG_{ij}^{(1)} + WG_{ij}^{(2)} + WG_{ij}^{(3)})\Theta^T(a)\bar{P}\Theta(a)]\zeta_{ij}(k,a)\}, \quad (23)
\end{aligned}$$

where the matrix inequality is

$$\Phi_{ij}^{(2)} = \begin{bmatrix} \Pi_{11}^{(2)} & \Pi_{12}^{(1)} & \Pi_{13}^{(1)} & \Pi_{14}^{(1)} & 0 & 0 \\ * & \Pi_{22}^{(2)} & \Pi_{23}^{(1)} & 0 & \Pi_{25}^{(1)} & 0 \\ * & * & \Pi_{33}^{(2)} & 0 & 0 & \Pi_{36}^{(1)} \\ * & * & * & \Pi_{44}^{(1)} & 0 & 0 \\ * & * & * & * & \Pi_{55}^{(1)} & 0 \\ * & * & * & * & * & \Pi_{66}^{(1)} \end{bmatrix} \quad (24)$$

where

$$\begin{aligned}
\Pi_{11}^{(2)} &= \Pi_{11}^{(1)} + WG_{ij}^{(1)}D_a^{(1)^T}\bar{P}_aD_a^{(1)},\\
\Pi_{22}^{(2)} &= \Pi_{22}^{(1)} + WG_{ij}^{(2)}D_a^{(2)^T}\bar{P}_aD_a^{(2)},\\
\Pi_{33}^{(2)} &= \Pi_{33}^{(1)} + WG_{ij}^{(3)}D_a^{(3)^T}\bar{P}_aD_a^{(3)}.
\end{aligned}$$

Because of **Lemma 2**, the following inequality in system (23)

$$\Phi_{ij}^{(2)} + (1+WG_{ij}^{(1)} + WG_{ij}^{(2)} + WG_{ij}^{(3)})\Theta^T(a)\bar{P}\Theta(a) < 0,$$

is equals to the matrix inequality

$$\Phi_{ij}^{(3)} = \begin{bmatrix} \Pi_{11}^{(2)} & \Pi_{12}^{(1)} & \Pi_{13}^{(1)} & \Pi_{14}^{(1)} & 0 & 0 & -\tilde{J}^T\bar{P}_a \\ * & \Pi_{22}^{(2)} & \Pi_{23}^{(1)} & 0 & \Pi_{25}^{(1)} & 0 & 0 \\ * & * & \Pi_{33}^{(2)} & 0 & 0 & \Pi_{36}^{(1)} & 0 \\ * & * & * & \Pi_{44}^{(1)} & 0 & 0 & \tilde{A}^T\bar{P}_a \\ * & * & * & * & \Pi_{55}^{(1)} & 0 & \tilde{B}^T\bar{P}_a \\ * & * & * & * & * & \Pi_{66}^{(1)} & \tilde{C}^T\bar{P}_a \\ * & * & * & * & * & * & \Pi_{77}^{(1)} \end{bmatrix} < 0, \quad (25)$$

where $\Pi_{77}^{(1)} = \Pi_{77}$. Consider that $\tilde{J}_m = J_m + \Delta J_m$, $\tilde{A}_m = A_m + \Delta A_m$, $\tilde{B}_m = B_m + \Delta B_m$ and $\tilde{C}_m = C_m + \Delta C_m$, so the uncertain part $\Delta\Phi_{ij}^{(3)}$ in $\Phi_{ij}^{(3)}$ can be defined as

$$\begin{aligned}
\Delta\Phi_{ij}^{(3)} &= \bar{P}(a)\Delta\Theta(a) + \Delta\Theta^T(a)\bar{P}^T(a)\\
&= \bar{P}(a)\Delta\Theta(a) + (\bar{P}(a)\Delta\Theta(a))^T\\
&= \bar{P}(a)M_a\Upsilon_aE_a + (\bar{P}(a)M_a\Upsilon_aE_a)^T\\
&\leq \frac{1}{\lambda}\bar{P}(a)M_aM_a^T\bar{P}^T(a) + \lambda E_a^TE \quad (26)
\end{aligned}$$

where

$$\begin{aligned}
\bar{P}(a) &= (0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; \bar{P}_a)^T,\\
\Delta\Theta(a) &= (-\Delta J_a \;\; 0 \;\; 0 \;\; \Delta A_a \;\; \Delta B_a \;\; \Delta C_a),\\
E_a &= (-E_a^J \;\; 0 \;\; 0 \;\; E_a^A \;\; E_a^B \;\; E_a^C).
\end{aligned}$$

And the matrix inequality (26) with uncertain parameters is satisfied when the following matrix inequality without uncertain parameters is satisfied.

$$\Phi_{ij}^{(4)} + \frac{1}{\lambda}\bar{P}(a)M_aM_a^T\bar{P}^T(a) < 0, \quad (27)$$

where

$$\Phi_{ij}^{(4)} = \begin{bmatrix} \Pi_{11}^{(3)} & \Pi_{12}^{(1)} & \Pi_{13}^{(1)} & \Pi_{14}^{(2)} & \Pi_{15}^{(1)} & \Pi_{16}^{(1)} & -J^T\bar{P}_a \\ * & \Pi_{22}^{(2)} & \Pi_{23}^{(1)} & 0 & \Pi_{25}^{(1)} & 0 & 0 \\ * & * & \Pi_{33}^{(2)} & 0 & 0 & \Pi_{36}^{(1)} & 0 \\ * & * & * & \Pi_{44}^{(2)} & \Pi_{45}^{(1)} & \Pi_{46}^{(1)} & A^T\bar{P}_a \\ * & * & * & * & \Pi_{55}^{(2)} & \Pi_{56}^{(1)} & B^T\bar{P}_a \\ * & * & * & * & * & \Pi_{66}^{(2)} & C^T\bar{P}_a \\ * & * & * & * & * & * & \Pi_{77}^{(1)} \end{bmatrix}, \quad (28)$$

and

$$\begin{aligned} &\Pi_{11}^{(3)} = \Pi_{11}^{(2)} + \lambda(E_a^J)^T E_a^J, \quad \Pi_{44}^{(2)} = \Pi_{44}^{(1)} + \lambda(E_a^A)^T E_a^A, \\ &\Pi_{55}^{(2)} = \Pi_{55}^{(1)} + \lambda(E_a^B)^T E_a^B, \quad \Pi_{66}^{(2)} = \Pi_{66}^{(1)} + \lambda(E_a^C)^T E_a^C, \\ &\Pi_{14}^{(2)} = \Pi_{14}^{(1)} - \lambda(E_a^J)^T E_a^A, \quad \Pi_{15}^{(1)} = -\lambda(E_a^J)^T E_a^B, \\ &\Pi_{16}^{(1)} = -\lambda(E_a^J)^T E_a^C, \quad \Pi_{45}^{(1)} = \lambda(E_a^A)^T E_a^B, \\ &\Pi_{46}^{(1)} = \lambda(E_a^A)^T E_a^C, \quad \Pi_{56}^{(1)} = \lambda(E_a^B)^T E_a^C. \end{aligned}$$

It should be noticed that, by using **Lemma 2**, the inequality (27) equals to matrix inequality (3). And the expectation (23) is translated into

$$\begin{aligned} &E[V(k+1, r(k+1) = b|r(k) = a) - V(k, a)] \\ &\quad = \sum_{1\le i\le j\le W} \{\zeta_{ij}^T(k, a)\Phi_{ij}\zeta_{ij}(k, a)\}, \end{aligned} \quad (29)$$

which implies that

$$\begin{aligned} &E[V(k+1, r(k+1) = b|r(k) = a) - V(k, a)] \\ &\quad \le \lambda_{\max} \sum_{1\le i\le j\le W} E[\|x_i(k) - x_j(k)\|^2], \end{aligned} \quad (30)$$

where $\lambda_{\max}$ is the maximum of all maximum eigenvalues in different modes and $\lambda_{\max} < 0$, For $n_0$ iterations,

$$\begin{aligned} &E[V(k+1, r(k+1) = b|r(k) = a) - V(0, r(0))] \\ &\quad \le \lambda_{\max} \sum_{k=0}^{n_0} \sum_{1\le i\le j\le W} E[\|x_i(k) - x_j(k)\|^2], \end{aligned} \quad (31)$$

which means

$$\sum_{k=0}^{n_0} \sum_{1\le i\le j\le W} E[\|x_i(k) - x_j(k)\|^2] \le -\frac{1}{\lambda_{\max}} E[V(0, r(0))], \quad (32)$$

and then we can get the final conclusion that

$$\lim_{k\to\infty} E[\|x_i(k) - x_j(k)\|^2] = 0. \quad (33)$$

Then the proof is completed. ■

Similarly, when the neural networks are in other modes, *Theorem 1* is still estabilished.

## IV. NUMERICAL SIMULATIONS

In this section, example with (without) unknown parameters are provided to demonstrate the robustness and effective of our method. Consider the proposed discrete-time coupled neural networks (1) without unknown parameters, matrices will be used in the simulation are shown as follows:

$$\begin{aligned} &J_a = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}, \quad A_a = \begin{bmatrix} 0.1 & 0.01 \\ -0.02 & 0.2 \end{bmatrix}, \\ &B_a = \begin{bmatrix} 0.2 & 0.03 \\ -0.01 & 0.3 \end{bmatrix}, \quad C_a = \begin{bmatrix} 0.1 & 0.01 \\ 0 & 0.1 \end{bmatrix}, \\ &D_a^{(1)} = \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad D_a^{(2)} = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.2 \end{bmatrix}, \\ &D_a^{(3)} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad J_b = \begin{bmatrix} 0.03 & 0 \\ 0 & 0.02 \end{bmatrix}, \\ &A_b = \begin{bmatrix} 0.1 & 0.05 \\ 0.02 & 0.3 \end{bmatrix}, \quad B_b = \begin{bmatrix} 0.1 & 0.02 \\ -0.01 & 0.4 \end{bmatrix}, \\ &C_b = \begin{bmatrix} 0.1 & -0.01 \\ 0.02 & 0.2 \end{bmatrix}, \quad D_b^{(1)} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0 \end{bmatrix}, \\ &D_b^{(2)} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad D_b^{(3)} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}, \\ &\Pi = \begin{bmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{bmatrix}. \\ &G_1 = G_2 = G_3 = \begin{bmatrix} -0.2 & 0.1 & 0.1 \\ 0.1 & -0.2 & 0.1 \\ 0.1 & 0.1 & -0.2 \end{bmatrix}. \end{aligned}$$

In order to get better performance, we set the time delays as $\tau_{11} = 2$, $\tau_{12} = 6$, $\tau_{21} = 2$ and $\tau_{22} = 1$ and from $\Pi$, it is easy to get that $\pi = 0.4$. Functions in the neural networks are defined as

$$\begin{aligned} &f_1(x) = h_1(x) = o_1(x) = -0.6tanh(x), \\ &f_2(x) = h_2(x) = o_2(x) = tanh(0.2x), \end{aligned}$$

so the corresponding matrices are

$$\begin{aligned} &\Re_1 = \aleph_1 = \Im_1 = diag(0,0), \\ &\Re_2 = \aleph_2 = \Im_2 = diag(-0.3,0.1). \end{aligned}$$

By using the LMI toolbox, we can solve the LMI (3) with the proposed matrices and parameters and the feasible results are shown as

$$\begin{aligned} &P_a = \begin{bmatrix} 4.7934 & -0.1525 \\ -0.1525 & 1.5103 \end{bmatrix}, \quad P_b = \begin{bmatrix} 4.6450 & -0.1745 \\ -0.1745 & 1.7335 \end{bmatrix}, \\ &Q = \begin{bmatrix} 0.4609 & -0.0116 \\ -0.0116 & 0.1883 \end{bmatrix}, \quad R = \begin{bmatrix} 0.3735 & -0.0149 \\ -0.0149 & 0.0889 \end{bmatrix}, \\ &\lambda = 31.8040. \\ &\Psi_1 = diag(2.9537, 0.9240), \quad \Xi_1 = diag(1.7681, 2.0811), \\ &\Omega_1 = diag(1.0740, 0.5777), \quad \Psi_2 = diag(2.0079, 4.3599), \\ &\Xi_2 = diag(1.9934, 1.7133), \quad \Omega_2 = diag(1.7939, 1.1778). \end{aligned}$$

Based on the theorem 1, the discrete-time coupled neural networks with mixed time delays will get the synchronization and it is proved by numerical simulations. Fig. 1 and Fig. 2

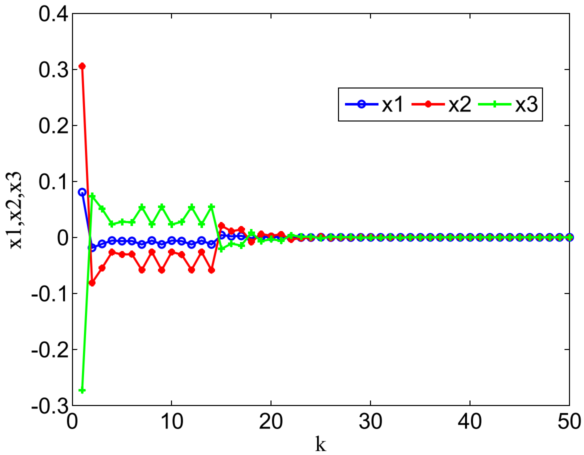


**FIGURE 1.** Synchronization state x in discrete-time coupled neural networks without unknown parameters.

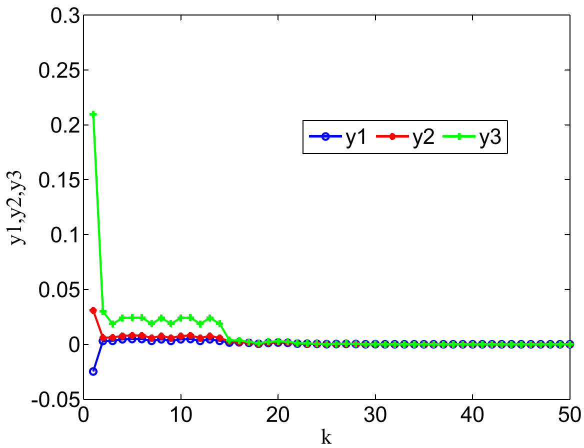


**FIGURE 2.** Synchronization state y in discrete-time coupled neural networks without unknown parameters.

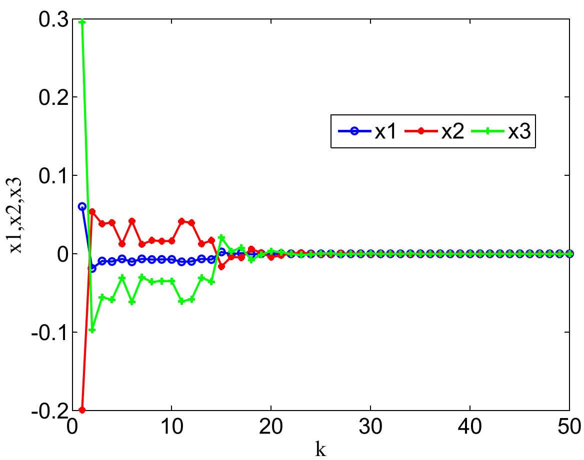


**FIGURE 3.** Synchronization state x in discrete-time coupled neural networks with unknown parameters.

shows the states of discrete-time coupled neural networks without unknown parameters. Similarly, Fig. 3 and Fig. 4 provides the states of discrete-time coupled neural networks with unknown parameters.

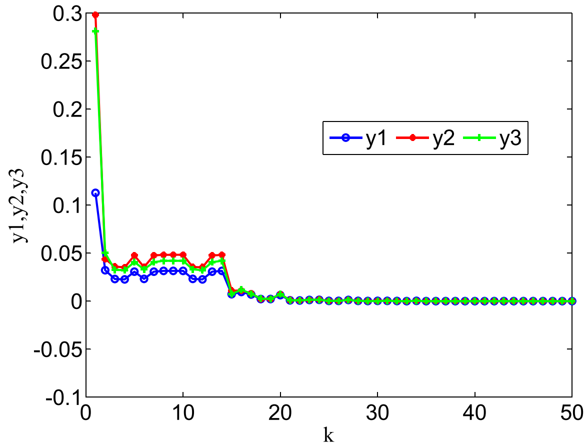


**FIGURE 4.** Synchronization state y in discrete-time coupled neural networks with unknown parameters.

## V. CONCLUSION

The synchronization problems in a new class of universal discrete-time coupled neural networks with inner mixed time delays and outer mixed time delays are studied in this paper. A novel discrete-time coupled markovian jumping neural networks with mixed time-delays is proposed. Based on the Lyapunov-Krasovskii functional method and Kronecker product, we complete the analysis of stability and get the sufficient conditions which can be easily solved by the Matlab LMI toolbox. In our study, we find that the synchronization process is related to the bounds of mixed time delays and unknown parameters problem can be solved by using the proposed method. In numerical simulations, feasible solutions of sufficient conditions are derived and synchronization results with(without) unknown parameters are achieved to demonstrate the effectiveness and robustness of our method. In the future, authors will study the continuous neural network with Markov jumping chain and extend the discrete neural network model by considering more complexities. In addition, more synchronization patterns for the proposed neural network model will be considered to enrich the study of the proposed neural network system.

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